สมบัติบางประการของโดเมนของตัวดำเนินการการคูณและตัวดำเนินการการหาอนุพันธ์ บนปริภูมิซีกัล-บาร์กแมนเชิงทั่วไป

นางสาวเบญจวรรณ สิงห์ภู่

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CERTAIN PROPERTIES OF THE DOMAINS OF MULTIPLICATION AND DIFFERENTIATION OPERATORS ON A GENERALIZED SEGAL-BARGMANN SPACE

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ปริภูมิซีกัล-บาร์กแมนเป็นปริภูมิของฟังก์ชันโฮโลมอร์ฟิกบน $\mathbf{C}^d$ ซึ่งเมื่อยกกำลังสองแล้วหาปริพันธ์ได้เทียบกับเมเชอร์เกาส์เซียน เราศึกษาสมบัติบางประการของตัวดำเนินการการคูณและตัวดำเนินการการหาอนุพันธ์บนปริภูมิซีกัล-บาร์กแมน จากนั้นเราจึงขยายสมบัติเหล่านี้ไปยังโดเมนของตัวดำเนินการการคูณและตัวดำเนินการการหาอนุพันธ์บนปริภูมิซีกัล-บาร์กแมนเชิงทั่วไป ซึ่งเราแทนเมเชอร์เกาส์เซียนด้วยเมเชอร์ที่มีฟังก์ชันความหนาแน่นซึ่งลดลงเร็วกว่าเมเชอร์เกาส์เซียนเมื่อเข้าใกล้อนันต์

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The Segal-Bargmann space is the space of all holomorphic functions on $\mathbf{C}^d$ which are square-integrable with respect to a Guassian measure. We study certain properties of the domains of multiplication and differentiation operators on the Segal-Bargmann space. Then we extend these results to those of a generalized Segal-Bargmann space, where we replace the Guassian measure by a measure with a density function which decays faster than the Guassian factor near infinity.

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Chapter 1

Statements of the results

In this work we consider the Gaussian measure μ on $\mathbb{C}^d$ defined by

$$d\mu(z) = \frac{1}{\pi^d} e^{-|z|^2} dz,$$

where $|z|^2 = |z_1|^2 + \cdots + |z_d|^2$. The Segal-Bargmann space, denoted by $\mathcal{H}L^2(\mathbb{C}^d, \mu)$, is the space of all holomorphic functions which are square-integrable with respect to the Gaussian measure μ on $\mathbb{C}^d$.

For each multi-index β, we define

$$\mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}} = \left\{ f \in \mathcal{H}L^2(\mathbb{C}^d, \mu) \;\middle|\; \frac{\partial^\beta f}{\partial z^\beta} \in L^2(\mathbb{C}^d, \mu) \right\}$$

and

$$\mathfrak{D}_{z^\beta} = \left\{ f \in \mathcal{H}L^2(\mathbb{C}^d, \mu) \;\middle|\; z^\beta f \in L^2(\mathbb{C}^d, \mu) \right\}.$$

In Theorem 3.1, we will show that polynomials are dense in $\mathcal{H}L^2(\mathbb{C}^d, \mu)$. Since $\mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}}$ and $\mathfrak{D}_{z^\beta}$ contain polynomials, it follows that $\mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}}$ and $\mathfrak{D}_{z^\beta}$ are dense in $\mathcal{H}L^2(\mathbb{C}^d, \mu)$. We prove the following results :

(1) $\mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} \subseteq \mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}}$ if β and γ are multi-indices such that $\beta \leq \gamma$.

(2) $\mathfrak{D}_{z^\gamma} \subseteq \mathfrak{D}_{z^\beta}$ if β and γ are multi-indices such that $\beta \leq \gamma$.

We use the above results to prove the following theorem, which is the main result of this work :

Theorem. $\mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}} = \mathfrak{D}_{z^\beta}$ for any multi-index β.

Next, we extend the Gaussian measure μ to the measure μ_α, $\alpha \geq 2$, given by

$$d\mu_\alpha(z) = C_\alpha e^{-|z|^\alpha} dz,$$

where $|z|^\alpha = |z_1|^\alpha + \cdots + |z_d|^\alpha$ and $C_\alpha = \left(\int_{\mathbb{C}^d} e^{-|z|^\alpha} dz\right)^{-1}$ is the normalizing factor. We call the space of all holomorphic functions which are square-integrable with respect to μ_α the generalized Segal-Bargmann space. We modify the proof of the previous results to the measure μ_α, $\alpha \geq 2$. Finally, we will obtain the following theorem :

Theorem. If $\alpha \geq 2$ is an integer, then $\mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}} = \mathfrak{D}_{z^{(\alpha-1)\beta}}$ for any multi-index β.

Chapter 2

Background and notation

For each $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$, let

$$\mu(z) = \frac{1}{\pi^d} e^{-|z|^2},$$

where $|z|^2 = |z_1|^2 + \cdots + |z_d|^2$, and define the Gaussian measure μ on $\mathbb{C}^d$ by

$$d\mu(z) = \mu(z)dz = \frac{1}{\pi^d} e^{-|z|^2} dz.$$

Denote by $\mathcal{H}L^2\left(\mathbb{C}^d, \mu\right)$ the space of all holomorphic functions which are square-integrable with respect to the Gaussian measure μ on $\mathbb{C}^d$. That is, $\mathcal{H}L^2\left(\mathbb{C}^d, \mu\right)$ consists of all holomorphic functions f on $\mathbb{C}^d$ for which

$$\int_{\mathbb{C}^d} |f(z)|^2 d\mu(z) < \infty.$$

Then $\mathcal{H}L^2\left(\mathbb{C}^d, \mu\right)$ is a Hilbert space, called the Segal-Bargmann space.

A multi-index $\beta = (\beta_1, \ldots, \beta_d)$ is a d-tuple of nonnegative integers. For such a multi-index β and for each $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$, we write

$$|\beta| = \beta_1 + \cdots + \beta_d,$$

$$\beta! = \beta_1! \cdots \beta_d!,$$

$$z^\beta = {z_1}^{\beta_1} \cdots {z_d}^{\beta_d},$$

and

$$\frac{\partial^\beta}{\partial z^\beta} = \frac{\partial^{|\beta|}}{\partial {z_1}^{\beta_1} \dots \partial {z_d}^{\beta_d}}.$$

For any multi-indices β and γ, we write

$$\beta \le \gamma \quad \text{if} \quad \beta_j \le \gamma_j \text{ for all } 1 \le j \le d.$$

For each multi-index β, define

$$\mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}} = \{f \in \mathcal{H}L^2\left(\mathbb{C}^d, \mu\right) \mid \frac{\partial^\beta f}{\partial z^\beta} \in L^2\left(\mathbb{C}^d, \mu\right)\}$$

and

$$\mathfrak{D}_{z^\beta} = \{f \in \mathcal{H}L^2\left(\mathbb{C}^d, \mu\right) \mid z^\beta f \in L^2\left(\mathbb{C}^d, \mu\right)\}.$$

Next, we consider a measure μ_α, $\alpha \ge 2$, on $\mathbb{C}^d$ given by

$$d\mu_\alpha(z) = C_\alpha e^{-|z|^\alpha} dz,$$

where $|z|^\alpha = |z_1|^\alpha + \cdots + |z_d|^\alpha$ and $C_\alpha = \left(\int_{\mathbb{C}^d} e^{-|z|^\alpha} dz\right)^{-1}$ is the normalizing factor. Notice that, for each $\alpha \ge 2$, μ_α looks like the Gaussian measure, but it decays faster than the Gaussian factor $\mu = \mu_2$ near infinity when $\alpha > 2$. We define $\mathcal{H}L^2\left(\mathbb{C}^d, \mu_\alpha\right)$ to be the space of all holomorphic functions which are square-integrable with respect to the measure μ_α on $\mathbb{C}^d$. We also have that $\mathcal{H}L^2\left(\mathbb{C}^d, \mu_\alpha\right)$ is a Hilbert space and we call it a generalized Segal-Bargmann space.

For the sake of completeness of this work, we will include the following theorem, [H, p.2].

Theorem 2.1. 1. *For all $z \in \mathbb{C}^d$, there exists a neighborhood V of z and a constant c_z such that*

$$|F(v)|^2 \le c_z \|F\|^2_{L^2(\mathbb{C}^d, \mu_\alpha)}$$

for all $v \in V$ *and all* $F \in \mathcal{H}L^2\left(\mathbb{C}^d, \mu_\alpha\right)$*. In particular, for all* $z \in \mathbb{C}^d$*, there exists a constant* c_z *such that*

$$|F(z)|^2 \leq c_z \|F\|^2_{L^2\left(\mathbb{C}^d, \mu_\alpha\right)}$$

for all $F \in \mathcal{H}L^2\left(\mathbb{C}^d, \mu_\alpha\right)$*.*

2. $\mathcal{H}L^2\left(\mathbb{C}^d, \mu_\alpha\right)$ *is a closed subspace of* $L^2\left(\mathbb{C}^d, \mu_\alpha\right)$*, and therefore a Hilbert space.*

Proof. (1) Let $P_s(z)$ be the "polydisk" of radius s, centered at z, that is

$$P_s(z) = \left\{ w \in \mathbb{C}^d \;\middle|\; \forall k \in \{1, \dots, d\}, |w_k - z_k| < s \right\}.$$

Given $z \in \mathbb{C}^d$ and $s > 0$ be arbitrary. Let $V = P_{\frac{s}{2}}(z)$. Then V is an open neighborhood of z. Let $v \in V$ and $F \in \mathcal{H}L^2\left(\mathbb{C}^d, \mu_\alpha\right)$. Claim that

$$F(v) = \frac{1}{\left(\pi \frac{s^2}{4}\right)^d} \int_{P_{\frac{s}{2}}(v)} F(w)\, dw. \tag{2.1}$$

Provided we can prove it for the case $d = 1$, for the general case $d > 1$, we factor the integration as a product of 1-dimensional integration in each variable. Thus

$$\begin{aligned} \int_{P_{\frac{s}{2}}(v)} F(w)\, dw &= \int_{B(v_1, \frac{s}{2}) \times \cdots \times B(v_d, \frac{s}{2})} F(w)\, dw \\ &= \int_{B(v_1, \frac{s}{2})} \cdots \int_{B(v_d, \frac{s}{2})} F(w_1, \dots, w_d)\, dw_1 \cdots dw_d\,. \end{aligned}$$

We will apply the 1-dimensional result d times, so we have done for all d integrations and get just $F(v)$.

