ทฤษฎีบทของกลุ่มย่อยนูนของกึ่งสนาม
และปริภูมิเวกเตอร์บนกึ่งสนาม

นางสาวศิริจันทร์ พหุพงศ์ทรัพย์

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THEOREMS OF CONVEX SUBGROUPS OF SEMIFIELDS
AND VECTOR SPACES OVER SEMIFIELDS

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ศิริจันทร์ พหุพงศ์ทรัพย์ : ทฤษฎีบทของกลุ่มย่อยนูนของกึ่งสนามและปริภูมิเวกเตอร์บนกึ่งสนาม (THEOREMS OF CONVEX SUBGROUPS OF SEMFIELDS AND VECTOR SPACES OVER SEMIFIELDS) อ. ที่ปรึกษา : ผู้ช่วยศาสตราจารย์ ดร. อมร วาสนาวิจิตร์, 45 หน้า ISBN 974-13-0923-6

เราจะเรียกสิ่งทั้งสามที่เป็นอันดับ $(K,+,\cdot)$ ว่า กึ่งสนาม ก็ต่อเมื่อ (1) $(K,\cdot)$ เป็นกลุ่มสลับที่ที่มี 0, (2) $(K,+)$ เป็นกึ่งกลุ่มสลับที่ที่มี 0 เป็นเอกลักษณ์ และ (3) $x(y+z)=xy+xz$ สำหรับทุกๆ $x,y,z\in K$ เราจะเรียกสับเซต $C\neq\{0\}$ ของ K ซึ่งเป็นเซตไม่ว่างว่า กลุ่มย่อยนูนของ K ก็ต่อเมื่อ (1) สำหรับทุกๆ $x,y\in C$ ซึ่ง $y\neq 0$ จะได้ $\frac{x}{y}\in C$ และ (2) สำหรับทุกๆ $x,y\in C$, $\alpha,\beta\in K$ ซึ่ง $\alpha+\beta=1$ จะได้ $\alpha x+\beta y\in C$ เราจะเรียกโซ่ของกึ่งสนามย่อยของ K , $K=K_0\rhd K_1\rhd\ldots\rhd K_n$,ว่า อนุกรมกลุ่มย่อยนูนจำกัดโดยแท้ใน K ก็ต่อเมื่อ K_{i+1} เป็นกลุ่มย่อยนูนของ K_i และ $K_l\neq K_j$ สำหรับ $l\neq j$ ให้ C และ C' เป็นอนุกรมกลุ่มย่อยนูนจำกัดโดยแท้ใน K เราจะกล่าวว่า C' ละเอียดกว่า C ถ้าทุกๆพจน์ของ C ปรากฏอยู่ใน C' และถ้า $C\neq C'$ แล้วเราจะกล่าวว่า C' ละเอียดกว่า C โดยแท้ เราจะเรียกอนุกรมกลุ่มย่อยนูนจำกัดโดยแท้ใน K , $K=K_0\rhd K_1\rhd\ldots\rhd K_n\rhd\{1\}$, ว่า อนุกรมผลประกอบ ก็ต่อเมื่ออนุกรมนั้นไม่มีอนุกรมที่ละเอียดกว่าโดยแท้ เราจะเรียกกลุ่มสลับที่ M ซึ่งมี 0 เป็นเอกลักษณ์ว่า ปริภูมิเวกเตอร์บนกึ่งสนาม K ถ้ามีฟังก์ชัน $(k,m)\mapsto km$ จาก $K\times M$ ไปยัง M ซึ่งสำหรับทุกๆ $k_1,k_2\in K$ และ $m_1,m_2\in M$ ได้ว่า (1) $(k_1k_2)m_1=k_1(k_2m_1)$ (2) $k_1(m_1+m_2)=k_1m_1+k_1m_2$ (3) $(k_1+k_2)m_1=k_1m_1+k_2m_1$ และ (4) $1_M\, m_1=m_1$ ให้ B เป็นสับเซตของปริภูมิเวกเตอร์ M บน K และ $<B>$ เป็นกลุ่มย่อยของ M ที่ถูกก่อกำเนิดโดยเซต $KB=\{kb \mid k\in K \text{ and } b\in B\}$ เราจะกล่าวว่า B แผ่ทั่ว M ถ้า $<B>=M$ เราจะเรียกสับเซต B ว่า อิสระเชิงเส้น ถ้า B สอดคล้องข้อใดข้อหนึ่งของเงื่อนไขต่อไปนี้ (1) $B=\phi$ หรือ (2) $|B|=1$ และ $B\neq\{0\}$ หรือ (3) $|B|>1$ และ $b\notin<B\setminus\{b\}>$ สำหรับทุกๆ $b\in B$ เราจะเรียกเซต B ว่าเป็นฐานหลักของปริภูมิเวกเตอร์ M บน K ถ้า B เป็นเซตอิสระเชิงเส้นที่แผ่ทั่ว M และเราจะกล่าวว่า M เป็นปริภูมิเวกเตอร์ที่มีมิติจำกัด ถ้า M มีฐานหลักเป็นเซตจำกัด

ผลสำคัญของงานวิจัยมีดังนี้

ทฤษฎีบท ให้ K เป็นกึ่งสนามที่มีอนุกรมผลประกอบ ดังนั้นทุกๆสองอนุกรมผลประกอบจะสมมูลกัน

ทฤษฎีบท ให้ A และ B เป็นสับเซตจำกัดของปริภูมิเวกเตอร์ M บนกึ่งสนาม K ซึ่งสอดคล้องสมบัติ $(*)$ ถ้าทั้ง A และ B เป็นฐานของ M แล้ว $|A|=|B|$ โดยที่สมบัติ $(*)$ คือ สำหรับทุกๆ $\alpha,\beta\in K$ จะมี $\gamma\in K$ ซึ่ง $\alpha+\gamma=\beta$ หรือ $\beta+\gamma=\alpha$

บทตั้งซัสเซนเฮาส์ ทฤษฎีบทไชเออร์และทฤษฎีบทของปริภูมิเวกเตอร์บนกึ่งสนามที่สอดคล้องสมบัติ $(*)$ ซึ่งขยายมาจากทฤษฎีบทต่างๆในปริภูมิเวกเตอร์บนสนาม

ภาควิชา **คณิตศาสตร์** ลายมือชื่อนิสิต..

สาขาวิชา **คณิตศาสตร์** ลายมือชื่ออาจารย์ที่ปรึกษา....................................

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SIRICHAN PAHUPONGSAB : THEOREMS OF CONVEX SUBGROUPS OF SEMIFIELDS AND VECTOR SPACES OVER SEMIFIELDS. THESIS ADVISOR : ASSISTANT PROFESSOR AMORN WASANAWICHIT, Ph.D. 45 pp. ISBN 974-13-0923-6

A triple $(K,+,\cdot)$ is called a semifield if (1) $(K,\cdot)$ is an abelian group with zero 0, (2) $(K,+)$ is a commutative semigroup with identity 0, and (3) for all $x,y,z \in K$, $x(y+z)=xy+xz$. A nonempty subset $C \neq \{0\}$ is a convex subgroup of K if (1) for all $x,y \in C$, $y \neq 0$ implies $\frac{x}{y} \in C$, and (2) for all $x,y \in C$, $\alpha,\beta \in K$, with $\alpha+\beta=1$, $\alpha x+\beta y \in C$. A strictly finite subconvex series in K is a chain of subsemifields of K, $K=K_0 \rhd K_1 \rhd \ldots \rhd K_n$, such that K_{i+1} is a convex subgroup of K_i and $K_l \neq K_j$ for $l \neq j$. Let C and C' be two strictly finite subconvex series in K. C' is a refinement of C if every term of C appears in C'. Moreover, if $C \neq C'$, then C' is a proper refinement of C. A strictly finite subconvex series in K, $K=K_0 \rhd K_1 \rhd \ldots \rhd K_n \rhd \{1\}$, is a composition series if it has no proper refinement. A vector space over a semifield K is an abelian additive group M with identity 0, for which there is a function $(k,m) \mapsto km$ from $K \times M$ into M such that for all $k_1,k_2 \in K$ and $m_1,m_2 \in M$, (1) $(k_1k_2)m_1=k_1(k_2m_1)$, (2) $k_1(m_1+m_2)=k_1m_1+k_1m_2$, (3) $(k_1+k_2)m_1=k_1m_1+k_2m_1$ and (4) $1_M m_1=m_1$. Let B be a subset of a vector space M over K and $<B>$ is the subgroup of M generated by $KB=\{kb \mid k \in K \text{ and } b \in B\}$. We call that B spans M if $<B>=M$. A set B is said to be a linearly independent set if it satisfies one of the following conditions: (1) $B=\phi$, or (2) $|B|=1$ and $B \neq \{0\}$, or (3) $|B|>1$ and $b \notin <B \setminus \{b\}>$ for all $b \in B$. A set B is said to be a basis of a vector space M over K if B is a linearly independent set which spans M and we say that M is finite-dimensional if M has a finite basis.

The main results of this research are follows:

Theorem Let K be a semifield which has a composition series. Then any two composition series are equivalent.

Theorem Let A and B be finite subsets of a vector space M over a semifield K which satisfies the property $(*)$, i.e., for all $\alpha,\beta \in K$ there exists a $\gamma \in K$ such that $\alpha+\gamma=\beta$ or $\beta+\gamma=\alpha$. If they are bases of M, then $|A|=|B|$.

Zassenhaus Lemma, Schreier's Theorem and standard theorems in vector spaces over a field can be extended in vector spaces over a semifield which satisfies the property $(*)$.

Department **Mathematics**	Student's signature................................
Field of Study **Mathematics**	Advisor's signature..............................
Academic year **2000**	Co-advisor's signature –

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CONTENTS

CHAPTER I

INTRODUCTION

In [5] Pornthip Sinutoke studied and generalized theorems from field theory to semifields. Also, in [2] Chaiwat Namnak generalized some fundamental theorems of partially ordered semigroups, partially orderings, partially ordered fields and partially ordered ratio semirings to positive ordered 0-semifields.

In this research we are interested in only semifields which are not fields. We study convex subgroups of semifields and obtain similar theorems in group theory. Moreover, we consider vector spaces over a semifield and obtain some theorems that similar to theorems in vector spaces over a field.

In Chapter II, we introduce some notations and definitions that will be used throughout this thesis.

In Chapter III, we study convex subgroups of a semifield and strictly finite subconvex series in a semifield.

In Chapter IV, we study vector spaces over a semifield which satisfies some property and linear transformations of vector spaces over a semifield.

CHAPTER II

PRELIMINARIES

In this chapter, we give some notation, definitions and examples. In this thesis, the following notation we will use:

$\mathbb{Z}$ is the set of all integers.

$\mathbb{Z}^+$ is the set of all positive integers.

$\mathbb{Q}$ is the set of all rational numbers.

$\mathbb{Q}^+$ is the set of all positive rational numbers.

$\mathbb{Q}_0^+ = \mathbb{Q}^+ \cup \{0\}$.

$\mathbb{R}^+$ is the set of all positive real numbers.

$\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$.

Definition 2.1. A nonempty set K is said to be a *semifield* if there are two binary operators, $+$ (addition) and $\cdot$ (multiplication) on K such that

(1) $(K, +)$ is a commutative semigroup with identity 0,

(2) $(K \setminus \{0\}, \cdot\,)$ is an abelian group and $k \cdot 0 = 0 \cdot k = 0$ for all $k \in K$, and

(3) $x(y + z) = xy + xz$ for all $x, y, z \in K$.

We always denote the identity of the group $(K \setminus \{0\}, \cdot\,)$ by 1.

Definition 2.2. Let K be a semifield. A nonempty subset L of K is said to be a *subsemifield* of K if

(1) $0 \in L$ and $L \neq \{0\}$,

(2) for all $x, y \in L$, with $y \neq 0$, implies $xy^{-1} \in L$, and

(3) for all $x, y \in L,\ x+y \in L$.

Remark 2.3. The intersection of a family of subsemifields of a semifield is a subsemifield.

Example 2.4. (1) $(\mathbb{Q}_0^+, +, \cdot\ )$, $(\mathbb{R}_0^+, +, \cdot\ )$ are semifields.

(2) If we define a binary operation $*$ on $\mathbb{Q}_0^+$ by $x*y = \max\{x, y\}$ for all $x, y \in \mathbb{Q}_0^+$. Then $(\mathbb{Q}_0^+, *, \cdot\ )$ is a semifield but not a subsemifield of all fields.

(3) If we define two binary operations on $\mathbb{Z}\cup\{\varepsilon\}$ by $x\odot y = x+y$, $x\odot\varepsilon = \varepsilon\odot x = \varepsilon$, $\varepsilon\odot\varepsilon = \varepsilon$ and $x \oplus y = \max\{x, y\}$, $x\oplus\varepsilon = \varepsilon\oplus x = x$ and $\varepsilon\oplus\varepsilon = \varepsilon$ for all $x, y \in \mathbb{Z}$. Then $(\mathbb{Z}\cup\{\varepsilon\}, \oplus, \odot)$ is a semifield but not a subsemifield of all fields.

(4) $(\mathbb{Q}^+ \times \mathbb{Q}^+ \cup \{(0,0)\}, +, \cdot\ )$ is a semifield.

In the remain of this thesis, we consider a semifield which is not a field. By [5], we have for every nonzero element in a semifield has no additive inverse.

Definition 2.5. Let K be a semifield. Then K is *additively cancellative* if and only if $x+z = y+z$ implies that $x = y$ for all $x, y, z \in K$.

Remark 2.6. Let K be a semifield such that $1+x = 1+y$ implies $x = y$ for all $x, y \in K$. Then K is additively cancellative.

Definition 2.7. Let K and L be semifields. A function $f : K \to L$ is a *homomorphism* of K into L if

(1) $f(x) = 0$ if and only if $x = 0$,

(2) for all $x, y \in K,\ f(x+y) = f(x)+f(y)$, and

(3) for all $x, y \in K,\ f(xy) = f(x)f(y)$.

The *multiplicative kernel* of f is the set $\{\ x \in K \mid f(x) = 1\}$, denoted by $\ker f$.

Note that if $f : K \to L$ is a homomorphism of semifields, then $\ker f$ is a subgroup of $(K \setminus \{0\}, \cdot\,)$.

Definition 2.8. A homomorphism $f : K \to L$ of semifields K and L is called a *monomorphism* if f is injective, an *epimorphism* if f is surjective and an *isomorphism* if f is bijective. Moreover, semifields K and L are *isomorphic*, denoted by $K \cong L$, if there exists an isomorphism of K onto L.

Definition 2.9. Let K be a semifield and $C \subseteq K$. Then C is said to be a *convex subset* of K if for all $x, y \in C$ and $\alpha, \beta \in K$ such that $\alpha + \beta = 1$, $\alpha x + \beta y \in C$.

Proposition 2.10. If C_1 and C_2 are convex subsets of a semifield K, then $C_1 + C_2$, $C_1 \cap C_2$ and $C_1C_2 = \{c_1c_2 \mid c_1 \in C_1 \text{ and } c_2 \in C_2\}$ are convex subsets of K.

Proof. Obviously, $C_1 + C_2$ and $C_1 \cap C_2$ are convex subsets of K.

Let $x, y \in C_1C_2$. Then $x = a_1b_1$ and $y = a_2b_2$ for some $a_1, a_2 \in C_1$ and $b_1, b_2 \in C_2$. Let $\alpha, \beta \in K$ be such that $\alpha + \beta = 1$.
If $a_1 = 0$, then $\beta a_2 = \alpha a_1 + \beta a_2 \in C_1$, so $\alpha x + \beta y = \beta a_2 b_2 \in C_1C_2$.
If $a_2 = 0$, then $\alpha a_1 = \alpha a_1 + \beta a_2 \in C_1$, so $\alpha x + \beta y = \alpha a_1 b_1 \in C_1C_2$.

