CHAPTER 2

THE SHORTEST ARC JOINING TWO POINTS.

If the points (x_o, y_o) and (x_n, y_n) are two given points, and $y = y(x)$ is a curve joining them, then the length of the curve is given by

$$I = \int_{x_o}^{x_n} \sqrt{1 + (y')^2}\, dx. \qquad \text{.........} \ (2.1)$$

We want to find the curve $y = y(x)$ which minimizes the integral I. Generally the functional I depends on the argument function $y = y(x)$. In the direct method for solving this problem, we consider the variation of I not along arbitrary curves $y = y(x)$, but along polygonal curves. In other words the functional I depends on the argument function whose graphs are polygons having vertices on the lines $x = x_o$, $x = x_1, \ldots$ $\ldots, x = x_n$, where $x_i = x_o + i\,\Delta x$, $\Delta x = \frac{x_n - x_o}{n}$, $i = 0, 1, \ldots$ $\ldots, n - 1$. The value of the integral I may then be written in the form

$$I = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \sqrt{1 + \left(\frac{y_{i+1} - y_i}{\Delta x}\right)^2}\, dx$$

$$= \sum_{i=0}^{n-1} \sqrt{1 + \left(\frac{y_{i+1} - y_i}{\Delta x}\right)^2} \cdot \Delta x \qquad \text{.........} \ (2.2)$$

where y' is the slope of the straight line segments joining the consecutive vertices of the polygon, that is $y' = \frac{y_{i+1} - y_i}{\Delta x}$.

Initially we choose the polygon P^o which has as the ordinates of its vertices, y_o^o, y_1^o, ... y_{n-1}^o, y_n^o. Let the value of the integral I along P^o be I_o. We shall try to construct a polygon P^1 which makes the corresponding integral I_1 less than I_o. Then we try to construct the polygon P^2 that makes $I_2 < I_1$, and so on.

By this method we will obtain a sequence of polygonal curves to which correspond values of the integral in a monotonic decreasing sequence I_o, I_1, ... I_k .... which is bounded below by zero. Then the sequence must converge to its greatest lower bound.* In other words we may say that the sequence of polygons converges to the polygon that makes the integral I a minimum.

Then as $n \longrightarrow \infty$ and $\Delta x \longrightarrow 0$ the limiting polygon will approach the smooth curve which we require.

Lemma 1

Let P_o (x_o, y_o) and P_2 (x_2, y_2) be two points, and let the interval $[x_o, x_2]$ be divided into two equal parts at x_1. If P_1 (x_1, y_1) is an arbitrary point on the line $x = x_1$ then the distance $P_oP_1 + P_1P_2$ is shortest

when $$y_1 = \frac{1}{2}(y_o + y_2) .$$ (see Fig. 1)

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* Mathematical Analysis
by Tom, M Apostol
Addison Wesley publication company Inc. 1957.
page 355

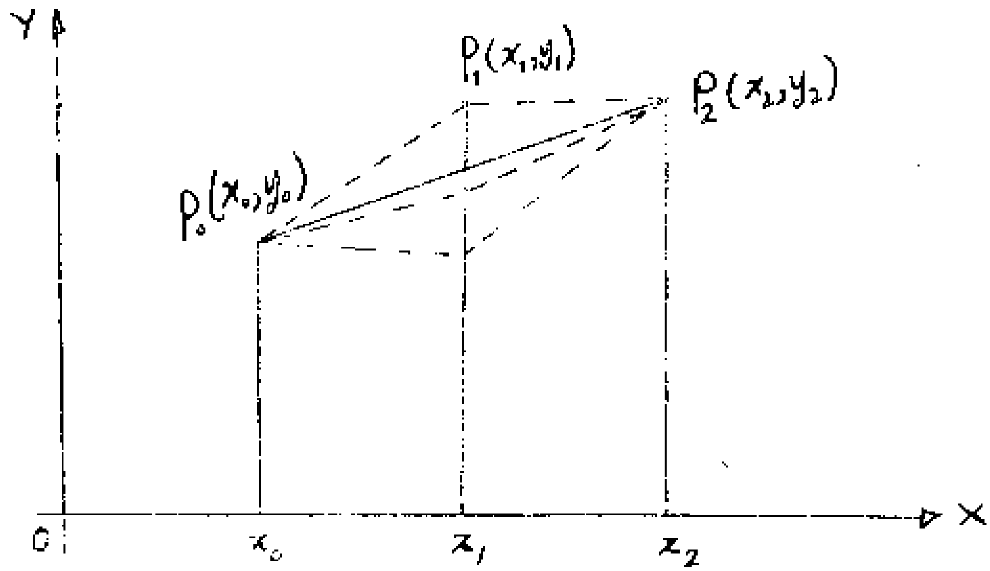


Fig. 1 Figure of lemma 1.

