CHAPTER II

MAXIMA AND MINIMA

<u>Definition II.1</u>

The function $f(x)$ in the domain of real numbers is said to be maximum at $x = h$, if there exists $\mu > o$ such that if $o < \partial < \mu$,

$$\text{then } f(h) > f(h + \partial)$$

$$\text{and } f(h) > f(h - \partial),$$

<u>Definition II.2</u>

The function $f(x)$ is said to be minimum at $x = h$, if there exists $\mu > o$ such that if, $o < \partial < \mu$,

$$\text{then } f(h) < f(h + \partial)$$

$$\text{and } f(h) < f(h - \partial)$$

<u>Example II.3</u>

Prove that the function $f(x) = |x|$ is minimum at $x = o$.

$$\text{We have } f(o) = o$$

$$\text{and } f(o + \partial) = |\partial|$$

$$= \partial$$

$$f(o - \partial) = |-\partial|$$

$$= \partial$$

$\therefore$ $f(o) < f(o + \partial)$ and $f(o) < f(o - \partial)$ for all $\partial > o$

$\therefore$ μ can be chosen.

Example II.4

Prove that the function $f(x) = x^3 - 6x^2 + 9x$ is maximum at $x = 1$.

We have
$$f(1) = 1 - 6 + 9$$
$$= 4$$
$$f(1 + \partial) = (1 + \partial)^3 - 6(1 + \partial)^2 + 9(1 - \partial)$$
$$= 4 - 3\partial^2 + \partial^3$$
$$= 4 - \partial^2(3 - \partial)$$
$$f(1 - \partial) = (1 - \partial)^3 - 6(1 + \partial)^2 + 9(1 - \partial)$$
$$= 4 - 3\partial^2 - \partial^3$$
$$= 4 - \partial^2(3 + \partial)$$

Clearly see that we can make both $f(1 + \partial)$ and $f(1 - \partial)$ less than $f(1)$ by making both $\partial^2(3 - \partial)$ and $\partial^2(3 + \partial)$ positive.

i.e. by choosing $\mu = 3$.

Lemma II.5

If $o \leq \partial < 1$ and $\partial < \dfrac{|a_k|}{|a_{k+1}| + \ldots\ldots + |a_n|}$, then the polynomials $P_1 = a_k + a_{k+1}\partial + \ldots\ldots\ldots + a_n \partial^{n-k}$

and $P_2 = a_k + a_{k+1}(-\partial) + \ldots\ldots + a_n(-\partial)^{n-k}$

have the same sign as a_k.

Proof. $P_1 - a_k = \partial \left[a_{k+1} + a_{k+2}\,\partial + \ldots\ldots + a_n\,\partial^{n-k-1} \right]$

$$\therefore \left| P_1 - a_k \right| \leq \left| \partial \right| \left(\left| a_{k+1} \right| + \left| a_{k+2} \right| \left| \partial \right| + \ldots\ldots + \left| a_n \right| \left| \partial^{n-k-1} \right| \right)$$

$$\therefore \left| P_1 - a_k \right| < \partial \left(\left| a_{k+1} \right| + \left| a_{k+2} \right| + \ldots\ldots + \left| a_n \right| \right)$$

$$\left| P_1 - a_k \right| < \left| a_k \right| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

If $a_k > 0$, (1) implies

$$\left| P_1 - a_k \right| < a_k$$

$$\therefore \quad -a_k < P_1 - a_k < a_k$$

$$\therefore \quad 0 < P_1 < 2\,a_k$$

$$\therefore \quad P_1 > 0.$$

If $a_k < 0$, (1) implies

$$\left| P_1 - a_k \right| < -a_k$$

$$\therefore \quad -(-a_k) < P_1 - a_k < -a_k$$

$$\therefore \quad P_1 < 0$$

Again $P_2 - a_k = (-\partial) \left[a_{k+1} + a_{k+2}(-\partial) + \ldots + a_n (-\partial)^{n-k-1} \right]$

$$\therefore \left| P_2 - a_k \right| \leq \left| (-\partial) \right| \left[\left| a_{k+1} \right| + \left| a_{k+2} \right| \left| (-\partial) \right| + \ldots + \left| a_n \right| \left| (-\partial)^{n-k-1} \right| \right]$$

$$\therefore \left| P_2 - a_k \right| < \partial \left(\left| a_{k+1} \right| + \ldots\ldots\ldots + \left| a_n \right| \right)$$

$$\therefore \left| P_2 - a_k \right| < \left| a_k \right| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