Assume that $a_1 \neq 0$ and $a_2 \neq 0$. Since C_1 is a convex subset of K, we have $\alpha a_1 + \beta a_2 \in C_1$. Since C_2 is a convex subset of K, $\frac{\alpha a_1 b_1}{\alpha a_1 + \beta a_2} + \frac{\beta a_2 b_2}{\alpha a_1 + \beta a_2} \in C_2$. Hence $\alpha x + \beta y = \alpha a_1 b_1 + \beta a_2 b_2 = (\alpha a_1 + \beta a_2)(\frac{\alpha a_1 b_1}{\alpha a_1 + \beta a_2} + \frac{\beta a_2 b_2}{\alpha a_1 + \beta a_2}) \in C_1C_2$. Therefore C_1C_2 is a convex subset of K. $\square$

CHAPTER III

CONVEX SUBGROUPS OF A SEMIFIELD

In this chapter, we study convex subgroups of a semifield, strictly finite subconvex series and composition series in a semifield.

Definition 3.1. Let K be a semifield. A nonempty subset $C \neq \{0\}$ of K is a *convex subgroup* of K if

(1) for all $x, y \in C$, $y \neq 0$ implies $\frac{x}{y} \in C$, and

(2) for all $x, y \in C, \alpha, \beta \in K$ with $\alpha + \beta = 1$, $\alpha x + \beta y \in C$.

We write $C \lhd K$ or $K \rhd C$ for saying that C is a convex subgroup of K.

Example 3.2. Let K be a semifield.

(1) $\{1\}$, $K \setminus \{0\}$ and K are convex subgroups of K.

(2) Let S be a multiplicative subsemigroup of $K \setminus \{0\}$. Then $C = \big\{(\sum\limits_{i=1}^{m} a_i x_i)(\sum\limits_{j=1}^{n} b_j y_j)^{-1} \mid m, n \in \mathbb{Z}^+,\ a_i, b_j \in K,\ x_i, y_j \in S$ and $\sum\limits_{i=1}^{m} a_i = \sum\limits_{j=1}^{n} b_j = 1$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}\big\}$ is a convex subgroup of K.

Proof. (2) Let $\big(\sum\limits_{i=1}^{n} a_i x_i\big)\big(\sum\limits_{i=1}^{n} b_i y_i\big)^{-1}$, $\big(\sum\limits_{j=1}^{m} c_j z_j\big)\big(\sum\limits_{j=1}^{m} d_j w_j\big)^{-1} \in C$. Then

$$\begin{aligned}
&\big(\sum_{i=1}^{n} a_i x_i\big)\big(\sum_{i=1}^{n} b_i y_i\big)^{-1}\Big[\big(\sum_{j=1}^{m} c_j z_j\big)\big(\sum_{j=1}^{m} d_j w_j\big)^{-1}\Big]^{-1} \\
&\qquad = \big(\sum_{i=1}^{n}\sum_{j=1}^{m} a_i d_j x_i w_j\big)\big(\sum_{i=1}^{n}\sum_{j=1}^{m} b_i c_j y_i z_j\big)^{-1} \\
&\qquad \in C.
\end{aligned}$$

Let $a, b \in K$ be such that $a+b=1$. Then

$$\begin{aligned}
a\Big[\big(\sum_{i=1}^{n} a_i x_i\big)\big(\sum_{i=1}^{n} b_i y_i\big)^{-1}\Big] &+ b\Big[\big(\sum_{j=1}^{m} c_j z_j\big)\big(\sum_{j=1}^{m} d_j w_j\big)^{-1}\Big] \\
&= \Big[a\big(\sum_{i=1}^{n} a_i x_i\big)\big(\sum_{j=1}^{m} d_j w_j\big) + b\big(\sum_{j=1}^{m} c_j z_j\big)\big(\sum_{i=1}^{n} b_i y_i\big)\Big]\big(\sum_{i=1}^{n}\sum_{j=1}^{m} b_i d_j y_i w_j\big)^{-1} \\
&= \Big[\sum_{i=1}^{n}\sum_{j=1}^{m} a a_i d_j x_i w_j + \sum_{i=1}^{n}\sum_{j=1}^{m} b b_i c_j y_i z_j\Big]\big(\sum_{i=1}^{n}\sum_{j=1}^{m} b_i d_j y_i w_j\big)^{-1} \\
&\in C.
\end{aligned}$$

Hence C is a convex subgroup of K. $\square$

Remark 3.3. Let K be a semifield. Then the following statements hold.

If C_1 and C_2 are convex subgroups of K, then $C_1C_2 = \{c_1c_2 \mid c_1 \in C_1 \text{ and } c_2 \in C_2\}$ and $C_1 \cap C_2$ are convex subgroups of K.

Proof. Let $x, y \in C_1C_2$ be such that $y \neq 0$. Then $x = c_1c_2$ and $y = \bar{c}_1\bar{c}_2$ where $c_1, \bar{c}_1 \in C_1$ and $c_2, \bar{c}_2 \in C_2 \setminus \{0\}$. Since C_1 and C_2 are convex subgroups of K, we have $\frac{c_1}{\bar{c}_1} \in C_1$ and $\frac{c_2}{\bar{c}_2} \in C_2$. So $\frac{x}{y} = \frac{c_1c_2}{\bar{c}_1\bar{c}_2} = \frac{c_1}{\bar{c}_1}\frac{c_2}{\bar{c}_2} \in C_1C_2$. By Proposition 2.10, C_1C_2 is a convex subgroup of K.

Clearly, $1 \in C_1 \cap C_2$, so $C_1 \cap C_2 \neq \{0\}$. Let $u, v \in C_1 \cap C_2$ be such that $v \neq 0$. Since C_1 and C_2 are convex subgroups of K, $\frac{u}{v} \in C_1 \cap C_2$. By Proposition 2.10, we have $C_1 \cap C_2$ is a convex subgroup of K. $\square$

Definition 3.4. Let K be a semifield and C a convex subgroup of K and let K/C is the set $\{\ xC \mid x \in K\ \}$. Define two operations $+$ and $\cdot$ on K/C as follow:

for all $x, y \in K$, $\quad xC + yC = (x+y)C$ and $xC \cdot yC = xyC$.

To show that $+$ and $\cdot$ are well-defined. Since $(x+y)C = \{(x+y)c \mid c \in C\}$ $= \{xc + yc \mid c \in C\}$, we have $(x+y)C \subseteq xC + yC$. Let $c_1, c_2 \in C$ be such that $xc_1 + yc_2 \in xC + yC$. If $x = 0$ or $y = 0$, then we have $xC + yC = (x+y)C$. Assume that $x \neq 0$ and $y \neq 0$. Then $xc_1 + yc_2 = (x+y)(\frac{xc_1}{x+y} + \frac{yc_2}{x+y})$. Since $\frac{x}{x+y} + \frac{y}{x+y} = 1$, $\frac{xc_1}{x+y} + \frac{yc_2}{x+y} \in C$. Hence $xC + yC \subseteq (x+y)C$. Therefore $xC + yC = (x+y)C$.

Clearly, $xyC \subseteq xCyC$. Let $c_1, c_2 \in C$ be such that $xc_1yc_2 \in xCyC$. Then $xc_1yc_2 = xyc_1c_2 \in xyC$. Thus $xCyC \subseteq xyC$. Hence $xCyC = xyC$. Therefore $+$ and $\cdot$ are well-defined.

We have K/C is a semifield and K/C is called the *quotient semifield of K by C.*

Theorem 3.5. Let K be a semifield and $C \subseteq K \setminus \{0\}$. Then C is a convex subgroup of K if and only if $C = \ker f$ for some homomorphism f with domain K.

Proof. Assume that C is a convex subgroup of K. Define $f : K \to K/C$ by $f(x) = xC$ for all $x \in K$. Then f is a homomorphism of K into K/C and $C = \ker f$.

Conversely, let $x, y \in C$ and $\alpha, \beta \in K$ be such that $\alpha + \beta = 1$. Since $C = \ker f$, we obtain that $f(x) = f(y) = 1$. Thus $f(\alpha x + \beta y) = f(\alpha)f(x) + f(\beta)f(y) = f(\alpha) + f(\beta) = f(\alpha + \beta) = f(1) = 1$. This implies that $\frac{x}{y}$, $\alpha x + \beta y \in C$. Therefore C is a convex subgroup of K. $\square$

Theorem 3.6. ([2]) Let K and L be semifields. If $f : K \to L$ an epimorphism, then $K/\ker f \cong L$.

Lemma 3.7. ([2]) Let H be a subsemifield of a semifield K and C a convex subgroup of K. Then $HC = \{hc \mid h \in H \text{ and } c \in C\}$ is a subsemifield of K and $H \cap C$ is a convex subgroup of H.

Theorem 3.8. ([2]) Let H be a subsemifield of a semifield K and C a convex subgroup of K. Then $H/(H \cap C) \cong (HC)/C$.

Lemma 3.9. ([2]) Let N and H be convex subgroups of a semifield K and $H \subseteq N$. Then N/H is a convex subgroup of K/H.

Theorem 3.10. ([2]) Let N and H be convex subgroups of a semifield K and $H \subseteq N$. Then $(K/H)/(N/H) \cong K/N$.

Lemma 3.11. ([2]) Let M and N be semifields and L a convex subgroup of N. If $f : M \to N$ is an epimorphism, then $f^{-1}(L)$ is a convex subgroup of M.

Theorem 3.12. ([2]) Let M and N be semifields and L a convex subgroup of N. If $f : M \to N$ is an epimorphism, then $M/f^{-1}(L) \cong N/L$.

Lemma 3.13. Let M and L be subsemifields of a semifield K. If N is a convex subgroup of M and H is a convex subgroup of L, then $(N \cap L)(H \cap M)$ is a convex subgroup of $M \cap L$.

Proof. By Remark 2.3, $M \cap L$ is a subsemifield of K. By Lemma 3.7, $N \cap L = (M \cap N) \cap L = (M \cap L) \cap N$ and $H \cap M = (L \cap H) \cap M = (M \cap L) \cap H$ are convex subgroups of $M \cap L$. By Remark 3.3, $(N \cap L)(H \cap M)$ is a convex subgroup of $M \cap L$. $\square$

Theorem 3.14. Let A and B be subsemifields of a semifield K, $A^* \subseteq A \smallsetminus \{0\}$ a convex subgroup of A and $B^* \subseteq B \smallsetminus \{0\}$ a convex subgroup of B. Then

(1) $A^*(A \cap B^*)$ is a convex subgroup of $A^*(A \cap B)$,

(2) $B^*(A^* \cap B)$ is a convex subgroup of $B^*(A \cap B)$, and

(3) $(A^*(A \cap B))/(A^*(A \cap B^*)) \cong (B^*(A \cap B))/(B^*(A^* \cap B))$.

Proof. First, we show that $A^*(A \cap B^*)$ is a convex subgroup of $A^*(A \cap B)$. Since B^* is a convex subgroup of B, we have B^* is a convex subgroup of $A \cap B$, by Lemma 3.7 $A \cap B^* = (A \cap B) \cap B^*$ is a convex subgroup of $A \cap B$. Since A^* is a convex subgroup of A, A^* is a convex subgroup of $A \cap B$, by Remark 3.3 we have $A^*(A \cap B^*)$ is a convex subgroup of $A \cap B$. Hence $A^*(A \cap B^*)$ is a convex subgroup of $A^*(A\cap B)$. Similarly, we have $B^*(A^*\cap B)$ is a convex subgroup of $B^*(A\cap B)$.

Next, we show that $(A^*(A\cap B))/(A^*(A\cap B^*)) \;\cong\; (B^*(A\cap B))/(B^*(A^*\cap B))$. By Lemma 3.13, $(A^*\cap B)(A\cap B^*)$ is a convex subgroup of $A\cap B$. To show that $(A^*(A\cap B))/(A^*(A\cap B^*)) \;\cong\; (A\cap B)/((A^*\cap B)(A\cap B^*))$, let $D=(A\cap B)/((A^*\cap B)(A\cap B^*))$ and define $f: A^*(A\cap B)\to D$ by

$$f(ac)=c((A^*\cap B)(A\cap B^*)) \quad \text{for all } a\in A^* \text{ and } c\in A\cap B.$$

Let $a_1,a_2\in A^*$ and $c_1,c_2\in (A\cap B)$ be such that $a_1c_1=a_2c_2$.
If $c_1=0$, then we are done. Assume that $c_1\neq 0$. Then $\frac{a_1}{a_2}=\frac{1}{a_2}(a_1c_1)\frac{1}{c_1}=\frac{1}{a_2}(a_2c_2)\frac{1}{c_1}=\frac{c_2}{c_1}\in A^*\cap(A\cap B)=A^*\cap B\subseteq (A^*\cap B)(A\cap B^*)$, so $\frac{c_2}{c_1}\in (A^*\cap B)(A\cap B^*)$. Hence $c_1((A^*\cap B)(A\cap B^*))=c_2((A^*\cap B)(A\cap B^*))$. Therefore f is well-defined. Clearly, f is an epimorphism.

Next, we show that $\ker f=A^*(A\cap B^*)$. Let $x\in \ker f$. Then $x\in A^*(A\cap B)$ and $f(x)=(A^*\cap B)(A\cap B^*)$. Thus $x=ab$ for some $a\in A^*$ and $b\in A\cap B$, so $b((A^*\cap B)(A\cap B^*))=(A^*\cap B)(A\cap B^*)$. Hence $b\in (A^*\cap B)(A\cap B^*)$. Thus $b=b_1b_2$ for some $b_1\in A^*\cap B$ and $b_2\in A\cap B^*$. So we have $x=ab=a(b_1b_2)=(ab_1)b_2$ $\in A^*(A\cap B^*)$. Hence $\ker f\subseteq A^*(A\cap B^*)$. Let $y\in A^*(A\cap B^*)$. Then $y=y_1y_2$ for some $y\in A^*$ and $y_2\in A\cap B^*$. Since $A\cap B^*\subseteq (A^*\cap B)(A\cap B^*)$, it follows that $y_2\in (A^*\cap B)(A\cap B^*)$. Then $y_2((A^*\cap B)(A\cap B^*))=(A^*\cap B)(A\cap B^*)$. Since $A\cap B^*\subseteq A\cap B$, we have $y_2\in A\cap B$. Hence $f(y)=f(y_1y_2)=y_2((A^*\cap B)(A\cap B^*))$ $=(A^*\cap B)(A\cap B^*)$. So $y\in\ker f$. Thus $A^*(A\cap B^*)\subseteq \ker f$. Therefore, we have $\ker f=A^*(A\cap B^*)$. By Theorem 3.6, $(A^*(A\cap B))/(A^*(A\cap B^*)) \;\cong\; D$. Similarly, we have $(B^*(A\cap B))/(B^*(A^*\cap B)) \;\cong\; D$. Hence

$$(A^*(A\cap B))/(A^*(A\cap B^*)) \;\cong\; (B^*(A\cap B))/(B^*(A^*\cap B)). \qquad \square$$

Definition 3.15. Let K be a semifield. A *strictly finite subconvex series in* K is a chain of subsemifields of K, i.e., $K=K_0\rhd K_1\rhd\cdots\rhd K_n$ such that K_{i+1} is a convex subgroup of K_i for all $0\le i<n$ and $K_l\neq K_j$ for all $l\neq j$ and $l,j\in\{0,1,\dots,n\}$.

The *factors* of the series are the quotient semifields K_i/K_{i+1}. Moreover, the *length* of the series is the number of nonidentity factors.

Definition 3.16. Let $C : K = K_0 \rhd K_1 \rhd \cdots \rhd K_n$ and $C' : K = K'_0 \rhd K'_1 \rhd \cdots \rhd K'_m$ be two strictly finite subconvex series in a semifield K. Then C' is said to be a *refinement* of C if every term of C appears in C'. Moreover, if $C \neq C'$, then C' is a *proper refinement* of C.

