CHAPTER I

ORTHODOX SEMIGROUPS AND QUASI-INVERSE SEMIGROUPS

In this chapter, we study orthodox semigroups and quasi-inverse semigroups in general. Many important properties satisfied by both or by one but not the other are introduced. Various examples are also given.

Let S be a semigroup, $a \in S$, we denote V(a) the set of all inverses of a in S; that is,

$$V(a) = \left\{ x \in S \,\middle|\, a = axa \text{ and } x = xax \right\}.$$

The first theorem gives equivalent definitions of a regular semigroup to be orthodox.

1.1 <u>Theorem</u> [7, Lemma 1.3]. For any regular semigroup S, the following three conditions are equivalent :

(i) S is orthodox.

(ii) For any elements a, b in S, $V(b)V(a) \subseteq V(ab)$.

(iii) Any inverse of an idempotent in S is an idempotent; that is, $V(e) \subseteq E(S)$ for all $e \in E(S)$.

The next proposition shows that any regular subsemigroup of an orthodox semigroup is orthodox.

1.2 <u>Proposition</u>. A regular subsemigroup of an orthodox semigroup is orthodox.

Proof : Let A be a regular subsemigroup of an orthodox semigroup S. Let $e \in E(A)$ and x be an inverse of e in A. Then $e \in E(S)$ and x is an inverse of e in S. Since x is an inverse of an idempotent e in S which is an orthodox semigroup, by Theorem 1.1, $x \in E(S)$, so $x \in E(A)$. Hence A is orthodox. #

Every ideal of a regular semigroup is clearly regular. Hence by Proposition 1.2, the following corollary follows :

1.3 Corollary. An ideal of an orthodox semigroup is orthodox.

Let ρ be a congruence on a semigroup S. It is clear that $\{e\rho \mid e \in E(S)\} \subseteq E(S/\rho)$. The two sets are equal if S is regular. To prove this, let S be a regular semigroup and ρ be a congruence on S. Let $a \in S$ such that $a\rho \in E(S/\rho)$. Let x be an inverse of a^2 in S. Then

$$(axa)(axa) = axa^2xa = axa,$$

$$(axa)\rho = a\rho x\rho a\rho = a^2\rho x\rho a^2\rho = a^2\rho = a\rho,$$

so $axa \in E(S)$ and $axa \in a\rho$.

A homomorphic image of a regular semigroup is clearly regular. Because of the corresponding between congruences and homomorphisms, it follows that if a semigroup T is a homomorphic image of a regular semigroup S by a homomorphism ψ, then T is regular and

$$E(T) = \{e\psi \mid e \in E(S)\},$$

which implies that E(T) is a subsemigroup of T if E(S) is a subsemigroup of S. Hence, the following proposition follows :

1.4 Proposition. A homomorphic image of an orthodox semigroup is orthodox.

Let I be an ideal of a semigroup S. If S is orthodox, then, by Corollary 1.3 and Proposition 1.4, I and the Rees quotient semigroup S/I are orthodox. It has been proved by Hall in [4] that this converse is also true.

The next proposition is a characterization of an orthodox semigroup in terms of its principal ideals. The following lemma is required first :

1.5 Lemma. Let S be a semigroup. Then S is regular if and only if every principal ideal of S is regular.

Proof : Assume that every principal ideal of a semigroup S is regular. Let $a \in S$. Then S^1aS^1 is regular. Since $a \in S^1aS^1$, there exists $x \in S^1aS^1$ such that $a = axa$ and $x = xax$. If $x = 1$, then $a = 1 \in S$, so a is a regular element of S. If $x \neq 1$, then $x \in S$ and hence a is regular in S. This proves S is regular.

The converse is obvious. #

1.6 Proposition. Let S be a semigroup. Then S is orthodox if and only if every principal ideal of S is orthodox.

Proof : If S is orthodox, then by Corollary 1.3, every principal ideal of S is orthodox.

Conversely, assume every principal ideal of S is orthodox.

By Lemma 1.5, S is regular. Let $e \in E(S)$ and $x \in V(e)$. Then $x = xex \in SeS$ which is an orthodox semigroup by assumption. Therefore $x \in E(SeS)$ by Theorem 1.1, so $x \in E(S)$. This proves $V(e) \subseteq E(S)$ for any $e \in E(S)$. Hence S is orthodox by Theorem 1.1. #

A regular semigroup S is called a right-inverse semigroup if every principal left ideal of S, has a unique idempotent generator. It has been shown in [3] that a regular semigroup S is right-inverse iff $efe = fe$ for all $e, f \in E(S)$.

If S is a right-inverse semigroup, then for any $e, f \in E(S)$, $(ef)^2 = (efe)f = fef = ef$, which implies E(S) is a subsemigroup of S. Hence, every right-inverse semigroup is an orthodox semigroup.

An orthodox semigroup need not be right-inverse. A nontrivial left zero semigroup is an orthodox semigroup but not a right-inverse semigroup. (Recall that a semigroup S is called a left zero semigroup if for any $a, b \in S$, $ab = a$.)

1.7 Proposition. (1) A regular subsemigroup of a right-inverse semigroup is right-inverse. Hence, an ideal of a right-inverse semigroup is right-inverse.

(2) A homomorphic image of a right-inverse semigroup is right-inverse.

