CHAPTER III

GENERAL SOLUTION OF $f(xoy)+f(xoy^{-1}) = 2f(x)+2f(y)$ ON ABELIAN GROUP G.

Let $(G,\circ)$ and $(G',+)$ be abelian groups such that G' has no element of order 2. In this chapter we shall determine all functions $f : G \longrightarrow G'$ satisfying the functional equation

(*) $$f(x \circ y)+ f(x \circ y^{-1}) = 2f(x)+ 2f(y),$$

for all x, y in G.

3.1 Some Useful formulae

<u>Proposition 3.1.1</u> Let $(G,\circ)$ and $(G',+)$ be abelian groups such that G' has no element of order 2. Let f be a function from G into G' satisfying

(*) $$f(x \circ y)+ f(x \circ y^{-1}) = 2f(x)+ 2f(y),$$

for all x, y in G. Then the followings hold,

(3.1.1.1) $f(e) = 0$,

(3.1.1.2) $f(x^{-1}) = f(x)$ for all x in G,

(3.1.1.3) $f(A\circ x^{n}) = nf(A\circ x)-(n-1)f(A)+ n(n-1)f(x)$ for all A, x in G, and for all integer n.

<u>Proof</u> Setting $x = y = e$ in (*) we get

$$f(e \circ e) + f(e \circ e^{-1}) = 2f(e) + 2f(e).$$

Thus

$$f(e) + f(e) = 2f(e) + 2f(e).$$

Therefore

(3.1.1.4) $$2f(e) = 0.$$

Since G' has no element of order 2, hence it follows from (3.1.1.4) that

$$f(e) = 0.$$

That is (3.1.1.1) holds.

Setting $x = e$ in (*) we get

$$f(e \circ y) + f(e \circ y^{-1}) = 2f(e) + 2f(y).$$

Thus

$$f(y) + f(y^{-1}) = 2f(e) + 2f(y).$$

Using (3.1.1.1), we obtain

$$f(y) + f(y^{-1}) = 2f(y).$$

Therefore

$$f(y^{-1}) = f(y)$$

for all y in G, i.e.(3.1.1.2) holds.

Let A, x be any elements in G. Clearly (3.1.1.3) holds for n=0 or 1. Let n>1. Assume that

$$(3.1.1.5) \qquad f(A \circ x^k) = kf(A \circ x)-(k-1)f(A) + k(k-1)f(x)$$

for all positive $k < n$.

Replacing x and y in (*) by $A \circ x^{n-1}$ and x, respectively, we have

$$f(A \circ x^{n-1} \circ x) + f(A \circ x^{n-1} \circ x^{-1}) = 2f(A \circ x^{n-1}) + 2f(x).$$

Hence

$$f(A \circ x^n) + f(A \circ x^{n-2}) = 2f(A \circ x^{n-1}) + 2f(x).$$

Therefore

$$f(A \circ x^n) = 2f(A \circ x^{n-1}) + 2f(x) - f(A \circ x^{n-2}).$$

By the induction hypothesis (3.1.1.5), we have

$$f(A\circ x^n) = 2\{(n-1)f(A\circ x)-(n-2)f(A)+(n-1)(n-2)f(x)\}$$

$$+2f(x)-\{(n-2)f(A\circ x)-(n-3)f(A)+(n-2)(n-3)f(x)\},$$

$$= nf(A\circ x)-(n-1)f(A)+n(n-1)f(x).$$

Hence

$$(3.1.1.6)\quad f(A\circ x^n) = nf(A\circ x)-(n-1)f(A)+n(n-1)f(x),$$

for all A, x in G and for all non-negative integer n.

For any negative integer n, we have $n = -m$ for some positive integer m. Then we have

$$f(A\circ x^n) = f(A\circ x^{-m}) = f(A\circ(x^{-1})^m).$$

It follows from (3.1.1.6) that

$$(3.1.1.7)\quad f(A\circ x^n) = mf(A\circ x^{-1})-(m-1)f(A)+m(m-1)f(x^{-1}).$$

Using (*), we find that

$$f(A\circ x^{-1}) = 2f(A)+2f(x)-f(A\circ x).$$

From this equation and (3.1.1.7), we have

$$f(A\circ x^n) = m(2f(A)+2f(x)-f(A\circ x))-(m-1)f(A)+m(m-1)f(x^{-1}).$$

Using (3.1.1.2), we obtain

$$f(A\circ x^n) = m(2f(A)+2f(x)-f(A\circ x))-(m-1)f(A)+m(m-1)f(x),$$

$$= -mf(A\circ x)-(-m-1)f(A)+(-m)(-m-1)f(x)$$

$$= nf(A\circ x)-(n-1)f(A)+n(n-1)f(x).$$

Therefore

$$f(A\circ x^n) = nf(A\circ x)-(n-1)f(A)+n(n-1)f(x),$$

for all A, x in G, and for all integer n, i.e.(3.1.1.3) holds.

<u>Lemma 3.1.2</u> Let $(G,\circ)$ and $(G',+)$ be abelian groups. Let $f : G \longrightarrow G'$ satisfy

$$(*) \qquad f(x\circ y) + f(x\circ y^{-1}) = 2f(x) + 2f(y)$$

for all x, y in G. Then for any A, B, C in G, we have

$$f(A\circ B\circ C) = f(A\circ B)+f(A\circ C)+f(B\circ C)-f(A)-f(B)-f(C).$$

<u>Proof</u> Let A, B, C be any elements in G.

By several applications of (3.1.1.3), we see that

$$f(A\circ B^2\circ C^2) = 2f((A\circ B^2)\circ C)- f(A\circ B^2)+ 2f(C) ,$$

$$= 2f((A\circ C)\circ B^2)- f(A\circ B^2)+ 2f(C) ,$$

$$= 2\{2f(A\circ C\circ B)- f(A\circ C)+ 2f(B)\}-\{2f(A\circ B)- f(A) +2f(B)\}+ 2f(C).$$

Hence

$$(3.1.2.1) \quad f(A\circ B^2\circ C^2) = 4f(A\circ B\circ C)- 2f(A\circ B)- 2f(A\circ C)+ f(A)+ 2f(B)+ 2f(C).$$

We also have

$$f(A\circ B^2\circ C^2) = f(A\circ (B\circ C)^2) ,$$

$$= 2f(A\circ B\circ C)- f(A)+ 2f(B\circ C).$$

Hence

$$(3.1.2.2) f(A\circ B^2\circ C^2) = 2f(A\circ B\circ C)- f(A)+ 2f(B\circ C).$$

From (3.1.2.1) and (3.1.2.2) we get

$$2f(A\circ B\circ C)-f(A)+2f(B\circ C) = 4f(A\circ B\circ C)-2f(A\circ B)-2f(A\circ C)+f(A)+2f(B)+2f(C).$$

Thus

$$2f(A\circ B\circ C) = 2\{f(A\circ B)+f(A\circ C)+f(B\circ C)-f(A)-f(B)-f(C)\} .$$

Since G' has no element of order 2, hence

$$f(A \circ B \circ C) = f(A \circ B) + f(A \circ C) + f(B \circ C) - f(A) - f(B) - f(C).$$

<u>Proposition 3.1.3</u> Let $(G, \circ)$ and $(G', +)$ be abelian groups such that G' has no element of order 2. Let $f : G \longrightarrow G'$ satisfy

$$(*) \qquad f(x \circ y) + f(x \circ y^{-1}) = 2f(x) + 2f(y)$$

for all x, y in G. Then for any positive integer m, any g_i in G and any integers n_i, $i = 1, \ldots, m$, we have

$$(3.1.3.1) \quad f\left(\prod_{i=1}^{m} g_i^{n_i}\right) = \sum_{1 \leq i < j \leq m} n_i n_j f(g_i \circ g_j) + 2\sum_{i=1}^{m} n_i^2 f(g_i) - \left(\sum_{i=1}^{m} n_i\right) \sum_{i=1}^{m} n_i f(g_i),$$

where $\sum_{1 \leq i < j \leq 1} n_i n_j f(g_i \circ g_j)$ is meant to be 0.

<u>Proof</u> We prove by induction on m.

Observe that the case m = 1 follows from (3.1.1.3) of proposition 3.1.1 by setting A = e and using (3.1.1.1).

Let m be any positive integer such that $m > 1$.

Assume that

$$(3.1.3.2) \quad f\left(\prod_{i=1}^{k} g_i^{n_i}\right) = \sum_{1 \leq i < j \leq k} n_i n_j f(g_i \circ g_j) + 2\sum_{i=1}^{k} n_i^2 f(g_i) - \left(\sum_{i=1}^{k} n_i\right) \sum_{i=1}^{k} n_i f(g_i),$$

for all positive integer $k < m$.

