CHAPTER V

FRACTIONAL INTEGRAL

5.1 Definition

In this chapter we extend the classical theorem on fractional integrals of functions in H^p to be n-dimensional case.

We define, for a function f in $L^p(\mathbb{R}_n)$, its fractional integral of order α to be

$$[J_\alpha(f)](X) = \frac{1}{\eta_\alpha} \int_{\mathbb{R}_n} \frac{f(X-Y)}{|Y|^{n-\alpha}} \, dY \qquad (5.1.1)$$

where $$\eta_\alpha = \frac{\pi^{\frac{1}{2}n} 2^\alpha \Gamma(\frac{1}{2}\alpha)}{\Gamma(\frac{1}{2}(n-\alpha))} , \quad 1 \leq p < \infty .$$

It can be checked easily that the integral in (5.1.1) converges for almost all X provided that $0 < \alpha < n/_p$.

5.2 On semigroup property of H^p

For the systems of conjugate harmonic function F in H^p, we will show that their fractional integral satisfy the "semigroup property" for appropriate values of p, i.e., $J_\alpha(J_\beta(F)) = J_{\alpha+\beta}(F)$.

Lemma 5.2.1 Let $s(X,y) \geq 0$ be a subharmonic function defined on $\mathbb{R}^+_{n+1}$ satisfying

$$\int_{\mathbb{R}_n} [s(X,y)]^p \, dx \leq c^p < \infty \qquad (5.2.2)$$

where $1 \leq p < \infty$ and c is independent of $y > 0$. Then

$$s(X,y) \leq cy^{-(\frac{n}{p})} . \qquad (5.2.3)$$

Furthermore, if $0 < \frac{1}{\gamma} < y \leq \gamma$, $\gamma \in \mathbb{R}$, then $s(X,y) \to 0$ uniformly in y as $|X| \to \infty$.

Proof. Since s^p, being a convex function of a subharmonic function, is subharmonic (see Helms, [4]). Thus letting ν_{n+1} be the volumn of the unit sphere in $\mathbb{R}_{n+1}$, we have for $(X,y) \in \mathbb{R}^+_{n+1}$

$$[s(X,y)]^p \leq \frac{1}{\nu_{n+1}y^{n+1}} \int_{B((X,y),y)} [s(Z,t)]^p \, dZ \, dt \qquad ([4])$$

where $B((X,y),y)$ is the ball in $\mathbb{R}_{n+1}$ with center at (X,y) and radius y.

Hence $$[s(X,y)]^p \leq \frac{1}{\nu_{n+1}y^{n+1}} \int_{|X-Z|^2+(y-t)^2 \, y^2} [s(Z,t)]^p dZ \, dt$$

$$\leq \frac{1}{\nu_{n+1}y^{n+1}} \int_{0<t<2y} [s(Z,t)]^p dZ \, dt$$

$$= \frac{1}{\nu_{n+1}y^{n+1}} \int_0^{2y} \{ \int_{\mathbb{R}_n} [s(Z,t)]^p dZ \} \, dt$$

$$\leq \frac{2c^p y}{\nu_{n+1}y^{n+1}} < c^p y^{-n}$$

and (5.2.3) is established.

In order to prove the last part of the theorem we observe that if

$$I_k = \{(X,y) : k-1 \leqslant |X| < k \ , \ 0 < y \leqslant \frac{1}{\gamma} + \gamma\}$$

$$k = 1,2,3,\ldots, \quad \text{then}$$

$$\sum_{k=1}^{\infty} \int_{I_k} [s(Z,t)]^p \, dZdt = \int_0^{\frac{1}{\gamma}+\gamma} \{ \int_{\mathbb{R}_n} [s(Z,t)]^p \, dZ\} \, dt$$

$$\leqslant c^p(\frac{\gamma^2+1}{\gamma}) \qquad < \infty .$$

Hence $\int_{I_k} [s(Z,t)]^p \, dZ \, dt \to 0$ as $k \to \infty$. (5.2.4)

Suppose that (X,y) is such that $0 < \frac{1}{\gamma} \leqslant y \leqslant \gamma$ then (X,y) must belongs to I_k for some k. We note that $B((X,y), \frac{1}{\gamma})$ is contained in $I_{k-1} \cup I_k \cup I_{k+1}$ (where I_0 is the null set). Thus

$$[s(X,y)]^p \leqslant \frac{\gamma^{n+1}}{\nu_{n+1}} \int_{|Z|^2+t^2<(\frac{1}{\gamma})^2} [s(X+Z,y+t)]^p \, dZ \, dt$$

$$\leqslant \frac{\gamma^{n+1}}{\nu_{n+1}} \sum_{j=k-1}^{k+1} \int_{I_j} [s(Z,t)]^p dZ \, dt .$$

But by (5.2.4), the last term tends to zero as $k \to \infty$ and then the last conclusion follows.

