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นายรณสรรพ์ ชินรัมย์

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GENERALIZED MATRIX RINGS

HAVING THE INTERSECTION PROPERTY OF QUASI-IDEALS

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ให้ R เป็นริง สำหรับ A, B ซึ่งเป็นเซตย่อยไม่ว่างของ R ให้ AB แทนเซตของผลบวกอันตะทั้งหมดที่อยู่ในรูปแบบ $\sum a_i b_i$ โดยที่ $a_i \in A$ และ $b_i \in B$ จะเรียกริงย่อย Q ของ R ว่า *ควอซี-ไอเดียล* ของ R ถ้า $RQ \cap QR \subseteq Q$ เป็นที่รู้กันว่า ส่วนร่วมของไอเดียลซ้ายและไอเดียลขวาของ R เป็นควอซี-ไอเดียล แต่ควอซี-ไอเดียลของ R ไม่จำเป็นต้องมีสมบัติเช่นนี้ จะเรียกริง R ว่า *มีสมบัติส่วนร่วมของควอซี-ไอเดียล* ถ้าทุกควอซี-ไอเดียลของ R เป็นส่วนร่วมของไอเดียลซ้ายและไอเดียลขวาของ R

ต่อจากนี้ไป ให้ R เป็นริงการหาร และ m และ n เป็นจำนวนเต็มบวก ให้ $M_{m,n}(R)$ แทนเซตของเมทริกซ์ขนาด $m \times n$ บน R ทั้งหมด สำหรับ $P \in M_{n,m}(R)$ ให้ $(M_{m,n}(R), +, P)$ เป็นริง $M_{m,n}(R)$ ภายใต้การบวกปกติ และการคูณ $*$ ซึ่งนิยามโดย $A * B = APB$ สำหรับทุก $A, B \in M_{m,n}(R)$ ให้ $M_{n,n}(R) = M_n(R)$ และ $SU_n(R)$ แทนเซตของเมทริกซ์สามเหลี่ยมบนโดยแท้ใน $M_n(R)$ ทั้งหมด สำหรับ P ที่เป็นเมทริกซ์สามเหลี่ยมบนขนาด $n \times n$ บน R นิยาม $(SU_n(R), +, P)$ ในทำนองเดียวกัน

ผลสำคัญของการวิจัยมีดังนี้

ทฤษฎีบท 1 สำหรับ $P \in M_{n,m}(R)$ ริง $(M_{m,n}(R), +, P)$ มีสมบัติส่วนร่วมของควอซี-ไอเดียล ก็ต่อเมื่อ ไม่ $P = 0$ ก็ค่าลำดับชั้น(P) = ค่าน้อยสุด$\{m, n\}$

บทแทรก 2 สำหรับ $P \in M_n(R)$ ริง $(M_n(R), +, P)$ มีสมบัติส่วนร่วมของควอซี-ไอเดียล ก็ต่อเมื่อ ไม่ $P = 0$ ก็ P เป็นเมทริกซ์ที่หาตัวผกผันได้

ทฤษฎีบท 3 สำหรับ P ที่เป็นเมทริกซ์สามเหลี่ยมบนขนาด $n \times n$ บน R ริง $(SU_n(R), +, P)$ มีสมบัติส่วนร่วมของควอซี-ไอเดียล ก็ต่อเมื่อ หนึ่งในข้อความต่อไปนี้เป็นจริง

(i) $n \le 3$

(ii) $n = 4$ และ $P_{22} = 0$ หรือ $P_{33} = 0$

(iii) $n > 4$, $P_{ij} = 0$ สำหรับทุก $i, j \in \{3, 4, \ldots, n-2\}$ และ

- (a) $P_{2j} = 0$ สำหรับทุก $j \in \{2, 3, \ldots, n-2\}$ หรือ
- (b) $P_{i,n-1} = 0$ สำหรับทุก $i \in \{3, 4, \ldots, n-1\}$

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Let R be a ring. For nonempty subsets $A, B \subseteq R$, let AB denote the set of all finite sums of the form $\sum a_i b_i$ where $a_i \in A$ and $b_i \in B$. A subring Q of R is called a *quasi-ideal* of R if $RQ \cap QR \subseteq Q$. It is known that the intersection of a left ideal and a right ideal of R is a quasi-ideal but a quasi-ideal of R need not be obtained in this way. The ring R is said to *have the intersection property of quasi-ideals* if every quasi-ideal of R is the intersection of a left ideal and a right ideal of R.

In the remainder, let R be a division ring and m and n positive integers. We denote by $M_{m,n}(R)$ the set of all m x n matrices over R. For $P \in M_{n,m}(R)$, let $(M_{m,n}(R), +, P)$ be the ring $M_{m,n}(R)$ under usual addition and the multiplication $*$ defined by $A * B = APB$ for all $A, B \in M_{m,n}(R)$. Let $M_{n,n}(R) = M_n(R)$ and we denote by $SU_n(R)$ the set of all strictly upper triangular matrices in $M_n(R)$. For an upper triangular n x n matrix P over R, let $(SU_n(R), +, P)$ be defined similarly.

The main results of this research are as follows:

Theorem 1. For $P \in M_{n,m}(R)$, the ring $(M_{m,n}(R), +, P)$ has the intersection property of quasi-ideals if and only if either $P = 0$ or rank $P = \min\{m, n\}$.

Corollary 2. For $P \in M_n(R)$, the ring $(M_n(R), +, P)$ has the intersection property of quasi-ideals if and only if either $P = 0$ or P is invertible.

Theorem 3. For an upper triangular n x n matrix P over R, the ring $(SU_n(R), +, P)$ has the intersection property of quasi-ideals if and only if one of the following statements holds.

(i) $n \leq 3$.

(ii) $n = 4$ and $P_{22} = 0$ or $P_{33} = 0$.

(iii) $n > 4$, $P_{ij} = 0$ for all $i, j \in \{3, 4, \ldots, n-2\}$ and

  (a) $P_{2j} = 0$ for all $j \in \{2, 3, \ldots, n-2\}$ or

  (b) $P_{i,n-1} = 0$ for all $i \in \{3, 4, \ldots, n-1\}$.

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CONTENTS

CHAPTER I

INTRODUCTION

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote respectively the set of all positive integers, the set of all integers and the set of all real numbers. For a ring R and $n \in \mathbb{N}$, let

$$\begin{aligned} M_n(R) &= \text{ the full } n \times n \text{ matrix ring over } R, \\ U_n(R) &= \text{ the ring of all upper triangular } n \times n \text{ matrices over } R \text{ and} \\ SU_n(R) &= \text{ the ring of all strictly upper triangular } n \times n \text{ matrices over } R. \end{aligned}$$

For $A \in M_n(R)$ and $i, j \in \{1, 2, \ldots, n\}$, let A_{ij} denote the entry of A in $i^{\underline{th}}$ row and $j^{\underline{th}}$ column.