Observe that

$$f\left(\prod_{i=1}^{m} g_i^{n_i}\right) = f\left(\prod_{i=1}^{m-2} g_i^{n_i} \circ g_{m-1}^{n_{m-1}} \circ g_m^{n_m}\right).$$

Here, when m = 2, we define $\prod_{i=1}^{m-2} g_i^{n_i}$ to be e.

Hence, by lemma 3.1.2, we have

$$f(\prod_{i=1}^{m} g_i^{n_i}) = f(\prod_{i=1}^{m-1} g_i^{n_i})+f(\prod_{i=1}^{m-2} g_i^{n_i}\circ g_m^{n_m})+f(g_{m-1}^{n_{m-1}}\circ g_m^{n_m})-f(\prod_{i=1}^{m-2} g_i^{n_i})$$

$$-f(g_{m-1}^{n_{m-1}})-f(g_m^{n_m}).$$

Apply the induction hypothesis (3.1.3.2) to each term on the right we have

$$f(\prod_{i=1}^{m} g_i^{n_i}) = [\sum_{1\leq i<j\leq m-1} n_i n_j f(g_i\circ g_j)+2\sum_{i=1}^{m-1} n_i^2 f(g_i)-(\sum_{i=1}^{m-1} n_i)\sum_{i=1}^{m-1} n_i f(g_i)]$$

$$+[\sum_{1\leq i<j\leq m-2} n_i n_j f(g_i\circ g_j)+\sum_{i=1}^{m-2} n_i n_m f(g_i\circ g_m)+2\sum_{i=1}^{m-2} n_i^2 f(g_i)$$

$$+ 2n_m^2 f(g_m)-(\sum_{i=1}^{m-2} n_i)\sum_{i=1}^{m-2} n_i f(g_i)-n_m\sum_{i=1}^{m-2} n_i f(g_i) -$$

$$-(\sum_{i=2}^{m-2} n_i)n_m f(g_m)- n_m^2 f(g_m)]+[n_{m-1}n_m f(g_{m-1}\circ g_m) +$$

$$2n_{m-1}^2 f(g_{m-1})+2n_m^2 f(g_m)-(n_{m-1}+ n_m)(n_{m-1}f(g_{m-1})+n_m f(g_m))]$$

$$-[\sum_{1\leq i<j\leq m-2} n_i n_j f(g_i\circ g_j)+2\sum_{i=1}^{m-2} n_i^2 f(g_i)-(\sum_{i=1}^{m-2} n_i)\sum_{i=1}^{m-2} n_i f(g_i)]$$

$$- n_{m-1}^2 f(g_{m-1}) - n_m^2 f(g_m) .$$

After simplification, we have (3.1.3.1).

<u>Theorem 3.1.4</u> Let $(G,\circ)$ and $(G',+)$ be abelian groups. Let H be subgroup of G. A function $\bar{f} : G/_H \longrightarrow G'$ satisfies

$$(\bar{*}) \qquad \bar{f}(x\circ H\circ y\circ H) + \bar{f}(x\circ H\circ (y\circ H)^{-1}) = 2\bar{f}(x\circ H) + 2\bar{f}(y\circ H) ,$$

for all $x\circ H$, $y\circ H$ in $G/_H$ if and only if $\bar{f}(x\circ H) = f(x)$ for some $f : G \longrightarrow G'$ satisfying

$$(*) \qquad f(x\circ y) + f(x\circ y^{-1}) = 2f(x) + 2f(y) ,$$

for all x, y in G and f is constant on each coset of H.

<u>Proof</u> Assume that $f : G \longrightarrow G'$ satisfies $(*)$ on G and f is constant on each coset of H. Hence $\bar{f} : G/_H \longrightarrow G'$ defined by $\bar{f}(x\circ H) = f(x)$ is well defined.

Furthermore, we have

$$\begin{aligned} \bar{f}(x\circ H\circ y\circ H) + \bar{f}(x\circ H\circ (y\circ H)^{-1}) &= \bar{f}(x\circ y\circ H) + \bar{f}(x\circ y^{-1}\circ H) , \\ &= f(x\circ y) + f(x\circ y^{-1}) , \\ &= 2f(x) + 2f(y) \\ &= 2\bar{f}(x\circ H) + 2\bar{f}(y\circ H). \end{aligned}$$

Conversely, assume that $\bar{f} : G/_H \longrightarrow G'$ satisfies $(\bar{*})$ on $G/_H$.

Let $f : G \longrightarrow G'$ be definedby

$$f(x) = \bar{f}(x\circ H) ,$$

for all x in G.

If x, y are in the same coset of H, then $x\circ H = y\circ H$. Hence we have

$$f(x) = \bar{f}(x\circ H) = \bar{f}(y\circ H) = f(y) ,$$

i.e. f is constant on each coset of H.

Furthermore, we have

$$\begin{aligned} f(x\circ y)+ f(x\circ y^{-1}) &= \bar{f}(x\circ y\circ H)+ \bar{f}(x\circ y^{-1}\circ H) ,\\ &= \bar{f}(x\circ H\circ y\circ H)+ \bar{f}(x\circ H\circ (y\circ H)^{-1}),\\ &= 2\bar{f}(x\circ H)+ 2\bar{f}(y\circ H) ,\\ &= 2f(x) + 2f(y). \end{aligned}$$

Remark 3.1.5 Let $(G_1,\circ)$, $(G_2,\circ)$ and $(G',+)$ be abelian groups. Let $\Psi : G_1 \longrightarrow G_2$ be an isomorphism and $f_2: G_2 \longrightarrow G'$ be any function. Then function $f_1 = f_2\circ \Psi : G_1 \longrightarrow G'$ satisfies

$$(*_1) \quad f_1(x_1\circ y_1)+ f_1(x_1\circ y_1^{-1}) = 2f_1(x_1)+ 2f_1(y_1)$$

for all x_1, y_1 in G_1 if and only if f_2 satisfies

$$(*_2) \quad f_2(x_2\circ y_2)+ f_2(x_2\circ y_2^{-1}) = 2f_2(x_2)+ 2f_2(y_2)$$

for all x_2, y_2 in G_2 .

Proof Assume that f_2 satisfies $(*_2)$.

Let x_1, y_1 be any elements in G_1. Hence we have

$$\begin{aligned} f_1(x_1\circ y_1)+ f_1(x_1\circ y_1^{-1}) &= f_2\circ \Psi(x_1\circ y_1)+ f_2\circ \Psi(x_1\circ y_1^{-1}) ,\\ &= f_2(\Psi(x_1\circ y_1))+ f_2(\Psi(x_1\circ y_1^{-1})),\\ &= f_2(\Psi(x_1)\circ\Psi(y_1))+ f_2(\Psi(x_1)\circ(\Psi(y_1))^{-1}),\\ &= 2f_2(\Psi(x_1))+ 2f_2(\Psi(y_1)),\\ &= 2f_1(x_1) + 2f_1(y_1). \end{aligned}$$

Conversely, assume that $f_1 = f_2 \circ \Psi : G_1 \longrightarrow G'$ satisfies $(*_1)$.
Let x_2, y_2 be any elements in G_2. Then there exist x_1, y_1 in G_1 such that $\Psi(x_1) = x_2$ and $\Psi(y_1) = y_2$. Hence we have

$$\begin{aligned}
f_2(x_2 \circ y_2) + f_2(x_2 \circ y_2^{-1}) &= f_2(\Psi(x_1) \circ \Psi(y_1)) + f_2(\Psi(x_1) \circ (\Psi(y_1))^{-1}), \\
&= f_2(\Psi(x_1 \circ y_1)) + f_2(\Psi(x_1 \circ y_1^{-1})), \\
&= f_1(x_1 \circ y_1) + f_1(x_1 \circ y_1^{-1}), \\
&= 2f_1(x_1) + 2f_1(y_1), \\
&= 2f_2 \circ \Psi(x_1) + 2f_2 \circ \Psi(y_1), \\
&= 2f_2(\Psi(x_1)) + 2f_2(\Psi(y_1)), \\
&= 2f_2(x_2) + 2f_2(y_2).
\end{aligned}$$

3.2 The Main Theorem

Our main result is the following theorem.