If $a_k > 0$, (2) implies

$$\left| P_2 - a_k \right| < a_k$$

$$\therefore \quad -a_k < P_2 - a_k < a_k$$

$$\therefore \quad 0 < P_2 < 2a_k$$

$$\therefore \quad P_2 > 0$$

If $a_k < 0$, (2) implies

$$\left| P_2 - a_k \right| < -a_k$$

$$\therefore \quad -(-a_k) < P_2 - a_k < -a_k$$

$$\therefore \quad 2a_k < P_2 < 0$$

$$\therefore \quad P_2 < 0$$

Theorem II.6

If the polynomial $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ is maximum or minimum at $x = 0$, then $a_1 = 0$.

Proof. Suppose $a_1 \neq 0$; i.e. either $a_1 > 0$ or $a_1 < 0$

case (i) if $a_1 > 0$

let $A_1 = a_1 + a_2\partial + a_3\partial^2 + \dots\dots + a_n\partial^{n-1}$

and $A_2 = a_1 + a_2(-\partial) + a_3(-\partial)^2 + \dots + a_n(-\partial)^{n-1}$

where $0 < \partial < 1$ and $\partial < \dfrac{|a_1|}{|a_2| + \dots\dots + |a_n|}$

then by lemma II.5 A_1 and A_2 have the same sign as a_1

i.e. $A_1 > o$ and $A_2 > o$

now because $f(o) = a_o$

and $f(\partial) = a_o + a_1\partial + a_2\partial^2 + \ldots\ldots + a_n\partial^n$

and $f(-\partial) = a_o + a_1(-\partial) + a_2(-\partial)^2 + \ldots + a_n(-\partial)^n$

$\therefore \quad f(\partial) = f(o) + \partial A_1$

and $f(-\partial) = f(o) + (-\partial)A_2$

$\therefore$ for $o < \partial < 1$ and $\partial < \dfrac{|a_1|}{|a_2| + \ldots\ldots + |a_n|}$

we have $f(-\partial) < f(o) < f(\partial)$

i.e. if $a_1 > o$, $f(x)$ **cannot** be maximum or minimum at $x = o$

case **(ii)** if $a_1 < o$

similar to case (i), we may prove that $f(x)$ **cannot** be maximum or minimum at $x = o$

i.e. if $a_1 \neq o$, $f(x)$ **cannot** be maximum or minimum at $x = o$, which is contradictory to the hypothesis

$\therefore \quad a_1 = o$

Theorem II.7

Let $f(x) = c + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$ if k is an even number such that all $a_i = o$ for $i < k$, then $f(x)$ has a maximum or minimum value at $x = o$, according as $a_k < o$ or $a_k > o$.

Proof. For $o < \partial < 1$ and $\partial < \dfrac{|a_k|}{|a_{k+1}| + \dots\dots\dots + |a_n|}$

$$f(o) = c$$

$$f(\partial) = c + a_k \partial^k + a_{k+1} \partial^{k+1} + \dots\dots\dots\dots + a_n \partial^n$$

$$= f(o) + \partial^k A_1 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (1)$$

where $A_1 = a_k + a_{k+1}\partial + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots + a_n \partial^{n-k}$

and $f(-\partial) = c + a_k (-\partial)^k + a_{k+1} (-\partial)^{k+1} + \dots\dots + a_n (-\partial)^n$

$$= f(o) + (-\partial)^k A_2 \dots\dots\dots\dots\dots\dots\dots\dots \quad (2)$$

where $A_2 = a_k + a_{k+1}(-\partial) + \dots\dots\dots\dots\dots\dots\dots\dots\dots + a_n (-\partial)^{n-k}$

If $a_k < o$, then by lemma II.5 $A_1 < o$ and $A_2 < o$

since k is even, ∂^k and $(-\partial)^k$ are positive.

