CHAPTER III

DERIVATIVES

III.1 Expansion of polynomials

From the Binomial Theorem we know that if $f(x)$ is a polynomial of degree n, then $f(x+h)$ is also a polynomial of degree n in x and

$$f(x+h) = f(h) + f_1(h)\,x + f_2(h)\,x^2 + \ldots + f_n(h)\,x^n,$$

where $f_1(h)$, $f_2(h)$, ... $f_n(h)$ are some polynomials in h.

Definition III.2

The derivative of $f(x)$ at $x = h$ (denoted by $D_x^h f(x)$) is defined to be the coefficient of x in the expansion of $f(x+h)$.

Note This definition obviously allows us to write

$$f(x+h) = f(h) + \left[D_x^h f(x)\right].x + \text{terms in higher powers of } x.$$

III.3 Algebra of derivatives.

According to the previous definition the following formulae are valid.

III.3.1 $D_x^h(c) = 0$, where c is constant

III.3.2 $D_x^h(x) = 1$

III.3.3 $D_x^h(x^n) = n\,h^{n-1}$, n being a positive integer

III.3.4 $D_x^h\left[f(x) + g(x)\right] = D_x^h f(x) + D_x^h g(x)$

III.3.5 $D_x^h \left[c\ f(x)\right] = c \cdot D_x^h f(x)$

III.3.6 $D_x^h \left[f(x)\ g(x)\right] = f(h)\ D_x^h g(x) + g(h)\ D_x^h f(x)$

III.3.7 $D_x^h \left[f(x)\right]^n = n\left[f(h)\right]^{n-1} \cdot D_x^h f(x)$, n being a positive integer.

To prove that $D_x^h (c) = o$

Proof. Let $\phi(x) = c$

$\therefore \phi(x + h) = c$

$\therefore \phi(h) + \left[D_x^h \phi(x)\right].x$ + terms in higher powers of $x = c$.

By equating the coefficient of x, we have

$$D_x^h \phi(x) = o$$

$$\text{i.e. } D_x^h (c) = o$$

To prove that $D_x^h (x) = 1$

Proof. Let $\phi(x) = x$

$\therefore \phi(x + h) = x + h$

L.S $= \phi(h) + \left[D_x^h \phi(x)\right].x$ + terms in higher powers of x.

R.S. $= h + x$

$\therefore$ By equating the coefficient of x, we have

$$D_x^h \phi(x) = 1$$

$$\text{i.e. } D_x^h (x) = 1$$

To prove that $D_x^h (x^n) = n h^{n-1}$

Proof. Let $\phi (x) = x^n$

$\therefore \phi (x + h) = (x + h)^n$

$= (h + x)^n$

$$L.S. = \phi (h) + \left[D_x^h \phi (x) \right] x + \text{terms in higher powers of } x.$$

$$R.S. = h^n + n h^{n-1} . x + \text{terms in higher powers of } x.$$

$\therefore$ By equating the coefficient of x, we have

$$D_x^h \phi (x) = n h^{n-1}$$

i.e. $$D_x^h (x^n) = n h^{n-1}$$

To prove that $D_x^h \left\{ f (x) + g (x) \right\} = D_x^h f (x) + D_x^h g (x)$

Proof. Let $\phi (x) = f (x) + g (x)$

$\therefore \phi (x + h) = f (x + h) + g (x + h)$

$$L.S. = \phi (h) + \left[D_x^h \phi (x) \right] x + \text{terms in higher powers of } x.$$

$$R.S. = f (h) + \left[D_x^h f (x) \right] x + \text{terms in higher powers of } x.$$

$$+ g (h) + \left[D_x^h g (x) \right] x + \text{terms in higher powers of } x.$$

$$= f (h) + g (h) + \left[D_x^h f (x) + D_x^h g (x) \right] x + \text{terms in higher powers of } x$$

$\therefore$ By equating the coefficient of x, we have

$$D_x^h \phi (x) = D_x^h f (x) + D_x^h g (x)$$

i.e. $$D_x^h \left[f (x) + g (x) \right] = D_x^h f (x) + D_x^h g (x)$$

To prove that $D_x^h \left[c\ f\ (x) \right] = c \cdot D_x^h f\ (x)$

Proof. Let $\phi\ (x) = c\ f\ (x)$

$\therefore \quad \phi\ (x + h) = c\ f\ (x + h)$

L.S. $= \phi\ (h) + \left[D_x^h \phi\ (x) \right] x$ + terms in higher powers of x

R.S. $= c \left\{ f\ (h) + \left[D_x^h f\ (x) \right] x + \text{terms in higher powers of } x \right\}$

$= c\ f\ (h) + c \left[D_x^h f\ (x) \right] x$ + terms in higher powers of x

$\therefore$ By equating the coefficient of x, we have

$$D_x^h \phi\ (x) = c \cdot D_x^h f\ (x)$$

$$\text{i.e. } D_x^h \left[c\ f\ (x) \right] = c \cdot D_x^h f\ (x)$$

To prove that $D_x^h \left[f\ (x)\ g\ (x) \right] = f\ (h)\ D_x^h g\ (x) + g\ (h)\ D_x^h f\ (x)$

Proof. Let $\phi\ (x) = f\ (x)\ g\ (x)$

$\therefore \quad \phi\ (x + h) = f\ (x + h)\ g\ (x + h)$

L.S. $= \phi\ (h) + \left[D_x^h \phi\ (x) \right] x$ + terms in higher powers of x.