A ring R is said to be a (*Von Neumann*) *regular ring* if for every $a \in R$, $a = axa$ for some $x \in R$. It is known that $M_n(R)$ is a regular ring if and only if R is a regular ring ([2], page 114-115). In particular, if R is a division ring, $M_n(R)$ is a regular ring.

For nonempty subsets A, B of a ring R, let $\mathbb{Z}A$ and AB denote respectively the set of all finite sums of the form $\sum k_i a_i$ where $k_i \in \mathbb{Z}$ and $a_i \in A$ and the set of all finite sums of the form $\sum a_i b_i$ where $a_i \in A$ and $b_i \in B$. A subring Q of a ring R is called a *quasi-ideal* of R if $RQ \cap QR \subseteq Q$. Then every left ideal and every right ideal of R is a quasi-ideal of R. In fact, quasi-ideals are a generalization of left ideals and right ideals. This can be seen from the following example.

Example. Let $n \in \mathbb{N}$ and $n > 1$. For $k, l \in \{1, 2, \ldots, n\}$, let $Q(k, l) \subseteq M_n(\mathbb{R})$ consisting of all matrices of the form

$$
\begin{array}{c} \\ \\ k^{\underline{th}} \text{ row} \rightarrow \end{array}
\begin{array}{c}
\qquad\qquad\quad l^{\underline{th}} \text{ column} \\
\qquad\qquad\quad \downarrow \\
\begin{bmatrix}
0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & x & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0
\end{bmatrix}
\end{array}.
$$

Then for $k, l \in \{1, 2, \ldots, n\}$, $Q(k, l)$ is a quasi-ideal of $M_n(\mathbb{R})$ but neither a left ideal nor a right ideal of $M_n(\mathbb{R})$.

The notion of quasi-ideal for rings was first introduced by O. Steinfled in [5]. It is known that the intersection of any set of quasi-ideals of R is a quasi-ideal of R ([6], page 10). For a nonempty subset X of R, the *quasi-ideal of R generated by X*, $(X)_q$ is defined to be the intersection of all quasi-ideals of R containing X. H. J. Weinert [7] has given the following fact.

Theorem 1.1 ([7]). *For a nonempty subset X of a ring R,*

$$(X)_q = \mathbb{Z}X + (RX \cap XR).$$

It is clearly seen that the intersection of a left ideal and a right ideal of R is a quasi-ideal of R. Also, this can be seen in [6], page 7. However, a quasi-ideal of R need not be obtained in this way. For examples, one can see in [6], page 8, [4] and [3]. Some examples can be seen from this research. It is observed that the examples we have seen are not obvious ones. We say that a quasi-ideal Q of a ring R *has the intersection property* if Q is the intersection of a left ideal and a right ideal of R and R is said to *have the intersection property of quasi-ideals* if every quasi-ideal of R has the intersection property. It is clearly seen that the following statements hold for any ring R. Every left ideal and every right ideal of R is a quasi-ideal of R having the intersection property. If R is commutative, every quasi-ideal of R is an ideal, so R has the intersection property of quasi-ideals. Moreover, the following two propositions are known.

Proposition 1.2 ([6], page 9). *If a ring R has a one-sided identity, then R has the intersection property of quasi-ideals.*

Proposition 1.3 ([6], page 73). *If R is a regular ring, then R has the intersection property of quasi-ideals.*

Hence if R is a ring with identity, by Proposition 1.2, for every $n \in \mathbb{N}$, $M_n(R)$ has the intersection property of quasi-ideals. As was mentioned, if R is a regular ring, then so is $M_n(R)$ for every $n \in \mathbb{N}$, and hence by Proposition 1.3, $M_n(R)$ has the intersection property of quasi-ideals. Observe that $SU_n(R)$ is a zero ring if $n \leqslant 2$ and if $n > 2$ and $|R| > 1$, $SU_n(R)$ has no one-sided identity and it is not regular.

Let Q be a quasi-ideal of a ring R. Then $RQ \cap QR \subseteq Q$. If $RQ \subseteq QR$, then $RQ = RQ \cap QR \subseteq Q$, so Q is a left ideal of R. Similarly, $QR \subseteq RQ$ implies that Q is a right ideal of R. Then the next proposition is obtained. It will be often referred in Chapter III.

Proposition 1.4. *If every quasi-ideal Q of a ring R has the property that $RQ \subseteq QR$ or $QR \subseteq RQ$, then R has the intersection property of quasi-ideals.*

In fact, characterizations of quasi-ideals having the intersection property were given by H.J. Weinert [7] as follows:

Theorem 1.5 ([7]). *For a quasi-ideal Q of R, the following statements are equivalent:*

1. *Q has the intersection property.*
2. $(RQ+Q)\cap(QR+Q)=Q$.
3. $RQ\cap(QR+Q)\subseteq Q$.
4. $QR\cap(RQ+Q)\subseteq Q$.

In [4], Z. Moucheng, C. Yuqun and L. Yonghau have given a result that strengthen Theorem 1.5 as follows:

Theorem 1.6 ([4]). *Let X be a nonempty subset of a ring R and $Q=(X)_q$. Then the following statements are equivalent:*

1. *Q has the intersection property.*
2. $(\mathbb{Z}X+XR)\cap(\mathbb{Z}X+RX)=Q$.
3. $RX\cap(\mathbb{Z}X+XR)\subseteq Q$.
4. $XR\cap(\mathbb{Z}X+RX)\subseteq Q$.

At this point, by using Theorem 1.1 and Theorem 1.6, we give an example of a quasi-ideal of $SU_4(\mathbb{R})$ which does not have the intersection property.

Example. Let $A,B\in SU_4(\mathbb{R})$ be defined by

$$A=\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix},\qquad B=\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

and set

$$Q=(\{A,B\})_q$$

in $SU_4(\mathbb{R})$. By Theorem 1.1,

$$Q = \mathbb{Z}(\{A, B\}) + SU_4(\mathbb{R})(\{A, B\}) \cap (\{A, B\})SU_4(\mathbb{R}).$$

Since for $C \in SU_4(\mathbb{R})$,

$$\begin{aligned}
&CA = \begin{bmatrix} 0 & 0 & 0 & C_{13} \\ 0 & 0 & 0 & C_{23} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad CB = \begin{bmatrix} 0 & 0 & 0 & C_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&AC = \begin{bmatrix} 0 & 0 & C_{23} & C_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ BC = \begin{bmatrix} 0 & 0 & 0 & C_{34} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\end{aligned} \qquad (*)$$

it follows that

$$Q = \left\{ \begin{bmatrix} 0 & k & m & x \\ 0 & 0 & 0 & m \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid k, m \in \mathbb{Z} \text{ and } x \in \mathbb{R} \right\}.$$

Therefore $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \notin Q$. From $(*)$, we have

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} A = B + A \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\in SU_4(\mathbb{R})(\{A, B\}) \cap (\mathbb{Z}(\{A, B\}) + (\{A, B\})SU_4(\mathbb{R})).$$

Therefore by that 1. $\Leftrightarrow$ 3. of Theorem 1.6, we have that Q does not have the intersection property.