Now we will prove the case $d = 1$. Since F is analytic on $\mathbb{C}^d$, we can expand F in a Taylor series at $w = v$, so we have

$$F(w) = F(v) + \sum_{n=1}^{\infty} a_n (w - v)^n$$

for all $w \in \mathbb{C}^d$. This series converges uniformly to F on the compact set $\overline{P_s(z)}$ and also on $P_{\frac{s}{2}}(v)$ since $P_{\frac{s}{2}}(v) \subseteq \overline{P_s(z)}$. Then

$$\begin{aligned}\int_{P_{\frac{s}{2}}(v)} F(w)\,dw &= \int_{P_{\frac{s}{2}}(v)} F(v)\,dw + \int_{P_{\frac{s}{2}}(v)} \sum_{n=1}^{\infty} a_n (w-v)^n\,dw \\ &= \pi\frac{s^2}{4}F(v) + \sum_{n=1}^{\infty} a_n \int_{P_{\frac{s}{2}}(v)} (w-v)^n\,dw\end{aligned}$$

since $P_{\frac{s}{2}}(v)$ is just a disk of radius $\frac{s}{2}$ when $d=1$. If we use polar coordinates with the origin at v, then $(w-v)^n = r^n e^{in\theta}$. So for $n \geq 1$, we have

$$\begin{aligned}\int_{P_{\frac{s}{2}}(v)} (w-v)^n\,dw &= \int_0^s \int_0^{2\pi} r^n e^{in\theta} r\,d\theta dr \\ &= \int_0^s r^{n+1}\,dr \int_0^{2\pi} e^{in\theta}\,d\theta \\ &= 0.\end{aligned}$$

Thus

$$\int_{P_{\frac{s}{2}}(v)} F(w)\,dw = \pi\frac{s^2}{4}F(v),$$

which gives

$$F(v) = \frac{1}{\left(\pi\frac{s^2}{4}\right)^d}\int_{P_{\frac{s}{2}}(v)} F(w)\,dw.$$

So now rewrite (2.1) in the form

$$\begin{aligned}F(v) &= \frac{1}{\left(\pi\frac{s^2}{4}\right)^d}\frac{1}{C_\alpha}\int_{\mathbb{C}^d} 1_{P_{\frac{s}{2}}(v)}(w)e^{|w|^\alpha}F(w)C_\alpha e^{-|w|^\alpha}\,dw \\ &= \frac{1}{\left(\pi\frac{s^2}{4}\right)^d}\frac{1}{C_\alpha}\left\langle 1_{P_{\frac{s}{2}}(v)}e^{|w|^\alpha}, F\right\rangle,\end{aligned}$$

where $1_{P_{\frac{s}{2}}(v)}$ is the function which is one on $P_{\frac{s}{2}}(v)$ and zero elsewhere. Thus by the Schwarz's inequality, we have

$$\begin{aligned}\|F(v)\|^2 &\leq \frac{1}{\left(\pi\frac{s^2}{4}\right)^{2d}}\frac{1}{{C_\alpha}^2}\|1_{P_{\frac{s}{2}}(v)}e^{|w|^\alpha}\|^2\|F\|^2 \\ &\leq \frac{1}{\left(\pi\frac{s^2}{4}\right)^{2d}}\frac{1}{{C_\alpha}^2}\|1_{P_s(z)}e^{|w|^\alpha}\|^2\|F\|^2\end{aligned}$$

since $P_{\frac{s}{2}}(v) \subseteq P_s(z)$. Note that $\overline{P_s(z)}$ is a compact subset of $\mathbb{C}^d$ and $1_{P_s(z)}e^{|w|^\alpha}$ is positive and continuous on $\overline{P_s(z)}$; thus $\|1_{P_s(z)}e^{|w|^\alpha}\|^2$ is finite. By choosing $c_z = \frac{1}{\left(\pi\frac{s^2}{4}\right)^{2d}}\frac{1}{C_\alpha{}^2}\|1_{P_s(z)}e^{|w|^\alpha}\|^2$, we have (1).

(2) Let (F_n) be a sequence in $\mathcal{H}L^2\left(\mathbb{C}^d, \mu_\alpha\right)$, and let $F \in L^2\left(\mathbb{C}^d, \mu_\alpha\right)$ be such that $F_n \to F$ in $L^2(\mu_\alpha)$. Then (F_n) is a Cauchy sequence in $L^2(\mu_\alpha)$. Given $z \in \mathbb{C}^d$, by (1) there exists a neighborhood V of z and a constant c_z such that

$$|F(v)|^2 \leq c_z\|F\|^2_{L^2(\mathbb{C}^d,\mu_\alpha)}$$

for all $v \in V$ and all $F \in \mathcal{H}L^2\left(\mathbb{C}^d, \mu_\alpha\right)$. Since $F_n - F_m \in \mathcal{H}L^2\left(\mathbb{C}^d, \mu_\alpha\right)$ for all $n, m \in \mathbb{N}$, we have

$$|F_n(v) - F_m(v)| = |(F_n - F_m)(v)| \leq \sqrt{c_z}\,\|F_n - F_m\|$$

for all $v \in V$. Therefore

$$\sup_{v\in V}|F_n(v) - F_m(v)| \leq \sqrt{c_z}\,\|F_n - F_m\| \to 0 \quad \text{as } n, m \to \infty.$$

This shows that the sequence (F_n) converges locally uniformly to some limit function, which must be F. But a standard theorem shows that a locally uniform limit of holomorphic functions is always holomorphic. So the limit function F is actually in $\mathcal{H}L^2\left(\mathbb{C}^d, \mu_\alpha\right)$, which shows that $\mathcal{H}L^2\left(\mathbb{C}^d, \mu_\alpha\right)$ is a closed subspace of $L^2\left(\mathbb{C}^d, \mu_\alpha\right)$. □

Finally, for each multi-index β, we define

$$\mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}} = \{f \in \mathcal{H}L^2\left(\mathbb{C}^d, \mu_\alpha\right) \mid \frac{\partial^\beta f}{\partial z^\beta} \in L^2\left(\mathbb{C}^d, \mu_\alpha\right)\}$$

and

$$\mathfrak{D}_{z^\beta} = \left\{f \in \mathcal{H}L^2\left(\mathbb{C}^d, \mu_\alpha\right) \mid z^\beta f \in L^2\left(\mathbb{C}^d, \mu_\alpha\right)\right\}.$$

Chapter 3

The Segal-Bargmann space

In this chapter we consider the case of the Gaussian measure $\mu = \mu_2$. Our objective is to show the relationship between the domain of differentiation operator and the domain of multiplication operator on the Segal-Bargmann space. We obtain the next theorem from Gross and Malliavin [G-M].

Theorem 3.1. *The set $\{z^\beta\}$ forms an orthogonal basis of $\mathcal{H}L^2(\mathbb{C}^d, \mu)$. Assume that f is a holomorphic function on $\mathbb{C}^d$ and has the pointwise convergent power series*

$$f(z) = \sum_\beta a_\beta z^\beta, \tag{3.1}$$

where $a_\beta \in \mathbb{C}$ for each multi-index β. Then

$$\int_{\mathbb{C}^d} |f(z)|^2 d\mu(z) = \sum_\beta |a_\beta|^2 \beta! \tag{3.2}$$

where the summation is taken over all multi-indices $\beta = (\beta_1, \ldots, \beta_d)$. The series (3.1) is also convergent in the $L^2(\mathbb{C}^d, \mu)$ sense if either side (hence both sides) of (3.2) is finite, and thus

$$\|f\|^2 = \sum_\beta |a_\beta|^2 \beta!.$$

Proof. Let $D(\sigma)$ be the polydisc $\left\{z \in \mathbb{C}^d \mid \max_{1\le j\le d} |z_j| \le \sigma\right\}$. Consider first the case $d = 1$. Let $M_\sigma(j,k) = \int_{|z|<\sigma} z^j \overline{z}^k e^{-|z|^2} dz$. Putting $z = re^{i\theta}$ and using polar coordinates, we have

$$\begin{aligned} M_\sigma(j,k) &= \int_0^\sigma \int_0^{2\pi} r^j e^{ij\theta} r^k e^{-ik\theta} e^{-r^2} r\, d\theta dr \\ &= \int_0^\sigma r^{j+k+1} e^{-r^2} dr \int_0^{2\pi} e^{i(j-k)\theta} d\theta. \end{aligned}$$

It follows that

(i) $M_\sigma(j,k) = 0$ if $j \neq k$ since $\int_0^{2\pi} e^{i(j-k)\theta} d\theta = 0$ if $j \neq k$;

(ii) if $j = k$ then

$$\begin{aligned} \lim_{\sigma\to\infty} M_\sigma(k,k) &= 2\pi \int_0^\infty r^{2k} e^{-r^2} r\, dr \\ &= \pi\Gamma(k+1) \\ &= \pi k!\ , \end{aligned}$$

where Γ is the real-valued function defined by

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1}\, dt, \quad x \in (0, +\infty);$$

(iii) from (i) and (ii), we have that

$$\begin{aligned} \langle z^j, z^k \rangle &= \frac{1}{\pi} \int_{\mathbb{C}} z^j \overline{z}^k e^{-|z|^2} dz \\ &= \frac{1}{\pi} \lim_{\sigma\to\infty} M_\sigma(j,k) \\ &= \delta_{jk} k!. \end{aligned}$$