R.S. $= \left\{ f\ (h) + \left[D_x^h f\ (x) \right] x + \ldots\ldots \right\} \left\{ g\ (h) + D_x^h g\ (x)\ \ x + \ldots\ldots \right\}$

$= f\ (h)\ g\ (h) + \left[f\ (h)\ D_x^h g\ (x) + g\ (h)\ D_x^h f\ (x) \right] x + \ldots\ldots\ldots$

By equating coefficient of x, we have

$$D_x^h \phi\ (x) = f\ (h)\ D_x^h g\ (x) + g\ (h)\ D_x^h f\ (x)$$

i.e. $D_x^h \left[f\ (x)\ g\ (x) \right] = f\ (h)\ D_x^h g\ (x) + g\ (h)\ D_x^h f\ (x)$

To prove that $D_x^h \left[f(x) \right]^n = n \left[f(h) \right]^{n-1} . D_x^h f(x)$

Proof. Let $\phi(x) = \left[f(x) \right]^n$

$$\therefore \quad \phi(x+h) = \left[f(x+h) \right]^n$$

$$\text{L.S.} = \phi(h) + \left[D_x^h \phi(x) \right] x + \text{terms in higher powers of } x$$

$$\text{R.S.} = \left\{ f(h) + \left[D_x^h f(x) \right] . x + \text{terms in higher powers of } x \right\}^n$$

$$= \left\{ f(h) + \left[D_x^h f(x) \right] x \right\}^n + \text{terms in higher powers of } x$$

$$= \left[f(h) \right]^n + \left\{ n \left[f(h) \right]^{n-1} . D_x^h f(x) \right\} . x + \text{terms in higher powers of } x.$$

$\therefore$ By equating the coefficient of x, we have

$$D_x^h \phi(x) = n \left\{ f(h) \right\}^{n-1} . D_x^h f(x)$$

i.e. $$D_x^h \left[f(x) \right]^n = n \left[f(h) \right]^{n-1} . D_x^h f(x)$$

<u>Theorem III.4</u>

If a polynomial $f(x)$ has a maximum or minimum value at $x = h$ then the derivative of $f(x)$ at $x = h$ is zero.

Proof. $f(x)$ has a maximum or minimum value at $x = h$, then evidently $g(x) = f(x+h)$ has a maximum or minimum value at $x = o$.

Since $$g(x) = f(x+h)$$

$$g(x) = f(h) + \left[D_x^h f(x) \right] x + \ldots\ldots\ldots$$

By theorem II.6

$$D_x^h f(x) = 0$$

<u>Example III.5</u>

Find the maximum and the minimum values of the function

$f(x) = x^3 - 6x^2 + 9x$

Let the function has a maximum or minimum value at $x = h$ then by theorem III.4

$$D_x^h \left[x^3 - 6x^2 + 9x \right] = 0$$

$$\therefore \quad 3h^2 - 12h + 9 = 0$$

$$\therefore \quad 3(h-1)(h-3) = 0$$

$$\therefore \quad h = 1 \text{ or } 3$$

Let $g(x) = f(x+1)$

$$f(x) = x^3 - 6x^2 + 9x$$

$$\therefore \quad g(x) = (x+1)^3 - 6(x+1)^2 + 9(x+1)$$

$$= x^3 - 3x^2 + 4$$

By theorem II.7, since the coefficient of x^2 is negative, $f(x)$ has a maximum value $f(1) = g(0) = 4$,

Again, let $h(x) = f(x+3)$

$$f(x) = x^3 - 6x^2 + 9x$$

$$\therefore \quad h(x) = (x+3)^3 - 6(x+3)^2 + 9(x+3)$$

$$= x^3 + 3x^2$$

By theorem II.7, since the coefficient of x^2 is positive, $f(x)$ has a minimum value $f(3) = h(o) = o$

Example III.6

Prove that the function $f(x) = x^3 + 3x^2 + 3x$ has no maximum or minimum value.

Suppose $f(x)$ has a maximum or minimum value at $x = h$, then by theorem III.4 $D_x^h f(x) = o$

$$\text{i.e.} \quad D_x^h \left\{ x^3 + 3x^2 + 3x \right\} = o$$

$$\therefore \quad 3h^2 + 6h + 3 = o$$

$$\therefore \quad 3(h^2 + 2h + 1) = o$$

$$\therefore \quad 3(h + 1)^2 = o$$

$$\therefore \quad h = -1$$

Let

$$g(x) = f(x - 1)$$

$$= (x - 1)^3 + 3(x - 1)^2 + 3(x - 1)$$

$$= x^3 - 1$$

By corollary II.8 $g(x)$ cannot have a maximum or minimum value at $x = o$

$\therefore$ $f(x)$ cannot have a maximum or minimum value.