In [4], Z. Moucheng, C. Yuqun and L. Yonghau used Theorem 1.1 and Theorem 1.6 to characterize rings having the intersection property of quasi-ideals as follows:

Theorem 1.7 ([4]). *A ring R has the intersection property of quasi-ideals if and only if for any finite nonempty subset X of R,*

$$RX \cap (\mathbb{Z}X + XR) \subseteq \mathbb{Z}X + (RX \cap XR).$$

Using Theorem 1.7 as a main tool, Y. Kemprasit and P. Juntarakhajorn [3] have characterized when $SU_n(R)$ has the intersection property of quasi-ideals if R is a field as follows:

Theorem 1.8 ([3]). *If R is a field, then $SU_n(R)$ has the intersection property of quasi-ideals if and only if $n \leqslant 3$.*

However, the given proof shows that the commutativity of the multiplication of R is not required. Then we have

Theorem 1.9. *If R is a division ring, then $SU_n(R)$ has the intersection property of quasi-ideals if and only if $n \leqslant 3$.*

It can be observed from the proof given for Theorem 1.8 that if $n \leqslant 3$, then every quasi-ideal of $SU_n(R)$ is a left ideal or a right ideal.

In the remainder of this research, let $m, n \in \mathbb{N}$ and R a division ring. As was mentioned previously, $M_n(R)$ has the intersection property of quasi-ideals. To generalize this fact, the following ring is considered. Let $M_{m,n}(R)$ denote the set of all $m \times n$ matrices over R. For $P \in M_{n,m}(R)$, let $(M_{m,n}(R), +, P)$ denote the ring $M_{m,n}(R)$ under usual addition and the multiplication $*$ defined by

$$A * B = APB$$

for all $A, B \in M_{m,n}(R)$. Then $M_{n,n}(R) = M_n(R)$ and $(M_n(R), +, I_n) \cong M_n(R)$ where I_n is the identity $n \times n$ matrix over R. The first main result of this research is given in Chapter II. It characterizes when $(M_{m,n}(R), +, P)$ has the intersection property of quasi-ideals. It will be proved that

$$\begin{aligned}&(M_{m,n}(R), +, P) \text{ has the intersection property}\\&\text{of quasi-ideals if and only if either } P = 0 \text{ or}\\&\operatorname{rank} P = \min\{m, n\}.\end{aligned} \tag{1.1}$$

To prove (1.1), a generalization of rings of all linear transformations on a vector space is provided and some basic knowledge of vector spaces and linear transformations are considered as follows:

Let V and W be vector spaces over R. The notation $L_R(V, W)$ denotes the set of all linear transformations $\alpha : V \to W$. For $\theta \in L_R(W, V)$, we denote by $(L_R(V, W), +, \theta)$ the ring $L_R(V, W)$ under usual addition and the multiplication $*$ defined by $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in L_R(V, W)$ where functions in this research are written on the right.

Assume that $\dim_R V = m$, $\dim_R W = n$, $B = \{v_1, v_2, \ldots, v_m\}$ is an ordered basis of V and $B' = \{w_1, w_2, \ldots, w_n\}$ is an ordered basis of W. For $\alpha \in L_R(V, W)$, let $[\alpha]_{B,B'}$ denote the $m \times n$ matrices (r_{ij}) where

$$v_1\alpha = r_{11}w_1 + r_{12}w_2 + \ldots + r_{1n}w_n$$

$$v_2\alpha = r_{21}w_1 + r_{22}w_2 + \ldots + r_{2n}w_n$$

$$\vdots$$

$$v_m\alpha = r_{m1}w_1 + r_{m2}w_2 + \ldots + r_{mn}w_n$$

and the matrix $[\alpha]_{B,B'}$ is called *the matrix of* α *relative to the ordered bases* B *and* B' ([1], page 329). Then

$$(L_R(V, W), +, \theta) \cong (M_{m,n}(R), +, [\theta]_{B',B}) \text{ by } \alpha \mapsto [\alpha]_{B,B'} \tag{1.2}$$

([1], page 329-330). Moreover, for every $\alpha \in L_R(V, W)$,

$$\operatorname{rank} \alpha = \operatorname{rank} [\alpha]_{B,B'} \tag{1.3}$$

([1], page 337 and 339).

In Chapter III, Theorem 1.9 is generalized. Since $SU_n(R)$ is an ideal of $U_n(R)$ ([1], page 335), $APB \in SU_n(R)$ for all $A, B \in SU_n(R)$ and $P \in U_n(R)$. For $P \in U_n(R)$, define $(SU_n(R), +, P)$ to be the ring $SU_n(R)$ under usual addition and the multiplication $*$ defined by

$$A * B = APB$$

for all $A, B \in SU_n(R)$. Then $(SU_n(R), +, I_n) = SU_n(R)$. We characterize when $(SU_n(R), +, P)$ has the intersection property of quasi-ideals. It will be proved in Chapter III that

$(SU_n(R), +, P)$ has the intersection property of quasi-ideals if and only if one of the following statements holds:

(i) $n \leqslant 3$.

(ii) $n = 4$ and $P_{22} = 0$ or $P_{33} = 0$.

(iii) $n > 4$, $P_{ij} = 0$ for all $i, j \in \{3, 4, \ldots, n-2\}$ and (1.4)

(a) $P_{2j} = 0$ for all $j \in \{2, 3, \ldots, n-2\}$ or

(b) $P_{i,n-1} = 0$ for all $i \in \{3, 4, \ldots, n-1\}$.

Then Theorem 1.9 becomes a corollary of (1.4).

CHAPTER II

GENERALIZED FULL MATRIX RINGS

In this chapter, we prove that $(M_{m,n}(R), +, P)$ has the intersection property of quasi-ideals if and only if either $P = 0$ or $\operatorname{rank} P = \min\{m, n\}$. In particular, for $P \in M_n(R)$, $(M_n(R), +, P)$ has the intersection property of quasi-ideals if and only if either $P = 0$ or P is invertible.