$\therefore$ from (1) $f(\partial) - f(o) = \partial^k A_1$

$\therefore$ $f(\partial) - f(o)$ is negative

$\therefore$ $f(o) > f(\partial)$ .... (3)

from (2) $f(-\partial) - f(o) = (-\partial)^k A_2$

$\therefore$ $f(-\partial) - f(o)$ is negative

$\therefore$ $f(o) > f(-\partial)$ .... (4)

The inequalities (3) and (4) show that

$f(x)$ is maximum at $x = o$

Similarly if $a_k > o$ we may prove that $f(x)$ is minimum at $x = o$.

Corollary II.8

If k is an odd number and $a_k \neq o$, then $f(x)$ has no maximum or minimum value at $x = o$.

Proof. For $o < \partial < 1$ and $\partial < \dfrac{|a_k|}{|a_{k+1}| + \cdots\cdots\cdots + |a_n|}$

$$f(o) = c$$

$$f(\partial) = c + a_k\partial^k + a_{k+1}\partial^{k+1} + \cdots\cdots\cdots\cdots + a_n\partial^n$$

$$= f(o) + \partial^k A_1 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (1)$$

where $A_1 = a_k + a_{k+1}\partial + \cdots\cdots\cdots\cdots\cdots\cdots\cdots + a_n\partial^{n-k}$

and $f(-\partial) = c + a_k(-\partial)^k + a_{k+1}(-\partial)^{k+1} + \cdots\cdots\cdots + a_n(-\partial)^n$

$$= f(o) + (-\partial)^k A_2 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (2)$$

where $A_2 = a_k + a_{k+1}(-\partial) + \cdots\cdots\cdots\cdots\cdots\cdots + a_n(-\partial)^{n-k}$

$\because$ $a_k \neq o$, then either $a_k < o$ or $a_k > o$

case (i) if $a_k < o$, then by lemma II.5

$A_1 < o$ and $A_2 < o$

since k is odd, ∂^k is potitive and $(-\partial)^k$ is negative

$\therefore$ from (1) $f(\partial) - f(o) = \partial^k A_1$

$\therefore$ $f(\partial) - f(o)$ is negative

$\therefore$ $f(\partial) < f(o) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (3)$

from (2) $$f(-\partial) - f(o) = (-\partial)^k A_2$$

$\therefore$ $f(-\partial) - f(o)$ is positive

$$\therefore \quad f(-\partial) < f(-\partial) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

from (3) and (4) we have

$$f(\partial) < f(o) < f(-\partial)$$

$\therefore$ $f(x)$ cannot have a maximum or a minimum value at $x = o$.

case (ii) if $a_k > o$, then by lemma II.5

$$A_1 > o \quad \text{and} \quad A_2 > o$$

since k is odd, ∂^k is positive and $(-\partial)^k$ is negative

$\therefore$ from (1) $$f(\partial) - f(o) = \partial^k A_1$$

$\therefore$ $f(\partial) - f(o)$ is positive

$$\therefore \quad f(o) < f(\partial) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

$\therefore$ from (2) $$f(-\partial) - f(o) = (-\partial)^k A_2$$

$\therefore$ $f(-\partial) - f(o)$ is negative

$$\therefore \quad f(-\partial) < f(o) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

from (5) and (6) we have

$$f(-\partial) < f(o) < f(\partial)$$

$\therefore$ $f(x)$ cannot have a maximum or a minimum value at $x = o$.

from case (i) and (ii) it is seen that if $a_k \neq 0$, and k is odd, then $f(x)$ has no maximum or minimum value at $x = 0$.

II.9 Extension of lemma II.5

If the series

$$a_k + a_{k+1}x + a_{k+2}x^2 + a_{k+3}x^3 + \cdots\cdots\cdots$$

is absolutely convergent for $0 \leqslant x \leqslant m$, then there exists $\mu > 0$ such that if $0 < \partial < \mu$, then the infinite polynomials

$$P_1 = a_k + a_{k+1}\partial + a_{k+2}\partial^2 + a_{k+3}\partial^3 + \cdots\cdots\cdots\cdots$$

and $$P_2 = a_k + a_{k+1}(-\partial) + a_{k+2}(-\partial)^2 + a_{k+3}(-\partial)^3 + \cdots\cdots$$

have the same sign as a_k

Proof. Choose $$\mu = \frac{m\,|a_k|}{a_k + m\,|a_{k+1}| + m^2\,|a_{k+2}| + m^3\,|a_{k+3}| + \cdots\cdots}$$