Proof The distance to be considered is

$$I = P_0P_1 + P_1P_2$$

$$= \sqrt{1 + \left(\frac{y_1 - y_0}{\Delta x}\right)^2} \cdot \Delta x + \sqrt{1 + \left(\frac{y_2 - y_1}{\Delta x}\right)^2} \cdot \Delta x$$

Then
$$\frac{dI}{dy_1} = \frac{(y_1 - y_0)}{\left((\Delta x)^2 + (y_1 - y_0)^2\right)^{\frac{1}{2}}} - \frac{(y_2 - y_1)}{\left((\Delta x)^2 + (y_2 - y_1)^2\right)^{\frac{1}{2}}}$$

For I to be the minimum, we require .

$$\frac{dI}{dy_1} = 0 \quad \text{which gives}$$

$$\frac{(y_1 - y_0)}{\left((\Delta x)^2 + (y_1 - y_0)^2\right)^{\frac{1}{2}}} = \frac{(y_2 - y_1)}{\left((\Delta x)^2 + (y_2 - y_1)^2\right)^{\frac{1}{2}}}$$

and hence

$$y_1 = \frac{1}{2}(y_0 + y_2) .$$

<u>Lemma 2</u> . Let P^o be the initial polygon joing the points (x_o , y_o) and (x_n , y_n) , and let the abscissae $x_o, x_1, \ldots\ldots x_n$ of the vertices be such that $x_i = x_o + i \Delta x$ and $\Delta x = \frac{x_n - x_o}{n}$. Let the points P_i^o (x_i, y_i^o) be the vertices of the polygonal curve P^o. Denote by $P_i^oP_j^o$ the distance between P_i^o and P_j^o and let $I_o = P_o^o P_1^o + P_1^o P_2^o + \ldots\ldots P_{n-1}^o P_n^o$. Then construct the first polygon P^1 with vertices denoted by P_i^1 (x_i, y_i^1) by means of the formula $y_i^1 = \frac{1}{2} (y_{i-1}^1 + y_{i+1}^o)$ $i = 1, 2, \ldots\ldots n - 1,$ and $y_o^1 = y_o^o$, $y_n^1 = y_n^o$. Let $I_1 = P_o^1 P_1^1 + P_1^1 P_2^1 + \ldots\ldots\ldots + P_{n-1}^1 P_n^1$. By the same method construct the polygons $P^2, P^3 \ldots\ldots P^k \ldots\ldots$ where if P_i^k (x_i, y_i^k) are the vertices of the polygon P^k, then $y_i^k = \frac{1}{2} (y_{i-1}^k + y_{i+1}^{k-1})$, $y_o^k = y_o$, $y_n^k = y_n$, and

$$I_k = P_o^k P_1^k + P_1^k P_2^k + \ldots\ldots P_{n-1}^k P_n^k , \text{ and so on.}$$

Then $I_o, I_1, I_2, \ldots\ldots I_k \ldots\ldots$ which is the value of the integral I in (2.2) along the polygon $P^o, P^1, \ldots\ldots P^k \ldots\ldots$ respectively is a monotonic decreasing sequence. (see Fig. 2)

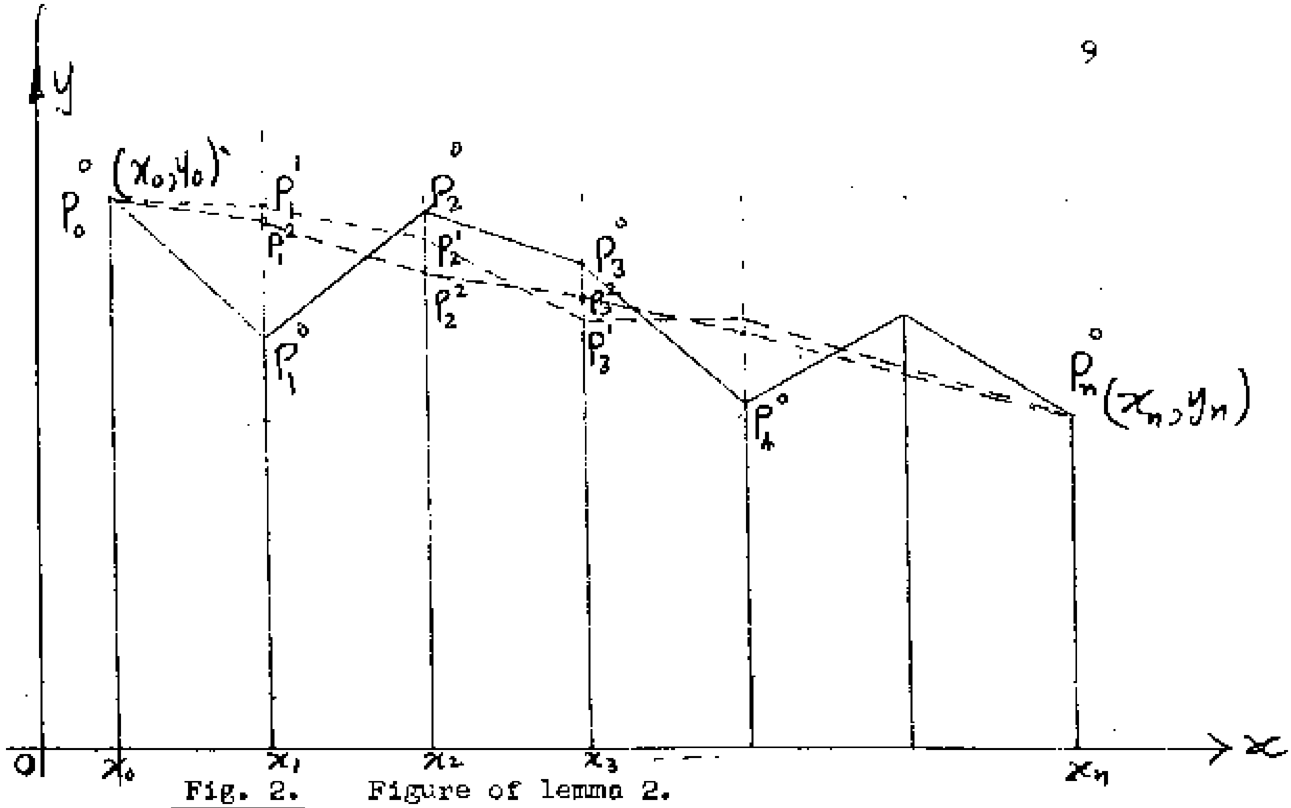


Fig. 2. Figure of lemma 2.