Now we consider the general case. Since the power series in (3.1) converges uniformly on the product set $D(\sigma)$ and $z^\beta \overline{z}^\gamma e^{-|z|^2}$ is itself a product of functions of $z_1, \ldots, z_d$, we obtain

$$\begin{aligned}
\int_{D(\sigma)} |f(z)|^2 d\mu(z) &= \frac{1}{\pi^d} \int_{D(\sigma)} |f(z)|^2 e^{-|z|^2} dz \\
&= \frac{1}{\pi^d} \int_{D(\sigma)} \sum_\beta a_\beta z^\beta \sum_\gamma \overline{a}_\gamma \overline{z}^\gamma e^{-|z|^2} dz \\
&= \frac{1}{\pi^d} \sum_\beta \sum_\gamma a_\beta \overline{a}_\gamma \int_{D(\sigma)} z^\beta \overline{z}^\gamma e^{-|z|^2} dz \\
&= \frac{1}{\pi^d} \sum_\beta \sum_\gamma a_\beta \overline{a}_\gamma \prod_{j=1}^d \left(\int_{|z_j|<\sigma} {z_j}^{\beta_j} \overline{z}_j^{\gamma_j} e^{-|z_j|^2} dz_j \right) \\
&= \frac{1}{\pi^d} \sum_\beta \sum_\gamma a_\beta \overline{a}_\gamma \prod_{j=1}^d M_\sigma(\beta_j, \gamma_j) \\
&= \frac{1}{\pi^d} \sum_\beta |a_\beta|^2 \prod_{j=1}^d M_\sigma(\beta_j, \beta_j),
\end{aligned}$$

where in the last sum we have written $\beta = (\beta_1, \ldots, \beta_d)$, and have used (i). Now let $\sigma \to \infty$ and use the monotone convergence theorem on both sides of the last equality. Then (3.2) follows from (ii). The set $\{z^\beta\}$ is an orthogonal set by (iii) and the functions $\frac{z^\beta}{\sqrt{\beta!}}$ are orthonormal since

$$\begin{aligned}
\|z^\beta\|^2 &= \frac{1}{\pi^d} \int_{\mathbb{C}^d} |z^\beta|^2 e^{-|z|^2} dz \\
&= \frac{1}{\pi^d} \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} |{z_1}^{\beta_1}|^2 \cdots |{z_d}^{\beta_d}|^2 e^{-|z_1|^2} \cdots e^{-|z_d|^2} dz_1 \cdots dz_d \\
&= \prod_{j=1}^d \left(\frac{1}{\pi} \int_{\mathbb{C}} |{z_j}^{\beta_j}|^2 e^{-|z_j|^2} dz_j \right) \\
&= \prod_{j=1}^d \frac{1}{\pi} \lim_{\sigma \to \infty} M_\sigma(\beta_j, \beta_j) \\
&= \prod_{j=1}^d \beta_j! \ = \ \beta!.
\end{aligned}$$

Assume that the right side of (3.2) is finite. Then the sequence of partial sum $f_N(z) := \sum_{|\beta| \leq N} a_\beta z^\beta$ converges in the $L^2(\mu)$ sense to some function g. Since a subsequence converges a.e. to g, we have $f = g$ a.e. So the series in (3.1) converges to f in the $L^2(\mu)$ sense. In particular, if f is in $\mathcal{H}L^2(\mathbb{C}^d, \mu)$ then the series in

(3.1) converges to f in the $L^2(\mu)$ sense; that is the sequence $\sum\limits_{|\beta|\le N} a_\beta z^\beta$, which is in span$\{z^\beta\}$, converges to f in $L^2(\mu)$, so $\overline{\text{span}\{z^\beta\}} = \mathcal{H}L^2(\mathbb{C}^d,\mu)$. Hence $\{z^\beta\}$ is an orthogonal basis of $\mathcal{H}L^2(\mathbb{C}^d,\mu)$. □

By applying Theorem 3.1, we obtain Theorem 3.2 and Theorem 3.3.

Theorem 3.2. $\mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} \subseteq \mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}}$ *if β and γ are multi-indices such that $\beta \le \gamma$.*

Proof. It suffices to assume that $\gamma = \beta + e_i$ for some $1 \le i \le d$. Thus

$$\gamma = (\beta_1, \ldots, \beta_{i-1}, \beta_i + 1, \beta_{i+1}, \ldots, \beta_d)$$

for some $1 \le i \le d$. First, we claim that $\mathfrak{D}_{\frac{\partial^{\beta_i+1}}{\partial z_i^{\beta_i+1}}} \subseteq \mathfrak{D}_{\frac{\partial^{\beta_i}}{\partial z_i^{\beta_i}}}$. Let $f \in \mathcal{H}L^2(\mathbb{C}^d,\mu)$, then $f(z) = \sum\limits_\nu a_\nu z^\nu$, so

$$\frac{\partial^{\beta_i} f}{\partial z_i^{\beta_i}}(z) = \sum_{\nu_i=\beta_i}^{\infty} \sum_{\substack{\nu_j \ge 0 \\ j\ne i}} a_\nu \left(\frac{\nu_i!}{(\nu_i-\beta_i)!}\right) z_i^{\nu_i-\beta_i} \prod_{\substack{1\le j\le d \\ j\ne i}} z_j^{\nu_j}$$

and

$$\frac{\partial^{\beta_i+1} f}{\partial z_i^{\beta_i+1}}(z) = \sum_{\nu_i=\beta_i+1}^{\infty} \sum_{\substack{\nu_j \ge 0 \\ j\ne i}} a_\nu \left(\frac{\nu_i!}{(\nu_i-\beta_i-1)!}\right) z_i^{\nu_i-\beta_i-1} \prod_{\substack{1\le j\le d \\ j\ne i}} z_j^{\nu_j}.$$

Applying Theorem 3.1, we have

$$\left\|\frac{\partial^{\beta_i} f}{\partial z_i^{\beta_i}}\right\|^2 = \sum_{\nu_i=\beta_i}^{\infty} \sum_{\substack{\nu_j \ge 0 \\ j\ne i}} |a_\nu|^2 \left(\frac{\nu_i!}{(\nu_i-\beta_i)!}\right)^2 (\nu_i-\beta_i)! \prod_{\substack{1\le j\le d \\ j\ne i}} \nu_j!$$

and

$$\left\|\frac{\partial^{\beta_i+1} f}{\partial z_i^{\beta_i+1}}\right\|^2 = \sum_{\nu_i=\beta_i+1}^{\infty} \sum_{\substack{\nu_j \ge 0 \\ j\ne i}} |a_\nu|^2 \left(\frac{\nu_i!}{(\nu_i-\beta_i-1)!}\right)^2 (\nu_i-\beta_i-1)! \prod_{\substack{1\le j\le d \\ j\ne i}} \nu_j!.$$

Let

$$M := \sum_{\nu_i=\beta_i} \sum_{\substack{\nu_j \ge 0 \\ j\ne i}} |a_\nu|^2 (\beta_i!)^2 \prod_{\substack{1\le j\le d \\ j\ne i}} \nu_j!.$$

Hence,

$$
\begin{aligned}
\left\|\frac{\partial^{\beta_i} f}{\partial z_i^{\beta_i}}\right\|^2 &= M + \sum_{\nu_i=\beta_i+1}^{\infty} \sum_{\substack{\nu_j \geq 0 \\ j\neq i}} |a_\nu|^2 \left(\frac{\nu_i!}{(\nu_i-\beta_i)!}\right)^2 (\nu_i-\beta_i)(\nu_i-\beta_i-1)! \prod_{\substack{1\leq j\leq d \\ j\neq i}} \nu_j! \\
&\leq M + \sum_{\nu_i=\beta_i+1}^{\infty} \sum_{\substack{\nu_j \geq 0 \\ j\neq i}} |a_\nu|^2 \left(\frac{\nu_i!}{(\nu_i-\beta_i)!}\right)^2 (\nu_i-\beta_i)^2(\nu_i-\beta_i-1)! \prod_{\substack{1\leq j\leq d \\ j\neq i}} \nu_j! \\
&= M + \sum_{\nu_i=\beta_i+1}^{\infty} \sum_{\substack{\nu_j \geq 0 \\ j\neq i}} |a_\nu|^2 \left(\frac{\nu_i!}{(\nu_i-\beta_i-1)!}\right)^2 (\nu_i-\beta_i-1)! \prod_{\substack{1\leq j\leq d \\ j\neq i}} \nu_j! \\
&= M + \left\|\frac{\partial^{\beta_i+1} f}{\partial z_i^{\beta_i+1}}\right\|^2 .
\end{aligned}
$$

Since $f \in \mathcal{H}L^2\left(\mathbb{C}^d, \mu\right)$, it follows that

$$\|f\|^2 = \sum_{\nu} |a_\nu|^2 \nu! < \infty,$$

and that

$$\sum_{\nu_i=\beta_i} \sum_{\substack{\nu_j \geq 0 \\ j\neq i}} |a_\nu|^2 \beta_i! \prod_{\substack{1\leq j\leq d \\ j\neq i}} \nu_j! < \infty.$$

Therefore

$$M := \sum_{\nu_i=\beta_i} \sum_{\substack{\nu_j \geq 0 \\ j\neq i}} |a_\nu|^2 (\beta_i!)^2 \prod_{\substack{1\leq j\leq d \\ j\neq i}} \nu_j! < \infty.$$

Thus $\left\|\frac{\partial^{\beta_i+1} f}{\partial z_i^{\beta_i+1}}\right\|^2 < \infty$ implies $\left\|\frac{\partial^{\beta_i} f}{\partial z_i^{\beta_i}}\right\|^2 < \infty$. That is $\frac{\partial^{\beta_i+1} f}{\partial z_i^{\beta_i+1}} \in L^2\left(\mathbb{C}^d, \mu\right)$ implies $\frac{\partial^{\beta_i} f}{\partial z_i^{\beta_i}} \in L^2\left(\mathbb{C}^d, \mu\right)$. Thus $\mathfrak{D}_{\frac{\partial^{\beta_i+1}}{\partial z_i^{\beta_i+1}}} \subseteq \mathfrak{D}_{\frac{\partial^{\beta_i}}{\partial z_i^{\beta_i}}}$.