Definition 3.17. A strictly finite subconvex series in a semifield K such that $K : K = K_0 \rhd K_1 \rhd \cdots \rhd K_n = \{1\}$ is called a *composition series* if it has no proper refinement.

Two strictly finite subconvex series C and C' in a semifield K are *equivalent* if there is a 1-1 correspondence between the nontrivial factors of C and the nontrivial factors of C' such that corresponding factors are isomorphic semifields.

Remark 3.18. If C is a composition series of a semifield K, then any refinements of C are equivalent to C.

Theorem 3.19. Any two strictly finite subconvex series in a semifield have refinements that are all equivalent.

Proof. Let K be a semifield, $K = K_0 \rhd K_1 \rhd \cdots \rhd K_n \rhd K_{n+1} = \{1\}$ and $K = L_0 \rhd L_1 \rhd \cdots \rhd L_m \rhd L_{m+1} = \{1\}$ be two strictly finite subconvex series in K. For $0 \leq i \leq n$, we have $K_i = K_{i+1}(K_i \cap L_0) \rhd K_{i+1}(K_i \cap L_1) \rhd \cdots \rhd K_{i+1}(K_i \cap L_m) \rhd K_{i+1}(K_i \cap L_{m+1}) = K_{i+1}$. Let $K_{(i,j)} = K_{i+1}(K_i \cap L_j)$ for all $0 \leq i \leq n$ and $0 \leq j \leq m$. Then we obtain a refinement $M : K = K_0 \rhd K_{(0,1)} \rhd K_{(0,2)} \rhd \cdots \rhd K_{(0,m)} \rhd K_1 \rhd K_{(1,1)} \rhd \cdots \rhd K_{(1,m)} \rhd K_2 \rhd \cdots \rhd K_n \rhd K_{(n,1)} \rhd \cdots \rhd K_{(n,m)} \rhd \{1\}$. Similarly, let $L_{(i,j)} = L_{j+1}(L_j \cap K_i)$ for all $0 \leq i \leq n$ and $0 \leq j \leq m$. Then we have $N : K = L_0 \rhd L_{(1,0)} \rhd L_{(2,0)} \rhd \cdots \rhd L_{(n,0)} \rhd L_1 \rhd L_{(1,1)} \rhd \cdots \rhd L_{(n,1)} \rhd L_2 \rhd \cdots \rhd$

$L_m \rhd L_{(1,m)} \rhd \cdots \rhd L_{(n,m)} \rhd \{1\}$. By Theorem 3.14, for $0 \leq i < n$ and $0 \leq j < m$,

$$\begin{aligned} K_{(i,j)}/K_{(i,j+1)} &= K_{i+1}(K_i \cap L_j)/K_{i+1}(K_i \cap Lj + 1) \\ &\cong L_{j+1}(L_j \cap K_i)/L_{j+1}(L_j \cap K_{i+1}) \\ &= L_{(i,j)}/L_{(i+1,j)} \end{aligned}$$

This implies that $K_{(i,j)} = K_{(i,j+1)}$ if and only if $L_{(i,j)} = L_{(i+1,j)}$. Let M_1 and N_1 be strictly finite subconvex series in K. Assume further that M_1 and N_1 are obtained from M and N, respectively, by dropping every term which is equal to its predecessor. Then M_1 and N_1 are equivalent. □

Theorem 3.20. Let K be a semifield which has a composition series. Then any two composition series are equivalent.

Proof. Let C and C' be two composition series in K. By Theorem 3.19, C and C' have refinements, say C_1 and C_1', respectively, and $C_1 \cong C_1'$. By Remark 3.18, C_1 is equivalent to C and C_1' is equivalent to C'. Hence C and C' are equivalent. □

CHAPTER IV

VECTOR SPACES OVER A SEMIFIELD AND LINEAR TRANSFORMATIONS

In this chapter, we divide the chapter into two parts. First part, we consider semifields satisfying some property and study vector spaces over a semifield. In the second part, we are interested in linear transformations of vector spaces over a semifield.

4.1 Vector Spaces over a Semifield

Definition 4.1.1. Let K be a semifield. A *vector space M over K* is an abelian additive group with identity 0, for which there is a function $(k, m) \mapsto km$ from $K \times M$ into M such that for all $k_1, k_2 \in K$ and $m_1, m_2 \in M$,

(1) $(k_1k_2)m_1 = k_1(k_2m_1)$,

(2) $k_1(m_1 + m_2) = k_1m_1 + k_1m_2$,

(3) $(k_1 + k_2)m_1 = k_1m_1 + k_2m_1$, and

(4) $1_Km_1 = m_1$.

Remark 4.1.2. If M is a vector space over a semifield K, then clearly the following statements hold:

(1) $0m = 0$ for all $m \in M$.

(2) $k0 = 0$ for all $k \in K$.

(3) $-(km) = k(-m)$ for all $k \in K$ and $m \in M$.

Definition 4.1.3. Let M be a vector space over a semifield K. A *subspace* of M is a subset of M which is, itself, a vector space over K with the operations of addition and scalar multiplication of M.

Example 4.1.4. (1) $\mathbb{Q}^n$ is a vector space over $\mathbb{Q}_0^+$ for all $n \in \mathbb{N}$.

(2) $\mathbb{R}^n$ is a vector space over $\mathbb{R}_0^+$ for all $n \in \mathbb{N}$.

(3) $\mathbb{Q} \times \mathbb{Q}$ is a vector space over $\mathbb{Q}_0^+$.

(4) $\mathbb{Q} \times \mathbb{R}$ is a vector space over $\mathbb{Q}_0^+$.

Theorem 4.1.5. Let N be a nonempty subset of a vector space M over a semifield K. Then the following statements are equivalent.

(1) N is a subspace of M.

(2) If $n_1, n_2 \in N$ and $k \in K$, then $n_1 + n_2, kn_1 \in N$.

(3) If $n_1, n_2 \in N$ and $k_1, k_2 \in K$, then $k_1 n_1 + k_2 n_2 \in N$.

(4) If $n_1, n_2 \in N$ and $k \in K$, then $kn_1 + n_2 \in N$.

Theorem 4.1.6. The intersection of any collection of subspaces of a vector space M over a semifield K is also a subspace of M.

Definition 4.1.7. Let M be a vector space over a semifield K. An element $m \in M$ is a *linear combination* of $m_1, m_2, \ldots, m_n \in M$ if $m = \alpha_1 m_1 + \cdots + \alpha_n m_n$ for some $\alpha_1, \ldots, \alpha_n \in K$. We denote $\alpha_1 m_1 + \cdots + \alpha_n m_n$ by $\sum\limits_{i=1}^{n} \alpha_i m_i$ and we simply write $\sum\limits_{u \in \{m_1, \ldots, m_n\}} \alpha_u u$.

Next, we simply denote a linear combination of finite elements in a set B, $\alpha_1 b_1 + \cdots + \alpha_n b_n$ where $\alpha_1, \ldots, \alpha_n \in K$, $b_1, \ldots, b_n \in B$, by $\sum\limits_{b \in B} \alpha_b b$.

Definition 4.1.8. Let M be a vector space over a semifield K and S a subset of M. Moreover, let $\{ N_i \mid i \in I \}$ be the family of all subspaces of M which contain S. Then $\bigcap\limits_{i\in I} N_i$ is the *subspace of M generated by S* and $\langle S\rangle$ is the subgroup of M generated by $KS = \{ ks \mid k \in K \text{ and } s \in S \}$.

If $\langle S\rangle = M$, then we say that S *spans* M.

For $s_1, \ldots, s_n \in S$, let $\langle s_1, \ldots, s_n\rangle$ denote $\big\langle \{s_1, \ldots, s_n\}\big\rangle$ and we simply call it the subspace of M generated by $s_1, \ldots, s_n$.

We denote the number of elements of S by $|S|$.

Definition 4.1.9. A subset S of M is *linearly independent* if it satisfies one of the following conditions:

(1) $S = \emptyset$,

(2) $|S| = 1$ and $S \neq \{0\}$,

(3) $|S| > 1$ and $s \notin \langle S \setminus \{s\}\rangle$ for all $s \in S$.

Moreover, S is said to be a *linearly dependent set* if S is not linearly independent.

Remark 4.1.10. If S is a subset of M and $0 \in S$, then S is linearly dependent.

Definition 4.1.11. Let S be a subset of a vector space M over a semifield. Then S is a *basis* of M if S is a linearly independent set which spans M. If $M = \{0\}$, then we have $\emptyset$ is a basis of M.

Next, we consider a semifield K which satisfies the following property :

$(*)$: for all $\alpha, \beta \in K$ there exists a $\gamma \in K$ such that $\alpha = \beta + \gamma$ or $\beta = \alpha + \gamma$.

Remark 4.1.12. Let M be a vector space over a semifield K which is not a field and satisfies the property $(*)$. Then the following statements hold:

(1) For all $\alpha, \beta \in K$ and $u \in M$ there exists a $\gamma \in K$ such that $\alpha u - \beta u = \gamma u$ or $\alpha u - \beta u = -\gamma u$.

(2) If B is a subset of M which spans M, then, for all $m \in M$, $m = \sum_{b \in B} \alpha_b \varepsilon_b b$ where $\alpha_b \in K$ and $\varepsilon_b b \in \{b, -b\}$ for all $n \in B$.

(3) for all $\alpha, \beta \in K$ and $u \in M$ there exists a $\gamma \in K$ such that $\gamma \varepsilon u = \alpha \varepsilon_1 u + \alpha \varepsilon_2 u$ where $\varepsilon u, \varepsilon_i u \in \{u, -u\}$ for all $i \in \{1, 2\}$. Moreover, if $\alpha \neq \beta$ and $u \neq 0$, then $\gamma \neq 0$.

(4) If B is a basis of $M \neq \{0\}$, then every element m of M can be written uniquely as $m = \sum_{b \in B} \alpha_b \varepsilon_b b$, that is, if $m = \sum_{b \in B} \alpha_b \varepsilon_b b = \sum_{b \in B} \beta_b \bar{\varepsilon}_b b$, then $\alpha_b = \beta_b$ and $\varepsilon_b b = \bar{\varepsilon}_b b$ for all $b \in B$.

Proof. (1) Let $\alpha, \beta \in K$ and $u \in M$. Since K satisfies the property $(*)$, we obtain that $\alpha = \beta + \gamma$ or $\beta = \alpha + \gamma$ for some $\gamma \in K$. If $\alpha = \beta + \gamma$, then $\alpha u - \beta u = (\beta + \gamma)u - \beta u = \gamma u$. Otherwise, $\beta = \alpha + \gamma$. Then $\alpha u - \beta u = \alpha u - (\alpha + \gamma)u = -\gamma u$.

(2) Let $m \in M$. Since B spans M, we have $m = \sum_{b \in B} \alpha_b b + \sum_{b \in B} \beta_b(-b)$ where $\alpha_b, \beta_b \in K$ for all $b \in B$. Then $m = \sum_{b \in B} (\alpha_b b - \beta_b b)$. By (1), for all $b \in B$ there exists $\gamma_b \in K$ such that $\alpha_b b - \beta_b b = \gamma_b b$ or $\alpha_b b - \beta_b b = -\gamma_b b$, so $\alpha_b b - \beta_b b = \gamma_b b$ or $\gamma_b(-b)$. Hence $m = \sum_{b \in B} \gamma_b \varepsilon_b b$ where $\varepsilon_b b \in \{b, -b\}$ for all $b \in B$.

(3) Let $\alpha, \beta \in K$ and $u \in M$.

If $\varepsilon_1 u = u = \varepsilon_2 u$, then $\alpha \varepsilon_1 u + \beta \varepsilon_2 u = \alpha u + \beta u = (\alpha + \beta)u$, choose $\gamma = \alpha + \beta$.

If $\varepsilon_1 u = u$ and $\varepsilon_2 u = -u$, then done by (1).

If $\varepsilon_1 = -u$ and $\varepsilon_2 u = u$, then also done by (1).

If $\varepsilon_1 u = -u = \varepsilon_2 u$, then $\alpha \varepsilon_1 u + \beta \varepsilon_2 u = \alpha(-u) + \beta(-u) = (\alpha + \beta)(-u)$, choose $\gamma = \alpha + \beta$.

Hence there exists a $\gamma \in K$ such that $\gamma \varepsilon u = \alpha \varepsilon_1 u + \beta \varepsilon_2 u$ where $\varepsilon u \in \{u, -u\}$.

Next, assume that $\alpha \neq \beta$ and $u \neq 0$. Suppose that $\gamma = 0$.

Case 1 Let $\varepsilon_1 u = u = \varepsilon_2 u$. Then $0 = \alpha u + \beta u = (\alpha+\beta)u$, so $\alpha + \beta = 0$, this is contradiction.

Case 2 Let $\varepsilon_1 u = u$ and $\varepsilon_2 u = -u$. Then $0 = \alpha u + \beta(-u)$. Since K satisfies the property $(*)$, there exists a $\eta \in K$ such that $\alpha = \beta + \eta$ or $\beta = \alpha + \eta$.

Case 2.1 Assume that $\alpha = \beta + \eta$, then $0 = \beta u + \eta u + \beta(-u) = \eta u$, so $\eta = 0$, this implies that $\alpha = \beta$, contradiction.

Case 2.2 Assume that $\beta = \alpha + \eta$. Then $0 = \alpha u + \alpha(-u) + \eta(-u) = \eta(-u)$, so $\eta = 0$. This implies that $\alpha = \beta$, contradiction.

Case 3 Let $\varepsilon_1 u = -u$ and $\varepsilon_2 u = u$. Similar Case 2.

Case 4 Let $\varepsilon_1 u = -u = \varepsilon_2 u$. Then $0 = \alpha(-u) + \beta(-u) = (\alpha+\beta)(-u)$. This implies that $\alpha + \beta = 0$, contradiction.

(4) Let $m \in M$ and $\sum\limits_{b\in B} \alpha_b \varepsilon_b b = m = \sum\limits_{b \in B} \beta_b \bar{\varepsilon}_b b$ where $\alpha_b, \beta_b \in K$, $\varepsilon_b b \in \{b, -b\}$ and $\bar{\varepsilon}_b b \in \{b, -b\}$ for all $b \in B$. To show that for all $b \in B$, $\alpha_b = \beta_b$ and $\varepsilon_b b = \bar{\varepsilon}_b b$ if $\alpha_b \neq 0$. First, suppose that there exists $b_0 \in B$ such that $\alpha_{b_0} \neq \beta_{b_0}$. Then $\sum\limits_{b\in B} \alpha_b \varepsilon_b b - \sum\limits_{b\in B} \beta_b \bar{\varepsilon}_b b = 0$. So $\sum\limits_{b\in B} \alpha_b \varepsilon_b b + \sum\limits_{b\in B} \beta_b \varepsilon_b' b = 0$ where $\varepsilon_b' b = -\bar{\varepsilon}_b b$. By (3), we have $\sum\limits_{b\in B} \eta_b \bar{\bar{\varepsilon}}_b b = 0$ where $\eta_b \in K$ and $\bar{\bar{\varepsilon}}_b b_b \in \{b, -b\}$. Since $\alpha_{b_0} \neq \beta_{b_0}$, we have $\eta_{b_0} \neq 0$.

If $\bar{\bar{\varepsilon}}_{b_0} b_0 = b_0$, then we have $b_0 = -\Big(\sum\limits_{\substack{b\in B\\ b\neq b_0}} \frac{\eta_b}{\eta_{b_0}} \bar{\bar{\varepsilon}}_b b\Big) \in \langle B \setminus \{b_0\}\rangle$, a contradiction.

If $\bar{\bar{\varepsilon}}_{b_0} b_0 = -b_0$, then $b_0 = \sum\limits_{\substack{b\in B\\ b\neq b_0}} \frac{\eta_b}{\eta_{b_0}} \bar{\bar{\varepsilon}}_b b \in \langle B \setminus \{b_0\}\rangle$, a contradiction.