To proof that $I_1 < I_0$.

Let $P_i^k \, P_j^l$ be the distance between P_i^k and P_j^l.

Since $\quad P_0^0 P_1^1 + P_1^1 P_2^0 \leq P_0^0 P_1^0 + P_1^0 P_2^0$ (by lemma 1) ....(1)

and $\quad P_1^1 P_2^1 + P_2^1 P_3^0 \leq P_1^1 P_2^0 + P_2^0 P_3^0$, (by lemma 1)

we have $\quad P_1^1 P_2^1 + P_2^1 P_3^0 - P_1^1 P_2^0 \leq P_2^0 P_3^0$ ....(2)

then from (1) and (2)

$$P_0^0 P_1^1 + P_1^1 P_2^1 + P_2^1 P_3^0 \leq P_0^0 P_1^0 + P_1^0 P_2^0 + P_2^0 P_3^0 \qquad \text{..........(3)}$$

Similary

$P_1^0 P_1^1 + P_1^1 P_2^1 + \ldots\ldots + P_{n-2}^1 P_{n-1}^0 \leq P_0^0 P_1^0 + P_1^0 P_2^0 + \ldots P_{n-2}^0 P_{n-1}^0$ ....(4)

and $P_{n-2}^1 P_{n-1}^1 + P_{n-1}^1 P_n^0 \leq P_{n-2}^1 P_{n-1}^0 + P_{n-1}^0 P_n^0$

Therefore $P^1_{n-2} P^1_{n-1} + P^1_{n-1} P^o_n - P^1_{n-2} P^o_{n-1} \leqq P^o_{n-1} P^o_n$ .....(5)

Then from (4) and (5)

$$P^o_o P^1_1 + P^1_1 P^1_2 + \dots + P^1_{n-1} P^o_n \leqq P^o_o P^o_1 + P^o_1 P^o_2 + \dots + P^o_{n-1} P^o_n \quad \dots (6)$$

Since $P^k_o = P^o_o$ and $P^k_n = P^o_n$ for all k, by putting $k = 1$ we obtain from (6) the relation

$$P^1_o P^1_1 + P^1_1 P^1_2 + \dots + P^1_{n-1} P^1_n \leqq P^o_o P^o_1 + P^o_1 P^o_2 + \dots + P^o_{n-1} P^o_n$$

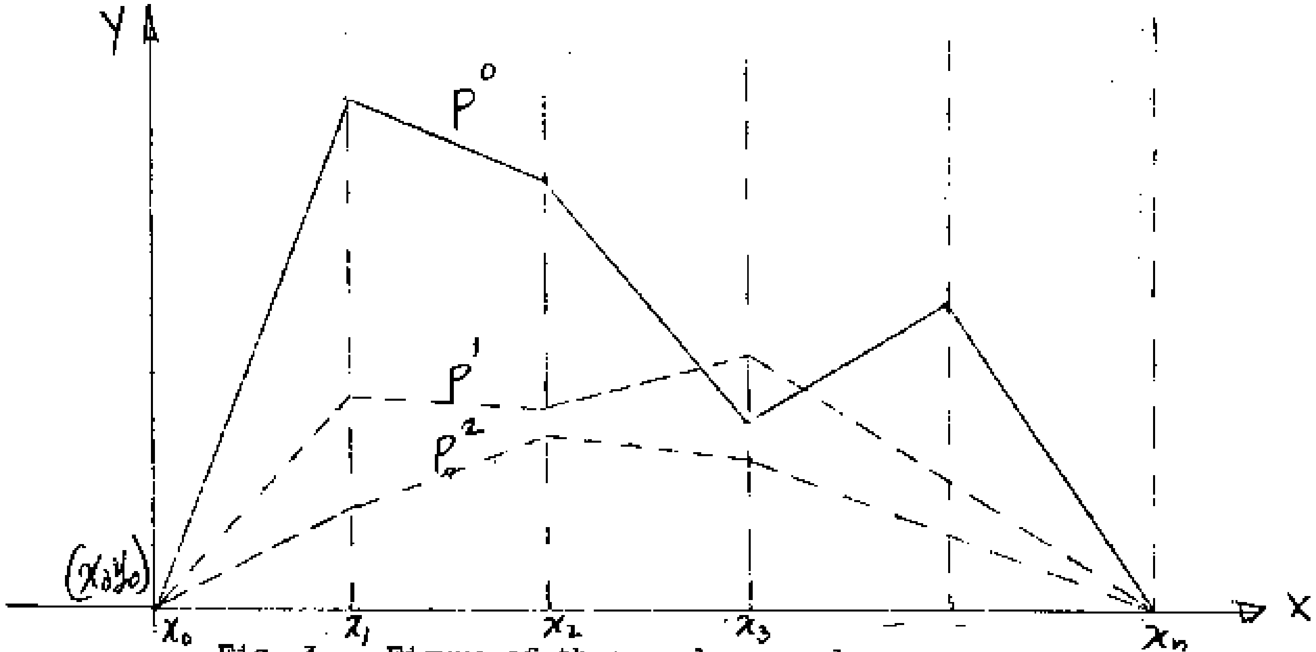
that is $I_1 \leqq I_o$.