Next, if $f \in \mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}}$, then

$$\frac{\partial^\gamma f}{\partial z^\gamma} = \frac{\partial^{\beta_i+1}}{\partial z_i^{\beta_i+1}}\left(\frac{\partial^{\beta-\beta_i e_i} f}{\partial z^{\beta-\beta_i e_i}}\right) \in L^2\left(\mathbb{C}^d, \mu\right).$$

So $\frac{\partial^{\beta-\beta_i e_i} f}{\partial z^{\beta-\beta_i e_i}} \in \mathfrak{D}_{\frac{\partial^{\beta_i+1}}{\partial z_i^{\beta_i+1}}}$, and then by the first part $\frac{\partial^{\beta-\beta_i e_i} f}{\partial z^{\beta-\beta_i e_i}} \in \mathfrak{D}_{\frac{\partial^{\beta_i}}{\partial z_i^{\beta_i}}}$. Thus $\frac{\partial^\beta f}{\partial z^\beta} \in L^2\left(\mathbb{C}^d, \mu\right)$, and hence $f \in \mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}}$. So $\mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} \subseteq \mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}}$. □

Theorem 3.3. *$\mathfrak{D}_{z^\gamma} \subseteq \mathfrak{D}_{z^\beta}$ if β and γ are multi-indices such that $\beta \leq \gamma$.*

Proof. Given $f(z) = \sum\limits_{\nu} a_\nu z^\nu \in \mathcal{H}L^2\left(\mathbb{C}^d, \mu\right)$. Then

$$z^\beta f(z) = \sum_\nu a_\nu z^{\nu+\beta}$$

and

$$z^\gamma f(z) = \sum_\nu a_\nu z^{\nu+\gamma}.$$

By Theorem 3.1, we have

$$\left\|z^\beta f\right\|^2 = \sum_\nu |a_\nu|^2 (\nu+\beta)!$$

and

$$\|z^\gamma f\|^2 = \sum_\nu |a_\nu|^2 (\nu+\gamma)!.$$

Hence

$$\begin{aligned}\left\|z^\beta f\right\|^2 &= \sum_\nu |a_\nu|^2 \prod_{j=1}^{d} (\nu_j+\beta_j)! \\ &\leq \sum_\nu |a_\nu|^2 \prod_{j=1}^{d} (\nu_j+\gamma_j)! \\ &= \sum_\nu |a_\nu|^2 (\nu+\gamma)! \\ &= \|z^\gamma f\|^2 .\end{aligned}$$

Hence, if $\|z^\gamma f\|^2$ is finite then $\left\|z^\beta f\right\|^2$ is finite, so $z^\gamma f \in L^2\left(\mathbb{C}^d, \mu\right)$ implies $z^\beta f \in L^2\left(\mathbb{C}^d, \mu\right)$. Therefore $\mathfrak{D}_{z^\gamma} \subseteq \mathfrak{D}_{z^\beta}$. □

By Theorem 3.2 and Theorem 3.3, we can prove the following theorem :

Theorem 3.4. $\mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}} = \mathfrak{D}_{z^\beta}$ *for any multi-index* β.

Proof. To prove this theorem, it suffices to prove the following two statements :

(i) $\mathfrak{D}_{\frac{\partial}{\partial z_i}} = \mathfrak{D}_{z_i}$ for each $i \in \{1, \ldots, d\}$

(ii) if $\mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}} = \mathfrak{D}_{z^\beta}$ for some multi-index β then $\mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} = \mathfrak{D}_{z^\gamma}$, where $\gamma = \beta + e_i$ for some $i \in \{1, \ldots, d\}$.

Let $i \in \{1, \ldots, d\}$. We will show that $\mathfrak{D}_{\frac{\partial}{\partial z_i}} = \mathfrak{D}_{z_i}$. Let $f \in \mathcal{H}L^2\left(\mathbb{C}^d, \mu\right)$ and write $f(z) = \sum\limits_{\nu} a_\nu z^\nu$. Then

$$\frac{\partial f}{\partial z_i}(z) = \sum_{\nu \geq e_i} \nu_i a_\nu z^{\nu - e_i}$$

and

$$z_i f(z) = \sum_{\nu} a_\nu z^{\nu + e_i}.$$

By Theorem 3.1, we have

$$\begin{aligned} \|f\|^2 &= \sum_{\nu} |a_\nu|^2 \nu_i! \prod_{\substack{1 \leq j \leq d \\ j \neq i}} \nu_j!, \\ \left\| \frac{\partial f}{\partial z_i} \right\|^2 &= \sum_{\nu \geq e_i} |a_\nu|^2 {\nu_i}^2 (\nu_i - 1)! \prod_{\substack{1 \leq j \leq d \\ j \neq i}} \nu_j!, \end{aligned}$$

and

$$\|z_i f\|^2 = \sum_{\nu} |a_\nu|^2 (\nu_i + 1)! \prod_{\substack{1 \leq j \leq d \\ j \neq i}} \nu_j!.$$

Hence,

$$\begin{aligned} \|z_i f\|^2 - \|f\|^2 &= \sum_{\nu} |a_\nu|^2 (\nu_i + 1)! \prod_{\substack{1 \leq j \leq d \\ j \neq i}} \nu_j! - \sum_{\nu} |a_\nu|^2 \nu_i! \prod_{\substack{1 \leq j \leq d \\ j \neq i}} \nu_j! \\ &= \sum_{\nu \geq e_i} |a_\nu|^2 \left[(\nu_i + 1)! - \nu_i!\right] \prod_{\substack{1 \leq j \leq d \\ j \neq i}} \nu_j! \\ &= \sum_{\nu \geq e_i} |a_\nu|^2 {\nu_i}^2 (\nu_i - 1)! \prod_{\substack{1 \leq j \leq d \\ j \neq i}} \nu_j! \\ &= \left\| \frac{\partial f}{\partial z_i} \right\|^2. \end{aligned}$$

Thus $\|z_i f\|^2$ is finite if and only if $\left\|\frac{\partial f}{\partial z_i}\right\|^2$ is finite. That is $z_i f \in L^2(\mathbb{C}^d, \mu)$ if and only if $\frac{\partial f}{\partial z_i} \in L^2(\mathbb{C}^d, \mu)$. Hence $\mathfrak{D}_{z_i} = \mathfrak{D}_{\frac{\partial}{\partial z_i}}$.

Now we assume that $\mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}} = \mathfrak{D}_{z^\beta}$ for some multi-index β. Let $\gamma = \beta + e_i$ for some $i \in \{1, \ldots, d\}$. We will show that $\mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} = \mathfrak{D}_{z^\gamma}$. If $\beta_i = 0$, then

$$\frac{\partial}{\partial z_i}\left(z^\beta f\right) = z^\beta \left(\frac{\partial f}{\partial z_i}\right),$$

so we have that

$$\begin{aligned}
f \in \mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} &\Longleftrightarrow \frac{\partial^\gamma f}{\partial z^\gamma} \in L^2(\mathbb{C}^d, \mu) \\
&\Longleftrightarrow \frac{\partial f}{\partial z_i} \in \mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}} \\
&\Longleftrightarrow z^\beta\left(\frac{\partial f}{\partial z_i}\right) \in L^2(\mathbb{C}^d, \mu) \quad \text{by the assumption} \\
&\Longleftrightarrow \frac{\partial}{\partial z_i}\left(z^\beta f\right) \in L^2(\mathbb{C}^d, \mu) \\
&\Longleftrightarrow z^\beta f \in \mathfrak{D}_{\frac{\partial}{\partial z_i}} \\
&\Longleftrightarrow z^\beta f \in \mathfrak{D}_{z_i} \quad \text{by part (i)} \\
&\Longleftrightarrow z^\gamma f \in L^2(\mathbb{C}^d, \mu) \\
&\Longleftrightarrow f \in \mathfrak{D}_{z^\gamma}.
\end{aligned}$$

Suppose that $\beta_i \geq 1$. First, note that

$$\frac{\partial}{\partial z_i}\left(z^\beta f\right) = \beta_i \cdot z^\nu f + z^\beta\left(\frac{\partial f}{\partial z_i}\right), \tag{1}$$

where $\nu = \beta - e_i$ and $f \in \mathcal{H}L^2(\mathbb{C}^d, \mu)$. Let $f \in \mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}}$. We will show that $\frac{\partial}{\partial z_i}\left(z^\beta f\right) \in L^2(\mathbb{C}^d, \mu)$ by showing that both terms on the right-hand-side of (1) are in $L^2(\mathbb{C}^d, \mu)$. Since $f \in \mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}}$, $\frac{\partial f}{\partial z_i} \in \mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}}$. So $\frac{\partial f}{\partial z_i} \in \mathfrak{D}_{z^\beta}$ since $\mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}} = \mathfrak{D}_{z^\beta}$ by assumption. Thus $z^\beta\left(\frac{\partial f}{\partial z_i}\right) \in L^2(\mathbb{C}^d, \mu)$. By Theorem 3.2, $f \in \mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} \subseteq \mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}} = \mathfrak{D}_{z^\beta} \subseteq \mathfrak{D}_{z^\nu}$, so we have that $z^\nu f \in L^2(\mathbb{C}^d, \mu)$. Hence, $\frac{\partial}{\partial z_i}\left(z^\beta f\right) \in L^2(\mathbb{C}^d, \mu)$. Then $z^\beta f \in \mathfrak{D}_{\frac{\partial}{\partial z_i}}$. Since $\mathfrak{D}_{\frac{\partial}{\partial z_i}} = \mathfrak{D}_{z_i}$, $z^\beta f \in \mathfrak{D}_{z_i}$. Hence, $z^\gamma f \in L^2(\mathbb{C}^d, \mu)$, so $f \in \mathfrak{D}_{z^\gamma}$.

Conversely, suppose that $f \in \mathfrak{D}_{z^\gamma}$. Then $z^\beta f \in \mathfrak{D}_{z_i}$, so $z^\beta f \in \mathfrak{D}_{\frac{\partial}{\partial z_i}}$ since $\mathfrak{D}_{z_i} = \mathfrak{D}_{\frac{\partial}{\partial z_i}}$. Thus $\frac{\partial}{\partial z_i}\left(z^\beta f\right) \in L^2\left(\mathbb{C}^d, \mu\right)$. By Theorem 3.3, $f \in \mathfrak{D}_{z^\gamma} \subseteq \mathfrak{D}_{z^\beta} \subseteq \mathfrak{D}_{z^\nu}$, so we have that $z^\nu f \in L^2\left(\mathbb{C}^d, \mu\right)$, and that $\beta_i \cdot z^\nu f \in L^2\left(\mathbb{C}^d, \mu\right)$. It follows from (1) that $z^\beta\left(\frac{\partial f}{\partial z_i}\right) \in L^2\left(\mathbb{C}^d, \mu\right)$. Therefore $\frac{\partial f}{\partial z_i} \in \mathfrak{D}_{z^\beta}$. By the assumption, we have $\frac{\partial f}{\partial z_i} \in \mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}}$. Hence, $\frac{\partial^\gamma f}{\partial z^\gamma} \in L^2\left(\mathbb{C}^d, \mu\right)$, which means $f \in \mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}}$. Thus, $\mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} = \mathfrak{D}_{z^\gamma}$ as desired. $\square$

Chapter 4

The generalized Segal-Bargmann space

In this chapter our measure is the measure μ_α, when $\alpha \geq 2$. We will prove analogous results to those in the previous chapter.