Let $h(X,y)$ be the Poisson integral of a function f in $L^p(\mathbb{R}_n)$, $p \geqslant 1$, i.e.,

$$h(X,y) = \int_{\mathbb{R}_n} P(X-Z,y)\, f(Z)dZ \quad , \quad \text{then}$$

$$h(X,y) \leqslant \left(\int_{\mathbb{R}_n} P(X-z,y)dZ\right)^{1/q} \left(\int_{\mathbb{R}_n} P(X-Z,y)|f(Z)|^p\, dZ\right)^{1/p}$$

where $(\frac{1}{p} + \frac{1}{q} = 1)$

$$= \left(\int_{\mathbb{R}_n} P(X-Z,y)|f(Z)|^p\, dZ\right)^{1/p} .$$

Therefore $\int_{\mathbb{R}_n} |h(X,y)|^p dX \leqslant \int_{\mathbb{R}_n} \left(\int_{\mathbb{R}_n} P(X,Z,y)|f(Z)|^p\, dZ\right) dX$

and by Fubini's Theorem we get

$$\int_{\mathbb{R}_n} \left(\int_{\mathbb{R}_n} P(X-Z,y)|f(Z)|^p dZ\right)dX = \int_{\mathbb{R}_n} |f(Z)|^p \int_{\mathbb{R}_n} P(X-Z,y)dXdZ$$

$$= \int_{\mathbb{R}_n} |f(Z)|^p\, dZ$$

so we get $\int_{\mathbb{R}_n} |h(X,y)|^p dX \leqslant \int_{\mathbb{R}_n} |f(Z)|^p\, dZ$. (5.2.5)

Similarly, if $h(X,y)$ is the Poisson integral of a Radon measure in $\mathbb{R}_n$, then

$$\int_{\mathbb{R}_n} |h(X,y)|dX \leqslant \int_{\mathbb{R}_n} d|\mu|(X) \quad . \qquad (5.2.6)$$

Lemma 5.2.7. Let $h(X,y)$ be a harmonic function defined in $\mathbb{R}^+_{n+1}$, satisfying

$$\int_{\mathbb{R}_n} |h(X,y)|^p\, dX \leqslant c^p < \infty \qquad (5.2.8)$$

for all $y > 0$, and c is independent of $y > 0$, then

$$h(X,y+s) = \int_{\mathbb{R}_n} h(Z,s)P(X-Z,y)dZ \qquad \text{for all} \quad s > 0.$$

<u>Proof</u> By (3.4.1), $w_s(X,y) = \int_{\mathbb{R}_n} h(Z,s)P(X-Z,y)dZ$ is harmonic in $\mathbb{R}^+_{n+1}$, By (3.1.2) we have

$$w_s(X,y)-h(X,s) = \int_{\mathbb{R}_n} [h(Z,s)-h(X,s)]P(X-Z,y)dZ$$

$$= \Big(\int_{|X-Z|<r} + \int_{|X-Z|\geq r}\Big)[h(Z,s)-h(X,s)]P(X-Z,y)dZ$$

$$= I_1 + I_2 \quad \text{say.}$$

Since $h(X,y)$ is harmonic, $|h(X,y)|$ is subharmonic and by (5.2.8) it satisfies the assumptions in lemma (5.2.1). Thus, by the last part of the lemma, $h(X,s)$ is uniformly continuous in $\mathbb{R}_n$, for each $s > 0$. From this it follows that, if r is small enough, I_1 is uniformly small.

And by using (5.2.3) and (5.2.8), we also have

$$|I_2| \leq \int_{|X-Z|\geq r} (|h(Z,s)|+|h(X,s)|)\, P(X-Z,y)\, dZ$$

$$\leq 2\, c\, s^{-(\frac{n}{p})} \int_{|X-Z|\geq r} P(X-Z,y)dZ \,.$$

By (3.1.3), the last integral tends to zero as $y \to 0$. This shows that $w_s(X,y) \to h(X,s)$ uniformly in X as $y \to 0$, i.e., for any ε, positive real number, given , there is a $\delta > 0$ such that for any y less than δ, $|w_s(X,y)-h(X,s)| < \varepsilon$ for all X in $\mathbb{R}_n$.