Hence $\alpha_b = \beta_b$ for all $b \in B$. Next, suppose that there exist a $b_k \in B$ such that $\varepsilon_{b_k} b_k \neq \bar{\varepsilon}_{b_k} b_k$. Without loss of generality, assume that $\varepsilon_{b_k} b_k = b_k$ and $\bar{\varepsilon}_{b_k} b_k = -b_k$. Then $\sum\limits_{\substack{b\in B\\ b\neq b_k}} \alpha_b \varepsilon_b b + \alpha_{b_k} b_k = \sum\limits_{\substack{b\in B\\ b\neq b_k}} \beta_b \bar{\varepsilon}_{b_k} b_k - \beta_{b_k} b_k$, so $(\alpha_{b_k} + \beta_{b_k}) b_k = \sum\limits_{\substack{b\in B\\ b\neq b_k}} \beta_b \bar{\varepsilon}_b b - \sum\limits_{\substack{b\in B\\ b\neq b_k}} \alpha_b \varepsilon_b b$. Thus $b_k = \sum\limits_{\substack{b\in B\\ b\neq b_k}} \frac{\beta_b}{\alpha_{b_k}+\beta_{b_k}} \bar{\varepsilon}_b b - \sum\limits_{\substack{b\in B\\ b\neq b_k}} \frac{\alpha_b}{\alpha_{b_k}+\beta_{b_k}} \varepsilon_b b \in \langle B \setminus \{b_k\}\rangle$, a contradiction. Hence $\varepsilon_b b = \bar{\varepsilon}_b b$ for all $b \in B$. $\square$

Example 4.1.13. (1) $\mathbb{Q}_0^+$ and $\mathbb{R}_0^+$ are semifields satisfying the property $(*)$.

(2) $(\mathbb{Q}_0^+, *, \cdot\ )$ and $(\mathbb{Z}\cup\{\varepsilon\}, \oplus, \odot)$ in Example 2.4 are semifields satisfying the property $(*)$.

(3) $(\mathbb{Q}^+ \times \mathbb{Q}^+) \cup \{(0,0)\}$ is a semifield but not satisfies the property $(*)$, since $(1,2) \neq (2,1) + (x,y)$ and $(2,1) \neq (1,2) + (x,y)$ for all $x, y \in \mathbb{Q}_0^+$.

From now on, we let $K^{(*)}$ denote a semifield K which satisfies the property $(*)$.

Theorem 4.1.14. Let M be a vector space over a semifield $K^{(*)}$. If A is a finite basis of M, then B is a linearly dependent set for all subsets B of M such that $|B| > |A|$.

Proof. Let $A = \{a_1, \ldots, a_n\}$. Since $|B| > |A|$, we let $b_1, \ldots, b_n, b_{n+1}$ be elements of B which are all distinct. Since $|B| > |A|$, there exists $b_{i_1} \in B$ such that $b_{i_1} \notin A$. Since A spans M, we have $b_{i_1} = \sum\limits_{k=1}^{n} \alpha_k \varepsilon_k a_k$ where $\alpha_k \in K^{(*)}$ and $\varepsilon_k a_k \in \{a_k, -a_k\}$ for all $k \in \{1, \ldots, n\}$. If $\alpha_k = 0$ for all k, then $b_{i_1} = 0$, so B is linearly dependent.

Assume that $\alpha_j \neq 0$ for some $j \in \{1, \ldots, n\}$.

If $\varepsilon_j a_j = a_j$, then $a_j = \frac{1}{\alpha_j} b_{i_1} - \frac{\alpha_1}{\alpha_j} a_1 - \cdots - \frac{\alpha_{j-1}}{\alpha_j} a_{j-1} - \frac{\alpha_{j+1}}{\alpha_j} a_{j+1} - \cdots - \frac{\alpha_n}{\alpha_j} a_n \in \langle (A \cup \{b_{i_1}\}) \setminus \{a_j\} \rangle$.

If $\varepsilon_j a_j = -a_j$, then $a_j = -\frac{1}{\alpha_j} b_{i_1} + \frac{\alpha_1}{\alpha_j} a_1 + \cdots + \frac{\alpha_{j-1}}{\alpha_j} a_{j-1} + \frac{\alpha_{j+1}}{\alpha_j} a_{j+1} + \cdots + \frac{\alpha_n}{\alpha_j} a_n \in \langle (A \cup \{b_{i_1}\}) \setminus \{a_j\} \rangle$. Hence we have

$$a_j \in \big\langle (A \cup \{b_{i_1}\}) \setminus \{a_j\} \big\rangle \tag{4.1}$$

<u>Case 1</u> For all $b_i \in B \setminus \{b_{i_1}\}$, $b_i \in A$, i.e., $|B| = |A| + 1$. By (4.1) implies that $M = \langle A \rangle = \big\langle (A \cup \{b_{i_1}\}) \setminus \{a_j\} \big\rangle$. But $a_j \in B$ and we have $a_j \in \langle B \setminus \{a_j\} \rangle$, so B is linearly dependent.

<u>Case 2</u> There exists $b_{i_2} \in B \setminus \{b_{i_1}\}$ such that $b_{i_2} \notin A$. Since A spans M, we have $b_{i_2} = \sum\limits_{k=1}^{n} \beta_k \varepsilon_k a_k$ where $\beta_k \in K^{(*)}$ and $\varepsilon_k a_k \in \{a_k, -a_k\}$ for all $k \in \{1, \ldots, n\}$. If

$\beta_k = 0$ for all k, then $b_{i_2} = 0$, so B is linearly dependent.

Assume that $\beta_r \neq 0$ for some $r \in \{1, \ldots, n\}$. By (4.1), $a_j = \sum\limits_{\substack{s=1 \\ s\neq j}}^{n} \gamma_s \varepsilon_s a_s + \gamma \varepsilon b_{i_1}$ where $\gamma, \gamma_s \in K^{(*)}$ and $\varepsilon_s a_s \in \{a_s, -a_s\}$, $\varepsilon b_{i_1} \in \{b_{i_1}, -b_{i_1}\}$ for all $s \in \{1, \ldots, n\} \backslash \{j\}$. Since A is linearly independent, $\gamma \neq 0$.

<u>Case 2.1</u> Let $r = j$. Then

$$\begin{aligned} b_{i_2} &= \beta_1 \varepsilon_1 a_1 + \cdots + \beta_r \varepsilon_r a_r + \cdots + \beta_n \varepsilon_n a_n \\ &= \beta_1 \varepsilon_1 a_1 + \cdots + \beta_j \varepsilon_j \left(\sum_{\substack{s=1 \\ s\neq j}}^{n} \gamma_s \varepsilon_s a_s + \gamma \varepsilon b_{i_1} \right) + \cdots + \beta_n \varepsilon_n a_n \\ &= \sum_{\substack{s=1 \\ s\neq j}}^{n} \eta_s \bar{\varepsilon}_s a_s + \eta \bar{\varepsilon} b_{i_1} \end{aligned}$$

where $\eta_s,\ \eta \in K^{(*)}$, $\bar{\varepsilon}_s a_s \in \{a_s, -a_s\}$, and $\bar{\varepsilon} b_{i_1} \in \{b_{i_1}, -b_{i_1}\}$ for all $i \in \{1, \ldots, n\} \setminus \{j\}$. Since $\beta_r \neq 0$ and $\gamma \neq 0$, we have $\eta \neq 0$.

If $\eta_s = 0$ for all $s \neq j$, then $b_{i_2} = \eta \bar{\varepsilon} b_{i_1} \in \langle B \setminus \{b_{i_2}\} \rangle$, so B is linearly dependent. Now, we assume that there exists $l \neq r$ such that $\eta_l \neq 0$.

If $\bar{\varepsilon}_l a_l = a_l$, then

$$a_l = \frac{1}{\eta_l} b_{i_2} - \frac{\eta}{\eta_l} \bar{\varepsilon} b_{i_1} - \sum_{\substack{s=1 \\ s\notin\{j,l\}}}^{n} \frac{\eta_s}{\eta_l} \bar{\varepsilon}_s a_s \in \big\langle (A \cup \{b_{i_1}, b_{i_2}\}) \setminus \{a_j, a_l\} \big\rangle.$$

If $\bar{\varepsilon}_l a_l = -a_l$, then

$$a_l = \frac{1}{\eta_l}(-b_{i_2}) + \frac{\eta}{\eta_l} \bar{\varepsilon} b_{i_1} + \sum_{\substack{s=1 \\ s\notin\{j,l\}}}^{n} \frac{\eta_s}{\eta_l} \bar{\varepsilon}_s a_s \in \big\langle (A \cup \{b_{i_1}, b_{i_2}\}) \setminus \{a_j, a_l\} \big\rangle.$$

Hence $a_l \in \big\langle (A \cup \{b_{i_1}, b_{i_2}\}) \setminus \{a_j, a_l\} \big\rangle$.

<u>Case 2.2</u> Let $r \neq j$. Without loss of generality, we assume that $r < j$. Then $b_{i_2} = \beta_1 \varepsilon_1 a_1 + \cdots + \beta_r \varepsilon_r a_r + \cdots + \beta_j \varepsilon_j \big(\sum\limits_{\substack{s=1 \\ s\neq j}}^{n} \gamma_s \varepsilon_s a_s + \gamma \varepsilon b_{i_1} \big) + \cdots + \beta_n \varepsilon_n a_n$. Since K satisfies the property $(*)$, we have $b_{i_2} = \eta_1 \bar{\bar{\varepsilon}}_1 a_1 + \cdots + \eta_r \bar{\bar{\varepsilon}}_r a_r + \cdots + \eta_{j-1} \bar{\bar{\varepsilon}}_{j-1} a_{j-1} + \eta_{j+1} \bar{\bar{\varepsilon}}_{j+1} a_{j+1} + \cdots + \eta_n \bar{\bar{\varepsilon}}_n a_n + \eta \bar{\bar{\varepsilon}} b_{i_1}$ where $\eta_i \in K^{(*)}$, $\eta = \beta_j \gamma$, $\bar{\bar{\varepsilon}}_s a_s \in \{a_s, -a_s\}$ and

$\bar{\bar{\varepsilon}} b_{i_1} \in \{b_{i_1}, -b_{i_1}\}$ for all $s \neq j$. So $b_{i_2} = \sum\limits_{\substack{s=1 \\ s \neq j}}^{n} \eta_s \bar{\bar{\varepsilon}}_s a_s + \eta \bar{\bar{\varepsilon}} b_{i_1}$. If $\eta_s = 0$ for all $s \neq j$, then $\eta \neq 0$ and $b_{i_2} = \eta \bar{\bar{\varepsilon}} b_{i_1} \in \langle B \setminus \{b_{i_2}\} \rangle$, so B is linearly dependent. Assume that there exists $v \neq r$ such that $\eta_v \neq 0$.

If $\bar{\bar{\varepsilon}}_v a_v = a_v$, then

$$a_v = \frac{1}{\eta_v} b_{i_2} - \left(\sum_{\substack{s=1 \\ s \neq r}}^{n} \frac{\eta_s}{\eta_v} \bar{\bar{\varepsilon}}_s a_s \right) - \frac{\eta}{\eta_v} \bar{\bar{\varepsilon}} b_{i_1} \in \big\langle (A \cup \{b_{i_1}, b_{i_2}\}) \setminus \{a_j, a_v\} \big\rangle.$$

If $\bar{\bar{\varepsilon}}_v a_v = -a_v$, then

$$a_v = \frac{1}{\eta_v}(-b_{i_2}) + \left(\sum_{\substack{s=1 \\ s \neq r}}^{n} \frac{\eta_s}{\eta_v} \bar{\bar{\varepsilon}}_s a_s \right) + \frac{\eta}{\eta_v} \bar{\bar{\varepsilon}} b_{i_1} \in \big\langle (A \cup \{b_{i_1}, b_{i_2}\}) \setminus \{a_j, a_v\} \big\rangle.$$

Hence $\quad a_v \in \langle (A \cup \{b_{i_1}, b_{i_2}\}) \setminus \{a_j, a_v\} \rangle$.

By Case 2.1 and Case 2.2, we obtain $a_k \in \langle A \cup \{b_{i_1}, b_{i_2}\}) \setminus \{a_j, a_k\} \rangle$ for some $k \neq j$.

Apply this method Case 2 to other element of B.

If $|B \setminus A| = m < n$, there exists an element x of $A \cap B$ such that $x \in \langle A \rangle = \langle (A \cup \{b_{i_1}, \ldots, b_{i_m}\}) \setminus \{a_{j_1}, \ldots, a_{j_m}, x\} \rangle = \langle B \setminus \{x\} \rangle$, so B is linearly dependent.

If $|B \setminus A| \geq n$, then we obtain

$$\begin{aligned}
&a_{j_1} \in \big\langle (A \cup \{b_{i_1}\}) \setminus \{a_{j_1}\} \big\rangle, \\
&a_{j_2} \in \big\langle (A \cup \{b_{i_1}, b_{i_2}\}) \setminus \{a_{j_1}, a_{j_2}\} \big\rangle, \ j_1 \neq j_2, \\
&a_{j_3} \in \big\langle (A \cup \{b_{i_1}, b_{i_2}, b_{i_3}\}) \setminus \{a_{j_1}, a_{j_2}, a_{j_3}\} \big\rangle, \ j_1, j_2 \text{ and } j_3 \text{ are distinct}, \\
&\quad \vdots \\
&a_{j_n} \in \big\langle (A \cup \{b_{i_1}, \ldots, b_{i_n}\}) \setminus \{a_{j_1}, \ldots, a_{j_n}\} \big\rangle = \langle b_{i_1}, \ldots, b_{i_n} \rangle, \ j_1, \ldots, j_n \text{ are distinct.}
\end{aligned}$$

This implies that $M = \langle b_{i_1}, \ldots, b_{i_n} \rangle$. Since $b_{i_{n+1}} \in B$ and $b_{i_{n+1}} \in \langle B \setminus \{b_{i_{n+1}}\} \rangle$, we have B is linearly dependent. $\square$

Theorem 4.1.15. Let A and B be finite subsets of a vector space M over a semifield $K^{(*)}$. If they are bases of M, then $|A| = |B|$.

Proof. This follows from Theorem 4.1.14. □

Theorem 4.1.16. Let $B = \{b_1, \dots, b_n\}$ be a maximal linearly independent subset of a vector space M over a semifield $K^{(*)}$. Then B is a basis of M.

Proof. Since B is linearly independent, $b_i \notin \langle B \setminus \{b_i\} \rangle$ for all $i \in \{1, \dots, n\}$.

Suppose that $\langle B \rangle \subsetneq M$. Then there exists an $m \in M \setminus \langle B \rangle$. Claim that $B \cup \{m\}$ is a linearly independent set. Suppose not. Then there exists a $b \in \langle B \cup \{m\} \setminus \{b\} \rangle$. If $b = m$, then $m \in \langle B \rangle$, this contradicts that $m \in M \setminus \langle B \rangle$. Assume that $b = b_j$ for some $j \in \{1, \dots, n\}$. Since $b_j \in \langle B \cup \{m\} \setminus \{b_j\} \rangle$, we have $b_j = \sum\limits_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i \varepsilon_i b_i + \alpha \varepsilon m$ where $\alpha_i, \alpha \in K^{(*)}$, $\varepsilon_i b_i \in \{b_i, -b_i\}$ and $\varepsilon m \in \{m, -m\}$. Since $m \in M \setminus \langle B \rangle$, we have $\alpha \neq 0$. If $\varepsilon m = m$, then $m = -\frac{1}{\alpha} \sum\limits_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i \varepsilon_i b_i + \frac{1}{\alpha} b_j \in \langle B \rangle$ which is a contradiction. Otherwise, if $\varepsilon m = -m$, then $m = \frac{1}{\alpha} \sum\limits_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i \varepsilon_i b_i - \frac{1}{\alpha} b_j \in \langle B \rangle$ which is, again, contradiction. Hence $B \cup \{m\}$ is a linearly independent set. But $\big| B \cup \{m\} \big| > |B|$, this contradicts the maximality of B. Hence B spans M. Therefore, B is a basis of M. □

Remark 4.1.17. Let M be a vector space over a semifield $K^{(*)}$ and B a linearly independent subset of M. If $m \in M \setminus \langle B \rangle$, then $B \cup \{m\}$ is also linearly independent.