Theorem 3.2.1 Let $(G, \circ)$ and $(G', +)$ be abelian groups such that G' has no element of order 2. Let $\mathcal{A} = \{a_\alpha : \alpha \in I\}$ be a set of generators of G with a system $\mathcal{R}$ of defining relations. Let $\mathcal{A}^{(1)} = \{\{a\} : a \in \mathcal{A}\}$ and $\mathcal{A}^{(2)} = \{\{a,b\} : a, b \in \mathcal{A},\ a \neq b\}$. A function $f : G \longrightarrow G'$ satisfies

$$(*) \qquad f(x \circ y) + f(x \circ y^{-1}) = 2f(x) + 2f(y),$$

for all x, y in G if and only if there exists a function $c : \mathcal{A}^{(1)} \cup \mathcal{A}^{(2)} \longrightarrow G'$ such that for any defining relation $\prod_{i=1}^{m} a_{\alpha_i}^{s_i} = e$ of $\mathcal{R}$, we have

(i) $\sum_{1 \leq i < j \leq m} s_i s_j c(\{a_{\alpha_i}, a_{\alpha_j}\}) + 2\sum_{i=1}^{m} s_i^2 c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} s_i) \sum_{i=1}^{m} s_i c(\{a_{\alpha_i}\}) = 0,$

(ii) $\sum_{i=1}^{m} s_i c(\{a_{\alpha_i}, a_\beta\}) - \sum_{i=1}^{m} s_i c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} s_i) c(\{a_\beta\}) = 0$

for all $a_\beta \neq a_{\alpha_i}$, $i = 1, \ldots, m$,

(iii) $\sum_{\substack{j=1 \\ j \neq i}}^{m} s_j c(\{a_{\alpha_j}, a_\beta\}) - \sum_{\substack{j=1 \\ j \neq i}}^{m} s_j c(\{a_{\alpha_j}\}) - (\sum_{\substack{j=1 \\ j \neq i}}^{m} s_j) c(\{a_\beta\}) + 2s_i c(\{a_\beta\}) = 0$

for all a_β such that $a_\beta = a_{\alpha_i}$ for some $i = 1, \ldots, m$, and for any $x = \prod_{i=1}^{m} a_{\alpha_i}^{n_i}$ in G, we have

$$f(x) = \sum_{1 \leq i < j \leq m} n_i n_j c(\{a_{\alpha_i}, a_{\alpha_j}\}) + 2\sum_{i=1}^{m} n_i^2 c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} n_i) \sum_{i=1}^{m} n_i c(\{a_{\alpha_i}\})$$

<u>Proof</u> Assume that $f : G \longrightarrow G'$ satisfies (*)
Since G is an abelian group having $\mathcal{A} = \{a_\alpha : \alpha \in I\}$ as its set of generators, hence by theorem 2.2.2, there exist a free abelian group F, with a basis W, and a subgroup H of F such that

(1) there exists a bijection $\Theta : \mathcal{A} \longrightarrow W$,

(2) there exists an isomorphism Ψ from $F/_H$ onto G such that

$$\Psi(\prod_{i=1}^{m} (\Theta(a_{\alpha_i})^{n_i} \circ H) = \prod_{i=1}^{m} a_{\alpha_i}^{n_i}$$

for all a_{α_i} and for all integer n_i.

Let $\bar{f} = f \circ \Psi$.

Hence by remark 3.1.5, $\bar{f} : F/_H \longrightarrow G'$ satisfies

(*) $\bar{f}(x \circ H \circ y \circ H) + \bar{f}(x \circ H \circ (y \circ H)^{-1}) = 2\bar{f}(x \circ H) + 2\bar{f}(y \circ H)$

for all $x \circ H$, $y \circ H$ in $F/_H$.

Let $f' : F \longrightarrow G'$ be defined by

$$f'(x) \;=\; \bar{f}(x \circ H) .$$

By theorem 3.1.4, f' satisfies

(*') $f'(x \circ y) + f'(x \circ y^{-1}) \;=\; 2f'(x) + 2f'(y)$

for all x, y in F and f' is constant on each coset of H.

Define $c : \mathcal{A}^{(1)} \cup \mathcal{A}^{(2)} \longrightarrow G'$ by

$$c(\{a_\alpha\}) \;=\; f'(\theta(a_\alpha))$$

for all α in I and

$$c(\{a_\alpha, a_\beta\}) \;=\; f'(\theta(a_\alpha) \circ \theta(a_\beta))$$

for all α, β in I such that $\alpha \neq \beta$.

For any defining relation $\prod_{i=1}^{m} a_{\alpha_i}^{s_i} = e$ of $\mathcal{R}$, we have $\prod_{i=1}^{m} \theta(a_{\alpha_i})^{s_i} \in H$.

Thus $\prod_{i=1}^{m} \theta(a_{\alpha_i})^{s_i}$ and e are in the coset $e \circ H$.

Since f' is constant on $e \circ H$, hence

$$f'\left(\prod_{i=1}^{m} \theta(a_{\alpha_i})^{s_i}\right) = f'(e).$$

Using (3.1.1.1), we obtain

$$f'\left(\prod_{i=1}^{m} \theta(a_{\alpha_i})^{s_i}\right) = 0.$$

Applying proposition 3.1.3 to the left handside of this equation, we get

$$\sum_{1\le i<j\le m} s_i s_j f'(\theta(a_{\alpha_i})\circ\theta(a_{\alpha_j}))+2\sum_{i=1}^{m} s_i^2 f'(\theta(a_{\alpha_i}))-(\sum_{i=1}^{m} s_i)\sum_{i=1}^{m} s_i f'(\theta(a_{\alpha_i})) = 0.$$

Therefore

$$\sum_{1\le i<j\le m} s_i s_j c(\{a_{\alpha_i},a_{\alpha_j}\})+2\sum_{i=1}^{m} s_i^2 c(\{a_{\alpha_i}\})-(\sum_{i=1}^{m} s_i)\sum_{i=1}^{m} s_i c(\{a_{\alpha_i}\}) = 0,$$

i.e. (i) holds.

For any generator a_β, f' is constant on the coset $\theta(a_\beta)\circ H$, hence for any defining relation $\prod_{i=1}^{m} a_{\alpha_i}^{s_i} = e$ of $\mathcal{R}$, we have

$$(3.2.1.1)\qquad f'(\prod_{i=1}^{m}\theta(a_{\alpha_i})^{s_i}\circ\theta(a_\beta)) = f'(\theta(a_\beta)).$$

If $a_\beta \ne a_{\alpha_i}$ for any $i = 1,\dots, m$, then (3.2.1.1) can be written

$$f'(\prod_{i=1}^{m+1}\theta(a_{\alpha_i})^{s_i}) = f'(\theta(a_\beta)),$$

where $s_{m+1} = 1$ and $\theta(a_{\alpha_{m+1}}) = \theta(a_\beta)$.

Applying proposition 3.1.3 to the left hand side of this equation, we get

$$\sum_{1\le i<j\le m+1} s_i s_j f'(\theta(a_{\alpha_i})\circ\theta(a_{\alpha_j}))+2\sum_{i=1}^{m+1} s_i^2 f'(\theta(a_{\alpha_i}))-(\sum_{i=1}^{m+1} s_i)\sum_{i=1}^{m+1} s_i f'(\theta(a_{\alpha_i}))$$

$$= f'(\theta(a_\beta)).$$

Thus we have

$$[\sum_{1\leq i<j\leq m} s_is_jf'(\theta(a_{\alpha_i})\circ\theta(a_{\alpha_j}))+\sum_{i=1}^{m} s_if'(\theta(a_{\alpha_i})\circ\theta(a_\beta))]+[2\sum_{i=1}^{m} s_i^2f'(\theta(a_{\alpha_i}))$$

$$+2f'(\theta(a_\beta))]-[(\sum_{i=1}^{m} s_i)\sum_{i=1}^{m} s_if'(\theta(a_{\alpha_i}))+(\sum_{i=1}^{m} s_i)f'(\theta(a_\beta))]-[\sum_{i=1}^{m} s_if'(\theta(a_{\alpha_i}))$$

$$+f'(\theta(a_\beta))] \quad = \quad f'(\theta(a_\beta)).$$

Hence

$$\sum_{1\leq i<j\leq m} s_is_jf'(\theta(a_{\alpha_i})\circ\theta(a_{\alpha_j}))+2\sum_{i=1}^{m} s_i^2f'(\theta(a_{\alpha_i}))-(\sum_{i=1}^{m} s_i)\sum_{i=1}^{m} s_if'(\theta(a_{\alpha_i}))$$

$$+\sum_{i=1}^{m} s_if'(\theta(a_{\alpha_i})\circ\theta(a_\beta))-\sum_{i=1}^{m} s_if'(\theta(a_{\alpha_i}))-(\sum_{i=1}^{m} s_i)f'(\theta(a_\beta)) = 0.$$

Therefore

$$\sum_{1\leq i<j\leq m} s_is_jc(\{a_{\alpha_i},a_{\alpha_j}\})+2\sum_{i=1}^{m} s_i^2c(\{a_{\alpha_i}\})-(\sum_{i=1}^{m} s_i)\sum_{i=1}^{m} s_ic(\{a_{\alpha_i}\})$$

$$+\sum_{i=1}^{m} s_ic(\{a_{\alpha_i},a_\beta\}) - \sum_{i=1}^{m} s_ic(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} s_i)c(\{a_\beta\}) = 0.$$

By (i), we see that the first three terms on the left handside of this equation add up to zero. Hence, we have

$$\sum_{i=1}^{m} s_ic(\{a_{\alpha_i},a_\beta\})-\sum_{i=1}^{m} s_ic(\{a_{\alpha_i}\})-(\sum_{i=1}^{m} s_i)c(\{a_\beta\}) = 0.$$

That is (ii) holds.