Proof : (1) Let A be a regular subsemigroup of a right-inverse semigroup S. Let $e, f \in E(A)$. Then $e, f \in E(S)$. Because S is right-inverse and $e, f \in E(S)$, $efe = fe$. Hence A is right-inverse.

(2) Let ψ be a homomorphism from a right-inverse semigroup S onto a semigroup T. Then T is regular. Because S is regular, $E(T) = \{e\psi \mid e \in E(S)\}$. Let e, f $\in$ E(S). Then

$$(e\psi)(f\psi)(e\psi) = (efe)\psi = (fe)\psi = f\psi e\psi.$$

Therefore, for any e', f' $\in$ E(T), e'f'e' = f'e'.

Hence, T is right-inverse. #

If I is an ideal of a semigroup S, we denote ρ_I, the Rees congruence on S induced by the ideal I; that is,

$$a\rho_I b \text{ if and only if either } a, b \in I \text{ or } a = b.$$

Let I be an ideal of a semigroup S. It follows from Proposition 1.7 that if S is right-inverse, then I and the Rees quotient semigroup S/I are also right-inverse. The next theorem shows that this converse is true, too.

1.8 Theorem. Let I be an ideal of a semigroup S. Then S is right-inverse if and only if I and the Rees quotient semigroup S/I are right-inverse.

Proof : Assume I and S/I are right-inverse. Then I and S/I are orthodox. From [4, Theorem 4], S is orthodox and therefore S is regular and E(S) is a subsemigroup of S. Let e, f $\in$ E(S). Because S/I is right-inverse, $(efe)\rho_I = (e\rho_I)(f\rho_I)(e\rho_I) = (f\rho_I)(e\rho_I) = (fe)\rho_I$. It follows that either efe = fe or efe, fe $\in$ I. If efe, fe $\in$ I, then

$$\begin{aligned} efe &= efefe && \text{(since } fe \in E(S)) \\ &= fe(efe)fe && \text{(since } efe, fe \in E(I) \text{ and } I \text{ is right-inverse)} \end{aligned}$$

$$= fefefe$$

$$= fe.$$

In any case, we have that efe = fe. Therefore S is right-inverse. #

The next proposition is a characterization of a right-inverse semigroup in terms of its principal ideals.

1.9 Proposition. A semigroup S is right-inverse if and only if every principal ideal of S is right-inverse.

Proof : If S is right-inverse, then by Proposition 1.7(1), every principal ideal of S is right-inverse.

Conversely, assume every principal ideal of S is right-inverse. Then every principal ideal of S is orthodox. By Proposition 1.6, S is orthodox, so S is regular. Let e, f $\in$ E(S). Then ef, fe $\in$ SfS. Since S is orthodox, ef, fe $\in$ E(S), and hence ef, fe $\in$ E(SfS) $\subseteq$ E(S). Thus efe = effe = feeffe = fe because SfS is right-inverse. This proves S is right-inverse. #

A regular semigroup S is called a generalized inverse semigroup if efgh = egfh for all e, f, g, h $\in$ E(S). If S is a generalized inverse semigroup, then for any e, f $\in$ E(S),

$$(ef)^2 = efef = eeff = ef.$$

If S is a generalized inverse semigroup with identity 1, then for any e, f $\in$ E(S), ef = 1(ef)1 = 1(fe)1 = fe. A regular semigroup S is an inverse semigroup if and only if ef = fe for all e, f $\in$ E(S) [1, Theorem 1.17]. Hence, we have

1.10 Proposition. (1) Every generalized inverse semigroup is orthodox.

(2) Every generalized inverse semigroup with identity is an inverse semigroup.

The following example shows that orthodox semigroups are a generalization of generalized inverse semigroups :

Example. Let $S = \left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\1&0\end{pmatrix}, \begin{pmatrix}1&0\\1&-1\end{pmatrix}\right\}$.

Then S is a semigroup with matrix multiplication and

$$E(S) = \left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\1&0\end{pmatrix}\right\}.$$

We can easily check that S is regular, E(S) is a subsemigroup of S. Hence S is orthodox. Since $\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&0\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix} \neq \begin{pmatrix}1&0\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}$, S is not an inverse semigroup. But S has $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ as its identity. Then by Proposition 1.10(2), S is not a generalized inverse semigroup. #

A nontrivial left zero semigroup is generalized inverse but not right-inverse. The following example shows that there is a right-inverse semigroup but not a generalized inverse semigroup :

Example. Let $S = \left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}1&1\\0&0\end{pmatrix}, \begin{pmatrix}1&1\\0&-1\end{pmatrix}\right\}$.

Under usual matrix multiplication, S is a regular semigroup having $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ as its identity, and $E(S) = \left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}1&1\\0&0\end{pmatrix}\right\}$.

Moreover, $\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix} = \begin{pmatrix}1&1\\0&0\end{pmatrix} = \begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix}$

$\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}$.

Therefore S is a right-inverse semigroup. Because

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

S is not a generalized inverse semigroup. #

It can be easily seen that a regular subsemigroup of a generalized inverse semigroup is generalized inverse and a homomorphic image of a generalized inverse semigroup is generalized inverse. Then, we have

1.11 <u>Proposition</u>. Let I be an ideal of a semigroup S. If S is generalized inverse, then I and the Rees quotient semigroup S/I are generalized inverse.