The following proposition is a general fact which will be referred.

Proposition 2.1. *Let $\alpha \in L_R(V, W)$.*

(i) If α is a monomorphism, then there exists $\beta \in L_R(W, V)$ such that $\alpha\beta = 1_V$ where 1_V is the identity map on V.

(ii) If α is an epimorphism, then there exists $\beta \in L_R(W, V)$ such that $\beta\alpha = 1_W$.

Proof. (i) Let B be a basis of V. Since α is a monomorphism, we have $B\alpha$ is a basis of $\operatorname{Im}\alpha$ and $u\alpha \neq u'\alpha$ for all distinct $u, u' \in B$. Let B' be a basis of W containing $B\alpha$. Let $\beta \in L_R(W, V)$ be defined by

$$v\beta = \begin{cases} u & \text{if } v = u\alpha \text{ for some } u \in B, \\ 0 & \text{if } v \in B' \smallsetminus B\alpha. \end{cases}$$

Then $u\alpha\beta = u$ for all $u \in B$ and hence $\alpha\beta = 1_V$.

(ii) Let B be a basis of W. Since $\operatorname{Im}\alpha = W$, for each $v \in B$, there exists $v' \in V$ such that $v'\alpha = v$. Define $\beta \in L_R(W, V)$ by

$$v\beta = v' \text{ for all } v \in B.$$

Then $v\beta\alpha = v$ for all $v \in B$, so $\beta\alpha = 1_W$. □

We first introduce two lemmas. The second lemma is a main tool to obtain our main result of this chapter. However, the first lemma gives a more general result and the second one becomes a special case.

Lemma 2.2. *For $\theta \in L_R(W,V)$, the ring $(L_R(V,W),+,\theta)$ has the intersection property of quasi-ideals if and only if*

(i) $\theta = 0$,

(ii) θ *is a monomorphism or*

(iii) θ *is an epimorphism.*

Proof. If $\theta = 0$, then $(L_R(V,W),+,\theta)$ is a zero ring, so it has the intersection property of quasi-ideals.

Assume that θ is a monomorphism. By Proposition 2.1(i), $\theta\theta' = 1_W$ for some $\theta' \in L_R(V,W)$. It follows that $\alpha\theta\theta' = \alpha$ for all $\alpha \in L_R(V,W)$. This implies that θ' is a right identity of the ring $(L_R(V,W),+,\theta)$. We deduce from Proposition 1.2 that $(L_R(V,W),+,\theta)$ has the intersection property of quasi-ideals.

Next, assume that θ is an epimorphism. By Proposition 2.1(ii), there is $\theta' \in L_R(V,W)$ such that $\theta'\theta = 1_V$. Then $\theta'\theta\alpha = \alpha$ for all $\alpha \in L_R(V,W)$, so θ' is a left identity of the ring $(L_R(V,W),+,\theta)$. Hence by Proposition 1.2, $(L_R(V,W),+,\theta)$ has the intersection property of quasi-ideals.

For the converse, assume that $\theta \neq 0$, θ is not a monomorphism and θ is not an epimorphism. It follows that

$$\{0\} \neq \operatorname{Ker}\theta \subsetneq W \text{ and } \{0\} \neq \operatorname{Im}\theta \subsetneq V.$$

Let $u \in \operatorname{Ker}\theta \smallsetminus \{0\}$, $w \in W \smallsetminus \operatorname{Ker}\theta$ and $z \in V \smallsetminus \operatorname{Im}\theta$. Then $w\theta \in \operatorname{Im}\theta \smallsetminus \{0\}$. Let B_1 be a basis of $\operatorname{Im}\theta$ containing $w\theta$. Since $z \in V \smallsetminus \operatorname{Im}\theta$, $B_1 \cup \{z\}$ is linearly independent over R. Let B be a basis of V containing $B_1 \cup \{z\}$. Let $\alpha, \beta, \gamma \in L_R(V, W)$ be defined by

$$v\alpha = \begin{cases} u & \text{if } v = w\theta, \\ w & \text{if } v = z, \\ 0 & \text{if } v \in B \smallsetminus \{w\theta, z\}, \end{cases}$$

$$v\beta = \begin{cases} w & \text{if } v = w\theta, \\ 0 & \text{if } v \in B \smallsetminus \{w\theta\} \end{cases}$$

and

$$v\gamma = \begin{cases} -w & \text{if } v = w\theta, \\ 0 & \text{if } v \in B \smallsetminus \{w\theta\}. \end{cases}$$

From what we define α, β and γ, we have

$$v\alpha\theta = 0 \text{ for all } v \in \operatorname{Im}\theta, \tag{2.2.1}$$

$$\begin{aligned} (w\theta)(\alpha + \alpha\theta\gamma) &= (w\theta)\alpha + (w\theta)\alpha\theta\gamma = u, \\ (w\theta)\beta\theta\alpha &= (w\theta)\alpha = u, \\ z(\alpha + \alpha\theta\gamma) &= z\alpha + z\alpha\theta\gamma \\ &= w + (w\theta)\gamma \\ &= w - w = 0, \\ z(\beta\theta\alpha) &= 0 \end{aligned}$$

and

$$v(\alpha + \alpha\theta\gamma) = 0 = v\beta\theta\alpha \text{ for all } v \in B \setminus \{w\theta, z\}.$$

Consequently, we have

$$\beta\theta\alpha = \alpha + \alpha\theta\gamma \in L_R(V,W)\theta\alpha \cap (\mathbb{Z}\alpha + \alpha\theta L_R(V,W)). \tag{2.2.2}$$

Suppose that $\beta\theta\alpha \in \mathbb{Z}\alpha + (L_R(V,W)\theta\alpha \cap \alpha\theta L_R(V,W))$. Then there exist $k \in \mathbb{Z}$ and $\lambda, \eta \in L_R(V,W)$ such that

$$\beta\theta\alpha = k\alpha + \lambda\theta\alpha = k\alpha + \alpha\theta\eta.$$

Then

$$u = (w\theta)\beta\theta\alpha = (w\theta)(k\alpha + \alpha\theta\eta) = ku = (k1_R)u$$

where 1_R is the identity of R. This implies that $k1_R = 1_R$ since $u \neq 0$. Therefore $k\alpha = \alpha$ and so

$$\beta\theta\alpha = \alpha + \lambda\theta\alpha.$$

Hence

$$0 = z(\beta\theta\alpha) = z(\alpha + \lambda\theta\alpha) = w + (z\lambda\theta)\alpha$$

and so $(z\lambda\theta)\alpha = -w$. It then follows from this equality and (2.2.1) that

$$-(w\theta) = (z\lambda\theta)\alpha\theta = 0$$

which is a contradiction since $w\theta \neq 0$. This shows that

$$\beta\theta\alpha \notin \mathbb{Z}\alpha + (L_R(V,W)\theta\alpha \cap \alpha\theta L_R(V,W)). \tag{2.2.3}$$

The statements (2.2.2), (2.2.3) and Theorem 1.7 yield the result that the ring $(L_R(V,W),+,\theta)$ does not have the intersection property of quasi-ideals.