If $a_\beta = a_{\alpha_i}$ for some $i = 1,\ldots,m$, then (3.2.1.1) can be written as

$$f'(\prod_{j=1}^{m} \theta(a_{\alpha_j})^{h_j}) = f'(\theta(a_\beta)),$$

where $h_j = s_j$ for all $j \neq i$, $h_j = s_i + 1$ if $j = i$

Applying proposition 3.1.3 to the left hand side of this equation, we get

$$\sum_{1\le j<k\le m} h_j h_k f'(\theta(a_{\alpha_j})\circ\theta(a_{\alpha_k})) + 2\sum_{j=1}^{m} h_j^2 f'(\theta(a_{\alpha_j})) - \Big(\sum_{j=1}^{m} h_j\Big)\sum_{j=1}^{m} h_j f'(\theta(a_{\alpha_j}))$$

$$= f'(\theta(a_\beta)).$$

Observe that $\displaystyle\sum_{1\le j<k\le m} h_j h_k f'(\theta(a_{\alpha_j})\circ\theta(a_{\alpha_k})) = \sum_{1\le j<k\le m} s_j s_k f'(\theta(a_{\alpha_j})\circ\theta(a_{\alpha_k}))$

$$+ \sum_{\substack{j=1\\ j\neq i}}^{m} s_j\, f'(\theta(a_{\alpha_j})\circ\theta(a_{\alpha_i})) ,$$

$$\sum_{j=1}^{m} h_j^2 f'(\theta(a_{\alpha_j})) = \sum_{j=1}^{m} s_j^2\, f'(\theta(a_{\alpha_j})) + (2s_i+1) f'(\theta(a_{\alpha_i})) \quad \text{and}$$

$$\Big(\sum_{j=1}^{m} h_j\Big)\sum_{j=1}^{m} h_j f'(\theta(a_{\alpha_j})) = \Big(\sum_{j=1}^{m} s_j\Big)\Big(\sum_{j=1}^{m} s_j f'(\theta(a_{\alpha_j}))\Big) + \Big(\sum_{j=1}^{m} s_j\Big) f'(\theta(a_{\alpha_i})) +$$

$$+ \sum_{j=1}^{m} s_j f'(\theta(a_{\alpha_j})) + f'(\theta(a_{\alpha_i})).$$

Hence, we have

$$\sum_{1\le j<k\le m} s_j s_k f'(\theta(a_{\alpha_j})\circ\theta(a_{\alpha_k})) + 2\sum_{j=1}^{m} s_j^2 f'(\theta(a_{\alpha_j})) - \Big(\sum_{j=1}^{m} s_j\Big)\sum_{j=1}^{m} s_j f'(\theta(a_{\alpha_j}))$$

$$+ \sum_{\substack{j=1\\ j\neq 1}}^{m} s_j f'(\theta(a_{\alpha_j})\circ\theta(a_{\alpha_i})) + 2(2s_i+1) f'(\theta(a_{\alpha_i})) - \Big(\sum_{j=1}^{m} s_j\Big) f'(\theta(a_{\alpha_i})) -$$

$$- \sum_{j=1}^{m} s_j f'(\theta(a_{\alpha_j})) - f'(\theta(a_{\alpha_i})) = f'(\theta(a_{\alpha_i})).$$

Thus

$$\sum_{1\le j<k\le m} s_j s_k f'(\theta(a_{\alpha_j})\circ\theta(a_{\alpha_k}))+2\sum_{j=1}^{m} s_j^2 f'(\theta(a_{\alpha_j}))-(\sum_{j=1}^{m} s_j)\sum_{j=1}^{m} s_j f'(\theta(a_{\alpha_j}))$$

$$+\sum_{\substack{j=1\\ j\neq i}}^{m} s_j f'(\theta(a_{\alpha_j})\circ\theta(a_{\alpha_i}))-(\sum_{\substack{j=1\\ j\neq i}}^{m} s_j) f'(\theta(a_{\alpha_i}))-\sum_{\substack{j=1\\ j\neq i}}^{m} s_j f'(\theta(a_{\alpha_j})) +$$

$$+ 2s_i\ f'(\theta(a_{\alpha_i})) = 0.$$

Therefore

$$\sum_{1\le j<k\le m} s_j s_k c(\{a_{\alpha_j}, a_{\alpha_k}\})+2\sum_{j=1}^{m} s_j^2 c(\{\theta(a_{\alpha_j})\})-(\sum_{j=1}^{m} s_j)\sum_{j=1}^{m} s_j c(\{\theta(a_{\alpha_j})\})$$

$$+\sum_{\substack{j=1\\ j\neq i}}^{m} s_j c(\{a_{\alpha_j}, a_{\alpha_i}\})-(\sum_{\substack{j=1\\ j\neq i}}^{m} s_j) c(\{a_{\alpha_i}\})-\sum_{\substack{j=1\\ j\neq i}}^{m} s_j c(\{a_{\alpha_j}\})+2s_i c(\{a_{\alpha_i}\}) = 0.$$

By (i), we see that the first three terms on the left hand side of this equation add up to zero. Hence we have

$$\sum_{\substack{j=1\\ j\neq i}}^{m} s_j c(\{a_{\alpha_i}, a_{\alpha_j}\})-\sum_{\substack{j=1\\ j\neq i}}^{m} s_j c(\{a_{\alpha_j}\})-(\sum_{\substack{j=1\\ j\neq i}}^{m} s_j) c(\{a_{\alpha_i}\})+2s_i c(\{a_{\alpha_i}\}) = 0.$$

That is (iii) holds

By proposition 3.1.3, for any $\prod_{i=1}^{m} \theta(a_{\alpha_i})^{n_i}$ in F, we have

$$f'(\prod_{i=1}^{m}\theta(a_{\alpha_i})^{n_i}) = \sum_{1\le i<j\le m} n_i n_j f'(\theta(a_{\alpha_i})\circ\theta(a_{\alpha_j}))+2\sum_{i=1}^{m} n_i^2\ f'(\theta(a_{\alpha_i})).$$

$$-(\sum_{i=1}^{m} n_i)\sum_{i=1}^{m} n_i\ f'(\theta(a_{\alpha_i})) .$$

Hence

$$f'\left(\prod_{i=1}^{m} \theta(a_{\alpha_i})^{n_i}\right) = \sum_{1\le i<j\le m} n_i n_j c(\{a_{\alpha_i}, a_{\alpha_j}\}) + 2\sum_{i=1}^{m} n_i^2 c(\{a_{\alpha_i}\}) - \left(\sum_{i=1}^{m} n_i\right)\sum_{i=1}^{m} n_i c(\{a_{\alpha_i}\}).$$

Therefore, for any $x = \prod_{i=1}^{m} a_{\alpha_i}^{n_i}$ in G, we have

$$f\left(\prod_{i=1}^{m} a_{\alpha_i}^{n_i}\right) = \bar{f}\left(\Psi^{-1}\left(\prod_{i=1}^{m} a_{\alpha_i}^{n_i}\right)\right) = \bar{f}\left(\prod_{i=1}^{m} \theta(a_{\alpha_i})^{n_i} \circ H\right) = f'\left(\prod_{i=1}^{m} \theta(a_{\alpha_i})^{n_i}\right) =$$

$$\sum_{1\le i<j\le m} n_i n_j\, c(\{a_{\alpha_i}, a_{\alpha_j}\}) + 2\sum_{i=1}^{m} n_i^2 c(\{a_{\alpha_i}\}) - \left(\sum_{i=1}^{m} n_i\right)\sum_{i=1}^{m} n_i c(\{a_{\alpha_i}\}).$$

Conversely, assume that we are given a function $c : \mathcal{A}^{(1)} \cup \mathcal{A}^{(2)} \longrightarrow G'$ such that (i)-(iii) hold for any defining relation $\prod_{i=1}^{m} a_{\alpha_i}^{s_i} = e$ of $\mathcal{R}$.