The converse of Proposition 1.11 is not true in general as shown in the following example :

<u>Example</u>. Let $S = \{I, E_1, E_2, E_3\}$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $E_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $E_3 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$. Then under the usual matrix multiplication; we have the following table :

$\cdot$	I	E_1	E_2	E_3
I	I	E_1	E_2	E_3
E_1	E_1	E_1	E_1	E_1
E_2	E_2	E_2	E_2	E_2
E_3	E_3	E_2	E_1	I

Thus S is a regular semigroup and $E(S) = \{I, E_1, E_2\}$. Because $IE_1E_2I = E_1 \neq E_2 = IE_2E_1I$, S is not generalized inverse. Let $A = \{E_1, E_2\}$. From the table, A is an ideal of S, so S/A is regular.

Because $E(A) = A$ and $E_2E_2E_1E_1 = E_2 = E_2E_1E_2E_1$, $E_2E_2E_1E_2 = E_2 = E_2E_1E_2E_2$, $E_1E_2E_1E_1 = E_1 = E_1E_1E_2E_1$ and $E_1E_2E_1E_2 = E_1 = E_1E_1E_2E_2$, it follows that A is generalized inverse. Because $S/A = \{E_1\rho_A, I\rho_A, E_3\rho_A\}$, from the table, $E(S/A) = \{E_1\rho_A, I\rho_A\}$. It is clear that S/A is generalized inverse. #

Arbitrary intersection of ideals of a semigroup S if nonempty is an ideal of S. Arbitrary intersection of congruences on a semigroup S is a congruence on S.

Let A and B be ideals of a semigroup S. It is easily seen that if $A \subsetneq B$, then $\rho_A \subsetneq \rho_B$, and hence S/B is a homomorphic image of S/A by the homomorphism ψ defined by $(a\rho_A)\psi = a\rho_B$.

Howie and Lallement has shown the existence of a minimum orthodox congruence on a regular semigroup in [5].

The intersection of all ideals of a semigroup S if nonempty is called the <u>kernel</u> of S. A nontrivial zero semigroup is a semigroup with kernel but it is not regular. (Recall that a semigroup S with zero 0 is called a <u>zero semigroup</u> if $xy = 0$ for all $x, y \in S$.) If we let S be the set of all negative integers and define an operation * on S by $m*n = \min\{m, n\}$, then S is a regular semigroup without kernel.

Let $\{\rho_\alpha\}_{\alpha\in\Lambda}$ be a collection of congruences on a semigroup S. Then $\bigcap_{\alpha\in\Lambda}\rho_\alpha$ is a congruence on S. For each $a \in S$, $a(\bigcap_{\alpha\in\Lambda}\rho_\alpha) = \bigcap_{\alpha\in\Lambda}a\rho_\alpha$. To show this, let $x \in a(\bigcap_{\alpha\in\Lambda}\rho_\alpha)$. Then $(x, a) \in \bigcap_{\alpha\in\Lambda}\rho_\alpha$, so $(x, a) \in \rho_\alpha$ for each $\alpha \in \Lambda$. Hence $x \in a\rho_\alpha$ for all $\alpha \in \Lambda$, so $x \in \bigcap_{\alpha\in\Lambda}a\rho_\alpha$

Conversely, let $y \in \bigcap_{\alpha\in\Lambda} a\rho_\alpha$. Then $(y, a) \in \rho_\alpha$ for each $\alpha \in \Lambda$, so $(y, a) \in \bigcap_{\alpha\in\Lambda}\rho_\alpha$. Hence $y \in a(\bigcap_{\alpha\in\Lambda}\rho_\alpha)$.

The next theorem shows that any semigroup S with kernel has the minimum orthodox Rees congruence and it is the intersection of all of its orthodox Rees congruences, and also, the intersection of all right-inverse Rees congruences on S is the minimum right-inverse Rees congruence on S. The three following lemmas are required to prove the theorem :

1.12 Lemma. Let S be a semigroup and $\{A_\alpha\}_{\alpha\in\Lambda}$ be a collection of ideals of S such that $\bigcap_{\alpha\in\Lambda} A_\alpha \neq \phi$. Then $\bigcap_{\alpha\in\Lambda}\rho_{A_\alpha} = \rho_{\bigcap_{\alpha\in\Lambda} A_\alpha}$. Hence, for each $a \in S$, $a\rho_{\bigcap_{\alpha\in\Lambda} A_\alpha} = \bigcap_{\alpha\in\Lambda}(a\rho_{A_\alpha})$.

Proof : Let $A = \bigcap_{\alpha\in\Lambda} A_\alpha$. Let $(a, b) \in \bigcap_{\alpha\in\Lambda}\rho_{A_\alpha}$. Then $(a, b) \in \rho_{A_\alpha}$ for all $\alpha \in \Lambda$.

Case $a \in A$. Then $a \in A_\alpha$ for all $\alpha \in \Lambda$, so $b \in A_\alpha$ for all $\alpha \in \Lambda$ since $(a, b) \in \rho_{A_\alpha}$ for all $\alpha \in \Lambda$. Therefore $(a, b) \in \rho_A$.

Case $a \notin A_\beta$ for some $\beta \in \Lambda$. Then $a\rho_{A_\beta} = \{a\}$. Since $(a, b) \in \rho_{A_\beta}$, $b = a$. Hence $a\rho_A = b\rho_A$.

Therefore $\bigcap_{\alpha\in\Lambda}\rho_{A_\alpha} \subseteq \rho_A$.