Definition 4.1.18. Let $M \neq \{0\}$ be a vector space over a semifield $K^{(*)}$. Then M is said to be *finite-dimensional* if M has a finite basis.

The *dimension* of M, denoted $\dim M$, is the number of elements in a basis of M.

Example 4.1.19. (1) $\{1\}$ is a basis of $\mathbb{Q}$ over $\mathbb{Q}_0^+$. Then $\dim \mathbb{Q} = 1$.

(2) Let $e_1, \ldots, e_n \in \mathbb{Q}^n$ be defined by

$$\begin{aligned} e_1 &= (1, 0, \ldots, 0, 0) \\ e_2 &= (0, 1, \ldots, 0, 0) \\ &\vdots \\ e_n &= (0, 0, \ldots, 0, 1). \end{aligned}$$

Then $\{e_1, \ldots, e_n\}$ is a basis of the vector spaces $\mathbb{Q}^n$ over the field $\mathbb{Q}$. In fact, by the definition of a vector space over a semifield, we also have that $\{e_1, \ldots, e_n\}$ is a basis of the vector space $\mathbb{Q}^n$ over the semifield $\mathbb{Q}_0^+$, hence $\dim \mathbb{Q}^n = n$. Also, this fact is true if we replace $\mathbb{Q}$ by $\mathbb{R}$ and $\mathbb{Q}_0^+$ by $\mathbb{R}_0^+$.

Theorem 4.1.20. Let M be a vector space over a semifield $K^{(*)}$ and S a linearly independent nonempty subset of M. Then there exists a subset B of M such that $S \subseteq B$ and B is a basis of M.

Proof. Let $\mathfrak{J} = \{C \subseteq M \mid C \text{ is linearly independent and } S \subseteq C\}$. Then $S \in \mathfrak{J}$. Recall that $\subseteq$ is a partial order on $\mathfrak{J}$. Let $\mathfrak{C}$ be a nonempty chain in $\mathfrak{J}$ and $D = \bigcup\limits_{C \in \mathfrak{C}} C$. Since $S \subseteq C$ for all $C \in \mathfrak{C}$, we obtain that $S \subseteq D$. We claim that D is linearly independent. Suppose not. Then $D \neq \emptyset$ and let $x \in \langle D \setminus \{x\} \rangle$ where $x \in D$. Thus $x = \sum\limits_{a \in D \setminus \{x\}} \alpha_a \varepsilon_a a$ where $\alpha_a \in K^{(*)}$ and $\varepsilon_a a \in \{a, -a\}$ for all $a \in D \setminus \{x\}$. Since $\mathfrak{C}$ is a chain, there exists a $C_0 \in \mathfrak{C}$ such that $x, a \in C_0$ for all $a \in D \setminus \{x\}$. Thus $x \in \langle C_0 \setminus \{x\} \rangle$, so C_0 is linearly dependent. This is a contradiction. Hence D is linearly independent. Thus $D \in \mathfrak{J}$, so D is an upper bound of $\mathfrak{C}$ in $\mathfrak{J}$. By Zorn's Lemma, $\mathfrak{J}$ has a maximal element, say N. Then N is linearly independent and $S \subseteq N$.

Next, we show that N spans M. Suppose that $\langle N \rangle \subsetneq M$. Then there exists $u \in M \setminus \langle N \rangle$. By Remark 4.1.17, we have $N \cup \{u\}$ is linearly independent. But

$N \subsetneq N \cup \{u\}$ which contradicts the maximality of N. Hence N spans M. Therefore N is a basis of M and $S \subseteq N$. □

Corollary 4.1.21. Every vector space M over a semifield $K^{(*)}$ has a basis.

Proof. If $M = \{0\}$, then $\emptyset$ is a basis of M.

Assume that $M \neq \{0\}$. By Theorem 4.1.20, let S be a singleton set of nonzero element in M. □

Theorem 4.1.22. Let M be a finite-dimensional vector space over a semifield $K^{(*)}$. If N is a proper subspace of M, then N is finite-dimensional and $\dim N < \dim M$.

Proof. Let B be a basis of N. By Theorem 4.1.20, there exists a subset C of M such that $B \subseteq C$ and C is a basis of M. Since M is finite-dimensional, C is finite, so B is finite. If $|B| = |C|$, then $C = B$ and $M = \langle C \rangle = \langle B \rangle = N$ which is a contradiction. □

Theorem 4.1.23. Let M be a vector space over a semifield $K^{(*)}$ and S a subset of M such that S spans M. Then there exists a subset B of S such that B is a basis of M.

Proof. Let $\mathfrak{J} = \{A \mid A \subseteq S \text{ and } A \text{ is linearly independent}\}$. Then $\emptyset \in \mathfrak{J}$. Let $\subseteq$ be a partially order on $\mathfrak{J}$. Let $\mathfrak{C}$ be a nonempty chain in $\mathfrak{J}$ and $C = \bigcup\limits_{A \in \mathfrak{C}} A$. Since $A \subseteq S$ for all $A \in \mathfrak{C}$, we have $C \subseteq S$. We claim that C is linearly independent. Suppose not. Then $C \neq \emptyset$ and let $x \in \langle C \setminus \{x\} \rangle$ where $x \in C$. Then $x = \sum\limits_{a \in C \setminus \{x\}} \alpha_a \varepsilon_a a$ where $\alpha_a \in K^{(*)}$ and $\varepsilon_a a \in \{a, -a\}$ for all $a \in C \setminus \{x\}$. Since $\mathfrak{C}$ is a chain, there exists an $A_0 \in \mathfrak{C}$ such that $x, a \in A_0$ for all $a \in C \setminus \{x\}$. Thus $x \in \langle A_0 \setminus \{x\} \rangle$ which implies that A_0 is linearly dependent. This is a contradiction. Hence C is linearly independent. So C is an upper bound of $\mathfrak{C}$ in $\mathfrak{J}$. By Zorn's Lemma, $\mathfrak{J}$ has a maximal element, say N. Thus $N \subseteq S$ and N is linearly independent.

Next, suppose that $\langle N\rangle \subsetneq M$. If $N = S$, then $\langle N\rangle = \langle S\rangle = M$, this is a contradiction, so $N \subsetneq S$. If $S \subseteq \langle N\rangle$, then $M = \langle S\rangle \subseteq \big\langle\langle N\rangle\big\rangle = \langle N\rangle$ which is a contradiction, so $\langle N\rangle \subsetneq S$. Thus there is a $u \in S$ such that $u \in S \setminus \langle N\rangle$. By Remark 4.1.17, we have $N \cup \{u\}$ is linearly independent and $N \cup \{u\} \subseteq S$. But $N \subsetneq N \cup \{u\}$ which contradicts the maximality of N. Hence N spans M. Therefore N is a basis of M. $\square$

Theorem 4.1.24. Let M be a finite-dimensional vector space over a semifield $K^{(*)}$, $\dim M = n$, and S a subset of M. Then

(1) S is linearly independent implies that $|S| \leq n$,

(2) $|S| < n$ implies that $\langle S\rangle \neq M$, and

(3) $|S| = n$ and S spans M implies that S is a basis of M.

Proof. The results (1) and (2) follow from Theorem 4.1.14 and Theorem 4.1.23, respectively.

(3) Let $S = \{s_1, \ldots, s_n\}$ and S spans M. Suppose that S is linearly dependent. Then $s_i \in \langle S \setminus \{s_i\}\rangle$ for some $i \in \{1, \ldots, n\}$. This implies that $S \setminus \{s_i\}$ spans M. But $\big|S \setminus \{s_i\}\big| < n$, by (2), we have $\langle S \setminus \{s_i\}\rangle \neq M$ which is a contradiction. Hence S is linearly independent. Thus S is a basis of M. $\square$

Definition 4.1.25. Let M_1 and M_2 be subspaces of a vector space over a semifield. Then $M_1 + M_2$ is defined to be the set of all elements of the form $m_1 + m_2$ where $m_1 \in M_1$ and $m_2 \in M_2$, i.e.,

$$M_1 + M_2 = \{m_1 + m_2 \mid m_1 \in M_1 \text{ and } m_2 \in M_2\}.$$

Lemma 4.1.26. Let M_1 and M_2 be subspaces of a vector space M over a semifield $K^{(*)}$. Then $M_1 + M_2$ is also a subspace of M.

Theorem 4.1.27. Let M be a vector space over a semifield $K^{(*)}$ and M_1, M_2 subspaces of M. Then $M_1 + M_2$ is the smallest subspace of M containing both M_1 and M_2, that is $M_1 + M_2 = \langle M_1 \cup M_2 \rangle$. Moreover, if B_1 spans M_1 and B_2 spans M_2, then $B_1 \cup B_2$ spans $M_1 + M_2$.

Proof. First, we prove that $M_1 \cup M_2$ spans $M_1 + M_2$. Since $0 \in M_2$, we have $M_1 \subseteq M_1 + M_2$. Similarly, $M_2 \subseteq M_1 + M_2$ since $0 \in M_1$. Clearly, $M_1 + M_2$ is a subspace of M containing $K(M_1 \cup M_2)$, so $\langle M_1 \cup M_2 \rangle \subseteq M_1 + M_2$.

Let $m \in M_1 + M_2$. Then $m = m_1 + m_2$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Since $m_1, m_2 \in \langle M_1 \cup M_2 \rangle$ and $\langle M_1 \cup M_2 \rangle$ is a subspace of M, we obtain that $m = m_1 + m_2 \in \langle M_1 \cup M_2 \rangle$. Hence $M_1 + M_2 \subseteq \langle M_1 \cup M_2 \rangle$. Therefore $M_1 \cup M_2$ spans $M_1 + M_2$. Clearly, $M_1 + M_2$ is the smallest subspace containing M_1 and M_2.

For the next part, we have $M_1 = \langle B_1 \rangle \subseteq \langle B_1 \cup B_2 \rangle$ and $M_2 = \langle B_2 \rangle \subseteq \langle B_1 \cup B_2 \rangle$. Then $M_1 \cup M_2 \subseteq \langle B_1 \cup B_2 \rangle \subseteq \langle M_1 \cup M_2 \rangle$. So $\langle M_1 \cup M_2 \rangle \subseteq \big\langle \langle B_1 \cup B_2 \rangle \big\rangle = \langle B_1 \cup B_2 \rangle$. Hence $\langle M_1 \cup M_2 \rangle = \langle B_1 \cup B_2 \rangle$. Therefore $M_1 + M_2 = \langle B_1 \cup B_2 \rangle$, this implies that $B_1 \cup B_2$ spans $M_1 + M_2$. $\square$

Theorem 4.1.28. Let M be a finite-dimensional vector space over a semifield $K^{(*)}$. If M_1 and M_2 are two subspace of M, then

$$\dim(M_1 + M_2) = \dim M_1 + \dim M_2 - \dim(M_1 \cap M_2).$$

Proof. Let B be a basis of $M_1 \cap M_2$. By Theorem 4.1.20, there exists a subset B_1 of M_1 such that $B \subseteq B_1$ and B_1 is a basis of M_1 and there exists a subset B_2 of M_2 such that $B \subseteq B_2$ and B_2 is a basis of M_2. By Theorem 4.1.27, we have $B_1 \cup B_2$ spans $M_1 + M_2$. We claim that $B_1 \cup B_2$ is linearly independent.

First, we consider $M_1 \cap M_2 = \{0\}$. Then $B = \emptyset$ and $B_1 \cap B_2 \subseteq M_1 \cap M_2 = \{0\}$, so $B_1 \cap B_2 = \emptyset$. Suppose that $B_1 \cup B_2$ is linearly dependent. Then there exists a $b \in B_1 \cup B_2$ such that $b \in \langle B_1 \cup B_2 \setminus \{b\} \rangle$.

Case 1 Let $b \in B_1$. Then $b = \sum\limits_{x \in B_1 \setminus \{b\}} \alpha_x \varepsilon_x x + \sum\limits_{y \in B_2} \beta_y \varepsilon_y y$ where $\alpha_x, \beta_y \in K^{(*)}$, $\varepsilon_x x \in \{x, -x\}$ and $\varepsilon_y y \in \{y, -y\}$ for all $x \in B_1 \setminus \{b\}$ and $y \in B_2$. If $\beta_y = 0$ for all $y \in B_2$, then $b \in \langle B_1 \setminus \{b\} \rangle$, this implies that B_1 is linearly dependent which is a contradiction. Hence $\beta_{y_0} \neq 0$ for some $y_0 \in B_2$ and we have

$$-\left(\sum_{y \in B_2} \beta_y \varepsilon_y y \right) = \sum_{x \in B_1 \setminus \{b\}} \alpha_x \varepsilon_x x - b.$$

The left-handed side is an element of M_2 while the right-handed side is an element of M_1. Thus the both sides belong to $M_1 \cap M_2 = \{0\}$. This implies that $\beta_{y_0} = 0$ which leads to a contradiction. Hence $B_1 \cup B_2$ is linearly independent.

Case 2 Let $b \in B_2$. The proof is similar to the Case 1. We have $B_1 \cup B_2$ is also linearly independent. Hence $B_1 \cup B_2$ is a basis of $M_1 + M_2$ and $\dim(M_1 + M_2) = |B_1 \cup B_2| = |B_1| + |B_2| = |B_1 \cap B_2| = \dim M_1 + \dim M_2 = \dim(M_1 \cap M_2)$.

Next, we assume that $M_1 \cap M_2 \neq \{0\}$. If $B_1 = B$ or $B_2 = B$, then we are done. Assume that $B \subsetneq B_1$ and $B \subsetneq B_2$. Suppose that $B_1 \cup B_2$ is linearly dependent. Then there exists a $b \in B_1 \cup B_2$ such that $b \in \langle B_1 \cup B_2 \setminus \{b\} \rangle$.

Let $B = \{b_1, \ldots, b_r\}$, $B_1 = \{b_1, \ldots, b_r, c_1, \ldots, c_s\}$ and $B_2 = \{b_1, \ldots, b_r, d_1, \ldots, d_t\}$. Then $B_1 \cup B_2 = \{b_1, \ldots, b_r, c_1, \ldots, c_s, d_1, \ldots, d_t\}$. Since $b \in B_1 \cup B_2$, we obtain that $b \in B$ or $b \in B_1 \setminus B$ or $b \in B_2 \setminus B$.

Case 1 Let $b \in B$. Then there exists $j \in \{1, \ldots, r\}$ such that $b = b_j$. Since $b \in \langle B_1 \cup B_2 \setminus \{b\} \rangle$, we have

$$b_j = \sum_{\substack{i=1 \\ i \neq j}}^{r} \alpha_i \varepsilon_i b_i + \sum_{k=1}^{s} \beta_k \varepsilon_k c_k + \sum_{l=1}^{t} \gamma_l \varepsilon_l d_l \tag{4.2}$$

where $\alpha_i, \beta_k, \gamma_l \in K^{(*)}$, $\varepsilon_i b_i \in \{b_i, -b_i\}$, $\varepsilon_k c_k \in \{c_k, -c_k\}$ and $\varepsilon_l d_l \in \{d_l, -d_l\}$. Thus

$$-\left(\sum_{l=1}^{t} \gamma_l \varepsilon_l d_l \right) = \sum_{\substack{i=1 \\ i \neq j}}^{r} \alpha_i \varepsilon_i b_i - b_j + \sum_{k=1}^{s} \beta_k \varepsilon_k c_k.$$

The left-handed side is an element of M_2 and the right-handed side is an element of M_1. So each side belongs to $M_1 \cap M_2 = \langle B \rangle$. Hence

$$-\left(\sum_{l=1}^{t} \gamma_l \varepsilon_l d_l\right) = \sum_{m=1}^{r} u_m \varepsilon_m b_m \tag{4.3}$$

$$\sum_{\substack{i=1 \\ i \neq j}}^{r} \alpha_i \varepsilon_i b_i - b_j + \sum_{k=1}^{s} \beta_k \varepsilon_k c_k = \sum_{n=1}^{r} v_n \varepsilon_n b_n \tag{4.4}$$

where $u_m, v_n \in K^{(*)}$, $\varepsilon_m b_m \in \{b_m, -b_m\}$ and $\varepsilon_n b_n \in \{b_n, -b_n\}$ for all m, n.