For any $x = \prod_{i=1}^{m} a_{\alpha_i}^{n_i}$ in G, let

$$f(x) = \sum_{1\le i<j\le m} n_i n_j c(\{a_{\alpha_i}, a_{\alpha_j}\}) + 2\sum_{i=1}^{m} n_i^2 c(\{a_{\alpha_i}\}) - \left(\sum_{i=1}^{m} n_i\right)\sum_{i=1}^{m} n_i c(\{a_{\alpha_i}\}).$$

To show that f is well-defined, first we show that for any relation $\prod_{i=1}^{m} a_{\alpha_i}^{k_{\alpha_i}} = e$ we have

(i') $$\sum_{1\le i<p\le m} k_{\alpha_i} k_{\alpha_p} c(\{a_{\alpha_i}, a_{\alpha_p}\}) + 2\sum_{i=1}^{m} k_{\alpha_i}^2 c(\{a_{\alpha_i}\}) - \left(\sum_{i=1}^{m} k_{\alpha_i}\right)\sum_{i=1}^{m} k_{\alpha_i} c(\{a_{\alpha_i}\}) = 0,$$

(ii') $$\sum_{i=1}^{m} k_{\alpha_i} c(\{a_{\alpha_i}, a_\beta\}) - \sum_{i=1}^{m} k_{\alpha_i} c(\{a_{\alpha_i}\}) - \left(\sum_{i=1}^{m} k_{\alpha_i}\right) c(\{a_\beta\}) = 0,$$

for all $\beta \neq \alpha_i$, $i = 1, \ldots, m$, (iii') $\sum_{\substack{p=1\\p\neq i}}^{m} k_{\alpha_p} c(\{a_{\alpha_p}, a_{\alpha_i}\}) - \sum_{\substack{p=1\\p\neq i}}^{m} k_{\alpha_p} c(\{a_{\alpha_p}\}) - (\sum_{\substack{p=1\\p\neq i}}^{m} k_{\alpha_p}) c(\{a_{\alpha_i}\})$

$$+ 2k_{\alpha_i} c(\{a_{\alpha_i}\}) = 0 ,$$

for all $i = 1,\ldots, m$.

Let $\prod_{i=1}^{m} a_{\alpha_i}^{k_{\alpha_i}} = e$ be any relation. Hence $\prod_{i=1}^{m} x_{\alpha_i}^{k_{\alpha_i}} \in H = \langle N \rangle$. Therefore

$$\prod_{i=1}^{m} x_{\alpha_i}^{k_{\alpha_i}} = \prod_{j=1}^{\ell} B_j^{n_j}$$

where $B_j = \prod_{i=1}^{m} x_{\alpha_i}^{s_{ji}} \in N.$ Thus

$$\prod_{i=1}^{m} x_{\alpha_i}^{k_{\alpha_i}} = \prod_{j=1}^{\ell} (\prod_{i=1}^{m} x_{\alpha_i}^{s_{ji}})^{n_j} = \prod_{j=1}^{\ell} (\prod_{i=1}^{m} x_{\alpha_i}^{n_j s_{ji}}) = \prod_{i=1}^{m} (\prod_{j=1}^{\ell} x_{\alpha_i}^{n_j s_{ji}}) = \prod_{i=1}^{m} x_{\alpha_i}^{\sum_{j=1}^{\ell} n_j s_{ji}} .$$

Since x_{α_i}, $i = 1,\ldots, m$, are elements of free abelian group, hence

$$k_{\alpha_i} = \sum_{j=1}^{\ell} n_j s_{ji} ,$$

$i = 1,\ldots,m$.

Since $\prod_{i=1}^{m} x_{\alpha_i}^{s_{ji}} \in N$, $j = 1,\ldots, \ell$, hence $\prod_{i=1}^{m} a_{\alpha_i}^{s_{ji}} = e$, $j = 1,\ldots, \ell$

are relations of $\mathfrak{R}$. Hence, from (ii) and (iii), we have

(ii-j) $\sum_{i=1}^{m} s_{ji} c(\{a_{\alpha_i}, a_\beta\}) - \sum_{i=1}^{m} s_{ji} c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} s_{ji}) c(\{a_\beta\}) = 0$

for all $\beta \neq \alpha_i$, $i = 1,\ldots m$,

(iii-j) $\sum_{\substack{p=1\\p\neq i}}^{m} s_{jp} c(\{a_{\alpha_p}, a_{\alpha_i}\}) - \sum_{\substack{p=1\\p\neq i}}^{m} s_{jp} c(\{a_{\alpha_p}\}) - (\sum_{\substack{p=1\\p\neq i}}^{m} s_{jp}) c(\{a_{\alpha_i}\}) + 2s_{ji} c(\{a_{\alpha_i}\}) = 0$

for all $i = 1,\ldots, m$.

It follows from (ii-j) that

$$\sum_{i=1}^{m} n_j s_{ji} c(\{a_{\alpha_i}, a_\beta\}) - \sum_{i=1}^{m} n_j s_{ji} c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} n_j s_{ji}) c(\{a_\beta\}) = 0,$$

$j = 1, \ldots, \ell$.

By adding these ℓ equations and rearranging terms, we have

$$\sum_{i=1}^{m} \sum_{j=1}^{\ell} n_j s_{ji} c(\{a_{\alpha_i}, a_\beta\}) - \sum_{i=1}^{m} \sum_{j=1}^{\ell} n_j s_{ji} c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} \sum_{j=1}^{\ell} n_j s_{ji}) c(\{a_\beta\}) = 0,$$

i.e. we have

$$\sum_{i=1}^{m} k_{\alpha_i} c(\{a_{\alpha_i}, a_\beta\}) - \sum_{i=1}^{m} k_{\alpha_i} c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} k_{\alpha_i}) c(\{a_\beta\}) = 0,$$

for all $\beta \neq \alpha_i$, $i = 1, \ldots, m$.

Hence for any relation $\prod_{i=1}^{m} a_{\alpha_i}^{k_{\alpha_i}} = e$, (ii') holds.

It follows from (iii-j) that

$$\sum_{\substack{p=1 \\ p\neq i}}^{m} n_j s_{jp} c(\{a_{\alpha_p}, a_{\alpha_i}\}) - \sum_{\substack{p=1 \\ p\neq i}}^{m} n_j s_{jp} c(\{a_{\alpha_p}\}) - (\sum_{\substack{p=1 \\ p\neq i}}^{m} n_j s_{jp}) c(\{a_{\alpha_i}\})$$

$$+ 2n_j s_{ji}\, c(\{a_{\alpha_i}\}) = 0,$$

for all $i = 1, \ldots, m$, $j = 1, \ldots \ell$.

By adding these ℓ equations and rearanging terms, we have

$$\sum_{\substack{p=1 \\ p\neq i}}^{m} \sum_{j=1}^{\ell} n_j s_{jp} c(\{a_{\alpha_p}, a_{\alpha_i}\}) - \sum_{\substack{p=1 \\ p\neq i}}^{m} \sum_{j=1}^{\ell} n_j s_{jp} c(\{a_{\alpha_p}\}) - (\sum_{\substack{p=1 \\ p\neq i}}^{m} \sum_{j=1}^{\ell} n_j s_{jp}) c(\{a_{\alpha_i}\})$$

$$+ 2 \sum_{j=1}^{\ell} n_j s_{ji}\, c(\{a_{\alpha_i}\}) = 0,$$

i.e. we have

$$\sum_{\substack{p=1\\p\neq i}}^{m} k_{\alpha_p} c(\{a_{\alpha_p}, a_{\alpha_i}\}) - \sum_{\substack{p=1\\p\neq i}}^{m} k_{\alpha_p} c(\{a_{\alpha_p}\}) - (\sum_{\substack{p=1\\p\neq i}}^{m} k_{\alpha_p}) c(\{a_{\alpha_i}\}) + 2k_{\alpha_i} c(\{a_{\alpha_i}\}) = 0,$$

for all $i = 1,\ldots, m$.

Hence for any relation $\prod_{i=1}^{m} a_{\alpha_i}^{k_{\alpha_i}} = e$, (iii') holds.

It follows from (iii') that

$$\sum_{\substack{p=1\\p\neq i}}^{m} k_{\alpha_i} k_{\alpha_p} c(\{a_{\alpha_p}, a_{\alpha_i}\}) - \sum_{\substack{p=1\\p\neq i}}^{m} k_{\alpha_i} k_{\alpha_p} c(\{a_{\alpha_p}\}) - (\sum_{\substack{p=1\\p\neq i}}^{m} k_{\alpha_i} k_{\alpha_p}) c(\{a_{\alpha_i}\})$$

$$+ 2k_{\alpha_i}^2 c(\{a_{\alpha_i}\}) = 0,$$

for all $i = 1,\ldots,m$.