For each $\beta \in \Lambda$, $A \subseteq A_\beta$, so $\rho_A \subseteq \rho_{A_\beta}$. Hence $\rho_A \subseteq \bigcap_{\alpha\in\Lambda}\rho_{A_\alpha}$.

Therefore $\bigcap_{\alpha\in\Lambda}\rho_{A_\alpha} = \rho_{\bigcap_{\alpha\in\Lambda} A_\alpha}$, as required. #

1.13 Lemma. Let S be a semigroup and $\{A_\alpha\}_{\alpha\in\Lambda}$ be a collection of ideals of S such that $\bigcap_{\alpha\in\Lambda} A_\alpha \neq \phi$. If for each $\alpha \in \Lambda$, ρ_{A_α} is a regular Rees congruence on S, then $\bigcap_{\alpha\in\Lambda}\rho_{A_\alpha}$ is also a regular Rees congruence on S.

Proof : Let $A = \bigcap_{\alpha\in\Lambda} A_\alpha$. By Lemma 1.12, $\bigcap_{\alpha\in\Lambda}\rho_{A_\alpha} = \rho_{\bigcap_{\alpha\in\Lambda} A_\alpha} = \rho_A$. Let $a \in S$. Then either $a \in A$ or $a \notin A_\beta$ for some $\beta \in \Lambda$. If $a \in A$, then $a\rho_A$ is the zero element of S/A and hence it is regular. Assume $a \notin A_\beta$ for some $\beta \in \Lambda$. Because $a\rho_{A_\beta} \in S/A_\beta$ which is regular, there is $x \in S$ such that

$$a\rho_{A_\beta} = a\rho_{A_\beta} x\rho_{A_\beta} a\rho_{A_\beta} = (axa)\rho_{A_\beta}.$$

But $a \notin A_\beta$. Then $a = axa$ and hence $a\rho_A = a\rho_A x\rho_A a\rho_A$. This shows that S/A is regular. Therefore $\bigcap_{\alpha\in\Lambda}\rho_{A_\alpha} = \rho_A$ is a regular Rees congruence on S. #

1.14 Lemma. Let S be a semigroup and $\{A_\alpha\}_{\alpha\in\Lambda}$ be a collection of ideals of S such that $\bigcap_{\alpha\in\Lambda} A_\alpha \neq \phi$. If ρ_{A_α} is an orthodox Rees congruence on S for all $\alpha \in \Lambda$, then $\bigcap_{\alpha\in\Lambda}\rho_{A_\alpha}$ is also an orthodox Rees congruence on S. Moreover, if ρ_{A_α} is a right-inverse Rees congruence on S for all $\alpha \in \Lambda$, then $\bigcap_{\alpha\in\Lambda}\rho_{A_\alpha}$ is also a right-inverse Rees congruence on S.

Proof : Assume S/A_α is orthodox for each $\alpha \in \Lambda$. Then S/A_α is regular for each $\alpha \in \Lambda$. Let $A = \bigcap_{\alpha\in\Lambda} A_\alpha$. From Lemma 1.12 and Lemma 1.13, S/ρ_A is regular. Let $a\rho_A \in E(S/A)$ and $x\rho_A \in V(a\rho_A)$. Then

$$x\rho_A = x\rho_A a\rho_A x\rho_A = (xax)\rho_A.$$

If $a \in A$, then $xax \in A$ and $x \in A$ which imply $a\rho_A = x\rho_A$ and it is the zero of S/A. Assume $a \notin A$. Then $a \notin A_\beta$ for some $\beta \in \Lambda$. Because $a\rho_A = (axa)\rho_A$, $x\rho_A = (xax)\rho_A$ and $A \subseteq A_\beta$. It follows that $a\rho_{A_\beta} = (axa)\rho_{A_\beta}$ and $x\rho_{A_\beta} = (xax)\rho_{A_\beta}$ which imply $x\rho_{A_\beta} \in V(a\rho_{A_\beta})$ in S/A_β. Since $a \notin A$ and $a\rho_A \in E(S/A)$, $a^2\rho_A = a\rho_A = \{a\}$, and so $a^2 = a$.

Hence $a\rho_{A_\beta} \in E(S/A_\beta)$. But S/A_β is orthodox. Then $x\rho_{A_\beta} \in E(S/A_\beta)$. Because $x \notin A_\beta$, $\{x\} = x\rho_{A_\beta} = x^2\rho_{A_\beta} = \{x^2\}$ and then $x = x^2$ which implies $x\rho_A \in E(S/A)$. This shows that ρ_A is an orthodox Rees congruence on S.

