ทฤษฎีบทก่อกำเนิดสำหรับแอนไอโซทรอปีแบบมีประจุและสมการโทลแมน-ออพเพนไฮม์เมอร์-
โวลคอฟฟ์ดัดแปร

นางสาวนภสร จงจิตตานนท์

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GENERATING THEOREMS FOR CHARGED ANISOTROPY AND MODIFIED TOLMAN-OPPENHEIMER-VOLKOV EQUATION

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หากพิจารณาคำตอบของสมการสนามของไอส์ไตน์สำหรับทรงกลมสถิตย์ คำตอบสามารถเขียนได้ในรูปแบบของเมตริกซ์และมวล (พลังงาน) สำหรับรูปแบบเมตริกซ์นั้น สมบัติเชิงเรขาคณิตของทรงกลมแสดงออกอย่างชัดเจน ในทางตรงกันข้ามความดันและความหนาแน่นอธิบายให้เห็นถึงโครงสร้างภายในของวัตถุได้อย่างชัดเจน ในวิทยานิพนธ์ฉบับนี้ได้สนใจทรงกลมของไหลแบบมีประจุโดยพิจารณาในรูปแบบของทั้งเมตริกซ์และมวล (พลังงาน) ทรงกลมของไหลแบบมีประจุนี้เป็นผลจากการปรับปรุงระบบทรงกลมของไหลสมบูรณ์โดยมีการเพิ่มสนามแม่เหล็กไฟฟ้าและสนามสเกลาร์ ลักษณะสำคัญของทรงกลมของไหลแบบมีประจุคือสมบัติแอนไอโซทรอปีของความดัน นั่นคือความดันในทิศรัศมีและความดันในทิศทแยงไม่เท่ากัน ในวิทยานิพนธ์ฉบับนี้ได้นำเสนอการสร้างทฤษฎีบทก่อกำเนิดสำหรับระบบทรงกลมของไหลแบบมีประจุ วิธีการนี้สามารถสร้างคำตอบใหม่จากคำตอบที่เราทราบอยู่แล้วโดยไม่ต้องผ่านการแก้สมการสนามของไอส์ไตน์โดยตรง ในส่วนของกาลอวกาศทฤษฎีได้ถูกสร้างขึ้นสำหรับคำตอบเมตริกซ์ที่อยู่ในระบบพิกัดชวาร์สชิลด์ จากนั้นทฤษฎีนี้ได้ถูกนำไปใช้กับเมตริกซ์โทลแมน-บายิน สำหรับคำตอบภายในสสาร สมการโทลแมน-ออพเพนไฮม์เมอร์-โวลคอฟฟ์ดัดแปรนำไปสู่คำตอบเริ่มต้นสำหรับการสร้างทฤษฎีบทก่อกำเนิดซึ่งอยู่ในรูปแบบของความดันและหนาแน่น นอกเหนือจากนั้นได้พิจารณาผลจากประจุไฟฟ้าที่มีต่อความดันของไหลสำหรับระบบในกรณีพิเศษ

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NAPASORN JONGJITTANON: GENERATING THEOREMS FOR CHARGED ANISOTROPY AND MODIFIED TOLMAN-OPPENHEIMER-VOLKOV EQUATION. ADVISOR: ASST. PROF. AUTTAKIT CHATRABHUTI, Ph.D., CO-ADVISOR: ASST. PROF. PETARPA BOONSERM, Ph.D., 77 pp.

If we consider a solution of the Einstein field equations for a static sphere, the solution can be written in terms of the metric or mass (energy). For the metric form, geometric properties of the sphere are explicit. On the other hand, the pressure and the density present clear explanations about the internal structure of an object. In this thesis, charged fluid spheres are of interest and considered in both the metric and mass (energy) form. The charged fluid sphere is a result of a modified perfect fluid sphere model by adding the electromagnetic field and scalar field. One particular feature of charged fluid sphere is the anisotropy of pressure, where the radial pressure and the transverse pressure are not equal. A construction of the solution generating theorems of charged fluid sphere is presented in the thesis. This technique can be used to generate new solutions from known solutions without having to solve the Einstein field equations directly. In the view of spacetime, the generating theorems for metric solutions are created in Schwarzschild coordinates. The theorems are then tested with the Tolman-Bayin metric. For the interior solution, the modified Tolman-Oppenheimer-Volkov (TOV) equation brings about the initial solution of the generating theorems in terms of pressure and density. Moreover, the effects of electric charge density on fluid pressure are also considered as a special case.

Department:	Physics	Student's Signature	....
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Introduction

From the idea brought about in physics regarding the relationship between mass (energy) and spacetime curvature, which has been presented through the Einstein field equations, in general relativity, cosmology as a field has grown up tremendously, producing a lot of new knowledge in the understanding of the universe. This thesis looks at the anisotropic pressure sphere problem in general relativity. The spherical symmetry is a condition to solve for a new solution of the Einstein field equations (Bondi, 1947; Boonserm, Visser, & Weinfurtner, 2005; Buchdahl, 1959). An anisotropic pressure system is interesting because it has characteristics that resemble a real spherical stellar object. However, solving the Einstein field equations for a new solution is complicated even though we have the symmetry and the required physical conditions. A useful technique that we use in the study to obtain a new solution, the anisotropic solution, is a solution generating theorem. The theorem can generate a new solution if we have an existing known solution.

