CHAPTER I

INTRODUCTION

Functional

The functionals discussed in this thesis are mappings from certain sets of function y(x) of a single real value x into the set of real numbers.

For instance, the length L of a curve joining two given points (x_0, y_0) and (x_1, y_1) on the plane given by

$$L = \int_{x_0}^{x_1} \sqrt{1 + (y')^2} \; dx$$

is a functional , because this length L is fully determined by the function y = y(x) which specifies the curves passing through the given points. The domain of a functional is a set of admissible functions y(x) which are called the argument functions.

The calculus of variations arises from the problem of finding maximum and minimum value of functionals. The name " The calculus of variations " was adopted as a result of notions which were introduced by Lagrange about the year 1760.* In order to compare the value of the integral

$$I(y) = \int_{x_1}^{x_2} F(x, y, y') \, dx$$ along an arbitrary arc y(x) with its value along a neighboring curve, he altered the function y(x)

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* Lectures on the calculus of variation, by G.A. Bliss.
The university of Chicago, Press 1964.
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by the addition of an increment $\epsilon\eta(x)$ which is called variation of function $y(x)$.

The following three problems had considerable influence on the development of the calculus of variations.

(1) The shortest arc joining two points.

The problem of determining the shortest arc joining two points furnishes a useful elementary introduction to the theory of the calculus of variations, because it is the simplest; the method of solution may be applied to many other problems.

If (x_1, y_1) and (x_2, y_2) are two given points, we want to find the curve $y = y(x)$ joining them which makes the integral

$$I = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx$$

a minimum.

(2) Surface of revolution of minimum area.

If two fixed points (x_1, y_1) and (x_2, y_2) are given we seek the arc $y = y(x)$ passing through them, whose rotation about the x-axis generates a surface with minimum area of revolution in the interval $x_1 \leq x \leq x_2$. We assume that $y_1 > 0$ and $y_2 > 0$ and that $y(x) \geqslant 0$ for $x_1 \leq x \leq x_2$. Thus we seek to find $y = y(x)$, which subject to the above restrictions minimized the integral

$$I = 2\pi \int_{x_1}^{x_2} y \sqrt{1 + (y')^2}\, dx.$$

(3) Brachistochrone Problems.

In June 1696, Johann Bernoulli set the following problem Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the vertical plane not lying on the same vertical line, find the path along which a particle M moves without friction in the shorest posible time from A to B under gravity acting in the y direction , (The term Brachisto-chrone derives from Greek brachistose, shortest, and Chronose, time)*. We suppose the y-axis directed vertically downward and the x-axis horizontal, so that the passage from A to B is marked by an increase in x. The initial velocity of the mass M at the point A is zero. After falling a distance y the particle has velocity $\sqrt{2gy}$ according to elementary Mechanics, where g is the acceleration due to gravity; It follows that the time of transit is given by the integral

$$I = \int_{x_1}^{x_2} \sqrt{\frac{1 + (y')^2}{2gy}} \, dx.$$

We want to find the function y = y(x) that makes the integral I a minimum.

There are many other problems in the Calculus of Variations, for example, the problem of finding Geodesics, the Isoperimetric problems, etc.

All of these problems were solved completely by Lagrange and Euler. The method used to solve them depends on the important equation.

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0,$$

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* Calculus of variations, by Robert Weinstock,

Mcgraw-Hill, 1952.

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which is called the Euler - Lagrange differential equation. This equation is often used in Mechanical physics.

This thesis contains an account of certain direct methods, different from the Euler method, for solving these problems. The basic idea upon which these methods depend may be applied to other branchesof study dealing with Euler's equation. The direct methods discussed are used to obtain numerical solutions, and are so designed as to be convenient for work with electronic computing machines.