Hence the lemma is completely proved. □

Lemma 2.3. *Assume that $dim_R V = m$, $dim_R W = n$ and $\theta \in L_R(W,V)$. Then the ring $(L_R(V,W),+,\theta)$ has the intersection property of quasi-ideals if and only if either $\theta = 0$ or $rank\,\theta = min\{m,n\}$.*

Proof. Assume that $(L_R(V,W),+,\theta)$ has the intersection property of quasi-ideals. By Lemma 2.2, $\theta = 0$, θ is a monomorphism or θ is an epimorphism. If $\theta : W \longrightarrow V$ is a monomorphism, then $W \cong \operatorname{Im}\theta$, so

$$m \geqslant \operatorname{rank}\theta = \dim_R \operatorname{Im}\theta = \dim_R W = n.$$

If $\theta : W \longrightarrow V$ is an epimorphism, then

$$m = \operatorname{rank}\theta = \dim_R V \leqslant \dim_R W = n.$$

This proves that if $\theta \neq 0$, then $\operatorname{rank}\theta = n \leqslant m$ or $\operatorname{rank}\theta = m \leqslant n$. Therefore either $\theta = 0$ or $\operatorname{rank}\theta = \min\{m,n\}$.

For the converse, assume that $\theta = 0$ or $\operatorname{rank}\theta = \min\{m,n\}$. Then $\theta = 0$, $\operatorname{rank}\theta = n$ or $\operatorname{rank}\theta = m$.

Case 1: $\operatorname{rank}\theta = n$. Then $\dim_R \operatorname{Im}\theta = n$. But

$$n = \dim_R W = \dim_R \mathrm{Im}\,\theta + \dim_R \mathrm{Ker}\,\theta = n + \dim_R \mathrm{Ker}\,\theta,$$

so $\mathrm{Ker}\,\theta = \{0\}$ which implies that θ is a monomorphism.

Case 2: $\mathrm{rank}\,\theta = m$. Then $\dim_R \mathrm{Im}\,\theta = m$. But $\mathrm{Im}\,\theta$ is a subspace of V and $\dim_R V = m$, so we have that $\mathrm{Im}\,\theta = V$. Therefore θ is an epimorphism.

This proves that $\theta = 0$, θ is a monomorphism or θ is an epimorphism. By Lemma 2.2, $(L_R(V,W), +, \theta)$ has the intersection property of quasi-ideals.

Hence the lemma is proved. $\square$

If $\dim_R V = \dim_R W = k < \infty$ and $\theta \in L_R(W,V)$, it is known that θ is an epimorphism if and only if θ is an isomorphism and hence $\mathrm{rank}\,\theta = k$ if and only if θ is an isomorphism.

The following corollary is an immediate consequence of the above fact and Lemma 2.3.

Corollary 2.4. *Assume that $\dim_R V = \dim_R W < \infty$ and $\theta \in L_R(W,V)$. Then the ring $(L_R(V,W), +, \theta)$ has the intersection property of quasi-ideals if and only if either $\theta = 0$ or θ is an isomorphism.*

Theorem 2.5. *For $P \in M_{n,m}(R)$, the ring $(M_{m,n}(R), +, P)$ has the intersection property of quasi-ideals if and only if either $P = 0$ or $\mathrm{rank}\,P = \min\{m, n\}$.*

Proof. Let V and W be finite dimensional vector spaces over R such that $\dim_R V = m$ and $\dim_R W = n$. Let B and B' be respectively ordered bases of V and W. By (1.2), there exists $\theta \in L_R(W,V)$ such that $[\theta]_{B',B} = P$. Therefore by (1.2)

$$(L_R(V,W),+,\theta) \cong (M_{m,n}(R),+,P) \text{ by } \alpha \mapsto [\alpha]_{B,B'}. \tag{2.5.1}$$

Also, by (1.3), we have

$$\operatorname{rank}\theta = \operatorname{rank}[\theta]_{B',B} = \operatorname{rank}P. \tag{2.5.2}$$

Assume that $(M_{m,n}(R),+,P)$ has the intersection property of quasi-ideals. By (2.5.1), $(L_R(V,W),+,\theta)$ has the intersection property of quasi-ideals. By Lemma 2.3, $\theta = 0$, $\operatorname{rank}\theta = n$ or $\operatorname{rank}\theta = m$. It then follows from (2.5.2) that $P = 0$, $\operatorname{rank}P = n$ or $\operatorname{rank}P = m$. Hence either $P = 0$ or $\operatorname{rank}P = \min\{m,n\}$.

Conversely, assume that $P = 0$ or $\operatorname{rank}P = \min\{m,n\}$. Then $P = 0$, rank $P = n$ or $\operatorname{rank}P = m$. From (2.5.2), we have $\theta = 0$, $\operatorname{rank}\theta = n$ or $\operatorname{rank}\theta = m$. Thus $\theta = 0$ or $\operatorname{rank}\theta = \min\{m,n\}$. It then follows from Lemma 2.3 that $(L_R(V,W),+,\theta)$ has the intersection property of quasi-ideals. Therefore by (2.5.1), $(M_{m,n}(R),+,P)$ has the intersection property of quasi-ideals.

Hence the theorem is proved, as required. □

It is known that for $P \in M_n(R)$, P is invertible if and only if rank $P = n$. Hence by Theorem 2.5, we have

Corollary 2.6. *For $P \in M_n(R)$, the ring $(M_n(R),+,P)$ has the intersection property of quasi-ideals if and only if either $P = 0$ or P is invertible.*

CHAPTER III

GENERALIZED RINGS OF STRICTLY UPPER TRIANGULAR MATRICES

The purpose of this chapter is to give necessary and sufficient conditions for n and the entries of P in order that the ring $(SU_n(R), +, P)$ has the intersection property of quasi-ideals where $P \in U_n(R)$.

Our aim is to prove the following.

Theorem 3.1. *The ring $(SU_n(R), +, P)$ has the intersection property of quasi-ideals if and only if one of the following statements holds.*

(i) $n \leqslant 3$.

(ii) $n = 4$ *and* $P_{22} = 0$ *or* $P_{33} = 0$.