Next, assume that ρ_{A_α} is a right-inverse Rees congruence on S for all $\alpha \in \Lambda$. Let $A = \bigcap_{\alpha\in\Lambda} A_\alpha$. To show S/A is right-inverse, let $a\rho_A, b\rho_A \in E(S/A)$. Then by Lemma 1.12, $a\rho_A = \bigcap_{\alpha\in\Lambda} a\rho_{A_\alpha}$ and $b\rho_A = \bigcap_{\alpha\in\Lambda} b\rho_{A_\alpha}$. Because $(a^2, a) \in \rho_A$, $(a^2, a) \in \rho_{A_\alpha}$ for all $\alpha \in \Lambda$. Similarly, $(b^2, b) \in \rho_{A_\alpha}$ for all $\alpha \in \Lambda$. Since for all $\alpha \in \Lambda$, S/A_α is right-inverse and $a\rho_{A_\alpha}, b\rho_{A_\alpha} \in E(S/A_\alpha)$, $(aba)\rho_{A_\alpha} = a\rho_{A_\alpha} b\rho_{A_\alpha} a\rho_{A_\alpha} = b\rho_{A_\alpha} a\rho_{A_\alpha} = (ba)\rho_{A_\alpha}$ for all $\alpha \in \Lambda$, so $a\rho_A b\rho_A a\rho_A = (aba)\rho_A = \bigcap_{\alpha\in\Lambda}(aba)\rho_{A_\alpha} = \bigcap_{\alpha\in\Lambda}(ba)\rho_{A_\alpha} = (ba)\rho_A = b\rho_A a\rho_A$. Because ρ_{A_α} is a right-inverse Rees congruence on S for each $\alpha \in \Lambda$, ρ_{A_α} is a regular Rees congruence on S for all $\alpha \in \Lambda$. Then, by Lemma 1.12 and Lemma 1.13, S/A is regular. Hence S/A is right-inverse. This proves ρ_A is a right-inverse Rees congruence on S. #

Let S be a semigroup with kernel. Let $\mathcal{C}$ be the collection of all ideals A of S such that the Rees congruences ρ_A are orthodox congruences on S, and let $\overline{\mathcal{C}}$ be the collection of all ideals A of S such that the Rees congruences ρ_A are right-inverse congruences on S. Then S belongs to $\mathcal{C}$ and $\overline{\mathcal{C}}$, so $\mathcal{C} \neq \phi$ and $\overline{\mathcal{C}} \neq \phi$. Because $\bigcap_{A\in\mathcal{C}} A \neq \phi$ and $\bigcap_{A\in\overline{\mathcal{C}}} A \neq \phi$, by Lemma 1.12 and Lemma 1.14, the following theorem follows directly :

1.15 Theorem. Any semigroup with kernel has the minimum orthodox Rees congruence and it is the intersection of all of its orthodox Rees congruences.

Moreover, any semigroup with kernel has the minimum right-inverse Rees congruence and it is the intersection of all of its right-inverse Rees congruences.

Also, the minimum generalized inverse Rees congruence on a semigroup with kernel always exists.

1.16 Proposition. Let S be a semigroup with kernel. Then the intersection of all of its generalized inverse Rees congruences on S is the minimum generalized inverse Rees congruence on S.

Proof : Let Λ be an index set such that $\{A_\alpha \mid \alpha \in \Lambda\}$ is the set of all ideals of S which give generalized inverse Rees congruences on S. Then S is an element of $\{A_\alpha \mid \alpha \in \Lambda\}$. Let $A = \bigcap_{\alpha\in\Lambda} A_\alpha$. Then A is an ideal of S because $A \neq \phi$. By Lemma 1.12 and Lemma 1.13, ρ_A is a regular congruence on S. Next, let $a\rho_A, b\rho_A, c\rho_A, d\rho_A \in E(S/A)$. Then by Lemma 1.12, $a\rho_A = \bigcap_{\alpha\in\Lambda} a\rho_{A_\alpha}$, $b\rho_A = \bigcap_{\alpha\in\Lambda} b\rho_{A_\alpha}$, $c\rho_A = \bigcap_{\alpha\in\Lambda} c\rho_{A_\alpha}$ and $d\rho_A = \bigcap_{\alpha\in\Lambda} d\rho_{A_\alpha}$. Because $(a^2, a), (b^2, b), (c^2, c)$ and $(d^2, d) \in \rho_A$, it follows that $(a^2, a), (b^2, b), (c^2, c)$ and $(d^2, d) \in \rho_{A_\alpha}$ for all $\alpha \in \Lambda$. Since for each $\alpha \in \Lambda$, S/A_α is generalized inverse and $a\rho_{A_\alpha}$, $b\rho_{A_\alpha}$, $c\rho_{A_\alpha}$ and $d\rho_{A_\alpha} \in E(S/A_\alpha)$, $(abcd)\rho_{A_\alpha} = a\rho_{A_\alpha} b\rho_{A_\alpha} c\rho_{A_\alpha} d\rho_{A_\alpha} = a\rho_{A_\alpha} c\rho_{A_\alpha} b\rho_{A_\alpha} d\rho_{A_\alpha} = (acbd)\rho_{A_\alpha}$ for all $\alpha \in \Lambda$, so $a\rho_A\, b\rho_A\, c\rho_A\, d\rho_A = (abcd)\rho_A = \bigcap_{\alpha\in\Lambda}(abcd)\rho_{A_\alpha} = \bigcap_{\alpha\in\Lambda}(acbd)\rho_{A_\alpha} = (acbd)\rho_A = a\rho_A c\rho_A b\rho_A d\rho_A$. Then S/A is generalized inverse. Therefore, ρ_A is a generalized inverse Rees congruence on S. Hence ρ_A is the minimum generalized inverse Rees congruence on S because $\rho_A = \bigcap_{\alpha\in\Lambda}\rho_{A_\alpha}$, so the theorem is completely proved. #

The kernel of any semigroup with zero always exists. Then the following corollary follows directly from Theorem 1.15 and Proposition 1.16.