By adding these m equations, we have

(3.2.1.2) $$\sum_{i=1}^{m} \sum_{\substack{p=1\\p\neq i}}^{m} k_{\alpha_i} k_{\alpha_p} c(\{a_{\alpha_p}, a_{\alpha_i}\}) - \sum_{i=1}^{m} \sum_{\substack{p=1\\p\neq i}}^{m} k_{\alpha_i} k_{\alpha_p} c(\{a_{\alpha_p}\}) - \sum_{i=1}^{m} (\sum_{\substack{p=1\\p\neq i}}^{m} k_{\alpha_i} k_{\alpha_p}) c(\{a_{\alpha_i}\})$$

$$+ 2 \sum_{i=1}^{m} k_{\alpha_i}^2 c(\{a_{\alpha_i}\}) = 0.$$

Observe that $\sum_{i=1}^{m} \sum_{\substack{p=1\\p\neq i}}^{m} k_{\alpha_i} k_{\alpha_p} c(\{a_{\alpha_p}, a_{\alpha_i}\}) = 2 \sum_{1\leq i<p\leq m} k_{\alpha_i} k_{\alpha_p} c(\{a_{\alpha_i}, a_{\alpha_p}\})$

and

$$\sum_{i=1}^{m} \sum_{\substack{p=1\\p\neq i}}^{m} k_{\alpha_i} k_{\alpha_p} c(\{a_{\alpha_p}\}) = \sum_{i=1}^{m} (\sum_{\substack{p=1\\p\neq i}}^{m} k_{\alpha_i} k_{\alpha_p}) c(\{a_{\alpha_i}\}) = (\sum_{i=1}^{m} k_{\alpha_i})(\sum_{i=1}^{m} k_{\alpha_i} c(\{a_{\alpha_i}\}))$$

$$- \sum_{i=1}^{m} k_{\alpha_i}^2 c(\{a_{\alpha_i}\}).$$

Applying these to (3.2.1.2), we have

$$2 \sum_{1\le i<p\le m} k_{\alpha_i} k_{\alpha_p} c(\{a_{\alpha_i}, a_{\alpha_p}\}) - 2\left(\sum_{i=1}^{m} k_{\alpha_i}\right) \sum_{i=1}^{m} k_{\alpha_i} c(\{a_{\alpha_i}\}) + 4 \sum_{i=1}^{m} k_{\alpha_i}^2 c(\{a_{\alpha_i}\}) = 0.$$

Since G' has no element of order 2, hence

$$\sum_{1\le i<p\le m} k_{\alpha_i} k_{\alpha_p} c(\{a_{\alpha_i}, a_{\alpha_p}\}) + 2 \sum_{i=1}^{m} k_{\alpha_i}^2 c(\{a_{\alpha_i}\}) - \left(\sum_{i=1}^{m} k_{\alpha_i}\right) \sum_{i=1}^{m} k_{\alpha_i} c(\{a_{\alpha_i}\}) = 0.$$

That is for any relation $\prod_{i=1}^{m} a_{\alpha_i}^{k_{\alpha_i}} = e$, (i') holds.

Now, we prove that f is well-defined. Consider any two representations of x in terms of the generators a_α's. Without loss of generality, we may assume that $x = \prod_{i=1}^{m} a_{\alpha_i}^{n_i}$, and $x = \prod_{i=1}^{m} a_{\alpha_i}^{n'_i}$ where a_{α_i}, $i = 1,\ldots,m$ are in $\mathcal{A}$ and n_i, n'_i, $i = 1,\ldots, m$, are integers, be two representations of x.

Then $\prod_{i=1}^{m} a_{\alpha_i}^{(n_i - n'_i)} = e$. With respect to this relation, (i') and (iii') become, respectively,

$$\text{(i")} \quad \sum_{1\le i<j\le m} (n_i - n'_i)(n_j - n'_j) c(\{a_{\alpha_i}, a_{\alpha_j}\}) + 2 \sum_{i=1}^{m} (n_i - n'_i)^2 c(\{a_{\alpha_i}\})$$

$$-\left(\sum_{i=1}^{m} (n_i - n'_i)\right) \sum_{i=1}^{m} (n_i - n'_i) c(\{a_{\alpha_i}\}) = 0,$$

and

$$\text{(iii")} \quad \sum_{\substack{j=1\\ j\ne i}}^{m} (n_j - n'_j) c(\{a_{\alpha_j}, a_{\alpha_i}\}) - \sum_{j=1}^{m} (n_j - n'_j) c(\{a_{\alpha_j}\}) -$$

$$-\left(\sum_{\substack{j=1\\ j\ne i}}^{m} (n_j - n'_j)\right) c(\{a_{\alpha_i}\}) + 2(n_i - n'_i) c(\{a_{\alpha_i}\}) = 0$$

for all a_{α_i} for some $i = 1,\ldots, m$.

By straightforward calculation, (i") gives

$$\sum_{1\le i<j\le m} n_i n_j c(\{a_{\alpha_i},a_{\alpha_j}\})+2\sum_{i=1}^{m} n_i^2 c(\{a_{\alpha_i}\})-(\sum_{i=1}^{m} n_i)\sum_{i=1}^{m} n_i c(\{a_{\alpha_i}\})$$

$$-[\sum_{1\le i<j\le m} n_i' n_j' c(\{a_{\alpha_i},a_{\alpha_j}\})+2\sum_{i=1}^{m} n_i'^2 c(\{a_{\alpha_i}\})-(\sum_{i=1}^{m} n_i')\sum_{i=1}^{m} n_i' c(\{a_{\alpha_i}\})]$$

$$-\sum_{i=1}^{m} n_i' \sum_{\substack{j=1\\ j\ne i}}^{m} (n_j-n_j')c(\{a_{\alpha_j},a_{\alpha_i}\})-2\sum_{i=1}^{m} n_i'(n_i-n_i')c(\{a_{\alpha_i}\})$$

$$+\sum_{i=1}^{m} n_i' \sum_{\substack{j=1\\ j\ne i}}^{m} (n_j-n_j')c(\{a_{\alpha_j}\})+\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\ne i}}^{m} n_j'(n_i-n_i')c(\{a_{\alpha_j}\}) = 0 .$$

Observe that $\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\ne i}}^{m} n_j'(n_i-n_i')c(\{a_{\alpha_j}\}) = \sum_{i=1}^{m} n_i' \sum_{\substack{j=1\\ j\ne i}}^{m} (n_j-n_j')c(\{a_{\alpha_i}\})$.

Hence, we have

$$\sum_{1\le i<j\le m} n_i n_j c(\{a_{\alpha_i},a_{\alpha_j}\})+2\sum_{i=1}^{m} n_i^2 c(\{a_{\alpha_i}\})-(\sum_{i=1}^{m} n_i)\sum_{i=1}^{m} n_i c(\{a_{\alpha_i}\})$$

$$-[\sum_{1\le i<j\le m} n_i' n_j' c(\{a_{\alpha_i},a_{\alpha_j}\})+2\sum_{i=1}^{m} n_i'^2 c(\{a_{\alpha_i}\})-(\sum_{i=1}^{m} n_i')\sum_{i=1}^{m} n_i' c(\{a_{\alpha_i}\})]$$

$$-\sum_{i=1}^{m} n_i' \sum_{\substack{j=1\\ j\ne i}}^{m} (n_j-n_j')c(\{a_{\alpha_j},a_{\alpha_i}\}) - 2\sum_{i=1}^{m} n_i'(n_i-n_i')c(\{a_{\alpha_i}\})$$

$$+\sum_{i=1}^{m} n_i' \sum_{\substack{j=1\\ j\ne i}}^{m} (n_j-n_j')c(\{a_{\alpha_j}\}) + \sum_{i=1}^{m} n_i' \sum_{\substack{j=1\\ j\ne i}}^{m} (n_j-n_j')c(\{a_{\alpha_i}\}) = 0.$$

Thus

$$\sum_{1\le i<j\le m} n_i n_j c(\{a_{\alpha_i}, a_{\alpha_j}\}) + 2\sum_{i=1}^{m} n_i^2 c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} n_i)\sum_{i=1}^{m} n_i c(\{a_{\alpha_i}\})$$

$$-[\sum_{1\le i<j\le m} n_i' n_j' c(\{a_{\alpha_i}, a_{\alpha_j}\}) + 2\sum_{i=1}^{m} n_i'^2 c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} n_i')\sum_{i=1}^{m} n_i' c(\{a_{\alpha_i}\})]$$

$$- \sum_{i=1}^{m} n_i' [\sum_{\substack{j=1\\ j\neq i}}^{m} (n_j - n_j') c(\{a_{\alpha_j}, a_{\alpha_i}\}) + 2(n_i - n_i') c(\{a_{\alpha_i}\}) -$$

$$- \sum_{\substack{j=1\\ j\neq i}}^{m} (n_j - n_j') c(\{a_{\alpha_j}\}) - \sum_{\substack{j=1\\ j\neq i}}^{m} (n_j - n_j') c(\{a_{\alpha_i}\})] = 0.$$

By (iii"), we see that the last term in the above equation becomes zero.