But the equality occurs only when $y^o_i = y^1_i$ for all i, that is only if the polygon P^o is the straight line. If the polygon P^o is not the straight line, then $I_1 < I_o$. We may prove in the same way that $I_2 < I_1$, and $I_3 < I_2$, and so on. That is the sequence $I_o, I_1, I_2, \dots I_k, \dots$ is a monotomic decreasing sequence.

<u>Theorem 1</u> The monotonic decreasing sequence I_k defined in lemma 2 converges to the straight line joining $P_0^0 (x_0, y_0^0)$ and $P_n^0(x_n, y_n^0)$.

<u>Proof</u>

Case 1 Assume that P_0^0, P_n^0 are on the x-axis, and suppose (I) that $y_i^0 \geqslant 0$ for all i. (see Fig. 3)



<u>Fig. 3.</u> Figure of theorem 1, case 1.

Since P_0^0, P_n^0 are on the x-axis, we have $y_0^k = y_n^k = 0$ for all k. Let y_i^k be the ordinate of the i-th vertex of the polygon P^k constructed as in lemma 2. Let the initial polygon P^0 has vertices $P_i^0 (x_i, y_i^0)$ with $y_i^0 \geqslant 0$ for all i = 0, 1, ... n. We shall prove that $y_i^k \geqslant 0$ for all i and for all k.

Since $y_i^0 \geqslant 0$ and $y_0^1 \geqslant 0$

We have $y_1^1 = \frac{1}{2}(y_0^1 + y_2^0) \geq 0$,

$y_2^1 = \frac{1}{2}(y_1^1 + y_3^0) \geq 0$, and so on.

Similary we can show that since $y_n^1 = 0$, we have $y_{n-1}^1 \geq 0$, and so on. Consequently $y_i^1 \geq 0$ for all i.

Again $y_0^2 = 0$

$y_1^2 = \frac{1}{2}(y_0^2 + y_2^1) \geq 0$

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$y_{n-1}^2 \geq 0$ and $y_n^2 = 0$

and it follows that $y_i^2 \geq 0$ for all i.

Since the argument may be repeated indefinitely, it follows that $y_i^k \geq 0$ for all i, and for all k. For given k, let Y^k be the maximum value of y_i^k and let α be the smallest value of i for which $y_i^k = Y^k$, then $y_{\alpha-1}^k < y_\alpha^k$.

We now want to show that $Y^{k+1} < Y^k$ for all k.

Since $y_i^k < Y^k$ for all $i < \alpha$

and $y_0^{k+1} = 0$, it follows that

$$y_1^{k+1} = \frac{1}{2}(y_0^{k+1} + y_2^k)$$

$$= \frac{1}{2}(y_2^k) < y_2^k < Y^k.$$

Now $y_2^{k+1} = \frac{1}{2}\left(y_1^{k+1} + y_3^k\right)$

$< \frac{1}{2}\left(y_2^k + y_3^k\right)$, (since $y_1^{k+1} < y_2^k$)

$< \frac{1}{2}(Y^k + Y^k)$

$< Y^k$,

and so on with the same result for $y_3^{k+1}, \ldots\ldots, y_{\alpha-2}^{k+1}$.

Next $y_{\alpha-1}^{k+1} = \frac{1}{2}\left(y_{\alpha-2}^{k+1} + y_{\alpha}^k\right)$

$< \frac{1}{2}(Y^k + Y^k)$

$< Y^k$

and $y_{\alpha}^{k+1} = \frac{1}{2}\left(y_{\alpha-1}^{k+1} + y_{\alpha+1}^k\right)$

$< \frac{1}{2}(Y^k + Y^k)$

$< Y^k$.

But $y_{\alpha+1}^k \leqslant Y^k$, $y_{\alpha+2}^k \leqslant Y^k, \ldots\ldots y_{n-1}^k \leqslant Y^k$ and so $y_i^{k+1} < Y^k$ for all $i = 0, 1, 2, \ldots\ldots n$. Let Y^{k+1} be the maximum value of y_i^{k+1} then $Y^{k+1} < Y^k$ for all k.

Hence the sequence $Y^0, Y^1, \ldots\ldots Y^k, Y^{k+1} \ldots\ldots\ldots$ is monotonic decreasing and is bounded below by Zero. Therefore the sequence has a limit Y. We need to show that $Y = 0$, that is to prove that $\forall\, \epsilon > 0\ \exists\ K$;

$$\forall k > K \qquad |Y^k - 0| < \varepsilon$$

or $\qquad |y_i^k - 0| < \varepsilon$ for all i. ......(1)

Proof

Suppose $Y \neq 0$, then (1) is false, that is $\exists\, \varepsilon > 0 :$

$$\forall K : \exists k > K : \exists i$$

$$|y_i^k - 0| \geqslant \varepsilon. \qquad .... (2)$$

It follows from (2) that $Y^k \geqslant \varepsilon$ (since Y^k is the maximum value of y_i^k). Let E denote the greatest lower bound of Y^k. Then

$$E = \lim_{k \to \infty} Y^k = Y.$$

Choose $\varepsilon_1 > E$, then there exists at least one k, say k', such that $\qquad E \leq Y^{k'} < \varepsilon_1$

that is $\qquad y_i^{k'} < \varepsilon_1$ for all i.