(iii) $n > 4$, $P_{ij} = 0$ *for all* $i, j \in \{3, 4, \ldots, n-2\}$ *and*

(a) $P_{2j} = 0$ *for all* $j \in \{2, 3, \ldots, n-2\}$ *or*

(b) $P_{i,n-1} = 0$ *for all* $i \in \{3, 4, \ldots, n-1\}$.

To be more clearly seen, (i), (ii) and (iii) can be illustrated as follows:

(i) $n = 1$ and $P = [P_{11}]$, $n = 2$ and $P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix}$,

$$n = 3 \text{ and } P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ 0 & P_{22} & P_{23} \\ 0 & 0 & P_{33} \end{bmatrix}.$$

$$\text{(ii) } n = 4 \text{ and } P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ 0 & 0 & P_{23} & P_{24} \\ 0 & 0 & P_{33} & P_{34} \\ 0 & 0 & 0 & P_{44} \end{bmatrix} \text{ or } \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ 0 & P_{22} & P_{23} & P_{24} \\ 0 & 0 & 0 & P_{34} \\ 0 & 0 & 0 & P_{44} \end{bmatrix}.$$

$$\text{(iii) } n > 4 \text{ and } P = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1,n-2} & P_{1,n-1} & P_{1n} \\ 0 & 0 & \dots & 0 & P_{2,n-1} & P_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & P_{n-1,n-1} & P_{n-1,n} \\ 0 & 0 & \dots & 0 & 0 & P_{nn} \end{bmatrix}$$

$$\text{or } P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & \dots & P_{1,n-1} & P_{1n} \\ 0 & P_{22} & P_{23} & \dots & P_{2,n-1} & P_{2n} \\ 0 & 0 & 0 & \dots & 0 & P_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & P_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 & P_{nn} \end{bmatrix}.$$

If P is the identity $n \times n$ matrix over R, P can satisfy neither (ii) nor (iii) of Theorem 3.1. Hence Theorem 1.9 becomes a corollary of this main theorem.

Corollary 3.2. *The ring $SU_n(R)$ has the intersection property of quasi-ideals if and only if $n \leqslant 3$.*

To prove the theorem, the following three lemmas are provided.

Lemma 3.3. *Assume that $n \geqslant 3$. If*

(1) $P_{ij} = 0$ for all $i \geqslant 2$ and $j \leqslant n-2$ or

(2) $P_{ij} = 0$ for all $i \geqslant 3$ and $j \leqslant n-1$,

then $(SU_n(R), +, P)$ has the intersection property of quasi-ideals.

Proof. For $A, B \in SU_n(R)$,

$$\begin{aligned} &\text{for } i = n \text{ or } j = 1, (APB)_{ij} = 0 \text{ and} \\ &\text{for } i < n \text{ and } j > 1, (APB)_{ij} = \sum_{k=1}^{j-1} \sum_{l=i+1}^{n} (A_{il} P_{lk} B_{kj}). \end{aligned} \tag{3.3.1}$$

Let Q be a quasi-ideal of $(SU_n(R), +, P)$.

First, assume that (1) holds. From (3.3.1) and (1), we have that for $A, B \in SU_n(R)$,

$$(APB)_{ij} = \begin{cases} 0 & \text{if } i = n \text{ or } j < n, \\ (\sum_{l=i+1}^{n} A_{il} P_{l,n-1}) B_{n-1,n} & \text{if } i < n \text{ and } j = n. \end{cases} \tag{3.3.2}$$

Case 1.1: $B_{n-1,n} = 0$ for all $B \in Q$ or $P_{l,n-1} = 0$ for all $l \geqslant 2$. By (3.3.2), we have $SU_n(R)PQ = \{0\} \subseteq QPSU_n(R)$.

Case 1.2: $B_{n-1,n} \neq 0$ for some $B \in Q$ and $P_{l,n-1} \neq 0$ for some $l \geqslant 2$. Then $l \leqslant n-1$. Let

$$m = \max\{i \in \{2, 3, \ldots, n-1\} | P_{i,n-1} \neq 0\}.$$

Since R is a division ring, by (3.3.2), we have

$$SU_n(R)PQ = \left\{ \begin{bmatrix} 0 & 0 & \dots & 0 & x_1 \\ 0 & 0 & \dots & 0 & x_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & x_{m-1} \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} | x_1, x_2, \dots, x_{m-1} \in R \right\}$$

$$= SU_n(R)PSU_n(R),$$

and so $QPSU_n(R) \subseteq SU_n(R)PQ$.

Next, assume that (2) holds. We have from (3.3.1) and (2) that

$$(APB)_{ij} = \begin{cases} 0 & \text{if } i > 1 \text{ or } j = 1, \\ A_{12}(\sum_{k=1}^{j-1} P_{2k}B_{kj}) & \text{if } i = 1 \text{ and } j > 1. \end{cases} \tag{3.3.3}$$

Case 2.1: $A_{12} = 0$ for all $A \in Q$ or P_{2k} for all $k \leqslant n-1$. By (3.3.3), $QPSU_n(R) = \{0\} \subseteq SU_n(R)PQ$.

Case 2.2: $A_{12} \neq 0$ for some $A \in Q$ and $P_{2k} \neq 0$ for some $k \leqslant n-1$. Then $k \geqslant 2$. Let

$$m = \min\{j \in \{2, 3, \dots, n-1\} | P_{2j} \neq 0\}.$$

From (3.3.3) and since R is a division ring, we get

$$QPSU_n(R) = \left\{ \begin{bmatrix} 0 & \dots & 0 & x_{m+1} & x_{m+2} & \dots & x_n \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix} | x_{m+1}, x_{m+2}, \dots, x_n \in R \right\}$$

$$= SU_n(R)PSU_n(R).$$

Hence $SU_n(R)PQ \subseteq QPSU_n(R)$.

By Proposition 1.4, Q has the intersection property. Therefore the lemma holds. □

From the given proof of Lemma 3.3, we remark here that under the assumption of Lemma 3.3, every quasi-ideal of $(SU_n(R), +, P)$ is a left ideal or a right ideal.