If $\gamma_l = 0$ for all l or $\beta_k = 0$ for all k, by (4.2) we have $b_j \in \langle B_1 \setminus \{b_j\} \rangle$ or $b_j \in \langle B_2 \setminus \{b_j\} \rangle$ which is a contradiction. Hence there exist $l_0 \in \{1, \ldots, t\}$ and $k_0 \in \{1, \ldots, s\}$ such that $\beta_{k_0} \neq 0$ and $\gamma_{l_0} \neq 0$. By (4.3) and (4.4) we have

$d_{l_0} \in \langle B_2 \setminus \{d_{l_0}\} \rangle$ and $c_{k_0} \in \langle B_1 \setminus \{c_{k_0}\} \rangle$ which is, again, a contradiction. Hence $B_1 \cup B_2$ is linearly independent.

Case 2 Let $b \in B_1 \setminus B$. Then $b = c_i$ for some $i \in \{1, \ldots, s\}$. Since

$b \in \langle B_1 \cup B_2 \setminus \{b\} \rangle$, we have

$$c_i = \sum_{j=1}^{r} \alpha_j \varepsilon_j b_j + \sum_{\substack{k=1 \\ k \neq i}}^{s} \beta_k \varepsilon_k c_k + \sum_{l=1}^{t} \gamma_l \varepsilon_l d_l \tag{4.5}$$

where $\alpha_j, \beta_k, \gamma_l \in K^{(*)}$, $\varepsilon_j b_j \in \{b_j, -b_j\}$, $\varepsilon_k c_k \in \{c_k, -c_k\}$ and $\varepsilon_l d_l \in \{d_l, -d_l\}$.
Thus $\quad -\big(\sum_{l=1}^{t} \gamma_l \varepsilon_l d_l\big) = \sum_{j=1}^{r} \alpha_j \varepsilon_j b_j + \sum_{\substack{k=1 \\ k \neq i}}^{s} \beta_k \varepsilon_k c_k - c_i.$

The left-handed side is an element of M_2 and the right-handed side is an element of M_1, so each side belongs to $M_1 \cap M_2 = \langle B \rangle$. Hence

$$-\left(\sum_{l=1}^{t} \gamma_l \varepsilon_l d_l\right) = \sum_{m=1}^{r} u_m \varepsilon_m b_m \tag{4.6}$$

$$\sum_{j=1}^{r} \alpha_j \varepsilon_j b_j + \sum_{\substack{k=1 \\ k \neq i}}^{s} \beta_k \varepsilon_k c_k - c_i = \sum_{n=1}^{r} v_n \varepsilon_n b_n \tag{4.7}$$

where $u_m, v_n \in K^{(*)}$, $\varepsilon_m b_m \in \{b_m, -b_m\}$ and $\varepsilon_n b_n \in \{b_n, -b_n\}$.

If $\gamma_l = 0$ for all l, from (4.5) we obtain $c_i \in \langle B_1 \setminus \{c_i\} \rangle$ which is a contradiction.

Hence $\gamma_{l_0} \neq 0$ for some $l_0 \in \{1, \ldots, t\}$. By (4.6) implies that $d_{l_0} \in \langle B_2 \setminus \{d_{l_0}\}\rangle$ and by (4.7) we have $c_i \in \langle B_1 \setminus \{c_i\}\rangle$ which is also a contradiction. Hence $B_1 \cup B_2$ is linearly independent.

<u>Case 3</u> Let $b \in B_2 \setminus B$. The proof is similar to Case 2. We have $B_1 \cup B_2$ is linearly independent. Thus $B_1 \cup B_2$ is a basis of $M_1 + M_2$ and $\dim(M_1 + M_2) = |B_1 \cup B_2| = r + s + t = (r + s) + (r + t) - r = \dim M_1 + \dim M_2 - \dim(M_1 \cap M_2)$. □

Definition 4.1.29. Let M be a vector space over a semifield K and $M_1, \ldots, M_n$ subspaces of M. We say that M is the *direct sum* of $M_1, \ldots, M_n$ if

(1) $M = M_1 + \cdots + M_n$ and

(2) $M_i \cap \big(\sum\limits_{j \neq i} M_j\big) = \{0\}$ for all $i \in \{1, \ldots, n\}$.

Moreover, we write $M = M_1 \oplus \cdots \oplus M_n$, the direct sum of $M_1, \ldots, M_n$.

Theorem 4.1.30. Let M be a finite-dimensional vector space over a semifield $K^{(*)}$ and M_1 a subspace of M. Then there exists a subspace M_2 such that $M = M_1 \oplus M_2$.

Proof. Let B be a basis of M_1. By Theorem 4.1.20, there exists a subset $B^{'}$ of M such that $B \subseteq B^{'}$ and $B^{'}$ is a basis of M. Let $M_2 = \langle B^{'} \setminus B\rangle$. If $M_1 = \{0\}$, then $M_2 = \langle B^{'}\rangle$ and, clearly, $M_1 \cap M_2 = \{0\}$, so $M = M_1 \oplus M_2$. Assume that $M_1 \neq \{0\}$. Let $B = \{b_1, \ldots, b_n\}$ and $B^{'} = \{b_1, \ldots, b_n, b_{n+1}, \ldots, b_m\}$. Then $M_2 = \langle b_{n+1}, \ldots, b_n\rangle$. Let $x \in M_1 \cap M_2$. Then $\sum\limits_{i=1}^{n} \alpha_i \varepsilon_i b_i = x = \sum\limits_{j=n+1}^{m} \beta_j \varepsilon_j b_j$ where $\alpha_i, \beta_j \in K^{(*)}$, $\varepsilon_i bi \in \{b_i, -b_i\}$ and $\varepsilon_j b_j \in \{b_j, -b_j\}$ for all $i \in \{1, \ldots, n\}$ and $j \in \{n+1, \ldots, m\}$. If $x \neq 0$, then there exists $i_0 \in \{1, \ldots, n\}$ such that $\alpha_{i_0} \neq 0$, thus we have $b_{i_0} \in \langle B^{'} \setminus \{b_{i_0}\}\rangle$ which leads to a contradiction. Hence $x = 0$ so that $M_1 \cap M_2 = \{0\}$. Therefore M is the direct sum of M_1 and M_2. □

Lemma 4.1.31. Let M be a vector space over a semifield. Then the following statements are equivalent.

(1) $M = M_1 \oplus \cdots \oplus M_n$.

(2) (2.1) $M = M_1 + \cdots + M_n$ and

(2.2) for $m_1 \in M_1, \ldots, m_n \in M_n,\ m_1 + \cdots + m_n = 0$ implies that $m_1 = 0, \ldots, m_n = 0$.

(3) For all $m \in M$ there exist unique $m_1 \in M_1, \ldots, m_n \in M_n$ such that $m = m_1 + \cdots + m_n$.

Lemma 4.1.32. For a sum of several subspace of a finite-dimensional vector space over a semifield $K^{(*)}$, to be direct it is necessary and sufficient that $\dim(M_1 + \cdots + M_n) = \dim M_1 + \cdots + \dim M_n$.

Lemma 4.1.33. Let A and B be linearly independent subsets of a vector space over a semifield $K^{(*)}$ and $A \cap B = \emptyset$. Then $A \cup B$ is linearly independent if and only if $\langle A \rangle \cap \langle B \rangle = \{0\}$.

Proof. Let $x \in \langle A \rangle \cap \langle B \rangle \setminus \{0\}$. Then $\sum\limits_{a \in A} \alpha_a \varepsilon_a a = x = \sum\limits_{b \in B} \beta_b \varepsilon_b b$ where $\alpha_a,\ \beta_b \in K^{(*)}$, $\varepsilon_a a \in \{a, -a\}$ and $\varepsilon_b b \in \{b, -b\}$. Since $x \neq 0$, there exist an $a_0 \in A$ and a $b_0 \in B$ such that $\alpha_{a_0} \neq 0$ and $\beta_{b_0} \neq 0$. This implies that $a_0 \in \langle A \cup B \setminus \{a_0\} \rangle$, so $A \cup B$ is linearly dependent.

Conversely, assume that $\langle A \rangle \cap \langle B \rangle = \{0\}$. Suppose that $A \cup B$ is linearly dependent. Then there exists an $x \in A \cup B$ such that $x \in \langle A \cup B \setminus \{x\} \rangle$. So $x = \sum\limits_{u \in A \cup B \setminus \{x\}} \alpha_u \varepsilon_u u$ where $\alpha_u \in K^{(*)}$ and $\varepsilon_u u \in \{u, -u\}$. Without loss of generality, assume that $x \in A$. Then $x = \sum\limits_{v \in A \setminus \{x\}} \alpha_v \varepsilon_v v + \sum\limits_{w \in B} \alpha_w \varepsilon_w w$. Thus

$$x - \sum_{v \in A \setminus \{x\}} \alpha_v \varepsilon_v = \sum_{w \in B} \alpha_w \varepsilon_w w \in \langle A \rangle \cap \langle B \rangle = \{0\}.$$

So $x - \sum_{v\in A\setminus\{x\}} \alpha_v\varepsilon_v = 0$. Hence $x = \sum_{v\in A\setminus\{x\}} \alpha_v\varepsilon_v \in \langle A\setminus\{x\}\rangle$ which is a contradiction. Therefore $A\cup B$ is linearly independent. $\square$

Remark 4.1.34. Let M be a vector space over a semifield and $C_1,\dots,C_n$ subsets of M. Then $\langle C_1\cup\cdots\cup C_n\rangle = \langle C_1\rangle + \cdots + \langle C_n\rangle$.

Theorem 4.1.35. Let M be a vector space over a semifield $K^{(*)}$ and $M_1,\dots,M_n$ subspaces of M. For each $i\in\{1,\dots,n\}$, let B_i be a basis of M_i. Then $M = M_1\oplus\cdots\oplus M_n$ if and only if (1) $B_i\cap B_j=\phi$ for $i\neq j$ and (2) $\bigcup_{i=1}^{n} B_i$ is a basis of M.

Proof. Assume that $M = M_1\oplus\cdots\oplus M_n$.

(1) Let $i,j\in\{1,\dots,n\}$ be such that $i\neq j$. Then $M_i\cap M_j\subseteq M_i\cap\big(\sum_{k\neq i} M_k\big)=\{0\}$. Since $B_i\cap B_j\subseteq M_i\cap M_j=\{0\}$, we obtain that $B_j\cap B_j=\emptyset$.

(2) By Remark 4.1.34, $\left\langle \bigcup_{i=1}^{n} B_i \right\rangle = \langle B_1\rangle+\cdots+\langle B_n\rangle = M_1+\cdots+M_n= M$, so $\bigcup_{i=1}^{n} B_i$ spans M. To show that $\bigcup_{i=1}^{n} B_i$ is linearly independent. We prove by induction. Assume that $B_1\cup\cdots\cup B_k$ is linearly independent for $k<n$. We claim that $B_1\cup\cdots\cup B_{k+1}$ is linearly independent. Since $\left\langle \bigcup_{i=1}^{k} B_i \right\rangle\cap\langle B_{k+1}\rangle = (M_1+\cdots+M_k)\cap M_{k+1}=\{0\}$ and by Lemma 4.1.33, we have $\bigcup_{i=1}^{k+1} B_i$ is linearly independent. Hence $\bigcup_{i=1}^{n} B_i$ is linearly independent. Therefore $\bigcup_{i=1}^{n} B_i$ is a basis of M.

Conversely, we show that $M = M_1\oplus\cdots\oplus M_n$. Clearly, $M=\left\langle \bigcup_{i=1}^{n} B_i \right\rangle = \langle B_1\rangle+\cdots+\langle B_n\rangle = M_1+\cdots+M_n$. Let $k\in\{1,\dots,n\}$. Since $B_k\cup\Big(\bigcup_{\substack{i=1\\ i\neq k}}^{n} B_i\Big)=\bigcup_{i=1}^{n} B_i$ is linearly independent and by Lemma 4.1.33, we have $\langle B_k\rangle\cap\left\langle \bigcup_{\substack{i=1\\ i\neq k}}^{n} B_i \right\rangle=\{0\}$, so $M_k\cap\Big(\sum_{\substack{i=1\\ i\neq k}}^{n} M_n\Big)=\{0\}$. Hence $M = M_1\oplus\cdots\oplus M_n$. $\square$

Theorem 4.1.36. Let M be a vector space over a semifield $K^{(*)}$ and $M_i \neq \{0\}$ a subspace of M for all $i \in \{1, \ldots, n\}$. If $M = M_1 \oplus \cdots \oplus M_n$ and $B \subseteq \bigcup_{i=1}^{n} M_i$ is a basis of M, then $B \cap M_i$ is a basis of M_i for all $i \in \{1, \ldots, n\}$.

Proof. Clearly, $M = \langle B \rangle = \left\langle B \cap \big(\bigcup_{i=1}^{n} M_i\big) \right\rangle = \left\langle \bigcup_{i=1}^{n} (B \cap M_i) \right\rangle = \langle B \cap M_1 \rangle + \cdots + \langle B \cap M_n \rangle$. Obviously, $\langle B \cap M_i \rangle \subseteq M_i$ for all $i \in \{1, \ldots, n\}$. Let $i \in \{1, \ldots, n\}$ and $m \in M_i$. Then $m = m_1 + \cdots + m_n$ for some $m_1 \in \langle B \cap M_1 \rangle, \ldots, m_n \in \langle B \cap M_n \rangle$. So $m - m_i = m_1 + \cdots + m_{i-1} + m_{i+1} + \cdots + m_n \in M_i \cap \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} M_j \right) = \{0\}$. This implies that $m = m_i \in \langle B \cap M_i \rangle$, so $M_i \subseteq \langle B \cap M_i \rangle$. Hence $M_i = \langle B \cap M_i \rangle$. Obviously, $B \cap M_i$ is linearly independent. Hence $B \cap M_i$ is a basis of M_i for all $i \in \{1, \ldots, n\}$. $\square$

Definition 4.1.37. Let M be a vector space over a semifield K and N a subspace of M. For $m \in M$, let $m + N$ be the set $\{m + n \mid n \in N\}$ and we call $m + N$ as a *coset* of N.

Lemma 4.1.38. Let M be a vector space over a semifield and N a subspace of M. Then

(1) for all $m_1, m_2 \in M$, $m_1 + N = m_2 + N$ if and only if $m_1 - m_2 \in N$, in particular, for $m \in M$, $m + N = N$ if and only if $m \in M$,

(2) for all $m_1, m_2 \in M$, $(m_1 + N) \cap (m_2 + N) = \emptyset$ or $m_1 + N = m_2 + N$, and

(3) for all $m_1, m_2 \in M$, $(m_1 + N) + (m_2 + N) = (m_1 + m_2) + N$.

Definition 4.1.39. Let M be a vector space over a semifield K and N a subspace of M. For $\alpha \in K$ and $m \in M$, let $\alpha(m + N) = \alpha m + N$.