1.17 Corollary. Any semigroup with zero has the minimum orthodox Rees congruence, the minimum right-inverse Rees congruence and the minimum generalized inverse Rees congruence.

We note that in any semigroup S, if the minimum orthodox Rees congruence, the minimum right-inverse Rees congruence and the minimum generalized inverse Rees congruence of S exist , then the minimum orthodox Rees congruence on S is contained in the intersection of the minimum right-inverse Rees congruence on S and the minimum generalized inverse Rees congruence on S.

Let A and B be ideals of a semigroup S such that $A \subseteq B$. Then S/B is a homomorphic image of S/A. Then, if ρ_A is an orthodox Rees congruence on S, then ρ_B is also an orthodox Rees congruence on S [Proposition 1.4], and if ρ_A is a right-inverse Rees congruence on S, then so is ρ_B [proposition 1.7(2)]. Moreover, if ρ_A is a generalized inverse Rees congruence on S, then ρ_B is also a generalized inverse Rees congruence on S. Hence, the following proposition follows directly :

1.18 Proposition. Let S be a semigroup with kernel K. Then the following hold :

(1) If ρ_K is an orthodox Rees congruence on S, then ρ_A is an orthodox Rees congruence on S for each ideal A of S and ρ_K is the minimum orthodox Rees congruence on S.

(2) If ρ_K is a right-inverse Rees congruence on S, then for each ideal A of S, ρ_A is a right-inverse Rees congruence on S and ρ_K is the minimum right-inverse Rees congruence on S.

(3) If ρ_K is a generalized inverse Rees congruence on S, then ρ_A is also a generalized inverse Rees congruence on S for all ideal A of S and ρ_K is the minimum generalized inverse Rees congruence on S.

A semigroup S is called quasi-inverse if for each $a \in S$, there exists an inverse subsemigroup of S containing a. Then a semigroup S is quasi-inverse if and only if it is a union of inverse subsemigroups of S. It clearly follows that every quasi-inverse semigroup is a regular semigroup.

An orthodox semigroup and a quasi-inverse semigroup are both regular and they are at the same time a generalization of the inverse semigroups and of the bands. However, they are not quite related. The following example shows that there exists an orthodox semigroup but not a quasi-inverse semigroup and vice versa :

Example. Let $X = \{1, 2, 3\}$. For $a, b, c \in X$, the notation $\alpha = (a, b, c) \in \mathcal{T}_X$ means the map on X with $1\alpha = a$, $2\alpha = b$ and $3\alpha = c$.

Schein has shown in [8] that $\mathcal{T}_X$ is quasi-inverse. Because (1 2 1), (1 3 3) $\in E(\mathcal{T}_X)$ and (1 2 1) (1 3 3) = (1 3 1) $\notin E(\mathcal{T}_X)$, it follows that $E(\mathcal{T}_X)$ is not a subsemigroup of $_X$ and hence $\mathcal{T}_X$ is not an orthodox semigroup.

Let $A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9\}$ where α_1 = (1 2 2), α_2 = (1 3 3), α_3 = (1 1 1), α_4 = (2 2 2), α_5 = (3 3 3), α_6 = (3 2 2), α_7 = (1 2 1), α_8 = (3 2 3) and α_9 = (1 3 1). Then, under the composition of maps, we have the following table :

$\circ$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9
α_1	α_1	α_2	α_3	α_4	α_5	α_6	α_1	α_6	α_2
α_2	α_1	α_2	α_3	α_4	α_5	α_6	α_3	α_5	α_3
α_3	α_3	α_3	α_3	α_4	α_5	α_5	α_3	α_5	α_3
α_4	α_4	α_5	α_3	α_4	α_5	α_4	α_4	α_4	α_5
α_5	α_4	α_5	α_3	α_4	α_5	α_4	α_3	α_5	α_3
α_6	α_4	α_5	α_3	α_4	α_5	α_4	α_1	α_6	α_2
α_7	α_7	α_9	α_3	α_4	α_5	α_8	α_7	α_8	α_9
α_8	α_4	α_5	α_3	α_4	α_5	α_4	α_7	α_8	α_9
α_9	α_7	α_9	α_3	α_4	α_5	α_8	α_3	α_5	α_3

Then from the table, A is a regular subsemigroup of $\mathcal{T}_X$. Let S = $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \quad \alpha_8, \alpha_9\}$. By checking directly from the table, we have that S is a regular subsemigroup of $\mathcal{T}_X$ and E(S) = $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_8\}$ forms a subsemigroup of S. Then S is orthodox.

Suppose S is quasi-inverse. Because $\alpha_9 \in S$, there exists an inverse subsemigroup T of S such that $\alpha_9 \in T$. From the table, α_6 is the only inverse of α_9 in S, so $\alpha_6 \in T$. Using the property of being subsemigroup of T in S, we have T = S, so S is an inverse semigroup which is a contradiction because $\alpha_3, \alpha_8 \in E(S)$ but $\alpha_3\alpha_8 = \alpha_5 \neq \alpha_3 = \alpha_8\alpha_3$. Therefore S is an orthodox semigroup which is not quasi-inverse. #