Lemma 3.4. *Assume that $n \geqslant 4$. If $(SU_n(R), +, P)$ has the intersection property of quasi-ideals, then $P_{2j} = 0$ for all $j \in \{2, 3, \dots, n-2\}$ or $P_{i,n-1} = 0$ for all $i \in \{3, 4, \dots, n-1\}$.*

Proof. Assume that $P_{2t} \neq 0$ and $P_{s,n-1} \neq 0$ for some $t \in \{2, 3, \dots, n-2\}$ and some $s \in \{3, 4, \dots, n-1\}$. Let $A, B \in SU_n(R)$ be defined by

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix}.$$

Then for $C \in SU_n(R)$,

$$CPA = C \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ 0 & P_{22} & \dots & P_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & P_{nn} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & C_{12} & C_{13} & \dots & C_{1n} \\ 0 & 0 & C_{23} & \dots & C_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & C_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & P_{11} & 0 & \dots & 0 & P_{1,n-1} \\ 0 & 0 & 0 & \dots & 0 & P_{2,n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & P_{n-1,n-1} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \dots & 0 & \sum_{k=2}^{n-1} C_{1k}P_{k,n-1} \\ 0 & \dots & 0 & \sum_{k=3}^{n-1} C_{2k}P_{k,n-1} \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & C_{n-2,n-1}P_{n-1,n-1} \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}, \qquad (3.4.1)$$

$$APC = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ 0 & P_{22} & \dots & P_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & P_{nn} \end{bmatrix} C$$

$$= \begin{bmatrix} 0 & P_{22} & P_{23} & \dots & P_{2n} \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & P_{nn} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & C_{12} & C_{13} & \dots & C_{1n} \\ 0 & 0 & C_{23} & \dots & C_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & C_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & P_{22}C_{23} & \sum_{k=2}^{3} P_{2k}C_{k4} & \dots & \sum_{k=2}^{n-1} P_{2k}C_{kn} \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{3.4.2}$$

$$CPB = C \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ 0 & P_{22} & \dots & P_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & P_{nn} \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & C_{12} & C_{13} & \dots & C_{1n} \\ 0 & 0 & C_{23} & \dots & C_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & C_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 & P_{11} & P_{12} \\ 0 & \dots & 0 & 0 & P_{22} \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \dots & 0 & C_{12}P_{22} \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

and

$$BPC = \begin{bmatrix} 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ 0 & P_{22} & \dots & P_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & P_{nn} \end{bmatrix} C$$

$$= \begin{bmatrix} 0 & \dots & 0 & P_{n-1,n-1} & P_{n-1,n} \\ 0 & \dots & 0 & 0 & P_{nn} \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & C_{12} & C_{13} & \dots & C_{1n} \\ 0 & 0 & C_{23} & \dots & C_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & C_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \dots & 0 & P_{n-1,n-1}C_{n-1,n} \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{bmatrix}.$$

Let $X = \{A, B\}$. Since $P_{s,n-1} \neq 0$ and $P_{2t} \neq 0$, from these equalities, we deduce that

$$SU_n(R)PX \cap XPSU_n(R) = \left\{ \begin{bmatrix} 0 & \dots & 0 & x \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{bmatrix} \mid x \in R \right\}. \qquad (3.4.3)$$

Define $D, E \in M_n(R)$ by

$$D = \begin{bmatrix} 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & -P_{2t}^{-1} & 0 \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow t^{\underline{th}} \text{ row} \\ \\ \\ \\ \end{matrix}$$

and

$$\begin{matrix} & & & & s^{\underline{th}} \text{ column} & & & \\ & & & & \downarrow & & & \end{matrix}$$

$$E = \begin{bmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & P_{s,n-1}^{-1} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Since $t \leqslant n-2 < n-1$ and $2 < 3 \leqslant s$, we have $D, E \in SU_n(R)$. From (3.4.1) and (3.4.2), we respectively obtain

$$EPA = \begin{bmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

and

$$APD = \begin{bmatrix} 0 & \dots & 0 & -1 & 0 \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

which imply that

$$EPA = B + APD = \begin{bmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

$$\in SU_n(R)PX \cap (\mathbb{Z}X + XPSU_n(R)).$$

From (3.4.3) and the definitions of A and B, the matrix

$$\begin{bmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

is not a member of $\mathbb{Z}X + (SU_n(R)PX \cap XPSU_n(R))$. It then follows from Theorem 1.7 that $(SU_n(R), +, P)$ does not have the intersection property of quasi-ideals.

Therefore the lemma is proved. □

Lemma 3.5. *Assume that $n > 4$. If $(SU_n(R), +, P)$ has the intersection property of quasi-ideals, then $P_{ij} = 0$ for all $i, j \in \{3, 4, \dots, n-2\}$.*

Proof. Assume that there exist $s, t \in \{3, 4, \ldots, n-2\}$ such that $P_{st} \neq 0$. Since P is upper triangular, we have $s \leqslant t$. Define $A, B \in SU_n(R)$ by

$$
\begin{array}{c}
\qquad\qquad s^{\underline{th}} \text{ column} \\
\qquad\qquad \downarrow
\end{array}
$$

$$
A = \begin{bmatrix}
0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & 0 & \ldots & 1 \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0
\end{bmatrix}
\begin{array}{l}
\\ \\ \\ \\ \leftarrow t^{\underline{th}} \text{ row} \\ \\ \\ \\
\end{array}
$$

and

$$
B = \begin{bmatrix}
0 & \ldots & 0 & 1 & 0 \\
0 & \ldots & 0 & 0 & 1 \\
0 & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 0 & 0 & 0
\end{bmatrix}.
$$

Then for $C \in SU_n(R)$, we have

$$
CPA = C \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ 0 & P_{22} & \dots & P_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & P_{nn} \end{bmatrix}
\begin{bmatrix} 0 & \dots & 0 & \overset{\overset{s^{\underline{th}}\text{ column}}{\downarrow}}{1} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix}
\begin{matrix} \\ \\ \\ \\ \leftarrow t^{\underline{th}}\text{ row} \\ \\ \\ \\ \end{matrix}
$$

$$
= \begin{bmatrix} 0 & C_{12} & C_{13} & \dots & C_{1n} \\ 0 & 0 & C_{23} & \dots & C_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & C_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}
\begin{bmatrix} 0 & \dots & 0 & \overset{\overset{s^{\underline{th}}\text{ column}}{\downarrow}}{P_{11}} & 0 & \dots & 0 & P_{1t} \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & P_{2t} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & P_{tt} \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}
\begin{matrix} \\ \\ \\ \leftarrow t^{\underline{th}}\text{ row} \\ \\ \\ \\ \end{matrix}
$$

$$= \begin{bmatrix} 0 & \dots & 0 & \sum_{k=2}^{t} C_{1k}P_{kt} \\ 0 & \dots & 0 & \sum_{k=3}^{t} C_{2k}P_{kt} \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & C_{t-1,t}P_{tt} \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{bmatrix}, \tag{3.5.1}$$

$$APC = \begin{array}{c} \\ \\ \\ \\ \rightarrow \\ t^{\underline{th}} \\ \text{row} \\ \\ \end{array} \overset{\begin{array}{c} s^{\underline{th}} \text{ column} \\ \downarrow \end{array}}{\begin{bmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix}} \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ 0 & P_{22} & \dots & P_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & P_{nn} \end{bmatrix} C$$

