CHAPTER III

PASCAL'S RULE AND THE GENERAL BINOMIAL SERIES

3.1 Generalization of Pascal's rule.

Pascal's rule may be generalized so as to apply anywhere in the r-n plane as follows :

$$f(r,n)+f(r+1,n) = f(r+1, n+1) \quad \ldots\ldots\ldots\ldots(1)$$

For example, it holds when $r = 1$, $n = -1.5$ because

$$f(r,n) = f(1, -1.5) = -1.5$$

$$f(r+1, n) = f(2, -1.5) = (2.5)(1.5)/2$$

and

$$f(r+1, n+1) = f(2, -0.5) = (1.5)(0.5)/2$$

and hence $f(1, -1.5)+f(2, -1.5) = f(2, -0.5)$.

We shall now prove (1) for all non-singular points in the r-n plane.

$$f(r,n)+f(r+1, n) = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)} + \frac{\Gamma(n+1)}{\Gamma(r+2)\Gamma(n-r)}$$

$$= \frac{(r+1)\Gamma(n+2)}{(n+1)\Gamma(r+2)\Gamma(n-r+1)} + \frac{(n-r)\Gamma(n+2)}{(n+1)\Gamma(r+2)\Gamma(n-r+1)}$$

$$= \frac{\Gamma(n+2)}{\Gamma(r+2)\Gamma(n-r+1)}\left[\frac{(r+1)}{n+1} + \frac{(n-r)}{n+1}\right]$$

$$= \frac{\Gamma(n+2)}{\Gamma(r+2)\Gamma(n-r+1)}$$

$$= f(r+1, n+1).$$

We can also show that if the singularities of $f(r,n)$ are removed at the lattice points of the third and fourth quadrants using a fixed value of m(see Chapter II), equation (1) holds between limiting values of $f(r,n)$ at these lattice points.

At the lattice points in the fourth quadrant, we have

$$\lim_{\varepsilon \to 0} f(r+\varepsilon, -k+m\varepsilon) = (-1)^r \; {}^{k+r-1}C_r (1-\tfrac{1}{m})$$

and

$$\lim_{\varepsilon \to 0} f(r+1+\varepsilon, -k+m\varepsilon) = (-1)^{r+1} \; {}^{k+r}C_{r+1} (1-\tfrac{1}{m}).$$

Therefore

$$\lim_{\varepsilon \to 0} f(r+\varepsilon, -k+m\varepsilon) + \lim_{\varepsilon \to 0} f(r+1+\varepsilon, -k+m\varepsilon)$$

$$= (-1)^r \; {}^{k+r-1}C_r (1-\tfrac{1}{m}) + (-1)^{r+1} \; {}^{k+r}C_{r+1} (1-\tfrac{1}{m})$$

$$= (-1)^{r+1} (1-\tfrac{1}{m}) \left[\frac{(k+r)!}{(r+1)!\,(k-1)!} - \frac{(k+r-1)!}{r!\;(k-1)!} \right]$$

$$= \frac{(-1)^{r+1} \; (k+r-1)!}{(r+1)!\,(k-2)!} (1-\tfrac{1}{m})$$

$$= \lim_{\varepsilon \to 0} f(r+1+\varepsilon, -(k-1) + m\varepsilon).$$

3.2 The Convergence of the General Binomial Series.

On each lattice point of the singular line $n = -1$ we first obtain the values of the limit of the function $f(r,n)$ taken along the line with slope m.

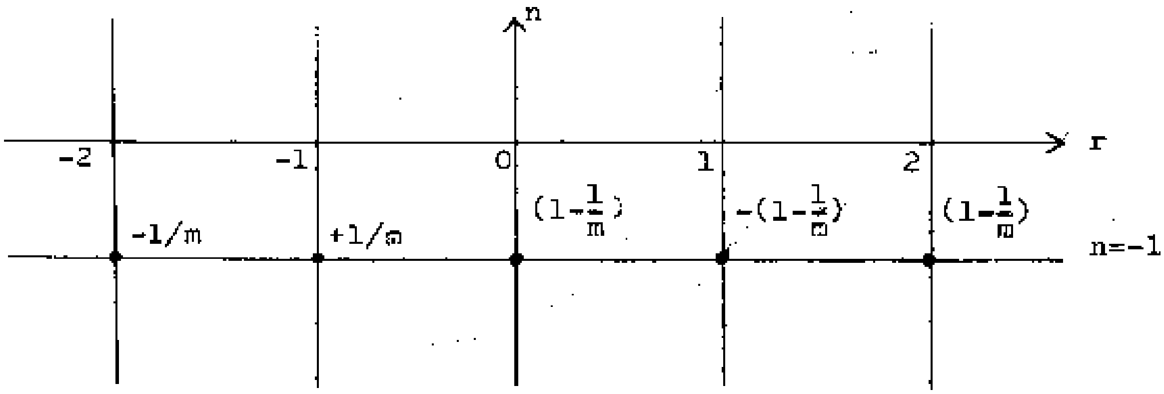

Figure 6 : Values of $\lim. f(r,n)$ on the lattice points $(r, -1)$

Using these values of the binomial coefficients, we now form the binomial series

$$\ldots\ldots + \frac{1}{m}\left(\frac{1}{a^3}\right) - \frac{1}{m}\left(\frac{1}{a^2}\right) + \frac{1}{m}\left(\frac{1}{a}\right) + \left(1-\frac{1}{m}\right)a^0 - \left(1-\frac{1}{m}\right)a + \ldots\ldots$$

$$= \frac{1}{m}\left(\frac{1}{a} - \frac{1}{a^2} + \frac{1}{a^3} - \frac{1}{a^4}\ \ldots\ldots\right) + \left(1 - \frac{1}{m}\right)(1-a+a^2-a^3+\ \ldots\ldots)$$

If we substitute $m = 1$, this series becomes

$$\frac{1}{a} - \frac{1}{a^2} + \frac{1}{a^3} - \frac{1}{a^4} + \ldots\ldots\ldots,$$ which is a convergent series

when $|a| > 1$ and equal to $(a+1)^{-1}$.