To show that the above operation is well-defined, let $m_1, m_2 \in M$ be such that $m_1 + N = m_2 + N$ and $\alpha \in K$. By Lemma 4.1.38, $m_1 - m_2 \in N$. Then $\alpha m_1 - (\alpha m_2) = \alpha m_1 + \alpha(-m_2) = \alpha(m_1 - m_1) \in N$, so $\alpha m_1 + N = \alpha m_2 + N$.

Definition 4.1.40. Let M be a vector space over a semifield K and N a subspace of M. Let $M/N = \{m + N \mid m \in M\}$. Then M/N is a vector space over K with respect to the operations: for all $m_1, m_2 \in M$ and $\alpha \in K$,

$$(m_1 + N) + (m_2 + N) = (m_1 + m_2) + N \quad \text{and} \quad \alpha(m_1 + N) = \alpha m_1 + N.$$

We have $0 + N$ is the zero element of M/N and $-(m + N) = -m + N$ for all $m \in M$. We call M/N as the *quotient space of* M *by* N.

Theorem 4.1.41. Let M be a finite-dimensional vector space over a semifield $K^{(*)}$ and N a subspace of M. Then M/N is finite-dimensional and $\dim M/N = \dim M - \dim N$.

Proof. Let B be a basis of N. By Theorem 4.1.20, there exists a subset B' of M such that $B \subseteq B'$ and B' is a basis of M. Since M is finite-dimensional, B' is finite. If $N = M$, then $M/N = \{m + N \mid m \in M\} = \{N\} = \langle \emptyset \rangle$, so $\dim M/N = 0 = \dim M - \dim N$.

Next, assume that $N \subsetneq M$. We show that $\{v + N \mid v \in B' \setminus B\}$ is a basis of M/N. Let $m \in M$. Since B' spans M, we have $m = \sum\limits_{u \in B'} \alpha_u \varepsilon_u u$ for some $\alpha_u \in K^{(*)}$ and $\varepsilon_u u \in \{u, -u\}$. Then $m + N = \Big(\sum\limits_{u \in B'} \alpha_u \varepsilon_u u\Big) + N =$ $\Big(\sum\limits_{v \in B' \setminus B} \alpha_v \varepsilon_v v + \sum\limits_{w \in B} \alpha_w \varepsilon_w w\Big) + N$. Since B spans N, we obtain that $\sum\limits_{w \in B} \alpha_w \varepsilon_w w \in N$. By Lemma 4.1.38, we have $\sum\limits_{w \in B} \alpha_w \varepsilon_w w + N = N$. So $m + N = \sum\limits_{v \in B' \setminus B} \alpha_v \varepsilon_v v + N$ $= \sum\limits_{v \in B' \setminus B} \alpha_v \varepsilon_v (v + N) \in \big\langle \{v + N \mid v \in B' \setminus B\} \big\rangle$ where $\varepsilon_v(v + N) \in \{v + N, -v + N\}$. Hence $\{v + N \mid v \in B' \setminus B\}$ spans M/N. We claim that $\{v + N \mid v \in B' \setminus B\}$ is a linearly independent set. Suppose not. Then there exists a $v_0 \in B' \setminus B$ such that $v_0 + N \in \langle \{v + N \mid v \in B' \setminus B\} \setminus \{v_0 + N\} \rangle$. Thus $v_0 + N = \sum\limits_{\substack{v \in B' \setminus B \\ v \neq v_0}} \beta_v \varepsilon_v (v + N)$ $= \Big(\sum\limits_{\substack{v \in B' \setminus B \\ v \neq v_0}} \beta_v \varepsilon_v v\Big) + N$ where $\beta_v \in K^{(*)}$, $\varepsilon_v(v + N) \in \{v + N, -v + N\}$ and

$\varepsilon_v v \in \{v, -v\}$ for all $v \in B^{'} \backslash B$. By Lemma 4.1.38, $v_0 - \Big(\sum\limits_{\substack{v \in B^{'} \backslash B \\ v \neq v_0}} \beta_v \varepsilon_v v\Big) \in N = \langle B \rangle$,

so $\quad v_0 - \Big(\sum\limits_{\substack{v \in B^{'} \backslash B \\ v \neq v_0}} \beta_v \varepsilon_v v\Big) = \sum\limits_{b \in B} \gamma_b \varepsilon_b b$ where $\gamma_b \in K^{(*)}$ and $\varepsilon_b b \in \{b, -b\}$. Thus

$v_0 = \Big(\sum\limits_{\substack{v \in B^{'} \backslash B \\ v \neq v_0}} \beta_v \varepsilon_v v\Big) + \sum\limits_{b \in B} \gamma_b \varepsilon_b b \in \langle B^{'} \setminus \{v_0\} \rangle$, this implies $B^{'}$ is linearly dependent which is a contradiction. Hence $\{v + N \mid v \in B^{'} \setminus B\}$ is linearly independent. Thus $\{v + N \mid v \in B^{'} \setminus B\}$ is a basis of M/N and we have $u_1 + N \neq u_2 + N$ if $u_1 \neq u_2$ in $B^{'} \backslash B$. Hence $\dim M/N = \big|\{v + N \mid v \in B^{'} \backslash B\}\big| = |B^{'}| - |B| = \dim M - \dim N$. $\square$

4.2 Linear Transformations

Definition 4.2.1. Let M and N be vector spaces over a semifield K and T a mapping from M into N. Then T is said to be a *linear transformation* if for all $m_1, m_2 \in M$ and $\alpha, \beta \in K$, $T(\alpha m_1 + \beta m_2) = \alpha T(m_1) + \beta T(m_2)$.

Example 4.2.2. Let n and m be positive integers, with $m < n$ and let $M = \mathbb{R}^n$, $N = \mathbb{R}^m$ be vector spaces over $\mathbb{R}_0^+$. Then we have the mapping $T : M \to N$ defined by $T(x_1, \ldots, x_n) = (x_1, \ldots, x_m)$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ is a linear transformation.

Remark 4.2.3. Let T be a linear transformation of a vector space M into a vector space N over the same semifield. Then $T(0) = 0$ and $T(-m) = -T(m)$ for all $m \in M$.

Lemma 4.2.4. Let M and N be vector spaces over a semifield K and $T : M \to N$. Then the following statements are equivalent.

(1) T is a linear transformation.

(2) For all $m_1, m_2 \in M$ and $\alpha \in K$, $T(m_1 + m_2) = T(m_1) + T(m_2)$ and $T(\alpha m_1) = \alpha T(m_1)$.

(3) For all $m_1, m_2 \in M$ and $\alpha \in K$, $T(\alpha m_1 + m_2) = \alpha T(m_1) + T(m_2)$.

Notation. For any function $T : M \to N$, we denote the range of T by $\operatorname{Im} T$.

Lemma 4.2.5. Let M and N be vector space over a semifield K and $T : M \to N$ a linear transformation. Then the following statements hold.

(1) T is injective if and only if $T(m) = 0$ implies that $m = 0$ for all $m \in M$.

(2) If M_1 is a subspace of M, then $T(M_1)$ is a subspace of N. Hence $\operatorname{Im} T$ is a subspace of N.

(3) If B is a subset of M which spans M, then $T(B)$ spans $\operatorname{Im} T$.

(4) If M is finite-dimensional and K satisfies the property $(*)$, then $\operatorname{Im} T$ is finite-dimensional.

(5) If N_1 is a subspace of N, then $T^{-1}[N_1]$ is a subspace of M. Hence $T^{-1}[0]$ is a subspace of M.

Proof. The proofs of (1), (2) and (5) are clear.

To proof (3), let $y \in \operatorname{Im} T$. Then $y = T(x)$ for some $x \in M$. Since B spans M, it follows that $x = \sum\limits_{b \in B} \alpha_b b + \sum\limits_{b \in B} \beta_b(-b)$ where $\alpha_b, \beta_b \in K$. Then

$$y = T(x) = T\Big(\sum_{b \in B} \alpha_b b + \sum_{b \in B} \beta_b(-b)\Big) = \sum_{b \in B} \alpha_b T(b) - \sum_{b \in B} \beta_b T(b) \in \langle T(B) \rangle.$$

Hence $T(B)$ spans $\operatorname{Im} T$.

(4) As a result of (3), $T(B)$ spans $\operatorname{Im} T$. By Theorem 4.1.23, there exists a subset B' of $T(B)$ such that B' is a basis of $\operatorname{Im} T$. Since $T(B)$ is finite, B' is finite. Hence $\operatorname{Im} T$ is finite-dimensional. $\square$

Theorem 4.2.6. Let M and N be vector spaces over a semifield $K^{(*)}$ and let $B = \{b_1, \ldots, b_n\}$ be a basis of M where $b_i \neq b_j$ for $i \neq j$. If $\{c_1, \ldots, c_n\}$ is a subset of N, then there exists a unique linear transformation $T : M \to N$ such that $T(b_i) = c_i$ for all $i \in \{1, \ldots, n\}$.

Proof. Since B is a basis of M, by Remark 4.1.12, every $m \in M$ can be written uniquely as $m = \sum_{i=1}^{n} \alpha_i \varepsilon_i b_i$ where $\alpha_i \in K^{(*)}$ and $\varepsilon_i b_i \in \{b_i, -b_i\}$ for all i. Define $T : M \to N$ by

$$T(m) = \sum_{i=1}^{n} \alpha_i \varepsilon_i c_i \qquad \text{for all } m \in M.$$

Clearly, $T(b_i) = c_i$ for all $i \in \{1, \ldots, n\}$. Let $T' : M \to N$ be a linear transformation such that $T'(b_i) = c_i$ for all $i \in \{1, \ldots, n\}$. Then $T\Big(\sum_{i=1}^{n} \alpha_i \varepsilon_i b_i\Big) = \sum_{i=1}^{n} \alpha_i \varepsilon_i T(b_i) = \sum_{i=1}^{n} \alpha_i \varepsilon_i c_i = \sum_{i=1}^{n} \alpha_i \varepsilon_i T'(b_i) = T'\Big(\sum_{i=1}^{n} \alpha_i \varepsilon_i b_i\Big)$. Thus $T = T'$. Hence T is the unique linear transformation from M into N such that $T(b_i) = c_i$ for all $i \in \{1, \ldots, n\}$. $\square$

Definition 4.2.7. Let M_1 and M_2 be vector spaces over the same semifield and T a linear transformation of M_1 into M_2. The *kernel* of T, denote by $\operatorname{Ker} T$, is the set $\{m \in M_1 \mid T(m) = 0\}$.

Remark 4.2.8. Let M, N and L be vector spaces over a semifield $K^{(*)}$. Then the following statements hold.

(1) If $T_1 : M \to N$ and $T_2 : M \to L$ are linear transformations, then $T_2 \circ T_1 : M \to L$ is also a linear transformation.

(2) If T is a 1-1 linear transformation of M onto N, then T^{-1} is a linear transformation of N onto M.

(3) $T_1 : M \to N$ and $T_2 : N \to L$ are 1-1 and onto linear transformations, then $T_2 \circ T_1 : M \to L$ is also a 1-1 and onto linear transformation.

(4) If T is a 1-1 linear transformation of M onto N and B is a basis of M, then $T(B)$ is a basis of N.

Proof. The proofs of (1), (2) and (3) are obvious.

(4) Clearly, $T(B)$ spans N. Next, we show $T(B)$ is linearly independent. If $B = \emptyset$, then $T(B) = \emptyset$ is linearly independent. Assume that $B \neq \emptyset$. Suppose that $T(B)$ is linearly dependent. Then there exists a $b_0 \in B$ such that $T(b_0) = \sum_{b \in B \setminus \{b_0\}} \alpha_b \varepsilon_b T(b)$ where $\alpha_b \in K^{(*)}$ and $\varepsilon_b T(b) \in \{T(b), -T(b)\}$. Thus $T\Big(b_0 - \sum_{b \in B \setminus \{b_0\}} \alpha_b \varepsilon_b b\Big) = 0$. Since T is 1-1, we have $b_0 - \sum_{b \in B \setminus \{b_0\}} \alpha_b \varepsilon_b b = 0$, so $b_0 = \sum_{b \in B \setminus \{b_0\}} \alpha_b \varepsilon_b b \in \langle B \setminus \{b_0\} \rangle$ which is a contradiction. Hence $T(B)$ is linearly independent. Therefore $T(B)$ is a basis of N. $\square$

Theorem 4.2.9. Let M and L be vector spaces over a semifield $K^{(*)}$ and $T : M \to L$ a linear transformation. If B is a basis of the kernel of T and B' is a basis of M such that $B \subseteq B'$, then

(1) for all $b_1, b_2 \in B' \setminus B$, $b_1 \neq b_2$ implies that $T(b_1) \neq T(b_2)$ and

(2) $T(B' \setminus B)$ is a basis of $\operatorname{Im} T$.

Proof. By Lemma 4.2.5, $T(B')$ spans $\operatorname{Im} T$. Since $T(b) = 0$ for all $b \in B$, we obtain that $T(B' \setminus B)$ spans $\operatorname{Im} T$. Next, we prove that $T(B' \setminus B)$ is linearly independent. Suppose not. Then there exists a $b_0 \in B' \setminus B$ such that $T(b_0) \in \langle T(B' \setminus B) \setminus \{T(b_0)\} \rangle$. Thus $T(b_0) = \sum_{\substack{u \in B' \setminus B \\ u \neq b_0}} \alpha_u \varepsilon_u T(u)$ where $\alpha_u \in K^{(*)}$ and $\varepsilon_u T(u) \in \{T(u), -T(u)\}$. So $T(b_0) = T\Big(\sum_{\substack{u \in B' \setminus B \\ u \neq b_0}} \alpha_u \varepsilon_u u\Big)$, we have $T\Big(b_0 - \sum_{\substack{u \in B' \setminus B \\ u \neq b_0}} \alpha_u \varepsilon_u u\Big) = 0$. Hence $b_0 - \sum_{\substack{u \in B' \setminus B \\ u \neq b_0}} \alpha_u \varepsilon_u u = \sum_{w \in B} \beta_w \varepsilon_w w$ where $\beta_w \in K^{(*)}$ and $\varepsilon_w w \in \{w, -w\}$. So we have $b_0 \in \langle B' \setminus \{b_0\} \rangle$ which is a contradiction. Thus $T(B' \setminus B)$ is linearly independent, so $T(B' \setminus B)$ is a basis of $\operatorname{Im} T$. $\square$

Theorem 4.2.10. Let M and L be vector spaces over a semifield $K^{(*)}$ and $T : M \to L$ a linear transformation. If M is finite-dimensional, then

$$\dim(\operatorname{Im} T) + \dim(\operatorname{Ker} T) = \dim M.$$

Proof. Let B be a basis of $\operatorname{Ker} T$. By Theorem 4.1.20, there exists a subset B' of M such that B' is a basis of M and $B \subseteq B'$. By Theorem 4.2.9, $T(B' \setminus B)$ is a basis of $\operatorname{Im} T$ and $\dim(\operatorname{Im} T) + \dim(\operatorname{Ker} T) = \left|T(B' \setminus B)\right| + |B| = \left|B' \setminus B\right| + |B| = |B'| = \dim M$. □

Definition 4.2.11. Let M and N be vector spaces over a semifield $K^{(*)}$ and let $L(M,N) = \{T : M \to N \mid T \text{ is a linear transformation}\}$. Then $L(M,N)$ is a vector space over $K^{(*)}$ with the operations defined as follows, for $T, U \in L(M,N)$, $m \in M$ and $\alpha \in K^{(*)}$,

$$(T+U)(m) = T(m) + U(m) \quad \text{and} \quad (\alpha T)(m) = \alpha T(m).$$

Remark 4.2.12. Let M_1, M_2 and M_3 be vector spaces over a semifield $K^{(*)}$. For $\alpha \in K^{(*)}, T_1, T_2 \in L(M_1, M_2)$ and $U_1, U_2 \in L(M_2, M_3)$,

$$U_1 \circ T_1 \in L(M_1, M_3),$$

$$U_1 \circ (T_1 + T_2) = U_1 \circ T_1 + U_1 \circ T_2,$$

$$(U_1 + U_2) \circ T_1 = U_1 \circ T_1 + U_2 \circ T_1, \text{ and}$$

$$\alpha(U_1 \circ T_1) = (\alpha U_1) \circ T_1 = U_1 \circ (\alpha T_1).$$

Theorem 4.2.13. Let M and N be finite-dimensional vector spaces over a semifield $K^{(*)}$, $\dim M = m$ and $\dim N = n$. Then $\dim L(M,N) = nm$.