Theorem 4.1. *The set $\{z^\beta\}$ forms an orthogonal basis of $\mathcal{H}L^2(\mathbb{C}^d, \mu_\alpha)$. Assume that f is a holomorphic function on $\mathbb{C}^d$ and has the pointwise convergent power series*

$$f(z) = \sum_\beta a_\beta z^\beta, \tag{4.1}$$

where $a_\beta \in \mathbb{C}$ for each multi-index β. Then

$$\int_{\mathbb{C}^d} |f(z)|^2 d\mu_\alpha(z) = C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_\beta |a_\beta|^2 \prod_{j=1}^{d} \Gamma(\frac{2(\beta_j+1)}{\alpha}) \tag{4.2}$$

where $\beta = (\beta_1, \ldots, \beta_d)$ is a multi-index and Γ is the Gamma function given by

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1}\, dt, \quad x \in (0, +\infty).$$

The series (4.1) is also convergent in the $L^2(\mathbb{C}^d, \mu_\alpha)$ sense if either side (hence both sides) of (4.2) is finite, and thus

$$\|f\|^2 = C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_\beta |a_\beta|^2 \prod_{j=1}^{d} \Gamma(\frac{2(\beta_j+1)}{\alpha}).$$

Proof. Let $D(\sigma)$ be the polydisc $\left\{z \in \mathbb{C}^d \mid \max_{1\le j\le d} |z_j| \le \sigma\right\}$. Consider first the case $d = 1$. Let $M_\sigma(j,k) = \int_{|z|<\sigma} z^j \overline{z}^k e^{-|z|^\alpha} dz$. Putting $z = re^{i\theta}$ and using polar coordinates, we have

$$\begin{aligned} M_\sigma(j,k) &= \int_0^\sigma \int_0^{2\pi} r^j e^{ij\theta} r^k e^{-ik\theta} e^{-r^\alpha} r\, d\theta dr \\ &= \int_0^\sigma r^{j+k+1} e^{-r^\alpha} dr \int_0^{2\pi} e^{i(j-k)\theta} d\theta. \end{aligned}$$

It follows that

(i) $M_\sigma(j,k) = 0$ if $j \ne k$ since $\int_0^{2\pi} e^{i(j-k)\theta} d\theta = 0$ if $j \ne k$;

(ii) if $j = k$ then

$$\begin{aligned} \lim_{\sigma\to\infty} M_\sigma(k,k) &= 2\pi \int_0^\infty r^{2k+1} e^{-r^\alpha} dr \\ &= \frac{2\pi}{\alpha} \Gamma(\frac{2(k+1)}{\alpha}); \end{aligned}$$

(iii) from (i) and (ii), we have that

$$\begin{aligned} \langle z^j, z^k \rangle &= C_\alpha \int_{\mathbb{C}} z^j \overline{z}^k e^{-|z|^\alpha} dz \\ &= C_\alpha \lim_{\sigma\to\infty} M_\sigma(j,k) \\ &= \delta_{jk} C_\alpha \frac{2\pi}{\alpha} \Gamma(\frac{2(k+1)}{\alpha}). \end{aligned}$$

Now we consider the general case. Since the power series (4.1) converges uniformly on the product set $D(\sigma)$ and $z^\beta \overline{z}^\gamma e^{-|z|^\alpha}$ is itself a product of functions of $z_1, \ldots, z_d$, we have

$$\begin{aligned} \int_{D(\sigma)} |f(z)|^2 d\mu_\alpha(z) &= C_\alpha \int_{D(\sigma)} |f(z)|^2 e^{-|z|^\alpha} dz \\ &= C_\alpha \int_{D(\sigma)} \sum_\beta a_\beta z^\beta \sum_\gamma \overline{a}_\gamma \overline{z}^\gamma e^{-|z|^\alpha} dz \end{aligned}$$

$$= C_\alpha \sum_\beta \sum_\gamma a_\beta \bar{a}_\gamma \int_{D(\sigma)} z^\beta \bar{z}^\gamma e^{-|z|^\alpha} dz$$

$$= C_\alpha \sum_\beta \sum_\gamma a_\beta \bar{a}_\gamma \prod_{j=1}^{d} \left(\int_{|z_j|<\sigma} {z_j}^{\beta_j} \bar{z}_j^{\gamma_j} e^{-|z_j|^\alpha} dz_j \right)$$

$$= C_\alpha \sum_\beta \sum_\gamma a_\beta \bar{a}_\gamma \prod_{j=1}^{d} M_\sigma(\beta_j, \gamma_j)$$

$$= C_\alpha \sum_\beta |a_\beta|^2 \prod_{j=1}^{d} M_\sigma(\beta_j, \beta_j),$$

where in the last sum we have written $\beta = (\beta_1, \ldots, \beta_d)$ and have used (i). Now let $\sigma \to \infty$ and use the monotone convergence theorem on both sides of the last equality. Then (4.2) follows from (ii). The set $\{z^\beta\}$ is an orthogonal set by (iii). Since

$$\|z^\beta\|^2 = C_\alpha \int_{\mathbb{C}^d} |z^\beta|^2 e^{-|z|^\alpha} dz$$

$$= C_\alpha \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} |{z_1}^{\beta_1}|^2 \cdots |{z_d}^{\beta_d}|^2 e^{-|z_1|^\alpha} \cdots e^{-|z_d|^\alpha} dz_1 \cdots dz_d$$

$$= C_\alpha \prod_{j=1}^{d} \left(\int_{\mathbb{C}} |{z_j}^{\beta_j}|^2 e^{-|z_j|^\alpha} dz_j \right)$$

$$= C_\alpha \prod_{j=1}^{d} \lim_{\sigma\to\infty} M_\sigma(\beta_j, \beta_j)$$

$$= C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \prod_{j=1}^{d} \Gamma\left(\frac{2(\beta_j+1)}{\alpha}\right),$$

we see that $\left\{\dfrac{z^\beta}{K_\beta}\right\}$ forms an orthonormal set, where

$$K_\beta = \sqrt{C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \prod_{j=1}^{d} \Gamma\left(\frac{2(\beta_j+1)}{\alpha}\right)}.$$

Assume that the right side of (4.2) is finite. Then the sequence of partial sum $f_N(z) := \sum_{|\beta|\le N} a_\beta z^\beta$ converges in the $L^2(\mu_\alpha)$ sense to some function g. Then there exists a subsequence of f_N which converges a.e. to g. This implies that $f = g$ a.e. So the series in (4.1) converges to f in the $L^2(\mu_\alpha)$ sense. In particular, if

f is in $\mathcal{H}L^2(\mathbb{C}^d, \mu_\alpha)$, then the series in (4.1) converges to f in the $L^2(\mu_\alpha)$ sense. That is the sequence of partial sums $\sum\limits_{|\beta|\leq N} a_\beta z^\beta$, which is in span$\{z^\beta\}$, converges to f in $L^2(\mu_\alpha)$, so $\overline{\text{span}\{z^\beta\}} = \mathcal{H}L^2(\mathbb{C}^d, \mu_\alpha)$. Hence $\{z^\beta\}$ is an orthogonal basis of $\mathcal{H}L^2(\mathbb{C}^d, \mu_\alpha)$. □

By applying Theorem 4.1, we can prove Theorem 4.2 and Theorem 4.3.

Theorem 4.2. $\mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} \subseteq \mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}}$ *if β and γ are multi-indices such that $\beta \leq \gamma$.*

Proof. It suffices to assume that $\gamma = \beta + e_i$ for some $1 \leq i \leq d$. Thus

$$\gamma = (\beta_1, \ldots, \beta_{i-1}, \beta_i + 1, \beta_{i+1}, \ldots, \beta_d)$$

for some $1 \leq i \leq d$. First, we will show that $\mathfrak{D}_{\frac{\partial^{\beta_i+1}}{\partial z_i^{\beta_i+1}}} \subseteq \mathfrak{D}_{\frac{\partial^{\beta_i}}{\partial z_i^{\beta_i}}}$. Let $f(z) = \sum\limits_\nu a_\nu z^\nu \in \mathcal{H}L^2(\mathbb{C}^d, \mu_\alpha)$. Then

$$\frac{\partial^{\beta_i} f}{\partial z_i^{\beta_i}}(z) = \sum_{\nu_i=\beta_i}^{\infty} \sum_{\substack{\nu_j \geq 0 \\ j\neq i}} a_\nu \left(\frac{\nu_i!}{(\nu_i - \beta_i)!}\right) z_i^{\nu_i - \beta_i} \prod_{\substack{1\leq j\leq d \\ j\neq i}} z_j^{\nu_j}$$

and

$$\frac{\partial^{\beta_i+1} f}{\partial z_i^{\beta_i+1}}(z) = \sum_{\nu_i=\beta_i+1}^{\infty} \sum_{\substack{\nu_j \geq 0 \\ j\neq i}} a_\nu \left(\frac{\nu_i!}{(\nu_i - \beta_i - 1)!}\right) z_i^{\nu_i - \beta_i - 1} \prod_{\substack{1\leq j\leq d \\ j\neq i}} z_j^{\nu_j}.$$

Let

$$P := \prod_{\substack{1\leq j\leq d \\ j\neq i}} \Gamma(\frac{2\nu_j}{\alpha} + \frac{2}{\alpha}).$$