We see by (5.2.5) that $|w_s(X,y)|$ satisfies condition (5.2.2). And by (3.4.1), $w_s(X,y)$ is harmonic and hence $|w_s(X,y)|$ is subharmonic. Thus both $|w_s(X,y)|$ and $|h(X,y+s)|$ satisfies the assumptions of lemma 5.2.1. Hence by (5.2.3), for y large enough say y_0, $|w_s(X,y)-h(X,y+s) < \varepsilon$ for all $y > y_0$, X in $\mathbb{R}_n$. Finally, the last part of lemma (5.2.1) implies that if $\delta \leq y \leq y_0$ there is a real number r such that for all X with $|X| \geq r$, $|w_s(X,y)-h(X,y+s)| < \varepsilon$

We see that on the boundary of a region

$D = \{(X,y) : |X| \leq r, \delta \leq y \leq y_0\}$ the harmonic function $w_s(X,y)-h(X,y+s)$ is small in absolute value, i.e., $|w_s(X,y)-h(X,y+s)| < \varepsilon$, by maximum principle, $|w_s(X,y)-h(X,y+s)| < \varepsilon$ throughout D.

Then, summing up, $|w_s(X,y)-h(X,y+s)| < \varepsilon$ throughout $\mathbb{R}^+_{n+1}$. Thus, we obtain $w_s(X,y) = h(X,y+s)$.

<u>Theorem 5.2.9</u> Let $f(X)$ be a function in $L^p(\mathbb{R}_n)$, $1 \leq p < \infty$, and $h(X,y)$ its Poisson integral. Then,

$$[J_\alpha(f)](X) = \frac{1}{\Gamma(\alpha)} \int_0^\infty h(X,y)y^{\alpha-1}\, dy \qquad (5.2.10)$$

where $0 < \alpha < \frac{n}{p}$.

Furthermore, $$h_\alpha(X,y) = \frac{1}{\Gamma(\alpha)} \int_0^\infty h(X,y+s)s^{\alpha-1}\, ds \qquad (5.2.11)$$

is the Poisson integral of $J_\alpha(f)$.

<u>Proof</u> The function $|h(X,y)|$ satisfies the hypothesis of lemma (5.2.1). Thus by (5.2.3) the integrand in (5.2.10) is absolutely integrable.

By considering the negative and positive parts of f we can reduce the proof of the theorem to the case $f \geq 0$. With this restriction on f, our various applications of Fubini's Theorem are justified.

Since
$$h(X,y) = \int_{\mathbb{R}_n} P(Z,y)f(X-Z)\,dZ ,$$

we have
$$\int_0^\infty h(X,y)y^{\alpha-1}dy = \int_{\mathbb{R}_n} \{\int_0^\infty P(Z,y)y^{\alpha-1}dy\}f(X-Z)\,dZ .$$

And
$$\int_0^\infty P(Z,y)y^{\alpha-1}dy = \frac{1}{c_n}\int_0^\infty \frac{y^\alpha}{(|Z|^2+y^2)^{\frac{1}{2}(n+1)}}\,dy$$

$$= \frac{|Z|^{\alpha-n}}{c_n}\int_0^\infty \frac{y^\alpha}{(1+y^2)^{\frac{1}{2}(n+1)}}\,dy.$$

(The last equality follows from the change of variable $s = y/|Z|$ and then replacing y for s).

Hence
$$\int_0^\infty h(X,y)y^{\alpha-1}\,dy = \frac{1}{c_n}\{\int_0^\infty \frac{y^\alpha}{(1+y^2)^{\frac{1}{2}(n+1)}}\,dy\}\int_{\mathbb{R}_n} \frac{f(X-Z)}{|Z|^{n-\alpha}}\,dZ .$$

By letting $y^2 = x/(1-x)$,

$$\int_0^\infty \frac{y^\alpha}{(1+y^2)^{\frac{1}{2}(n+1)}}\,dy = \int_0^1 \frac{\left(\frac{x}{(1-x)}\right)^{\frac{\alpha}{2}}}{\left(1+\left(\frac{x}{1-x}\right)\right)^{\frac{1}{2}(n+1)}} \frac{1}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}\,dx$$

$$= \int_0^1 x^{(\frac{\alpha}{2}-1)}(1-x)^{\frac{1}{2}(n-\alpha)-1}\,dx$$

$$= \frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{1}{2}(n-\alpha))}{\Gamma(\frac{n}{2})} .$$

Since $$c_n = \frac{\pi^{\frac{n}{2}}\,\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})} ,$$

then $$\frac{1}{c_n}\int_0^\infty \frac{y^\alpha}{(1+y^2)^{\frac{1}{2}(n+1)}}\,dy = \frac{\pi^{\frac{n}{2}}\Gamma(\frac{1}{2})\Gamma(\frac{\alpha}{2})\Gamma(\frac{1}{2}(n-\alpha))}{\Gamma(\frac{1}{2}(n+1))\Gamma(\frac{n}{2})} .$$

By using "duplication formula" (see[7], page 57) for Gamma function $\Gamma(2x)\Gamma(\frac{1}{2}) = 2^{2x-1}\,\Gamma(x)\Gamma(x+\frac{1}{2})$ we get

$$\int_0^\infty h(X,y)y^{\alpha-1}\,dy = \Gamma(\alpha)[J_\alpha(f)](X)$$

and then (5.2.10) is established .

