CHAPTER III

THE PROOF OF MAXWELL'S EQUATIONS FOR NON-ZERO CHARGE DENSITIES AND UNIFORM CURRENT DENSITIES.

Maxwell's equations involving charge and current densities ρ and j respectively are

$$\nabla \cdot \vec{E} = 4\pi\rho \quad .$$

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi\vec{J}}{c} \quad .$$

<u>Lemma</u>

<u>To prove</u> $\iint_S \frac{\vec{n}\cdot\vec{r}\,(1 - \frac{v^2}{c^2})\,dS}{s^3} = 4\pi$, where $\vec{r}$ is the position vector (x,y,z) , S is a closed surface that encloses the origin O, and $\vec{n}$ is the unit vector normal to the surface element of area dS directed outwards.

<u>Proof</u> Surround O by a small sphere s' of radius a that lies within S. Let τ denote the region bounded by S and s'. Then by the divergence theorem

$$\iint_S \frac{\vec{n}\cdot\vec{r}\,(1 - \frac{v^2}{c^2})\,dS}{s^3} - \iint_{s'} \frac{\vec{n}\cdot\vec{r}\,(1 - \frac{v^2}{c^2})\,dS}{s^3}$$

$$= \iiint_\tau \frac{\nabla\cdot\vec{r}\,(1 - \frac{v^2}{c^2})\,dv}{s^3}$$

$$= 0 ,$$

since $\frac{\nabla\cdot\vec{r}\,(1 - \frac{v^2}{c^2})}{s^3} = 0$ as already proved.

Since $r \neq 0$, $s \neq 0$, it follows that

$$\iint_S \frac{\vec{n} \cdot \vec{r}\,(1 - \frac{v^2}{c^2})\; dS}{s^3} = \iint_{s'} \frac{\vec{n} \cdot \vec{r}\,(1 - \frac{v^2}{c^2})\; dS}{s^3}$$

Now on s', $r = a$, $\vec{n} = \frac{\vec{r}}{a}$, and we have

$$\iint_{s'} \frac{\vec{n} \cdot \vec{r}\,(1 - \frac{v^2}{c^2})\; dS}{r^3 \left\{1 - \frac{v^2}{c^2}\left(\frac{y^2 + z^2}{r^2}\right)\right\}^{3/2}} = \iint_{s'} \frac{a^2(1 - \frac{v^2}{c^2})\; dS}{a^4 \left\{1 - \frac{v^2}{c^2} + \frac{v^2 x^2}{c^2 a^2}\right\}^{3/2}}$$

$$= \int_{\theta=0}^{\pi} \frac{(1 - \frac{v^2}{c^2})\, 2\pi a^2 \sin\theta \; d\theta}{a^2 \left\{1 - \frac{v^2}{c^2} + \frac{v^2 a^2 \cos^2\theta}{c^2 a^2}\right\}^{3/2}}$$

$$= 4\pi .$$

Hence $$\iint_S \frac{\vec{n} \cdot \vec{r}\;(1 - \frac{v^2}{c^2})\; dS}{s^3} = 4\pi .$$

<u>To prove</u> $\nabla \cdot \vec{E} = 4\pi\rho$

<u>Proof</u> Let the Surface S enclose a charge q. Then

$$\iint_S \vec{E} \cdot d\vec{S} = \iint_S \frac{q\;(1 - \frac{v^2}{c^2})\; \vec{r} \cdot d\vec{S}}{s^3}$$

$= 4\pi q$, by the lemma above,

$= \iiint_V \nabla \cdot \vec{E}\; dv$, by the divergence theorem.

But the total charge in the volume enclosed by S is

$$q = \iiint_V \rho \; dv$$

Since $\iint_S \vec{E} \cdot d\vec{S} = 4\pi q$, already proved, we have

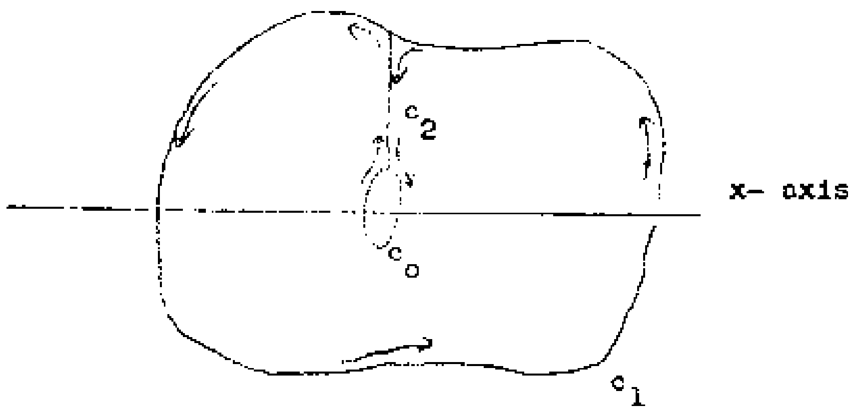
$$\therefore \iiint_v \nabla \cdot \vec{E} \, dv = 4\pi \iiint_v \rho \, dv \text{ , which implies that}$$

$$\nabla \cdot \vec{E} = 4\pi\rho$$

....

<u>Lemma</u> The circulation integral of the magnetic field $\vec{B}$ around any closed curve c_1 that goes round the x - axis once due to a steady current on the x-axis is equal to the circulation integral around a small circle c_o of radius a which is perpendicular to the x-axis, and has its centre on the x-axis.