Note that P depends on ν. Applying Theorem 4.1, we have

$$\left\|\frac{\partial^{\beta_i} f}{\partial z_i^{\beta_i}}\right\|^2 = C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=\beta_i}^{\infty} \sum_{\substack{\nu_j \geq 0 \\ j\neq i}} |a_\nu|^2 \left(\frac{\nu_i!}{(\nu_i - \beta_i)!}\right)^2 \Gamma(\frac{2(\nu_i - \beta_i)}{\alpha} + \frac{2}{\alpha})P$$

and

$$\left\|\frac{\partial^{\beta_i+1} f}{\partial z_i^{\beta_i+1}}\right\|^2 = C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=\beta_i+1}^{\infty} \sum_{\substack{\nu_j \geq 0 \\ j\neq i}} |a_\nu|^2 \left(\frac{\nu_i!}{(\nu_i - \beta_i - 1)!}\right)^2 \Gamma(\frac{2(\nu_i - \beta_i)}{\alpha})P.$$

Note that Γ is increasing on $[\frac{3}{2}, \infty)$. Let k_i be the smallest positive integer such that $k_i \geq \beta_i + 1$ and $\frac{2(k_i-\beta_i)}{\alpha} + \frac{2}{\alpha} \geq \frac{3}{2}$. Let

$$M := C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=\beta_i}^{k_i-1} \sum_{\substack{\nu_j \geq 0 \\ j \neq i}} |a_\nu|^2 \left(\frac{\nu_i!}{(\nu_i-\beta_i)!}\right)^2 \Gamma(\frac{2(\nu_i-\beta_i)}{\alpha} + \frac{2}{\alpha})P.$$

Hence,

$$\begin{aligned}
\left\|\frac{\partial^{\beta_i} f}{\partial z_i^{\beta_i}}\right\|^2 &= M + C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=k_i}^{\infty} \sum_{\substack{\nu_j \geq 0 \\ j \neq i}} |a_\nu|^2 \left(\frac{\nu_i!}{(\nu_i-\beta_i)!}\right)^2 \Gamma(\frac{2(\nu_i-\beta_i)}{\alpha} + \frac{2}{\alpha})P \\
&\leq M + C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=k_i}^{\infty} \sum_{\substack{\nu_j \geq 0 \\ j \neq i}} |a_\nu|^2 \left(\frac{\nu_i!}{(\nu_i-\beta_i)!}\right)^2 \Gamma(\frac{2(\nu_i-\beta_i)}{\alpha} + 1)P \\
&= M + C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=k_i}^{\infty} \sum_{\substack{\nu_j \geq 0 \\ j \neq i}} |a_\nu|^2 \left(\frac{\nu_i!}{(\nu_i-\beta_i)!}\right)^2 \frac{2}{\alpha}(\nu_i-\beta_i)\Gamma(\frac{2(\nu_i-\beta_i)}{\alpha})P \\
&\leq M + \frac{2}{\alpha} C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=k_i}^{\infty} \sum_{\substack{\nu_j \geq 0 \\ j \neq i}} |a_\nu|^2 \left(\frac{\nu_i!}{(\nu_i-\beta_i)!}\right)^2 (\nu_i-\beta_i)^2\Gamma(\frac{2(\nu_i-\beta_i)}{\alpha})P \\
&= M + \frac{2}{\alpha} C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=k_i}^{\infty} \sum_{\substack{\nu_j \geq 0 \\ j \neq i}} |a_\nu|^2 \left(\frac{\nu_i!}{(\nu_i-\beta_i-1)!}\right)^2 \Gamma(\frac{2(\nu_i-\beta_i)}{\alpha})P \\
&\leq M + \frac{2}{\alpha} C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=\beta_i+1}^{\infty} \sum_{\substack{\nu_j \geq 0 \\ j \neq i}} |a_\nu|^2 \left(\frac{\nu_i!}{(\nu_i-\beta_i-1)!}\right)^2 \Gamma(\frac{2(\nu_i-\beta_i)}{\alpha})P \\
&= M + \frac{2}{\alpha} \left\|\frac{\partial^{\beta_i+1} f}{\partial z_i^{\beta_i+1}}\right\|^2.
\end{aligned}$$

Since $f \in \mathcal{H}L^2(\mathbb{C}^d, \mu_\alpha)$,

$$\|f\|^2 = C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_\nu |a_\nu|^2 \prod_{j=1}^d \Gamma(\frac{2\nu_j}{\alpha} + \frac{2}{\alpha}) < \infty,$$

so we have that for each $\beta_i \leq \nu_i \leq k_i - 1$,

$$C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\substack{\nu_j \geq 0 \\ j \neq i}} |a_\nu|^2 \Gamma(\frac{2\nu_i}{\alpha} + \frac{2}{\alpha})P < \infty.$$

Then for each $\beta_i \leq \nu_i \leq k_i - 1$,

$$C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\substack{\nu_j \geq 0 \\ j \neq i}} |a_\nu|^2 \left(\frac{\nu_i!}{(\nu_i-\beta_i)!}\right)^2 \Gamma(\frac{2(\nu_i-\beta_i)}{\alpha} + \frac{2}{\alpha})P < \infty.$$

Therefore

$$M := C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=\beta_i}^{k_i-1} \sum_{\substack{\nu_j \geq 0 \\ j \neq i}} |a_\nu|^2 \left(\frac{\nu_i!}{(\nu_i-\beta_i)!}\right)^2 \Gamma(\frac{2(\nu_i-\beta_i)}{\alpha}+\frac{2}{\alpha})P < \infty.$$

Thus $\left\|\frac{\partial^{\beta_i+1} f}{\partial z_i^{\beta_i+1}}\right\|^2 < \infty$ implies $\left\|\frac{\partial^{\beta_i} f}{\partial z_i^{\beta_i}}\right\|^2 < \infty$, so $\frac{\partial^{\beta_i+1} f}{\partial z_i^{\beta_i+1}} \in L^2\left(\mathbb{C}^d, \mu_\alpha\right)$ implies $\frac{\partial^{\beta_i} f}{\partial z_i^{\beta_i}} \in L^2\left(\mathbb{C}^d, \mu_\alpha\right)$. Hence, $\mathfrak{D}_{\frac{\partial^{\beta_i+1}}{\partial z_i^{\beta_i+1}}} \subseteq \mathfrak{D}_{\frac{\partial^{\beta_i}}{\partial z_i^{\beta_i}}}$.

Next, if $f \in \mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}}$, then

$$\frac{\partial^{\beta_i+1}}{\partial z_i^{\beta_i+1}}\left(\frac{\partial^{\beta-\beta_i e_i} f}{\partial z^{\beta-\beta_i e_i}}\right) \in L^2\left(\mathbb{C}^d, \mu_\alpha\right).$$

So $\frac{\partial^{\beta-\beta_i e_i} f}{\partial z^{\beta-\beta_i e_i}} \in \mathfrak{D}_{\frac{\partial^{\beta_i+1}}{\partial z_i^{\beta_i+1}}}$, and then $\frac{\partial^{\beta-\beta_i e_i} f}{\partial z^{\beta-\beta_i e_i}} \in \mathfrak{D}_{\frac{\partial^{\beta_i}}{\partial z_i^{\beta_i}}}$ since $\mathfrak{D}_{\frac{\partial^{\beta_i+1}}{\partial z_i^{\beta_i+1}}} \subseteq \mathfrak{D}_{\frac{\partial^{\beta_i}}{\partial z_i^{\beta_i}}}$. Thus $\frac{\partial^\beta f}{\partial z^\beta} \in L^2\left(\mathbb{C}^d, \mu_\alpha\right)$, and hence $f \in \mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}}$. So $\mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} \subseteq \mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}}$. □

Theorem 4.3. *$\mathfrak{D}_{z^\gamma} \subseteq \mathfrak{D}_{z^\beta}$ if β and γ are multi-indices such that $\beta \leq \gamma$.*

Proof. It suffices to assume that $\gamma = \beta + e_i$ for some $1 \leq i \leq d$. Thus

$$\gamma = (\beta_1, \ldots, \beta_{i-1}, \beta_i + 1, \beta_{i+1}, \ldots, \beta_d)$$

for some $1 \leq i \leq d$. Let $f \in \mathcal{H}L^2\left(\mathbb{C}^d, \mu_\alpha\right)$. Then $f(z) = \sum_\nu a_\nu z^\nu$, so

$$z^\beta f(z) = \sum_\nu a_\nu z^{\nu+\beta}$$

and

$$z^\gamma f(z) = \sum_\nu a_\nu z^{\nu+\gamma}.$$

By applying Theorem 4.1, we have

$$\left\|z^\beta f\right\|^2 = C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_\nu |a_\nu|^2 \Gamma(\frac{2(\nu_i+\beta_i)}{\alpha}+\frac{2}{\alpha}) \prod_{\substack{1 \leq j \leq d \\ j \neq i}} \Gamma(\frac{2(\nu_j+\beta_j)}{\alpha}+\frac{2}{\alpha})$$

and

$$\|z^{\gamma}f\|^2 = C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu} |a_\nu|^2 \Gamma(\frac{2(\nu_i+\beta_i)}{\alpha} + \frac{4}{\alpha}) \prod_{\substack{1\le j\le d\\ j\ne i}} \Gamma(\frac{2(\nu_j+\beta_j)}{\alpha} + \frac{2}{\alpha}).$$

Let k_i be the smallest positive integer such that $k_i \geq \beta_i + 1$ and $\frac{2(k_i+\beta_i)}{\alpha} + \frac{2}{\alpha} \geq \frac{3}{2}$.