The previous example also shows that a regular subsemigroup of a quasi-inverse semigroup need not be quasi-inverse. In that example, A is a regular subsemigroup of $\mathcal{T}_X$. Suppose A is quasi-inverse. Because $\alpha_9 \in A$, there is an inverse subsemigroup T of $\mathcal{T}_X$ such that $\alpha_9 \in T$. Because α_6 and α_1 are the only inverses of α_9 in A, either α_6 or α_1 must belong to T. If $\alpha_6 \in T$, then $\{\alpha_9, \alpha_6, \alpha_8, \alpha_2, \alpha_3, \alpha_4, \alpha_5\} \subseteq T$ because T is a subsemigroup of $\mathcal{T}_X$, so T is not an inverse subsemigroup of $\mathcal{T}_X$ because $\alpha_3, \alpha_8 \in E(T)$ but $\alpha_3 \alpha_8 = \alpha_5 \neq \alpha_3 = \alpha_8 \alpha_3$. If $\alpha_1 \in T$, then $\{\alpha_9, \alpha_1, \alpha_7, \alpha_3, \alpha_2\} \subseteq T$, so T is not an inverse subsemigroup of $\mathcal{T}_X$ because $\alpha_7, \alpha_1 \in E(T)$ but $\alpha_7 \alpha_1 = \alpha_7 \neq \alpha_1 = \alpha_1 \alpha_7$. Hence A is not a quasi-inverse subsemigroup of $\mathcal{T}_X$.

A regular subsemigroup of an inverse semigroup S is clearly inverse.

We show in the next proposition that an ideal of a quasi-inverse semigroup is quasi-inverse.

1.19 Lemma. Let A be an ideal of a semigroup S. If T is an inverse subsemigroup of S such that $T \cap A \neq \phi$, then $T \cap A$ is an inverse subsemigroup of A.

Proof : Let $x \in T \cap A$. Then there is $x' \in T$ such that $x = xx'x$ and $x' = x'xx'$. Since A is an ideal of S and $x \in A$, $x' \in A$. Hence $x' \in T \cap A$. This shows that $T \cap A$ is a regular subsemigroup of T, so it is an inverse subsemigroup of T because T is inverse. Since $T \cap A \subseteq A$, $T \cap A$ is an inverse subsemigroup of A. #

1.20 Proposition. Every ideal of a quasi-inverse semigroup is quasi-inverse.

Proof : Let A be an ideal of a quasi-inverse semigroup S. Let $a \in A$. Then, there exists an inverse subsemigroup T of S such that $a \in T$ since S is quasi-inverse and $a \in S$. By Lemma 1.19, $T \cap A$ is an inverse subsemigroup of A containing a. Hence, A is quasi-inverse. #

A homomorphic image of an inverse semigroup is an inverse semigroup. Because a quasi-inverse semigroup is a union of inverse semigroups, it then follows that

1.21 Proposition. A homomorphic image of a quasi-inverse semigroup is quasi-inverse.

The following theorem follows directly from Proposition 1.20 and Proposition 1.21 :

1.22 <u>Theorem</u>. Let S be a semigroup and I be an ideal of S. If S is quasi-inverse, then I and the Rees quotient semigroup S/I are quasi-inverse.

The converse of Theorem 1.22 is not true in general. A counter example is given as follows :

<u>Example</u>. Let X = {1, 2, 3}. Let A be the subsemigroup of $\mathcal{T}_X$ as in the example, page 25. It has been shown that A is not quasi-inverse. Let I = $\{\alpha_3, \alpha_4, \alpha_5\}$. From the table, I is an ideal of A, and I is quasi-inverse because α_3, α_4 and α_5 are idempotents of A. Moreover, A/I = $\{\alpha_3\rho_I, \alpha_1\rho_I, \alpha_2\rho_I, \alpha_6\rho_I, \alpha_7\rho_I, \alpha_8\rho_I, \alpha_9\rho_I\}$. We can directly check from the table that $\{\alpha_3\rho_I, \alpha_2\rho_I, \alpha_6\rho_I, \alpha_8\rho_I, \alpha_9\rho_I\}$ is an inverse subsemigroup of A/I containing $\alpha_9\rho_I$ and containing $\alpha_6\rho_I$, and $\alpha_3\rho_I$, $\alpha_1\rho_I$, $\alpha_2\rho_I$, $\alpha_7\rho_I$, $\alpha_8\rho_I$ are idempotents of A/I. Then it follows that A/I is quasi-inverse. #

An ideal I of a semigroup S is said to be <u>completely prime</u> if for a, b $\in$ S, ab $\in$ I implies a $\in$ I or b $\in$ I.

The converse of Theorem 1.22 is true if I is completely prime.

1.23 <u>Theorem</u>. Let I be a completely prime ideal of a semigroup S. Then S is quasi-inverse if and only if I and the Rees quotient semigroup S/I are quasi-inverse.

<u>Proof</u> : The "only if" part follows from Theorem 1.22. Next, assume I and S/I are quasi-inverse. To show S is

quasi-inverse, let $a \in S$.

Case $a \in I$. Then there exists an inverse subsemigroup T of I such that $a \in T$, so T is an inverse subsemigroup of S containing a.