then obviously $\mu \leqslant m$ and $$\mu \leqslant \frac{|a_k|}{|a_{k+1}| + m\,|a_{k+2}| + m^2\,|a_{k+3}| + \cdots\cdots}$$

$$\therefore \quad \partial < m \quad \text{and} \quad \partial < \frac{|a_k|}{|a_{k+1}| + m\,|a_{k+2}| + m^2\,|a_{k+3}| + \cdots\cdots\cdots}$$

$$\therefore \quad \partial < m \quad \text{and} \quad \partial\left[|a_{k+1}| + m\,|a_{k+2}| + m^2\,|a_{k+3}| + \cdots\cdots\right] < |a_k|$$

we have

$$P_1 - a_k = \partial\,(a_{k+1} + a_{k+2}\partial + a_{k+3}\partial^2 + \cdots\cdots\cdots\cdots\cdots)$$

$$\therefore \quad |P_1 - a_k| \leqslant |\partial|\left[|a_{k+1}| + |a_{k+2}\partial| + |a_{k+3}\partial^2| + \cdots\cdots\cdots\right]$$

$\therefore \quad |P_1 - a_k| < \partial \left[|a_{k+1}| + m |a_{k+2}| + m^2 |a_{k+3}| + \ldots\ldots\ldots\ldots \right]$

$\therefore \quad |P_1 - a_k| < |a_k| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$

If $a_k > o$, (1) implies that

$$|P_1 - a_k| < a_k$$

$\therefore \quad -a_k < P_1 - a_k < a_k$

$\therefore \quad o < P_1 < 2a_k$

i.e. $P_1 > o$

If $a_k < o$, (1) implies that

$$|P_1 - a_k| < -a_k$$

$\therefore \quad -(-a_k) < P_1 - a_k < -a_k$

$\therefore \quad 2a_k < P_1 < o$

i.e. $P_1 < o$

Again $P_2 - a_k = (-\partial) \left[a_{k+1} + a_{k+2}(-\partial) + a_{k+3}(-\partial)^2 + \ldots\ldots \right]$

$\therefore \quad |P_2 - a_k| \leqslant |-\partial| \left[|a_{k+1}| + |a_{k+2}(-\partial)| + |a_{k+3}(-\partial)^2| + \ldots\ldots \right]$

$\therefore \quad |P_2 - a_k| < \partial \left[|a_{k+1}| + m |a_{k+2}| + m^2 |a_{k+3}| + \ldots\ldots\ldots \right]$

$\therefore \quad |P_2 - a_k| < |a_k| \quad \ldots\ldots\ldots\ldots\ldots \text{(2)}$

If $a_k > o$, (2) implies that

$$|P_2 - a_k| < a_k$$

$\therefore \quad -a_k < P_2 - a_k < a_k$

$\therefore \quad o < P_2 < 2\,a_k$

i.e. $\quad P_2 > o$

If $\quad a_k < o$, (2) implies that

$$|P_2 - a_k| < -a_k$$

$\therefore \quad -(-a_k) < P_2 - a_k < -a_k$

$\therefore \quad 2\,a_k < P_2 < o$

i.e. $\quad P_2 < o$

<u>Note</u> By this extension the theorems II.6 and II.7 and the corollary II.8 are automatically extended to be valid even for the infinite polynomials which represent most rational functions and transcendental functions.

<u>Example II.10</u>

Prove that the function $f(x) = e^x - x$ is minimum at $x = o$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots\dots\dots\dots\dots$$

$$\therefore \quad f(x) = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots\dots\dots\dots\dots$$

for $o \leqslant x \leqslant 1$;

$$1 = 1$$

$$\frac{x^2}{2!} \leqslant \frac{1}{2!}$$

$$\frac{x^3}{3!} \leqslant \frac{1}{3!}$$

. . . .

$$\frac{x^n}{n!} \leqslant \frac{1}{n!}$$

. . . .

and the series $1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots\ldots + \frac{1}{n!} + \ldots\ldots\ldots\ldots$

converges to $e - 1$

$\therefore$ By Weierstrass **M-test**

the series $1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots$

is absolutely convergent for $o \leqslant x \leqslant 1$

$\therefore$ By (the extended) theorem II-7, since the coefficient of x^2

is positive $f(x)$ is minimum at $x = o$.