Let

$$M := C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=0}^{k_i-1} \sum_{\substack{\nu_j\ge 0\\ j\ne i}} |a_\nu|^2 \Gamma(\frac{2(\nu_i+\beta_i)}{\alpha} + \frac{2}{\alpha}) \prod_{\substack{1\le j\le d\\ j\ne i}} \Gamma(\frac{2(\nu_j+\beta_j)}{\alpha} + \frac{2}{\alpha}).$$

Then

$$\begin{aligned}
\left\|z^{\beta}f\right\|^2 &= M + C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=k_i}^{\infty} \sum_{\substack{\nu_j\ge 0\\ j\ne i}} |a_\nu|^2 \Gamma(\frac{2(\nu_i+\beta_i)}{\alpha} + \frac{2}{\alpha}) \prod_{\substack{1\le j\le d\\ j\ne i}} \Gamma(\frac{2(\nu_j+\beta_j)}{\alpha} + \frac{2}{\alpha}) \\
&\le M + C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=k_i}^{\infty} \sum_{\substack{\nu_j\ge 0\\ j\ne i}} |a_\nu|^2 \Gamma(\frac{2(\nu_i+\beta_i)}{\alpha} + \frac{4}{\alpha}) \prod_{\substack{1\le j\le d\\ j\ne i}} \Gamma(\frac{2(\nu_j+\beta_j)}{\alpha} + \frac{2}{\alpha}) \\
&\le M + C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu} |a_\nu|^2 \Gamma(\frac{2(\nu_i+\beta_i)}{\alpha} + \frac{4}{\alpha}) \prod_{\substack{1\le j\le d\\ j\ne i}} \Gamma(\frac{2(\nu_j+\beta_j)}{\alpha} + \frac{2}{\alpha}) \\
&= M + \|z^{\gamma}f\|^2 .
\end{aligned}$$

If $\|z^{\gamma}f\|^2$is finite, we have that for each $0 \leq \nu_i \leq k_i - 1$,

$$C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\substack{\nu_j\ge 0\\ j\ne i}} |a_\nu|^2 \Gamma(\frac{2(\nu_i+\beta_i)}{\alpha} + \frac{4}{\alpha}) \prod_{\substack{1\le j\le d\\ j\ne i}} \Gamma(\frac{2(\nu_j+\beta_j)}{\alpha} + \frac{2}{\alpha}) < \infty.$$

Then if $\|z^{\gamma}f\|^2$is finite, we have that for each $0 \leq \nu_i \leq k_i - 1$,

$$C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\substack{\nu_j\ge 0\\ j\ne i}} |a_\nu|^2 \Gamma(\frac{2(\nu_i+\beta_i)}{\alpha} + \frac{2}{\alpha}) \prod_{\substack{1\le j\le d\\ j\ne i}} \Gamma(\frac{2(\nu_j+\beta_j)}{\alpha} + \frac{2}{\alpha}) < \infty.$$

Therefore

$$M := C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=0}^{k_i-1} \sum_{\substack{\nu_j\ge 0\\ j\ne i}} |a_\nu|^2 \Gamma(\frac{2(\nu_i+\beta_i)}{\alpha} + \frac{2}{\alpha}) \prod_{\substack{1\le j\le d\\ j\ne i}} \Gamma(\frac{2(\nu_j+\beta_j)}{\alpha} + \frac{2}{\alpha}) < \infty$$

if $\|z^{\gamma}f\|^2$ is finite. Thus if $\|z^{\gamma}f\|^2$is finite then $\left\|z^{\beta}f\right\|^2$ is finite. So $z^{\gamma}f \in L^2\left(\mathbb{C}^d, \mu_\alpha\right)$ implies $z^{\beta}f \in L^2\left(\mathbb{C}^d, \mu_\alpha\right)$. Hence $\mathfrak{D}_{z^\gamma} \subseteq \mathfrak{D}_{z^\beta}$. □

The next result is the main topic of this work. It is slightly different from Theorem 3.4 and we prove it by using similar idea. We obtain it by using the results from Theorem 4.2. In this theorem, we have to assume that α is an integer because of the technically in defining and proving results that involve the term z^γ, where z is a complex number and γ is an arbitrary real number.

Theorem 4.4. *If $\alpha \geq 2$ is an integer, then $\mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}} = \mathfrak{D}_{z^{(\alpha-1)\beta}}$ for any multi-index β.*

Proof. To prove this theorem it suffices to show that the following two statements are true :

(i) $\mathfrak{D}_{\frac{\partial}{\partial z_i}} = \mathfrak{D}_{z_i^{\alpha-1}}$ for each $i \in \{1, \ldots, d\}$

(ii) if $\mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}} = \mathfrak{D}_{z^{(\alpha-1)\beta}}$ for some multi-index β then $\mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} = \mathfrak{D}_{z^{(\alpha-1)\gamma}}$, where $\gamma = \beta + e_i$ for some $i \in \{1, \ldots, d\}$.

Let $i \in \{1, \ldots, d\}$. We will show that that $\mathfrak{D}_{\frac{\partial}{\partial z_i}} = \mathfrak{D}_{z_i^{\alpha-1}}$. Let $f \in \mathcal{H}L^2(\mathbb{C}^d, \mu_\alpha)$ and write $f(z) = \sum_\nu a_\nu z^\nu$. Then

$$z_i^{\alpha-1} f(z) = \sum_\nu a_\nu z^{\nu + e_i(\alpha-1)},$$

and

$$\frac{\partial f}{\partial z_i}(z) = \sum_{\nu_i=1}^{\infty} \sum_{\substack{\nu_j \geq 0 \\ j \neq i}} \nu_i \, a_\nu \, z^{\nu - e_i}.$$

By applying Theorem 4.1, we have

$$\left\| z_i^{\alpha-1} f \right\|^2 = C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_\nu |a_\nu|^2 \Gamma(\frac{2\nu_i}{\alpha} + 2) \prod_{\substack{1 \leq j \leq d \\ j \neq i}} \Gamma(\frac{2\nu_j}{\alpha} + \frac{2}{\alpha}),$$

and

$$\left\| \frac{\partial f}{\partial z_i} \right\|^2 = C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=1}^{\infty} \sum_{\substack{\nu_j \geq 0 \\ j \neq i}} |a_\nu|^2 \, \nu_i^2 \, \Gamma(\frac{2\nu_i}{\alpha}) \prod_{\substack{1 \leq j \leq d \\ j \neq i}} \Gamma(\frac{2\nu_j}{\alpha} + \frac{2}{\alpha}).$$

Let

$$L := C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_\nu |a_\nu|^2 \Gamma(\frac{2\nu_i}{\alpha}+1) \prod_{\substack{1\le j\le d \\ j\ne i}} \Gamma(\frac{2\nu_j}{\alpha}+\frac{2}{\alpha}).$$

Because $\Gamma(2) = \Gamma(1) = 1$ and $\Gamma(x+1) = x\Gamma(x)$, it follows that

$$\begin{aligned}
&\frac{\alpha^2}{4}\left\|{z_i}^{\alpha-1}f\right\|^2 - \frac{\alpha^2}{4}L \\
&= \frac{\alpha^2}{4} C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_\nu |a_\nu|^2 \Gamma(\frac{2\nu_i}{\alpha}+2) \prod_{\substack{1\le j\le d \\ j\ne i}} \Gamma(\frac{2\nu_j}{\alpha}+\frac{2}{\alpha}) \\
&\quad - \frac{\alpha^2}{4} C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_\nu |a_\nu|^2 \Gamma(\frac{2\nu_i}{\alpha}+1) \prod_{\substack{1\le j\le d \\ j\ne i}} \Gamma(\frac{2\nu_j}{\alpha}+\frac{2}{\alpha}) \\
&= C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \frac{\alpha^2}{4} \sum_{\nu\ge e_i} |a_\nu|^2 \left[\left(\frac{2\nu_i}{\alpha}+1\right) - 1\right] \left(\frac{2\nu_i}{\alpha}\right) \Gamma(\frac{2\nu_i}{\alpha}) \prod_{\substack{1\le j\le d \\ j\ne i}} \Gamma(\frac{2\nu_j}{\alpha}+\frac{2}{\alpha}) \\
&= C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu\ge e_i} |a_\nu|^2\, \nu_i^2\, \Gamma(\frac{2\nu_i}{\alpha}) \prod_{\substack{1\le j\le d \\ j\ne i}} \Gamma(\frac{2\nu_j}{\alpha}+\frac{2}{\alpha}) \\
&= \left\|\frac{\partial f}{\partial z_i}\right\|^2.
\end{aligned}$$

Since

$$\begin{aligned}
L &= C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_\nu |a_\nu|^2 \Gamma(\frac{2\nu_i}{\alpha}+1) \prod_{\substack{1\le j\le d \\ j\ne i}} \Gamma(\frac{2\nu_j}{\alpha}+\frac{2}{\alpha}) \\
&\le C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_\nu |a_\nu|^2 (\frac{2\nu_i}{\alpha}+1)\Gamma(\frac{2\nu_i}{\alpha}+1) \prod_{\substack{1\le j\le d \\ j\ne i}} \Gamma(\frac{2\nu_j}{\alpha}+\frac{2}{\alpha}) \\
&= C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_\nu |a_\nu|^2 \Gamma(\frac{2\nu_i}{\alpha}+2) \prod_{\substack{1\le j\le d \\ j\ne i}} \Gamma(\frac{2\nu_j}{\alpha}+\frac{2}{\alpha}) \\
&= \left\|{z_i}^{\alpha-1}f\right\|^2,
\end{aligned}$$

we have that if $\|{z_i}^{\alpha-1}f\|^2$ is finite then L is finite. Thus if $\|{z_i}^{\alpha-1}f\|^2$ is finite then $\left\|\frac{\partial f}{\partial z_i}\right\|^2$ is finite. So ${z_i}^{\alpha-1}f \in L^2\left(\mathbb{C}^d, \mu_\alpha\right)$ implies $\frac{\partial f}{\partial z_i} \in L^2\left(\mathbb{C}^d, \mu_\alpha\right)$. Hence $\mathfrak{D}_{{z_i}^{\alpha-1}} \subseteq \mathfrak{D}_{\frac{\partial}{\partial z_i}}$.