The equation (5.2.11) is, then, an immediate consequence of the "semigroup property" of the Poisson integral transform :

$$h(X,y+s) = \int_{\mathbb{R}_n} h(Z,s)P(X-Z,y)dZ , \quad \text{for all } y,\ s > 0$$

in lemma (5.2.7) .

Theorem (5.2.9) motivates the following definition :
If $F(X,y)$ is a system of conjugate harmonic function in $\mathbb{R}^+_{n+1}$ we defined its (real-valued) "fractional integral of order α", $\alpha > 0$, to be

$$F_\alpha(X,y) = [J_\alpha(F)](X,y) = \frac{1}{\Gamma(\alpha)} \int_0^\infty F(X,y+s)s^{\alpha-1}ds \qquad (5.2.12)$$

whenever this integral exists.

Theorem 5.2.13

a) The integral in (5.2.12) converges absolutely for each (X,y) in $\mathbb{R}^+_{n+1}$ provided F is in H^p and $\frac{(n-1)}{n} \leq p < \frac{n}{\alpha}$.

b) If F is in H^p, $\alpha > 0$, $\beta > 0$ and $\frac{n-1}{n} \leq p < \frac{n}{\alpha+\beta}$,

then
$$J_\alpha(J_\beta(F)) = J_{\alpha+\beta}(F) .$$

Proof

a) Let $q = \frac{pn}{n-1}$. Then since $|F(X,y)|^{\frac{n-1}{n}}$ is subharmonic, we have

$$\int_{\mathbb{R}_n} |F(X,y)|^{(\frac{n-1}{n})q} dX = \int_{\mathbb{R}_n} |F(X,y)|^p dX \leq c^q < \infty.$$

Let $h(X,y)$ be the harmonic majorant of $|F(X,y)|^{\frac{n-1}{n}}$ obtained in theorem 3.2.3. And from (5.2.5) and (5.2.6), we have

$$\int_{\mathbb{R}_n} [h(X,y)]^q dX \leq c^q .$$

Thus, by (5.2.3) $h(X,y) \leq cy^{-n/q}$

and $$|F(X,y)| \leq c^{n/n-1} y^{n/p} .$$

Hence $$\int_0^\infty |F(X,y+s)| s^{\alpha-1} ds \leq c^{n/n-1} \int_0^\infty (y+s)^{-n/p} s^{\alpha-1} ds .$$

Since $p < n/_\alpha$ and $\alpha > 0$, the last integral is finite.

b) To show $J_\alpha(J_\beta(F)) = J_{\alpha+\beta}(F)$, we must show that

$$\frac{1}{\Gamma(\alpha+\beta)} \int_0^\infty F(X,y+s)s^{\alpha+\beta-1}\, ds = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty r^{\alpha-1}\{\int_0^\infty F(X,y+s+t)t^{\beta-1}dt\}dr .$$

On the orther hand, the last integral is equal to

$$\int_0^\infty r^{\alpha-1}\{\int_r^\infty F(X,y+t)(t-r)^{\beta-1}dt\}dr = \int_0^\infty F(X,y+t)\{\int_0^t (t-r)^{\beta-1}r^{\alpha-1}dr\}dt .$$

Thus, we need only verify that

$$\frac{1}{\Gamma(\alpha+\beta)} s^{\alpha+\beta-1} = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^s (s-r)^{\beta-1}\, r^{\alpha-1}\, dr$$

or

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \frac{1}{s^{\alpha+\beta-1}} \int_0^s (s-r)^{\beta-1}\, r^{\alpha-1}\, dr . \quad (5.2.14)$$

Consider the right side of (5.2.14),

let $t = \frac{r}{s}$ we then get

$$\frac{1}{s^{\alpha+\beta-1}} \int_0^s (s-r)^{\beta-1}\, r^{\alpha-1}dr = \int_0^1 (1-t)^{\beta-1}t^{\alpha-1}\, dt$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

and we get (5.2.14), b) is now proved.

We can not extend this result to the class of G^p. The following example will justify that the conclusions of the theorem do not hold for the case of G^p.

Example 5.16 Let $F(x,y) = (u(x,y), v(x,y))$

$$= (x,y), \quad x \in \mathbb{R}, \quad y > 0.$$

Let $\alpha = \frac{1}{2}$ we see that $F \in G' - H'$.

But the integral in (5.2.12) diverges.