$$
\begin{array}{c}
s^{\underline{th}} \text{ column} \\
\downarrow
\end{array}
$$

$$
= \begin{array}{r} \\ \\ \\ \\ \Rsh \\ t^{\underline{th}} \\ \text{row} \\ \\ \end{array}
\begin{bmatrix}
0 & \dots & 0 & P_{ss} & \dots & P_{sn} \\
0 & \dots & 0 & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & \dots & 0 & 0 & \dots & 0 \\
0 & \dots & 0 & 0 & \dots & P_{nn} \\
0 & \dots & 0 & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & \dots & 0 & 0 & \dots & 0
\end{bmatrix}
\begin{bmatrix}
0 & C_{12} & C_{13} & \dots & C_{1n} \\
0 & 0 & C_{23} & \dots & C_{2n} \\
\dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & C_{n-1,n} \\
0 & 0 & 0 & \dots & 0
\end{bmatrix}
$$

$$
= \begin{bmatrix}
0 & \dots & 0 & P_{ss}C_{s,s+1} & \sum_{k=s}^{s+1} P_{sk}C_{k,s+2} & \dots & \sum_{k=s}^{n-1} P_{sk}C_{kn} \\
0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & \dots & 0 & 0 & 0 & \dots & 0
\end{bmatrix}, \quad (3.5.2)
$$

$$
CPB = C
\begin{bmatrix}
P_{11} & P_{12} & \dots & P_{1n} \\
0 & P_{22} & \dots & P_{2n} \\
\dots & \dots & \dots & \dots \\
0 & 0 & \dots & P_{nn}
\end{bmatrix}
\begin{bmatrix}
0 & \dots & 0 & 1 & 0 \\
0 & \dots & 0 & 0 & 1 \\
0 & \dots & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots \\
0 & \dots & 0 & 0 & 0
\end{bmatrix}
$$

$$
= \begin{bmatrix}
0 & C_{12} & C_{13} & \dots & C_{1n} \\
0 & 0 & C_{23} & \dots & C_{2n} \\
\dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & C_{n-1,n} \\
0 & 0 & 0 & \dots & 0
\end{bmatrix}
\begin{bmatrix}
0 & \dots & 0 & P_{11} & P_{12} \\
0 & \dots & 0 & 0 & P_{22} \\
0 & \dots & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots \\
0 & \dots & 0 & 0 & 0
\end{bmatrix}
$$

$$= \begin{bmatrix} 0 & \dots & 0 & C_{12}P_{22} \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

and

$$BPC = \begin{bmatrix} 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ 0 & P_{22} & \dots & P_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & P_{nn} \end{bmatrix} C$$

$$= \begin{bmatrix} 0 & \dots & 0 & P_{n-1,n-1} & P_{n-1,n} \\ 0 & \dots & 0 & 0 & P_{nn} \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & C_{12} & C_{13} & \dots & C_{1n} \\ 0 & 0 & C_{23} & \dots & C_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & C_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \dots & 0 & P_{n-1,n-1}C_{n-1,n} \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{bmatrix}.$$

Let $X = \{A, B\}$. Since $P_{st} \neq 0$, these equalities yield

$$SU_n(R)PX \cap XPSU_n(R) = \left\{ \begin{bmatrix} 0 & \dots & 0 & x \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{bmatrix} \mid x \in R \right\}. \qquad (3.5.3)$$

Define $D, E \in M_n(R)$ by

$$D = \begin{bmatrix} 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & -P_{st}^{-1} & 0 \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow t^{\underline{th}} \text{ row} \\ \\ \\ \\ \end{matrix}$$

and

$$\begin{matrix} & & & s^{\underline{th}} \text{ column} & & & \\ & & & \downarrow & & & \end{matrix}$$

$$E = \begin{bmatrix} 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & P_{st}^{-1} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Because $t \leqslant n-2 < n-1$ and $2 < 3 \leqslant s$, we have $D, E \in SU_n(R)$. From (3.5.1) and (3.5.2), we respectively obtain

$$EPA = \begin{bmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

and

$$APD = \begin{bmatrix} 0 & \dots & 0 & -1 & 0 \\ 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix}.$$

Therefore

$$EPA = B + APD = \begin{bmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

$$\in SU_n(R)PX \cap (\mathbb{Z}X + XPSU_n(R)).$$

From (3.4.3) and the definitions of A and B, the matrix

$$\begin{bmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

is not a member of $\mathbb{Z}X + (SU_n(R)PX \cap XPSU_n(R))$. It then follows from Theorem 1.7 that $(SU_n(R), +, P)$ does not have the intersection property of quasi-ideals.

Therefore the lemma is proved. □

Now we are ready to prove our main result of this chapter.

Proof of Theorem 3.1.

First, assume that $(SU_n(R), +, P)$ has the intersection property of quasi-ideals and $n > 3$. If $n = 4$, by Lemma 3.4, we have $P_{22} = 0$ or $P_{33} = 0$ and hence (ii) holds. If $n > 4$, then (iii) holds by Lemma 3.5 and Lemma 3.4.

For the converse, assume that (i), (ii) or (iii) holds. If $n \leqslant 2$, $(SU_n(R), +, P)$ is clearly a zero ring, so we are done. If $n = 3$, then P satisfies (1) and (2) of Lemma 3.3 since P is upper triangular, so by Lemma 3.3, $(SU_3(R), +, P)$ has the intersection property of quasi-ideals. Next, assume that $n = 4$ and $P_{22} = 0$ or $P_{33} = 0$. This implies that (1) or (2) of Lemma 3.3 holds. Thus from Lemma 3.3, $(SU_4(R), +, P)$ has the intersection property of quasi-ideals. Finally, assume that (iii) holds. Then $n > 4$ and

(1) $P_{ij} = 0$ for all $i, j \in \{3, 4, \ldots, n-2\}$ and $P_{2j} = 0$ for all $j \in \{2, 3, \ldots, n-2\}$

or

(2) $P_{ij} = 0$ for all $i, j \in \{3, 4, \ldots, n-2\}$ and $P_{i,n-1} = 0$ for all $i \in \{3, 4, \ldots, n-1\}$.

It is clear that (1) and (2) are respectively equivalent to

(1′) $P_{ij} = 0$ for all $i \geqslant 2$ and $j \leqslant n-2$ and

(2′) $P_{ij} = 0$ for all $i \geqslant 3$ and $j \leqslant n-1$.

Therefore we have that (1′) or (2′) holds. We then deduce from Lemma 3.3 that $(SU_n(R), +, P)$ has the intersection property of quasi-ideals.

Hence the theorem is completely proved. □

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