<u>Proof</u>



Construct a cross-cut c_2 connecting the closed curve c_1 and the small circle c_o. Then the circulation integral of the magnetic field $\vec{B}$ around the simple closed curve c consisting of $c_2, c_o, -c_2$ and $-c_1$ as shown in the figure is as follows

$$\int_c \vec{B} . d\vec{S} = \int_{c_2} \vec{B} \cdot d\vec{S} + \int_{c_o} \vec{B} . d\vec{S} - \int_{c_2} \vec{B} . d\vec{S} - \int_{c_1} \vec{B} . d\vec{S} . \qquad .....(1)$$

By Stokes' theorem

$$\int_c \vec{B} \cdot d\vec{S} = \iint_S (\nabla \times \vec{B}) \cdot d\vec{a} , \qquad \ldots\ldots\ldots(2)$$

where S is a finite surface bounded by c.

S may be chosen so as not to cut the x-axis.

Since the current on the x-axis is steady the electric field $\vec{E}$ at all points in space off the x-axis is constant. Therefore by equation II, chapter II, $\nabla \times \vec{B} = 0$ on S . Therefore by (2) the circulation integral of $\vec{B}$ over c is zero, and by (1)

$$\int_{c_1} \vec{B} \cdot d\vec{S} = \int_{c_o} \vec{B} \cdot d\vec{S}$$

To prove $\quad \nabla \times \vec{B} = \dfrac{4\pi \vec{J}}{c}$

Proof Let λ be the charge per unit length moving on the x-axis. Then the magnetic field at (x,y,z) in space due to the small element of charge $\lambda d\bar{x}$ at $\bar{x}$ is

$$dB_x = 0$$

$$dB_y = \frac{-\lambda\, d\bar{x}}{s^3} \left(1 - \frac{v^2}{c^2}\right) \frac{v}{c} z, \text{ where } s^2 = \left\{ (x - \bar{x})^2 + \left(1 - \frac{v^2}{c^2}\right)(y^2 + z^2) \right\} .$$

$$dB_z = \frac{\lambda\, d\bar{x}}{s^3} \left(1 - \frac{v^2}{c^2}\right) \frac{v}{c} y .$$

The total field at (x,y,z) is

$$B_x = \int_{\bar{x}=-\infty}^{\infty} dB_x , \quad B_y = \int_{\bar{x}=-\infty}^{\infty} dB_y , \quad B_z = \int_{\bar{x}=-\infty}^{\infty} dB_z .$$

We now have

$$B_x = 0$$

and $$B_y = -\int_{\bar{x}=-\infty}^{\infty} \frac{\lambda d\bar{x}}{s^3}\left(1-\frac{v^2}{c^2}\right)\frac{v}{c}z$$

$$= -\lambda\left(1-\frac{v^2}{c^2}\right)\frac{v}{c}\, z\int_{\bar{x}=-\infty}^{\infty}\frac{d\bar{x}}{\left[(x-\bar{x})^2+\left(1-\frac{v^2}{c^2}\right)(y^2+z^2)\right]^{3/2}}$$

$$= \frac{-2\lambda vz}{c(y^2+z^2)} \quad .$$

Similarly $$B_z = \int_{\bar{x}=-\infty}^{\infty}\frac{\lambda d\bar{x}}{s^3}\left(1-\frac{v^2}{c^2}\right)\frac{v}{c}\cdot y$$

$$= \frac{2\lambda vy}{c(y^2+z^2)} \quad .$$

$$\int_{c_1} \vec{B}\cdot d\vec{S} = \int_{c_o} \vec{B}\cdot d\vec{S} \quad \text{by the preceding lemma}$$

$$= \int_{c_o} B_y\, dy + \int_{c_o} B_z dz$$

$$= \int_{c_o}\frac{-2\lambda vz\ \ dy}{c\ (y^2+z^2)} + \int_{c_o}\frac{2\lambda vy\ \ dz}{c\ (y^2+z^2)}$$

$$= \int_{\theta=0}^{2\pi}\frac{-2\lambda va\sin\theta\ (-a)\ \sin\theta\ d\theta}{ca^2} + \int_{\theta=0}^{2\pi}\frac{2\lambda va\cos\theta\ a\cos\theta\ d\theta}{ca^2}$$

$$= \frac{4\pi\lambda v}{c} \quad .$$

We wish to calculate $\nabla \times \vec{B}$ in terms of the current density. Suppose the space is filled with uniform charge density ρ moving with the same velocity $\vec{v}$ in the x-direction.

The circulation integral around any curve c_1 is equal to $\frac{4\pi i}{c}$, where i is the current through c_1 given by

$$i = \rho\vec{v} \cdot (\text{area of } c_1 \text{ projected on a plane perpendicular to } \vec{v})$$

$$= \rho\vec{v}A \text{ , say.}$$

Then

$$\nabla \times \vec{B} = \lim_{A \to 0} \frac{\text{circulation integral}}{A}$$

$$= \lim_{A \to 0} \frac{\frac{4\pi\rho\vec{v}}{c} \cdot A}{A}$$

$$= \frac{4\pi\vec{j}}{c} \text{ , where } \vec{j} \text{ is the current density.}$$

<u>The equation $\nabla \times \vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi\vec{j}}{c}$</u>

The curl of magnetic field at any point in space is divided to two parts. One is the effect of the changing of the electric field and the other one is the effect of the current density. So that we have

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi\vec{j}}{c} .$$

We have proved it here only for a uniform steady current throughout space and a changing electric field $\vec{E}$ due to other charge movements at distinct points. It can be proved that the same equation holds for non-uniform steady currents.