Hence $\qquad y_0^{k'+1} = 0,$

$$y_1^{k'+1} = \frac{1}{2}(y_0^{k'+1} + y_2^{k'})$$

$$= \frac{1}{2}(0 + y_2^{k'})$$

$$< \frac{1}{2}\varepsilon_1$$

$$y_2^{k'+1} = \frac{1}{2}(y_1^{k'+1} + y_3^{k'})$$

$$< \frac{1}{2}(\varepsilon_1 + \varepsilon_1) < \varepsilon_1$$

$$y_{n-1}^{k+1'} = \frac{1}{2}\left(y_{n-2}^{k+1'} + y_{n}^{k'}\right)$$

$$< \frac{1}{2}\left(\varepsilon_1 + \varepsilon_1\right)$$

$$< \varepsilon_1$$

$$y_{n}^{k+1'} = 0$$

$$< \varepsilon_1$$

Now Y^{k} and hence $Y^{k+1'}$ are functions of ε_1. (see Fig. 4)

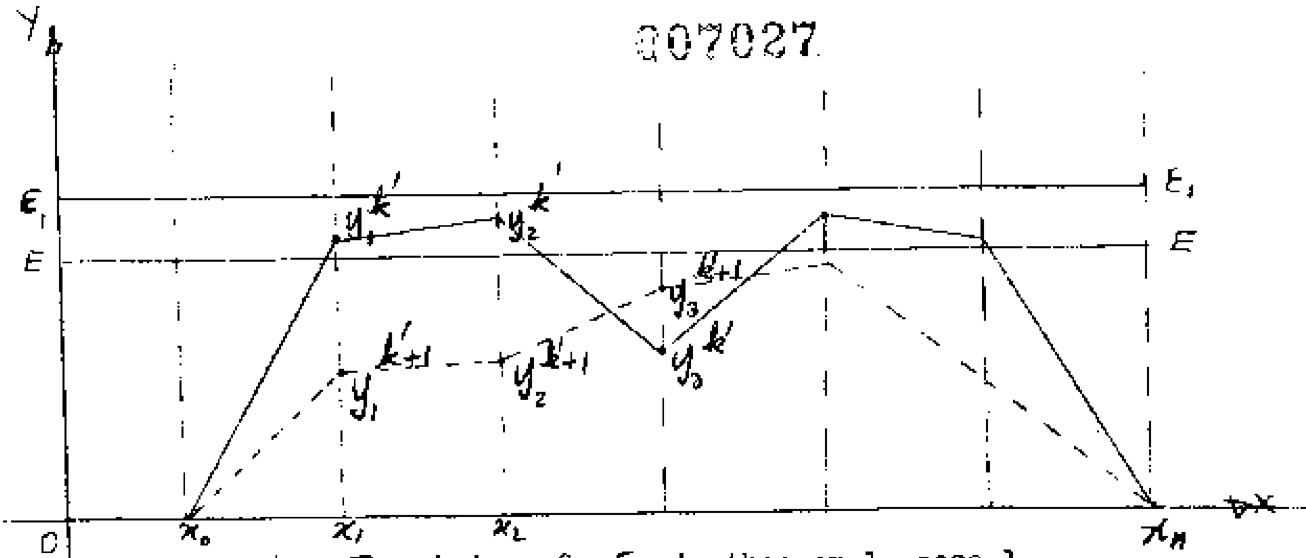

Fig. 4. The choice of ε_1 in theorem 1, case 1.

Choose ε_1 so that all $y_i^{k+1'}$ are less than E (see appendix (a)). This will make $Y^{k+1'} < Y^{k'}$ which contradicts the fact that Y^k is a monotonic decreasing sequence with limit Y.

Therefore the proof implies $Y = 0$ that is $\lim_{k\to\infty} y_i^k = 0$.

Now suppose (II) that $y_i^o \leqslant 0$ for all i. Then we may use $|y_i^o|$ instead of y_i^o, and the proof is similar to the case $y_i^o \geqslant 0$, but instead we finally obtain

$$\lim_{k\to\infty} |y_i^k| = 0, \text{ and hence}$$

$$\lim_{k\to\infty} y_i^k = 0.$$

<u>Case II</u> If P_o^o, P_n^o are on the x - axis and the y_i^o are positive, negative or zero, then we must prove that

$$\forall \varepsilon > 0 \ \exists K : \forall k > K$$

$$|y_i^k - 0| < \varepsilon \qquad \text{for all } i \text{ ....}$$

<u>Proof</u>

Let Y^k be the maximum value of $|y_i^k|$ for fixed k, and let α be the smallest value of i for which $|y_\alpha^k| = Y^k$. Then $|y_i^k| < |y_\alpha^k|$, for all $i < \alpha$.

We require to prove that

Y^o, Y^1, Y^2, ... Y^k .... is a monotonic decreasing sequence.

Since $|y_o^{k+1}| = 0$,

We have $\left| y_1^{k+1} \right| = \frac{1}{2} \left| y_0^{k+1} + y_2^k \right|$

$$= \frac{1}{2} \left| 0 + y_2^k \right|$$

$$= \frac{1}{2} \left| y_2^k \right|$$

$$< Y^k$$

Also $\left| y_2^{k+1} \right| = \frac{1}{2} \left| y_1^{k+1} + y_3^k \right|$

$$\leqslant \frac{1}{2} \left| y_1^{k+1} \right| + \frac{1}{2} \left| y_3^k \right|$$

$$< \frac{1}{2} Y^k + \frac{1}{2} Y^k$$

$$< Y^k ;$$

Similary $\left| y_3^{k+1} \right| < Y^k ,$

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$$\left| y_{\alpha-1}^{k+1} \right| < Y^k ,$$

$$\left| y_\alpha^{k+1} \right| = \frac{1}{2} \left| y_{\alpha-1}^{k+1} + y_{\alpha+1}^k \right|$$

$$\leq \frac{1}{2} \left| y_{\alpha-1}^{k+1} \right| + \frac{1}{2} \left| y_{\alpha+1}^k \right|$$

$$< \frac{1}{2} Y^k + \frac{1}{2} Y^k$$

$$< Y^k ,$$

But $\left| y^k_{\alpha+1} \right| \leq Y^k$, $\left| y^k_{\alpha+2} \right| \leq Y^k \ldots\ldots \left| y^k_{n-1} \right| \leq Y^k$,

and so $\left| y^{k+1}_i \right| < Y^k$ for all $i = 0, 1, \ldots\ldots n$. Let Y^{k+1} be the maximum value of $\left| y^{k+1}_i \right|$ then $Y^{k+1} < Y^k$, and $Y^o, Y^1, \ldots\ldots Y^k \ldots\ldots$ is a monotonic decreasing sequence bounded below by zero. It follows that $Y^o, Y^1, Y^2, \ldots\ldots Y^k$ has a limit Y.