Hence, we have

$$\sum_{1\le i<j\le m} n_i n_j c(\{a_{\alpha_i}, a_{\alpha_j}\}) + 2\sum_{i=1}^{m} n_i^2 c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} n_i)\sum_{i=1}^{m} n_i c(\{a_{\alpha_i}\})$$

$$-[\sum_{1\le i<j\le m} n_i' n_j' c(\{a_{\alpha_i}, a_{\alpha_j}\}) + 2\sum_{i=1}^{m} n_i'^2 c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} n_i')\sum_{i=1}^{m} n_i' c(\{a_{\alpha_i}\})] = 0.$$

Therefore

$$\sum_{1\le i<j\le m} n_i n_j c(\{a_{\alpha_i}, a_{\alpha_j}\}) + 2\sum_{i=1}^{m} n_i^2 c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} n_i)\sum_{i=1}^{m} n_i c(\{a_{\alpha_i}\})$$

$$= \sum_{1\le i<j\le m} n_i' n_j' c(\{a_{\alpha_i}, a_{\alpha_j}\}) + 2\sum_{i=1}^{m} n_i'^2 c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} n_i')\sum_{i=1}^{m} n_i' c(\{a_{\alpha_i}\}).$$

Hence

$$f(\prod_{i=1}^{m} a_{\alpha_i}^{n_i}) = f(\prod_{i=1}^{m} a_{\alpha_i}^{n_i'})$$

That is f is well defined.

Next, let x, y be any elements of G. Without loss of generality we may write $x = \prod_{i=1}^{m} a_{\alpha_i}^{n_i}$ and $y = \prod_{i=1}^{m} a_{\alpha_i}^{n'_i}$ where a_{α_i}, $i = 1,\ldots m$ are in $\mathcal{A}$ and n_i, n'_i, $i = 1,\ldots, m$ are integers.

Then we have

$$f(x\circ y) = f(\prod_{i=1}^{m} a_{\alpha_i}^{(n_i+n'_i)}) = \sum_{1\le i<j\le m} (n_i+n'_i)(n_j+n'_j)c(\{a_{\alpha_i},a_{\alpha_j}\})$$

$$+2\sum_{i=1}^{m}(n_i+n'_i)^2 c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m}(n_i+n'_i))\sum_{i=1}^{m}(n_i+n'_i)c(\{a_{\alpha_i}\}).$$

Thus

$$f(x\circ y) = \sum_{1\le i<j\le m}(n_in_j+n_in'_j+n'_in_j+n'_in'_j)c(\{a_{\alpha_i},a_{\alpha_j}\})$$

$$+2\sum_{i=1}^{m}(n_i^2+2n_in'_i+n'^2_i)c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} n_i)\sum_{i=1}^{m} n_i c(\{a_{\alpha_i}\})$$

$$-(\sum_{i=1}^{m} n_i)\sum_{i=1}^{m} n'_i c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} n'_i)\sum_{i=1}^{m} n_i c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} n'_i)\sum_{i=1}^{m} n'_i c(\{a_{\alpha_i}\}).$$

Similarly, we have

$$f(x\circ y^{-1}) = \sum_{1\le i<j\le m}(n_in_j-n_in'_j-n'_in_j+n'_in'_j)c(\{a_{\alpha_i},a_{\alpha_j}\})$$

$$+ 2\sum_{i=1}^{m}(n_i^2-2n_in'_i+n'^2_i)c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} n_i)\sum_{i=1}^{m} n_i c(\{a_{\alpha_i}\})$$

$$+ (\sum_{i=1}^{m} n_i)\sum_{i=1}^{m} n'_i c(\{a_{\alpha_i}\}) + (\sum_{i=1}^{m} n'_i)\sum_{i=1}^{m} n_i c(\{a_{\alpha_i}\}) - (\sum_{i=1}^{m} n'_i)\sum_{i=1}^{m} n'_i c(\{a_{\alpha_i}\}).$$

Hence

$$f(x\circ y)+f(x\circ y^{-1}) = \sum_{1\leq i<j\leq m} (2n_i n_j+2n_i' n_j')c(\{a_{\alpha_i},a_{\alpha_j}\})+2\sum_{i=1}^{m}(2n_i^2+2n_i'^2)c(\{a_{\alpha_i}\})$$

$$-2(\sum_{i=1}^{m} n_i)\sum_{i=1}^{m} n_i c(\{a_{\alpha_i}\})-2(\sum_{i=1}^{m} n_i')\sum_{i=1}^{m} n_i' c(\{a_{\alpha_i}\}),$$

$$= 2[\sum_{1\leq i<j\leq m} n_i n_j c(\{a_{\alpha_i},a_{\alpha_j}\})+2\sum_{i=1}^{m} n_i^2 c(\{a_{\alpha_i}\})-(\sum_{i=1}^{m} n_i)\sum_{i=1}^{m} n_i c(\{a_{\alpha_i}\})].$$

$$+2[\sum_{1\leq i<j\leq m} n_i' n_j' c(\{a_{\alpha_i},a_{\alpha_j}\})+2\sum_{i=1}^{m} n_i'^2 c(\{a_{\alpha_i}\})-(\sum_{i=1}^{m} n_i')\sum_{i=1}^{m} n_i' c(\{a_{\alpha_i}\})].$$

Therefore

$$f(x\circ y)+f(x\circ y^{-1}) = 2f(x)+2f(y)$$

for all x, y in G.

3.3 Examples

In this section we illustrate how our main theorem can be applied to particular cases.

Example 3.3.1 Let $G = \mathbb{Z}_4 \oplus \mathbb{Z}_6$, $G' = \mathbb{Z}$. To find all $f : G \longrightarrow G'$ satisfying

(*) $\quad f(x+y)+f(x-y) = 2f(x)+2f(y)$

for all x, y ε G, we first find a set of generators of G. Observe that G can be generated by two generators a and b with defining relations

$$4a = 0, \quad 6b = 0.$$

Let $\mathcal{A} = \{a,b\}$, $\mathcal{A}^{(1)} = \{\{a\},\{b\}\}$ and $\mathcal{A}^{(2)} = \{\{a,b\}\}$.

Hence, according to theorem 3.2.1, $f : G \longrightarrow G'$ satisfies (*) if and only if there exists a function $c : \mathcal{A}^{(1)} \cup \mathcal{A}^{(2)} \longrightarrow G'$ such that for the defining relations

(1) $4a = 0,$

and

(2) $6b = 0,$

we have, respectively,

(i-1) $2(4^2c(\{a\}))-4(4c(\{a\})) = 0,$

(ii-1) $4c(\{a,b\})-4c(\{a\})-4c(\{b\}) = 0,$

(iii-1) $2.4c(\{a\}) = 0,$

and

(i-2) $2(6^2c(\{b\}))-6(6c(\{b\})) = 0,$

(ii-2) $6c(\{a,b\})-6c(\{b\})-6c(\{a\}) = 0,$

(iii-2) $2.6c(\{b\}) = 0,$

and for any $x = ma + nb$, we have

(3.3.1.1) $f(x) = mnc(\{a,b\})+2(m^2c(\{a\})+n^2c(\{b\}))-(m+n)(mc(\{a\})+nc(\{b\})).$

By (iii-1) and (iii-2), we have

$$8c(\{a\}) = 0 \quad \text{and}$$

$$12c(\{b\}) = 0.$$

Hence

(3.3.1.2) $c(\{a\}) = 0$ and

(3.3.1.3) $c(\{b\}) = 0.$

Replacing $c(\{a\})$ and $c(\{b\})$ in (ii-1) by zero, we get

$$4c(\{a,b\}) = 0.$$

Hence

(3.3.1.4) $c(\{a,b\}) = 0.$

It is clear that the values of $c(\{a\})$, $c(\{b\})$, $c(\{a,b\})$ given in (3.3.1.2),(3.3.1.3),(3.3.1.4) satisfy (i-1),(ii-1),(iii-1),(i-2), (ii-2) and (iii-2). Substituting these values in (3.3.1.1) we find that

$$f(x) = 0$$

is the only solution of (*) from G to G'

<u>Example 3.3.2</u> Let $G = \mathbb{Z}_4 \oplus \mathbb{Z}_6$, $G' = \mathbb{Z}_3$. We shall find all $f : G \longrightarrow G'$ satisfying

(*) $$f(x+y)+f(x-y) = 2f(x)+2f(y)$$

for all x, y in G.

Observe that G can be generated by two generators a and b with defining relations

$$4a = 0, \quad 6b = 0.$$

Let $\mathcal{A} = \{a,b\}$, $\mathcal{A}^{(1)} = \{\{a\},\{b\}\}$ and $\mathcal{A}^{(2)} = \{\{a,b\}\}$.