Proof. Let $B = \{u_1, \ldots, u_m\}$ and $B' = \{b_1, \ldots, b_n\}$ be bases of M and N, respectively. By Theorem 4.2.6, for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, there exists a $T_{ij} \in L(M,N)$ such that $T_{ij}(u_k) = \begin{cases} b_i, & \text{if } k = j, \\ 0, & \text{if } k \neq j. \end{cases}$

So $T_{ij}(u_k) = \delta_{jk} b_i$ where $\delta_{jk} = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \neq j. \end{cases}$

We show that $C = \{T_{ij} \mid i \in \{1, \dots, n\} \text{ and } j \in \{1, \dots, m\}\}$ is a basis of $L(M, N)$. Since $T(u_j) \in N = \langle B' \rangle$ for $j \in \{1, \dots, m\}$, we have $T(u_j) = \sum_{i=1}^{n} \alpha_{ij} \varepsilon_{ij} b_i$ where $\alpha_{ij} \in K^{(*)}$ and $\varepsilon_{ij} b_i \in \{b_i, -b_i\}$ for all $j \in \{1, \dots, m\}$. Thus for $j \in \{1, \dots, m\}$,

$$\begin{aligned} T(u_j) &= \sum_{i=1}^{n} \alpha_{ij} \varepsilon_{ij} b_i \\ &= \sum_{i=1}^{n} \left(\sum_{k=1}^{m} \alpha_{ik} \delta_{kj} \right) \varepsilon_{ij} b_i \\ &= \sum_{i=1}^{n} \sum_{k=1}^{m} \alpha_{ik} \varepsilon_{ij} T_{ik}(u_j) \\ &= \left(\sum_{i=1}^{n} \sum_{k=1}^{m} \alpha_{ik} \varepsilon_{ij} T_{ik} \right) (u_j). \end{aligned}$$

Hence $T = \sum_{i=1}^{n} \sum_{k=1}^{m} \alpha_{ik} \varepsilon_{ij} T_{ik} \in \langle C \rangle$. Therefore, C spans $L(M, N)$.

Next, we show that C is linearly independent. Suppose not. Then there exist $k \in \{1, \dots, n\}$ and $l \in \{1, \dots, m\}$ such that $T_{kl} = \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_{ij} \varepsilon_{ij} T_{ij}(u_r)$ where $\beta_{ij} \in K^{(*)}$, $\varepsilon_{ij} T_{ij} \in \{T_{ij}, -T_{ij}\}$ and $\beta_{kl} = 0$. But

$$T_{ij}(u_r) = \begin{cases} b_i, & \text{if } j = r, \\ 0, & \text{if } j \neq r. \end{cases}$$

So we have $b_k = T_{kl}(u_l) = \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_{ij} \varepsilon_{ij} T_{ij}(u_l) = \sum_{i=1}^{n} \beta_{il} \varepsilon_{il} b_i$. Since $\beta_{kl} = 0$, we have $b_k = \sum_{\substack{i=1 \\ i \neq k}}^{n} \beta_{il} \varepsilon_{il} b_i \in \langle B' \setminus \{b_k\} \rangle$. This is a contradiction. Hence C is linearly independent. Therefore C is a basis of $L(M, N)$ and $\dim L(M, N) = |C| = nm$. $\square$

Theorem 4.2.14. Let M and L be finite-dimensional vector spaces over a semifield $K^{(*)}$ and $T : M \to L$ a linear transformation. If $\dim M = \dim L$, then T is 1-1 if and only if T is onto.

Proof. By Theorem 4.2.10, $\dim(\operatorname{Ker} T) + \dim(\operatorname{Im} T) = \dim L$. Assume that T is 1-1. By Lemma 4.2.5, $\operatorname{Ker} T = \{0\}$, so $\dim(\operatorname{Ker} T) = 0$. Then $\dim L = \dim(\operatorname{Im} T)$. By Theorem 4.1.22, $L = \operatorname{Im} T$. Hence T is onto.

Conversely, assume that T is onto. Then $L = \operatorname{Im}\, T$, so $\dim(\operatorname{Ker} T) = 0$. Thus $\operatorname{Ker} T = \{0\}$. Hence T is 1-1. □

Definition 4.2.15. Let M and N be vector spaces over the same semifield. We say that M is *isomorphic to* N, denoted by $M \cong N$, if there exists a 1-1 linear transformation from M onto N.

Theorem 4.2.16. Let M be a finite-dimensional vector space over $K^{(*)}$ and $\dim M = m$, then $M \cong K^m$.

Proof. Let $B = \{b_1, \ldots, b_m\}$ be a basis of M. Then, for $x \in M$, we have $x = \sum\limits_{i=1}^{m} \alpha_i \varepsilon_i b_i$ where $\alpha_i \in K^{(*)}$ and $\varepsilon_i b_i \in \{b_i, -b_i\}$. Define $T : M \to K^m$ by

$$T(x) = T\Big(\sum_{i=1}^{m} \alpha_i \varepsilon_i b_i\Big) = (\alpha_1, \alpha_2, \ldots, \alpha_m) \quad \text{for all } x \in M.$$

Since Remark 4.1.12, T is a well-defined. Clearly, T is and onto linear transformation. □

Corollary 4.2.17. Let M and L be finite-dimensional vector spaces over a semifield $K^{(*)}$. Then $\dim M = \dim L$ if and only if $M \cong L$.

Now, we consider a semifield K which satisfies the property $(*)$ and there exists a field F_K such that K is a subsemifield of F_K.

Definition 4.2.18. Let K be a semifield which satisfies the property $(*)$ and F_K a field containing a subsemifield K. A linear transformation from a vector space M over K into F_K is called a *linear functional.* Moreover, let $M^* = L(M, F_K)$ and $M^{**} = (M^*)^*$. Then M^* is the *dual space of* M and M^{**} the *double dual of* M.

Remark 4.2.19. If M is a finite-dimensional vector space over a semifield $K^{(*)}$, then $\dim M = \dim M^* = \dim M^{**}$.

Theorem 4.2.20. Let M be a finite-dimensional vector space over a semifield $K^{(*)}$, $\dim M = n$, and $B = \{b_1, \ldots, b_n\}$ a basis of M. For each $i \in \{1, \ldots, n\}$, let $f_i \in M^*$ be such that

$$f_i(b_j) = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \neq i. \end{cases}$$

Then the following statements hold.

(1) $\{f_1, \ldots, f_n\}$ is a basis of M^* which is called the *dual basis* of B.

(2) For all $f \in M^*$, $f = \sum\limits_{i=1}^{n} f(b_i) f_i$.

(3) For all $m \in M$, $m = \sum\limits_{i=1}^{n} f_i(m) b_i$.

Proof. (1) This follows from the proof of Theorem 4.2.13.

(2) Let $f \in M^*$. Then $\left(\sum\limits_{i=1}^{n} f(b_i) f_i\right)(b_j) = \sum\limits_{i=1}^{n} f(b_i) f_i(b_j) = f(b_j)$ for all $j \in \{1, \ldots, n\}$. Hence $f = \sum\limits_{i=1}^{n} f(b_i) f_i$.

(3) Let $m \in M$. Then $m = \sum\limits_{j=1}^{n} \alpha_j \varepsilon_j b_j$ where $\alpha_j \in K^{(*)}$ and $\varepsilon_j b_j \in \{b_j, -b_j\}$. Thus

$$\begin{aligned} \sum_{i=1}^{n} f_i(m) b_i &= \sum_{i=1}^{n} f_i\left(\sum_{j=1}^{n} \alpha_j \varepsilon_j b_j\right) b_i \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_j \varepsilon_j f_i(b_j) b_i \\ &= \sum_{i=1}^{n} \alpha_i \varepsilon_i b_i \\ &= m. \end{aligned}$$

Hence $m = \sum\limits_{i=1}^{n} f_i(m) b_i$. $\square$

Theorem 4.2.21. Let M be a vector space over a semifield $K^{(*)}$. For $m \in M$, if $f(m) = 0$ for all $f \in M^*$, then $m = 0$.

Proof. Let $m \in M \setminus \{0\}$. Then $\{m\}$ is linearly independent. By Theorem 4.1.20, there exists a subset B of M such that B is linearly independent and $\{m\} \subseteq B$. By Theorem 4.2.6, there exists a unique linear transformation $f : M \to F_K$ such that $f(b) = 1$ for all $b \in B$ where F_K is a field containing a subsemifield K. Since $\{m\} \subseteq B$, we have $m \in B$. Thus $f(m) \neq 0$. $\square$

Let F_K be a field containing a semifield K and M a vector space over K. For $m \in M$, define $L_m : M^* \to F_K$ by

$$L_m(f) = f(m) \qquad \text{for all } f \in M^*.$$

Then $L_m \in M^{**}$ for all $m \in M$ and hence $\{L_m \mid m \in M\}$ is a subset of M^{**}.

Theorem 4.2.22. Let M be a vector space over a semifield $K^{(*)}$. Then

(1) the mapping $m \mapsto L_m$ is a 1-1 linear transformation of M into M^{**} and

(2) if M is finite-dimensional, then

(2.1) the mapping $m \mapsto L_m$ is a 1-1 linear transformation of M onto M^{**} and

(2.2) for all $L \in M^{**}$ there exists a unique $m \in M$ such that $L = L_m$.

Proof. Let φ denote the mapping $m \mapsto L_m$.

(1) Let $m_1, m_2 \in M$ and $\alpha, \beta \in K^{(*)}$. Then, for all $f \in M^*$,

$$\begin{aligned} L_{\alpha m_1 + \beta m_2}(f) &= f(\alpha m_1 + \beta m_2) \\ &= \alpha f(m_1) + \beta f(m_2) \\ &= \alpha L_{m_1}(f) + \beta L_{m_2}(f) \\ &= (\alpha L_{m_1} + \beta L_{m_2})(f). \end{aligned}$$

Hence $\varphi(\alpha m_1 + \beta m_2) = L_{\alpha m_1 + \beta m_2} = \alpha L_{m_1} + \beta L_{m_2} = \alpha\varphi(m_1) + \beta\varphi(m_2)$. So φ is a linear transformation. Let $m \in M$ be such that $L_m = \varphi(m) = 0$. Then for all $f \in M^*$, we obtain that $0 = L_m(f) = f(m)$. By Theorem 4.2.21, $m = 0$. Hence φ is 1-1.

(2.1) Since M is finite-dimensional, $\dim M = \dim M^{**}$. By (1) and Theorem 4.2.14, φ is onto.

(2.2) This follows from (2.1). □

Theorem 4.2.23. Let M be a finite-dimensional vector space over a semifield$K^{(*)}$. Then each basis of M^* is the dual basis of some basis of M.

Proof. Let $\dim M = n$ and $\overline{B} = \{f_1, \ldots, f_n\}$ be a basis of M^*. Moreover, let $\{L_1, \ldots, L_n\}$ be the dual basis of $\overline{B}$ where, for $i, j \in \{1, \ldots, n\}$,

$$L_i(f_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

By Theorem 4.2.22, for $i \in \{1, \ldots, n\}$ there exists an $m_i \in M$ such that $L_i = L_{m_i}$. Since the mapping $m_i \mapsto L_{m_i}$ is a 1-1 linear transformation of M onto M^{**} and $\{L_{m_1}, \ldots, L_{m_n}\}$ is a basis of M^{**}, we have $\{m_1, \ldots, m_n\}$ is a basis of M. By Theorem 4.2.20, $\{f_1, \ldots, f_n\}$ is the dual basis of $\{m_1, \ldots, m_n\}$. □

Definition 4.2.24. Let M be a vector space over a semifield. For $S \subseteq M$, let S° be the set $\{f \in M^* \mid f(m) = 0 \text{ for all } m \in S\}$ and $S^{\circ\circ} = (S^\circ)^\circ$. Then S° is called the *annihilator of S*.

Remark 4.2.25. Let M be a vector space over a semifield. Then

(1) S° is a subspace of M^* for all subset S of M,

(2) $\{0\}^\circ = M^*$,

(3) $M^\circ = \{0\}$,

(4) for all subsets S_1, S_2 of M, $S_1 \subseteq S_2$ implies that $S_2^\circ \subseteq S_1^\circ$,

(5) for all subsets S_1, S_2 of M, $S_1^\circ + S_2^\circ \subseteq (S_1 \cap S_2)^\circ$, and

(6) for all subsets S_1, S_2 of M, $S_1^\circ \cap S_2^\circ \subseteq (S_1 + S_2)^\circ$ and they are equal if $0 \in S_1 \cap S_2$.

Theorem 4.2.26. Let M be a finite-dimensional vector space over a semifield $K^{(*)}$ and N a subspace of M. Then $\dim N + \dim N^\circ = \dim M$.

Proof. Let B be a basis of N. By Theorem 4.1.20, there exists a subset B' of M such that $B \subseteq B'$ and B' is a basis of M. For $b \in B'$, let $f^b \in M^*$ be such that

$$f^b(u) = \begin{cases} 1, & \text{if } u = b, \\ 0, & \text{if } u \neq b. \end{cases}$$

By Theorem 4.2.20, $\{f^b \mid b \in B'\}$ is a basis of M^*. If $B = B'$, then $N^\circ = M^\circ = \{0\}$, so $\dim M = \dim N + 0 = \dim N + \dim N^\circ$. Assume that $B \subsetneq B'$. Let $f \in N^\circ$. By Theorem 4.2.20, $f = \sum\limits_{b \in B'} f(b) f^b$. Since $f \in N^\circ$, we have $f(u) = 0$ for all $u \in B$. Thus $f = \sum\limits_{b \in B' \setminus B} f(b) f^b \in \langle \{f^b \mid b \in B' \setminus B\} \rangle$. Hence $\{f^b \mid b \in B' \setminus B\}$ spans N°. Clearly, $\{f^b \mid b \in B' \setminus B\}$ is linearly independent. Hence $\{f^b \mid b \in B' \setminus B\}$ is a basis of N° and $\dim M = |B'| = |B| + |B' \setminus B| = \dim N + \dim N^\circ$. $\square$

Theorem 4.2.27. Let M be a finite-dimensional vector space over a semifield $K^{(*)}$. Then

(1) N is a subspace of M implies that $\dim M = \dim N^{\circ\circ}$ and

(2) for any subset S of M, $\dim S^\circ + \dim S^{\circ\circ} = \dim M$.

Proof. This is clear by applying Theorem 4.2.26. $\square$

Theorem 4.2.28. Let M be a vector space over a semifield $K^{(*)}$ and N a subspace of M. Then the following statements hold.

(1) For all $n \in N$, $L_n \in N^{\circ\circ}$.

(2) The mapping $n \mapsto L_n$ is a 1-1 linear transformation of N into $N^{\circ\circ}$.

(3) If M is finite-dimensional, then the mapping in (2) is 1-1 and onto.

Proof. (1) Let $n \in N$ and $f \in N^{\circ}$. Then $f(u) = 0$ for all $u \in N$, so $f(n) = 0$. Hence $L_n(f) = f(n) = 0$. Therefore $L_n \in N^{\circ}$.

(2) This follows from (1) and Theorem 4.2.22.

(3) Let ψ be the mapping $n \mapsto L_n$. By (2), $\dim(\operatorname{Im}\psi) = \dim N = \dim N^{\circ\circ}$. By Theorem 4.1.22, $\operatorname{Im}\psi = N^{\circ\circ}$. Thus ψ is onto. Hence ψ is 1-1 and onto. $\square$

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