We must prove that $Y = 0$. That is we must prove that

$$\forall \varepsilon > 0 : \exists K : \forall k > K, \left| Y^k - 0 \right| < \varepsilon, \text{ or}$$

$$\left| y^k_i - 0 \right| < \varepsilon \quad \text{for all } i \qquad \ldots\ldots\ldots\ldots (1)$$

<u>Proof</u> Suppose Y is not equal to zero. Then (1) is false that is

$$\exists \varepsilon > 0 : \forall K : \exists k > K : \exists i : \left| y^k_i - 0 \right| \geq \varepsilon . \qquad \ldots\ldots (2)$$

It follows from (2) that $Y^k \geq \varepsilon$. (since Y^k is the maximum value of y^k_i).

Let E denote the greatest lower bound of Y^k that satisfies (2).

Then

$$E = \lim_{k \to \infty} Y^k = Y$$

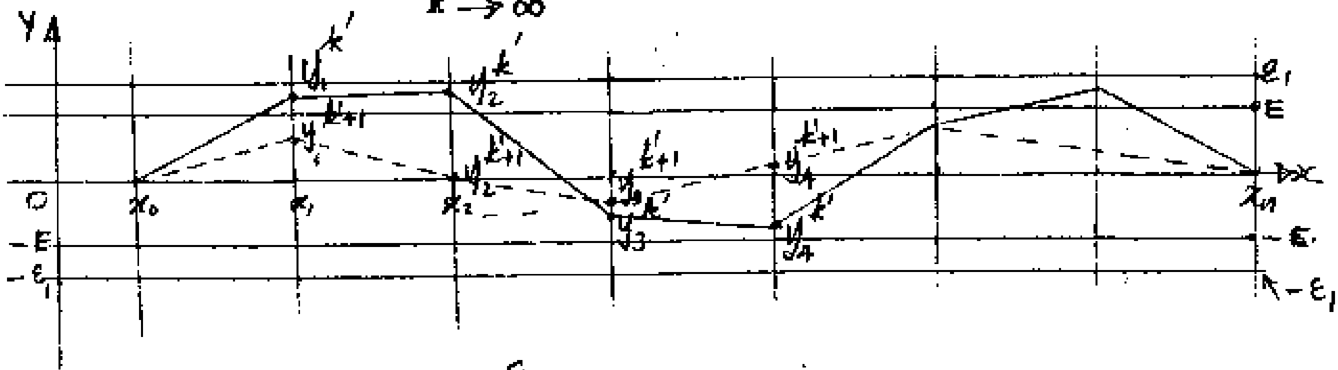

Fig. 5. The choice of ε_1 in theorem 1, case 2.

Choose $\varepsilon_1 > E$, then there exists at least one k, say k', such that $E \leqslant y^{k'} < \varepsilon_1$, that is

$$\left| y_i^{k'} \right| < \varepsilon_1 \quad \text{for all } i.$$

Since $\left| y_0^{k'+1} \right| = 0$

We have
$$\left| y_1^{k'+1} \right| = \frac{1}{2} \left| y_0^{k'+1} + y_2^{k'} \right|$$
$$\leqslant \frac{1}{2} \left| y_0^{k'+1} \right| + \frac{1}{2} \left| y_2^{k'} \right|$$
$$< \varepsilon_1 .$$

Also
$$\left| y_2^{k'+1} \right| = \frac{1}{2} \left| y_1^{k'+1} + y_3^{k'} \right|$$
$$\leqslant \frac{1}{2} \left| y_1^{k'+1} \right| + \frac{1}{2} \left| y_3^{k'} \right|$$
$$< \frac{1}{2} (\varepsilon_1 + \varepsilon_1)$$
$$< \varepsilon_1 ,$$

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$$\left| y_{n-1}^{k'+1} \right| < \varepsilon_1$$
$$\left| y_n^{k'+1} \right| = 0$$

Let Y'^{k+1} be the maximum value of $\left| y'^{k+1}_i \right|$ for all i, then

$$Y'^{k+1} < \mathcal{E}_1$$

Therefore Y'^k and hence Y'^{k+1} are functions of $\mathcal{E}_1$ (see Fig.5)

Choose $\mathcal{E}_1$ so that all $\left| y'^{k+1}_i \right|$ are less than E. This will make $Y'^{k+1} < Y'^k$, which contradicts the fact that Y^k is a monotonic decreasing with limit Y.

Therefore $\quad Y = 0.$

That is $\quad \lim\limits_{k \to \infty} y^k_i = 0.$

Case III Let $P^o_o\ (x_o, y^o_o)$ and $P^o_n\ (x_n, y^o_n)$ be any two points in the xy – plane, and let y^o_i be the ordinates of the vertices of the polygon P^o at x_i.