Hence, according to theorem 3.2.1, $f : G \longrightarrow G'$ satisfies (*) if and only if there exists a function $c : \mathcal{A}^{(1)} \cup \mathcal{A}^{(2)} \longrightarrow G'$ such that for the defining relations

(1) $$4a = 0,$$

and

(2) $$6b = 0.$$

we have, respectively,

(i-1) $$2(4^2c(\{a\}))-4(4c(\{a\}) = 0$$

(ii-1) $$4c(\{a,b\})-4c(\{a\})-4c(\{b\}) = 0,$$

(iii-1) $$2.4c(\{a\}) = 0,$$

and

$$(i\text{-}2) \quad 2(6^2c(\{b\}))-6(6c(\{b\})) = 0,$$

$$(ii\text{-}2) \quad 6c(\{a,b\})-6c(\{b\})-6c(\{a\}) = 0,$$

$$(iii\text{-}2) \quad 2.6c(\{b\}) = 0,$$

and for any $x = ma+nb$, we have

$$(3.3.2.1) \quad f(x) = mnc(\{a,b\})+2(m^2c(\{a\})+n^2c(\{b\}))-(m+n)(mc(\{a\})+nc(\{b\})).$$

By (iii-2), we see that $c(\{b\})$ is an arbitrary element of G'.

By (iii-1), we have

$$2c(\{a\}) = 0.$$

Hence

$$(3.3.2.2) \quad c(\{a\}) = 0.$$

By (ii-1), we have

$$c(\{a,b\})-c(\{a\})-c(\{b\}) = 0,$$

$$(3.3.2.3) \quad c(\{a,b\}) = c(\{a\})+c(\{b\}).$$

Substituting the value of $c(\{a\})$ from (3.3.2.2) in (3.3.2.3), we get

$$c(\{a,b\}) = c(\{b\})$$

We see that

$$c(\{a\}) = 0,$$

$$c(\{b\}) = K,$$

$$c(\{a,b\}) = K,$$

where K is any element of $\mathbb{Z}_3$, satisfy (i-1),(ii-1),(iii-1),(i-2),(ii-2) and (iii-2). Substituting these values in (3.3.2.1) we find that

$$f(x) = n^2K,$$

for any $x = ma+nb$ in G.

<u>Example 3.3.3</u> Let $G = \mathbb{Z} \oplus \mathbb{Z}_6$, $G' = \mathbb{Z}$. We shall find all $f : G \longrightarrow G'$ satisfying

$$(*) \qquad f(x+y)+f(x-y) = 2f(x)+2f(y)$$

for all x, y in G.

Observe that G can be generated by two generators a and b with only one defining relation

$$(1) \qquad 6b = 0.$$

Let $\mathcal{A} = \{a,b\}$, $\mathcal{A}^{(1)} = \{\{a\},\{b\}\}$ and $\mathcal{A}^{(2)} = \{\{a,b\}\}$.

Hence, according to theorem 3.2.1, $f : G \longrightarrow G'$ satisfies (*) if and only if there exists a function $c : \mathcal{A}^{(1)} \cup \mathcal{A}^{(2)} \longrightarrow G'$ such that for defining relation (1) we have

$$(i) \qquad 2(6^2c(\{b\}))-6(6c(\{b\})) = 0,$$

$$(ii) \qquad 6c(\{a,b\})-6c(\{b\})-6c(\{a\}) = 0,$$

$$(iii) \qquad 2.6c(\{b\}) = 0,$$

and for any $x = ma+nb$, we have

$$(3.3.3.1) \quad f(x) = mnc(\{a,b\})+2(m^2c(\{a\})+n^2c(\{b\}))-(m+n)(mc(\{a\})+nc(\{b\})).$$

By (iii) we have

$$12c(\{b\}) = 0.$$

Hence

$$(3.3.3.2) \qquad c(\{b\}) = 0.$$

Substituting the value of $c(\{b\})$ from (3.3.3.2) in (ii), we get

$$6c(\{a,b\})-6c(\{a\}) = 0.$$

Hence

$$c(\{a,b\})-c(\{a\}) = 0.$$

Therefore

(3.3.3.3) $$c(\{a,b\}) = c(\{a\}).$$

We see that

$$c(\{a\}) = K,$$
$$c(\{b\}) = 0,$$
$$c(\{a,b\}) = K,$$

where K is any element of $\mathbb{Z}$, satisfy (i), (ii) and (iii).

Substituting these values in (3.3.3.1) we find that

$$f(x) = m^2K,$$

for any $x = ma+nb$ in G.

Example 3.3.4 let $G = Q$, $G' = \mathbb{R}$. We shall find all $f : G \longrightarrow G'$ satisfying

(*) $$f(x+y)+f(x-y) = 2f(x)+2f(y)$$

for all x, y in G.

Observe that the additive group of rational number Q can be generated by generators $a_n = \frac{1}{n}$, $n = 1,2,\ldots$, with defining relations

$$(n) \quad a_1 - na_n = 0, \quad n = 2,3,\ldots\ldots$$

Let $\mathcal{A}$ be the set of all these generators of Q and let

$$\mathcal{A}^{(1)} = \{\{a_i\}: a_i \in \mathcal{A}\}, \quad \mathcal{A}^{(2)} = \{\{a_i,a_j\}: a_i, a_j \in \mathcal{A},\ i \neq j\}.$$

Hence, according to theorem 3.2.1, $f: G \to G'$ satisfies (*) if and only if there exists a function $c : \mathcal{A}^{(1)} \cup \mathcal{A}^{(2)} \longrightarrow G'$ such that for any defining relation

$$(n) \quad a_1 - na_n = 0,$$

we have

(i-n) $-nc(\{a_1,a_n\})+2(c(\{a_1\})+n^2c(\{a_n\}))-(1-n)(c(\{a_1\})-nc(\{a_n\})) = 0,$

(ii-n) $c(\{a_1,a_\beta\})-nc(\{a_n,a_\beta\})-(c(\{a_1\})-nc(\{a_n\}))-(1-n)c(\{a_\beta\}) = 0,$

for all $\beta \neq 1, n$,

(iii-n)
$$\begin{cases} -nc(\{a_1,a_n\})+nc(\{a_n\})+nc(\{a_1\})+2c(\{a_1\}) = 0, \\ c(\{a_1,a_n\})-c(\{a_1\})-c(\{a_n\})+2(-n)c(\{a_n\}) = 0, \end{cases}$$

and for any $\gamma \in Q$, $\gamma = \frac{p}{q} = p.a_q$ where p, q are integers and $q \neq 0$,

(3.3.4.1)
$$f(\gamma) = 2p^2c(\{a_q\})-p(pc(\{a_q\})),$$
$$= p^2c(\{a_q\}).$$

By the second equation in (iii-n) we find that

(3.3.4.2)
$$c(\{a_1,a_n\}) = c(\{a_1\})+(2n+1)c(\{a_n\}).$$

Substituting the value of $c(\{a_1,a_n\})$ from (3.3.4.2) in (i-n) and simplify, we get

$$c(\{a_1\})-n^2c(\{a_n\}) = 0.$$

Therefore

(3.3.4.3)
$$c(\{a_n\}) = \frac{1}{n^2}\, c(\{a_1\}).$$

Substituting the value $c(\{a_n\})$ from (3.3.4.3) in (3.3.4.2) we get

(3.3.4.4)
$$c(\{a_1,a_n\}) = \frac{(1+n)^2}{n^2}\, c(\{a_1\}).$$

Applying (3.3.4.3) and (3.3.4.4) to the left hand side of equation (ii-n) and simplify, we get

(3.3.4.5)
$$c(\{a_n,a_\beta\}) = \frac{(n+\beta)^2}{n^2\beta^2}\, c(\{a_1\}),$$

for all $\beta \neq 1, n$.

We see that $c(\{a_1\})$ can be chosen arbitrary. Let

$$c(\{a_1\}) = K,$$

where K is an arbitrary element of $\mathbb{R}$.

Hence, from (3.3.4.3),(3.3.4.4),(3.3.4.5), we have

$$c(\{a_n\}) = \frac{K}{n^2},$$

$$c(\{a_1, a_n\}) = \frac{K(1+n)^2}{n^2}$$

$$c(\{a_n, a_\beta\}) = \frac{K(n+\beta)^2}{n^2\beta^2}.$$

It can be verified that the values of $c(\{a_n\})$, $c(\{a_1, a_n\})$, $c(\{a_n, a_\beta\})$ given above satisfy (i-n), (ii-n) and (iii-n). Substituting

$c(\{a_q\}) = \frac{K}{q^2}$ in (3.3.4.1), we get

$$f(\gamma) = \gamma^2 K .$$

Therefore $f(\gamma) = \gamma^2 K$ for all $\gamma \in Q$, where K is a real number.