If we substitute $m = \infty$, it becomes

$$1 - a + a^2 - a^3 + a^4 - a^5 \ldots\ldots,$$ which is convergent

when $|a| < 1$ and equal to $(1+a)^{-1}$. These two values of m give the convergent series that are the same as Wanida's results.

If we substitute other values of m than $m = \infty$ and $m = 1$, we get a divergent series. For example, when $m = \frac{1}{2}$ the series is

$$2\left(\frac{1}{a} - \frac{1}{a^2} + \frac{1}{a^3} - \frac{1}{a^4} + \ldots\ldots\right) - (1-a+a^2-a^3 + \ldots\ldots)$$

when $|a| < 1$, the first term diverges, and when $|a| > 1$ the second term diverges. Therefore it is divergent for all non-zero a.

Thus on the singular line $n = -1$, the binomial series is divergent for every value of m except $m = \infty$ and 1, provided $a \neq 0$.

We shall now prove that on any singular line the series converges only when $m = \infty$ or 1.

As shown proviously (section 2.2 Chapter II), the values of the limits of $f(r,n)$ on the lattice points of the general singular line are,

$$\lim_{\varepsilon \to 0} f(r+\varepsilon, -k+m\varepsilon) = (-1)^r \; {}^{k+r-1}C_r(1-\tfrac{1}{m}),$$

when $r \geq 0$, and $n = -k$,

$$\lim_{\varepsilon \to 0} f(-j+\varepsilon, -k+m\varepsilon) = (-1)^{j-k} \; {}^{j-1}C_{k-1}(\tfrac{1}{m}),$$

when $r < 0$, $r = -j$,

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$n = -k$, and $j \geq k$,

and

$$\lim_{\varepsilon \to 0} f(-j+\varepsilon, -k+m\varepsilon) = 0, \quad \text{when } j < k.$$

Therefore on the singular line $n = -k$ the binomial series is,

$$\sum_{j \geq k} (-1)^{j-k} \; {}^{j-1}C_{k-1}(\tfrac{1}{m}) \; a^{-j} \; + \sum_{j < k} 0 \; a^{-j} + \sum_{r=0}^{\infty} (-1)^r \; {}^{k+r-1}C_r (1 - \tfrac{1}{m}) a^r,$$

which is

$$\frac{1}{m} \sum_{j \geq k} (-1)^{j-k} \; {}^{j-1}C_{k-1} a^{-j} + (1 - \frac{1}{m}) \sum_{r=0}^{\infty} (-1)^r \; {}^{k+r-1}C_r a^r \; ....(2)$$

When we substitute $m = 1$, we get

$$\sum_{j \geq k} (-1)^{j-k} \; {}^{j-1}C_{k-1} \; a^{-j}.$$

Using the Ratio test, we shall show that this series converges when $|a| > 1$.

Here $\qquad U_n = (-1)^{n-k} \; {}^{n-1}C_{k-1} \; a^{-n},$

and $\qquad U_{n+1} = (-1)^{n-k+1} \; {}^{n}C_{k-1} \; a^{-(n+1)}$

Therefore

$$\lim_{n \to \infty} \left| \frac{U_{n+1}}{U_n} \right| = \lim_{n \to \infty} \left| \frac{{}^{n}C_{k-1} \; a^{-(n+1)}}{{}^{n-1}C_{k-1} \; a^{-n}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{1}{a} \; \frac{n}{n-k+1} \right|$$

$$= \left| \frac{1}{a} \right|.$$

Hence this series converges when $\left|\frac{1}{a}\right| < 1$ or $|a| > 1$.

When we substitute $m = \infty$, the series becomes

$$\sum_{r=0}^{\infty} (-1)^r \; {}^{k+r-1}C_r \; a^r.$$

It follows in the same way that this series converges when $|a| < 1$.

For, $\quad U_n = (-1)^n \; {}^{k+n-1}C_n \; a^n ,$

$$U_{n+1} = (-1)^{n+1} \; {}^{k+n}C_{n+1} \; a^{n+1} ,$$

and $$\lim_{n\to\infty} \left| \frac{U_{n+1}}{U_n} \right| = \lim_{n\to\infty} \left| \frac{k+n}{n+1} \, a \right| = |a| .$$

Therefore this series converges when $|a| < 1$.

When m is neither 1 nor ∞ one of the two series in (2) diverges. The first sum diverges for $|a| < 1$, and the second sum diverges for $|a| > 1$; neither sum converges when $|a| = 1$. Therefore the complete expression (2) diverges for all a when m is neither 1 nor ∞ .