Denote by (x , y^*) the points on the straight line joining P^o_o and P^o_n. (see Fig. 6)

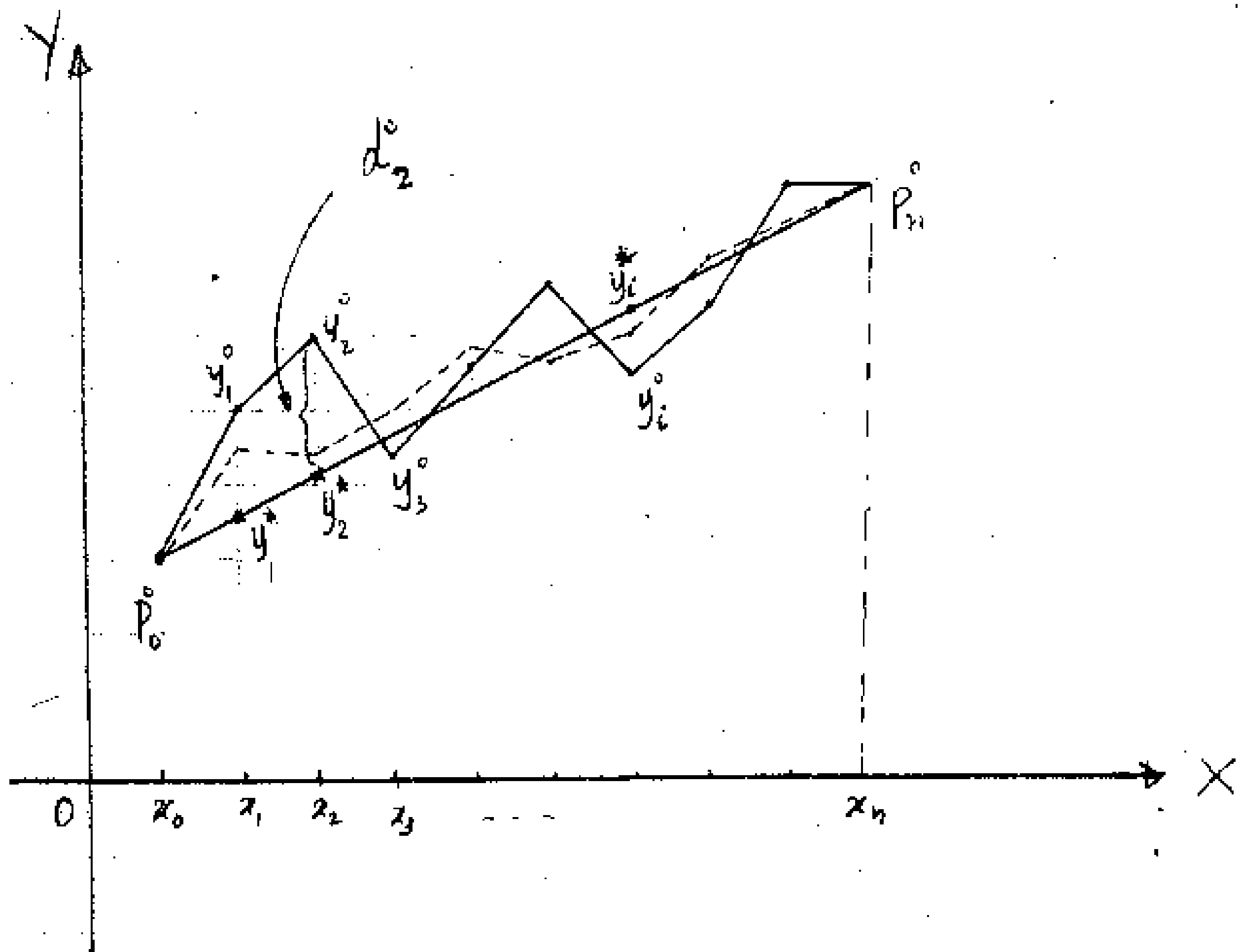

Fig. 6, figure of Theorem I, case 3.

Then $P_o^o P_n^o$ is the line

$$y^* - y_o^o = \frac{y_n^o - y_o^o}{x_n - x_o} (x - x_o) ,$$

and $$y_i^* - y_o = \frac{y_n^o - y_o^o}{x_n - x_o} (x_i - x_o) .$$

Let $d_i^o = y_i^o - y_i^*$.

Similarly if y_i^k be the ordinates of vertices of P^k at x_i.

Let $d_i^k = y_i^k - y_i^*$.

Now Lemma 2 holds when y_o^o , y_1^o , y_2^o are replaced by d_o^o , d_1^o , d_2^o . We may therefore apply case II to $\left| d_i^k \right|$, and we will get

$$\lim_{k \to \infty} \left| d_i^k \right| = 0 \quad \text{for all } i,$$

or $$\lim_{k \to \infty} y_i^k - y_i^* = 0 ,$$

$$\lim_{k \to \infty} y_i^k = y_i^* .$$

From the proof of the case I, II and III, we conclude that the polygonal curve that make the integral I in (2.2) a minimum is a straight line joining two given points.

Example 1.