Case $a \notin I$. Then $a\rho_I = \{a\}$. Since S/I is quasi-inverse, there exists an inverse subsemigroup T^* of S/I such that $a\rho_I \in T^*$. Let $L = \{x \in S \mid x\rho_I \in T^*\}$. Then $a \in L \setminus I$. Claim that $L \setminus I$ is an inverse subsemigroup of S. Let $x, y \in L \setminus I$. Then $x\rho_I$ and $y\rho_I \in T^*$, so $(xy)\rho_I = x\rho_I y\rho_I \in T^*$. Therefore $xy \in L$. If $xy \in I$, then either $x \in I$ or $y \in I$ because I is a completely prime ideal of S. Then $xy \notin I$. Hence $xy \in L \setminus I$. Since $x\rho_I \in T^*$ which is regular, there exists $x'\rho_I \in T^*$ such that $x\rho_I = (xx'x)\rho_I$. But $x \notin I$, so $x' \notin I$ and $xx'x \notin I$ which imply $x = xx'x$ and $x' \in L \setminus I$. Hence $L \setminus I$ is a regular subsemigroup of S. Next, let $e, f \in E(L \setminus I)$. Because I is a completely prime ideal of S, ef and $fe \notin I$, so $ef, fe \in L \setminus I$. Since T^* is an inverse subsemigroup of S/I and $e\rho_I, f\rho_I \in E(T^*)$,

$$(ef)\rho_I = e\rho_I f\rho_I = f\rho_I e\rho_I = (fe)\rho_I.$$

Because ef and $fe \notin I$, $ef = fe$. Therefore $L \setminus I$ is an inverse subsemigroup of S containing a.

Hence S is quasi-inverse, as desired. #

The next proposition gives a characterization of a quasi-inverse semigroup in terms of its principal ideals.

1.24 Proposition. A semigroup S is quasi-inverse if and only if every principal ideal of S is quasi-inverse.

Proof : If S is quasi-inverse, then by Proposition 1.20, every principal ideal of S is quasi-inverse.

Conversely, assume every principal ideal of S is quasi-inverse. Let $a \in S$. Then $a \in S^1aS^1$ which is quasi-inverse, there exists an inverse subsemigroup T of S^1aS^1 such that $a \in T$. Because T is an inverse subsemigroup of S^1aS^1 which is a subsemigroup of S, T is an inverse subsemigroup of S containing a. Hence S is quasi-inverse. #

Every principal ideal of a regular semigroup has an idempotent generator [Introduction, page 6]. In any regular semigroup S, $S^1aS^1 = SaS$. Hence, a regular semigroup S is quasi-inverse if and only if SeS is quasi-inverse for each $e \in E(S)$.

Orthodox semigroups and quasi-inverse semigroups are both a generalization of inverse semigroups. Any noncommutative band is both orthodox and quasi-inverse but not inverse. However, some relationships of orthodox semigroups and quasi-inverse semigroups can be given as follows :

1.24 Proposition. Every quasi-inverse semigroup has a maximal orthodox subsemigroup.

Proof : Let S be a quasi-inverse semigroup. Let $\mathscr{C}$ be the collection of all orthodox subsemigroups of S. Since S is regular, then $E(S) \neq \phi$. If $e \in E(S)$, then $\{e\} \in \mathscr{C}$. Then $\mathscr{C} \neq \phi$. Partially order $\mathscr{C}$ by inclusion. Let $\mathscr{S}$ be a chain of $\mathscr{C}$. Then $M = \bigcup_{T \in \mathscr{S}} T$ is clearly a regular subsemigroup of S. Let $e, f \in E(M)$. Then $e \in T_1$, $f \in T_2$

for some T_1, $T_2 \in \mathcal{S}$. Because $\mathcal{S}$ is a chain, either $T_1 \subseteq T_2$ or $T_2 \subseteq T_1$, say $T_1 \subseteq T_2$. Then e, f $\in T_2$. Since T_2 is orthodox, ef $\in E(T_2) \subseteq E(M)$. Therefore M is orthodox, so $M \in \mathcal{C}$ and it is an upper bound of $\mathcal{S}$. By Zorn's lemma, $\mathcal{C}$ has a maximal element. Hence S has a maximal orthodox subsemigroup. #

A subsemigroup T of a semigroup S is said to be <u>full</u> if $E(T) = E(S)$.

1.25 <u>Proposition</u>. Every orthodox semigroup has a maximal quasi-inverse subsemigroup. Moreover, every orthodox semigroup has a maximal full quasi-inverse subsemigroup.

<u>Proof</u> : Let S be an orthodox semigroup and let $\mathcal{C}$ be the collection of all quasi-inverse subsemigroups of S. Then $\mathcal{C} \neq \phi$ because $E(S) \in \mathcal{C}$. Partially order $\mathcal{C}$ by inclusion. Let $\mathcal{S}$ be a chain of $\mathcal{C}$. Let $M = \bigcup_{T \in \mathcal{S}} T$. Because $\mathcal{S}$ is a chain, M is a subsemigroup of S. Let $a \in M$. Then $a \in T$ for some $T \in \mathcal{C}$. Because T is quasi-inverse, there is an inverse subsemigroup T_1 of T such that $a \in T_1$. Then T_1 is an inverse subsemigroup of M containing a. Therefore $M \in \mathcal{C}$ and it is an upper bound of $\mathcal{S}$. By Zorn's lemma, $\mathcal{C}$ has a maximal element. Therefore, S has a maximal quasi-inverse subsemigroup.

If we let $\mathcal{C}^*$ be the collection of all full quasi-inverse subsemigroups of S. Then $E(S) \in \mathcal{C}^*$. The same proof as above, $\mathcal{C}^*$ has a maximal element, and hence S has a maximal full quasi-inverse subsemigroup. #