On the other hand, let

$$M := C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=0} \sum_{\substack{\nu_j \geq 0 \\ j \neq i}} |a_\nu|^2 \prod_{\substack{1 \leq j \leq d \\ j \neq i}} \Gamma(\frac{2\nu_j}{\alpha} + \frac{2}{\alpha}).$$

we have that

$$\begin{aligned}
L &= C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu} |a_\nu|^2 \Gamma(\frac{2\nu_i}{\alpha} + 1) \prod_{\substack{1 \leq j \leq d \\ j \neq i}} \Gamma(\frac{2\nu_j}{\alpha} + \frac{2}{\alpha}) \\
&= M + C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=1}^{\infty} \sum_{\substack{\nu_j \geq 0 \\ j \neq i}} |a_\nu|^2 \left(\frac{2\nu_i}{\alpha}\right) \Gamma(\frac{2\nu_i}{\alpha}) \prod_{\substack{1 \leq j \leq d \\ j \neq i}} \Gamma(\frac{2\nu_j}{\alpha} + \frac{2}{\alpha}) \\
&\leq M + \frac{2}{\alpha} \left[C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=1}^{\infty} \sum_{\substack{\nu_j \geq 0 \\ j \neq i}} |a_\nu|^2 \nu_i^2 \Gamma(\frac{2\nu_i}{\alpha}) \prod_{\substack{1 \leq j \leq d \\ j \neq i}} \Gamma(\frac{2\nu_j}{\alpha} + \frac{2}{\alpha}) \right] \\
&= M + \frac{2}{\alpha} \left\| \frac{\partial f}{\partial z_i} \right\|^2.
\end{aligned}$$

Since $f \in \mathcal{H}L^2(\mathbb{C}^d, \mu_\alpha)$,

$$\|f\|^2 = C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu} |a_\nu|^2 \prod_{j=1}^{d} \Gamma(\frac{2\nu_j}{\alpha} + \frac{2}{\alpha}) < \infty,$$

$$C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=0} \sum_{\substack{\nu_j \geq 0 \\ j \neq i}} |a_\nu|^2 \Gamma(\frac{2}{\alpha}) \prod_{\substack{1 \leq j \leq d \\ j \neq i}} \Gamma(\frac{2\nu_j}{\alpha} + \frac{2}{\alpha}) < \infty.$$

So

$$M := C_\alpha \left(\frac{2\pi}{\alpha}\right)^d \sum_{\nu_i=0} \sum_{\substack{\nu_j \geq 0 \\ j \neq i}} |a_\nu|^2 \prod_{\substack{1 \leq j \leq d \\ j \neq i}} \Gamma(\frac{2\nu_j}{\alpha} + \frac{2}{\alpha}) < \infty.$$

Thus if $\left\|\frac{\partial f}{\partial z_i}\right\|^2$ is finite, then L is finite. Thus $\left\|\frac{\partial f}{\partial z_i}\right\|^2$ is finite implies $\|{z_i}^{\alpha-1} f\|^2$ is finite. So $\frac{\partial f}{\partial z_i} \in L^2(\mathbb{C}^d, \mu_\alpha)$ implies ${z_i}^{\alpha-1} f \in L^2(\mathbb{C}^d, \mu_\alpha)$. Hence $\mathfrak{D}_{\frac{\partial}{\partial z_i}} \subseteq \mathfrak{D}_{{z_i}^{\alpha-1}}$. Therefore $\mathfrak{D}_{\frac{\partial}{\partial z_i}} = \mathfrak{D}_{{z_i}^{\alpha-1}}$.

Now we assume that $\mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}} = \mathfrak{D}_{z^{(\alpha-1)\beta}}$ for some multi-index β. Fix $i \in \{1, \ldots, d\}$ and let $\gamma = \beta + e_i$. We will show that $\mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} = \mathfrak{D}_{z^{(\alpha-1)\gamma}}$. If $\beta_i = 0$, then

$$\frac{\partial}{\partial z_i}\left(z^{(\alpha-1)\beta} f\right) = z^{(\alpha-1)\beta}\left(\frac{\partial f}{\partial z_i}\right),$$

so we have that

$$\begin{aligned} f \in \mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} &\iff \frac{\partial^\gamma f}{\partial z^\gamma} \in L^2\left(\mathbb{C}^d, \mu_\alpha\right) \\ &\iff \frac{\partial f}{\partial z_i} \in \mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}} \\ &\iff \frac{\partial f}{\partial z_i} \in \mathfrak{D}_{z^{(\alpha-1)\beta}} \quad \text{by the assumption} \\ &\iff z^{(\alpha-1)\beta}\left(\frac{\partial f}{\partial z_i}\right) \in L^2\left(\mathbb{C}^d, \mu_\alpha\right) \\ &\iff \frac{\partial}{\partial z_i}\left(z^{(\alpha-1)\beta} f\right) \in L^2\left(\mathbb{C}^d, \mu_\alpha\right) \\ &\iff z^{(\alpha-1)\beta} f \in \mathfrak{D}_{\frac{\partial}{\partial z_i}} \\ &\iff z^{(\alpha-1)\beta} f \in \mathfrak{D}_{z_i^{\alpha-1}} \quad \text{by part(i)} \\ &\iff z^{(\alpha-1)\gamma} f \in L^2\left(\mathbb{C}^d, \mu_\alpha\right) \\ &\iff f \in \mathfrak{D}_{z^{(\alpha-1)\gamma}}. \end{aligned}$$

Now, assume that $\beta_i \geq 1$. We have that

$$\begin{aligned} f \in \mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} &\implies \frac{\partial f}{\partial z_i} \in \mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}} \\ &\implies \frac{\partial f}{\partial z_i} \in \mathfrak{D}_{z^{(\alpha-1)\beta}} \quad \text{by the assumption} \\ &\implies z^{(\alpha-1)\beta}\left(\frac{\partial f}{\partial z_i}\right) \in L^2\left(\mathbb{C}^d, \mu_\alpha\right). \end{aligned} \tag{1}$$

Let $\nu = (\alpha-1)\beta - e_i$. By Theorem 4.2 and the assumption, we have that

$$\mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} \subseteq \mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}} = \mathfrak{D}_{z^{(\alpha-1)\beta}} \subseteq \mathfrak{D}_{z^\nu}.$$

Thus

$$\begin{aligned} f \in \mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} &\implies f \in \mathfrak{D}_{z^\nu} \\ &\implies z^\nu f \in L^2\left(\mathbb{C}^d, \mu_\alpha\right) \\ &\implies (\alpha-1)\beta_i \cdot z^\nu f \in L^2\left(\mathbb{C}^d, \mu_\alpha\right). \end{aligned} \tag{2}$$

So by (1) and (2), we have

$$f \in \mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} \implies (\alpha-1)\beta_i \cdot z^\nu f + z^{(\alpha-1)\beta}\left(\frac{\partial f}{\partial z_i}\right) \in L^2\left(\mathbb{C}^d, \mu_\alpha\right).$$

Since

$$\frac{\partial}{\partial z_i}\left(z^{(\alpha-1)\beta}f\right) = (\alpha-1)\beta_i \cdot z^{\nu} f + z^{(\alpha-1)\beta}\left(\frac{\partial f}{\partial z_i}\right),$$

we have

$$\begin{aligned} f \in \mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} &\Longrightarrow \frac{\partial}{\partial z_i}\left(z^{(\alpha-1)\beta}f\right) \in L^2\left(\mathbb{C}^d, \mu_\alpha\right) \\ &\Longrightarrow z^{(\alpha-1)\beta}f \in \mathfrak{D}_{\frac{\partial}{\partial z_i}} \\ &\Longrightarrow z^{(\alpha-1)\beta}f \in \mathfrak{D}_{z_i^{\alpha-1}} \quad \text{by part(i)} \\ &\Longrightarrow z^{(\alpha-1)\gamma}f \in L^2\left(\mathbb{C}^d, \mu_\alpha\right) \\ &\Longrightarrow f \in \mathfrak{D}_{z^{(\alpha-1)\gamma}}. \end{aligned}$$

Conversely,

$$\begin{aligned} f \in \mathfrak{D}_{z^{(\alpha-1)\gamma}} &\Longrightarrow z^{(\alpha-1)\beta}f \in \mathfrak{D}_{z_i^{\alpha-1}} \\ &\Longrightarrow z^{(\alpha-1)\beta}f \in \mathfrak{D}_{\frac{\partial}{\partial z_i}} \quad \text{by part(i)} \\ &\Longrightarrow \frac{\partial}{\partial z_i}\left(z^{(\alpha-1)\beta}f\right) \in L^2\left(\mathbb{C}^d, \mu_\alpha\right). \end{aligned} \tag{3}$$

Let $\nu = (\alpha-1)\beta - e_i$. By Theorem 4.3 and the assumption,

$$\mathfrak{D}_{z^{(\alpha-1)\gamma}} \subseteq \mathfrak{D}_{z^{(\alpha-1)\beta}} \subseteq \mathfrak{D}_{z^{\nu}}.$$

Thus

$$\begin{aligned} f \in \mathfrak{D}_{z^{(\alpha-1)\gamma}} &\Longrightarrow f \in \mathfrak{D}_{z^{\nu}} \\ &\Longrightarrow z^{\nu} f \in L^2\left(\mathbb{C}^d, \mu_\alpha\right) \\ &\Longrightarrow (\alpha-1)\beta_i \cdot z^{\nu} f \in L^2\left(\mathbb{C}^d, \mu_\alpha\right). \end{aligned} \tag{4}$$

Therefore by (3) and (4), we have that

$$f \in \mathfrak{D}_{z^{(\alpha-1)\gamma}} \Longrightarrow \frac{\partial}{\partial z_i}\left(z^{(\alpha-1)\beta}f\right) - (\alpha-1)\beta_i \cdot z^{\nu} f \in L^2\left(\mathbb{C}^d, \mu_\alpha\right)$$

Since

$$z^{(\alpha-1)\beta}\left(\frac{\partial f}{\partial z_i}\right) = \frac{\partial}{\partial z_i}\left(z^{(\alpha-1)\beta}f\right) - (\alpha-1)\beta_i \cdot z^{\nu} f,$$

we have

$$\begin{aligned} f \in \mathfrak{D}_{z^{(\alpha-1)\gamma}} &\Longrightarrow z^{(\alpha-1)\beta}\left(\frac{\partial f}{\partial z_i}\right) \in L^2\left(\mathbb{C}^d, \mu_\alpha\right) \\ &\Longrightarrow \frac{\partial f}{\partial z_i} \in \mathfrak{D}_{z^{(\alpha-1)\beta}} \\ &\Longrightarrow \frac{\partial f}{\partial z_i} \in \mathfrak{D}_{\frac{\partial^\beta}{\partial z^\beta}} \quad \text{by the assumption} \\ &\Longrightarrow \frac{\partial^\gamma f}{\partial z^\gamma} \in L^2\left(\mathbb{C}^d, \mu_\alpha\right) \\ &\Longrightarrow f \in \mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}}. \end{aligned}$$

Thus

$$f \in \mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} \Longleftrightarrow f \in \mathfrak{D}_{z^{(\alpha-1)\gamma}},$$

so we have $\mathfrak{D}_{\frac{\partial^\gamma}{\partial z^\gamma}} = \mathfrak{D}_{z^{(\alpha-1)\gamma}}$ as desired. □

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