To find a curve $y(x)$, where $y(0) = 0 \quad y(7) = 0$, that makes the integral

$$I = \int_0^7 \sqrt{1 + (y')^2}\, dx \text{ , a minimum.}$$

Choose the initial arbitrary polygonal curve P^0 with vertices $(0, 0)$, $(1, -1)$, $(2, 1)$, $(3, 4)$, $(4, 5)$, $(5, 3)$, $(6, 4)$, $(7, 0)$. Construct the polygons P^k, $k = 1, 2, \ldots$ by the method discribed in chapter 2 ,

Thus $$y_1^1 = \frac{1}{2}\left(y_0^1 + y_2^0\right) = \frac{1}{2}(0 + 1.0)$$

$$= 0.5$$

$$y_2^1 = \frac{1}{2}\left(y_1^1 + y_3^0\right) = \frac{1}{2}(0.5 + 4.0)$$

$$= 2.25 ,$$

and so on. The results are given in table 1 and illustrated in figure 7. Table 1 also shows the value of I_k .

Table 1 Example 1, Values of y_i^k .

k \ i	0	1	2	3	4	5	6	7	I_k
0	0	-1.00	1.00	4.00	5.00	3.00	4.00	0	15.178
1	0	.5	2.25	3.68	3.34	3.67	1.84	0	11.155
2	0	1.13	2.41	2.88	3.28	2.56	1.28	0	9.211
3	0	1.21	2.05	2.67	2.98	2.13	1.07	0	9.026
4	0	1.03	1.85	2.42	2.28	1.68	0.84	0	8.664
5	0	0.93	1.68	1.98	1.83	1.34	0.67	0	8.367
6	0	0.84	1.41	1.62	1.48	1.08	0.54	0	7.835
7	0	0.17	1.17	1.33	1.21	0.88	0.44	0	7.605
8	0	0.59	0.96	1.09	0.94	0.69	0.35	0	7.393
9	0	0.48	0.79	0.87	0.78	0.57	0.29	0	7.248
10	0	0.40	0.64	0.71	0.64	0.47	0.24	0	7.1855
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∞	0	0	0	0	0	0	0	0	7.00

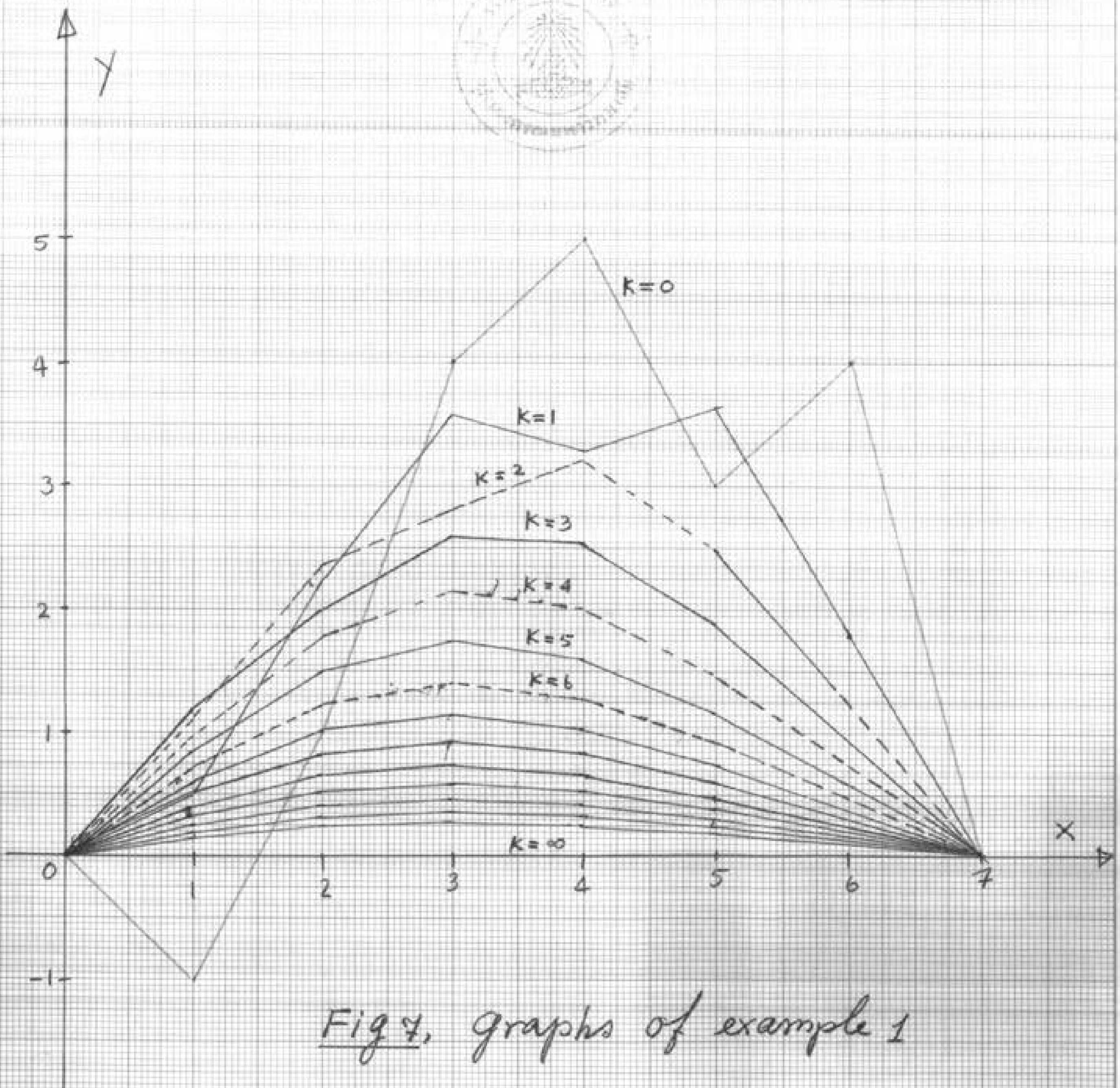

Fig 7, graphs of example 1