1.1 Introduction to general relativity

The background topics that we need in the understanding of this thesis are presented in Chapter 2. The general theory of relativity is a field that many physicists are interested in, which was first published in 1915 by the famous physicist Albert Einstein. The theory introduces a different perspective in the understanding of the universe from classical mechanics, particularly focusing on ideas about gravity (Carrol, 2014; Hartle, 2014). In classical physics, gravity is a result of an attraction of one mass to another mass. We can detect this attraction in term of force. If the mass disappears, the force is lost simultaneously. However, according to general relativity, the classical idea of gravitational force is not a completely correct concept. The gravity is instead expressed as a consequence of a mass on spacetime. According to the theory, the spacetime region at which matter is located will be curved because of the presence of mass. This curvature is recognized as gravity. Any other mass in proximity is also aware of curvature, and therefore, aware of the presence of gravity in spacetime. Anything that travels through spacetime has its speed limit not exceeding the speed of light. If the mass suddenly disappears, the other mass has to take the time to realize the absence of the gravity. The important concept of the general theory of relativity is the Einstein field equations, which express the relationship between mass (or energy) and the geometry of spacetime,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.1}$$

where $G_{\mu\nu}$ is an Einstein tensor, G is a gravitational constant, and $T_{\mu\nu}$ is energy momentum stress tensor. The Einstein field equations are a set of nonlinear differential equations with coupling variables. It is hard to solve the equations directly without any assumptions or symmetry. The first physicist who solved for the exact solution was Karl Schwarzschild. He applied spherical symmetry to Einstein's theory to explain the gravitational field produced by some objects in a model of fluid spheres (Carrol, 2014; Hartle, 2014). After that, perfect fluid spheres became known as the first approximation in a construction of a realistic model for a general relativistic star (Boonserm et al., 2005; Boonserm, Visser, & Weinfurtner, 2007; Delgaty & Lake, 1998; Lake, 2003; Rahman & Visser, 2002).

1.2 The model of perfect fluid spheres

We assume that all spherical objects can be equated to a perfect fluid sphere, allowing for symmetry that makes the Einstein field equations simpler. Perfect fluid spheres can represent a spherically symmetric ideal object, which is filled by mass or energy. The definitions for perfect fluid in this work differ from fluid mechanics. They are expressed in terms of energy-momentum-stress tensor $T_{\mu\nu}$ (Carrol, 2014);

$$T_{\mu\nu} = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix}. \tag{1.2}$$

The quantity $T_{\mu\nu}$ describes four momentum p_{μ} across the surface constant x_{ν}.

In general relativity, the three properties associated with perfect fluid are: no shear stress, no heat (energy) conduction, and isotropic pressure. In Veilbein formalism, these properties direct the energy-momentum-stress tensor of perfect fluid to the following diagonal matrix;

$$T_{\hat{a}\hat{b}} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{1.3}$$

where p is the pressure and ρ is the density of the fluid. For the isotropic property, the pressure in every direction must be equal. Using equation (1.1), we obtain a constraint for perfect fluid spheres in terms of the Einstein tensor $G_{\mu\nu}$,

$$G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}}. \tag{1.4}$$

The hat symbols above the coordinates are used to represent the quantities in Veilbein formalism. This form has a non-coordinate basis and agrees with the observer's view (Boonserm et al., 2005; Carrol, 2014; Delgaty & Lake, 1998). The constraint will be used to construct a solution generating theorem in the later section.

1.3 The Tolman-Oppenheimer-Volkov (TOV) equation

To express a system in physics, we may have to be familiar with physical observations such as mass and energy. In general relativity, the system, which is considered as a spherical object in this thesis, is described by the metric. Using the properties of perfect fluid sphere and the Einstein field equations, the interior solution of a perfect fluid spherical object obeys the Tolman-Oppenheiner-Volkov (TOV) equation as follows,

$$\frac{dp(r)}{dr} = -\frac{[\rho(r)+p(r)][m(r)+4\pi p(r)r^3]}{r^2[1-2m(r)/r]}, \tag{1.5}$$

$$\frac{dm(r)}{dr} = 4\pi\rho(r)r^2.$$

The TOV equation provides the relationship between the pressure profile p and the density profile $\rho,$ the m term refers to the mass. For astronomy related calculations, information of an object can more easily be obtained in terms of pressure and density rather than the geometry of spacetime around it.

1.4 Solution generating theorems

The method used in studying anisotropic fluid spheres in this work is referred to as a solution generating theorem. The prime concept of the theorem is to generate a new solution from an initial solution. In reference to 'Generating perfect fluid spheres in general relativity' by P. Boonserm et. al. (Boonserm et al., 2005), the theorem transformations are based on spacetime geometry of perfect fluid spheres in terms of the static Schwarzschild coordinates metric,

$$ds^2 = -\zeta_0(r)^2 dt^2 + \frac{dr^2}{B_0(r)} + r^2 d\Omega^2. \tag{1.6}$$

The solution is denoted by $\left\{\zeta_0(r), B_0(r)\right\}$, where it satisfies the perfect fluid sphere constraint $G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}}.$

For theorem 1, the $B_0(r)$ term is transformed to $B_0(r)+\lambda\Delta_0(r).$ The solution after applying the theorem is $\left\{\zeta_0(r), B_0(r)+\lambda\Delta_0(r)\right\}.$ The new solution is also a solution of perfect fluid spheres, where the $\Delta_0(r)$ term should be in the form

$$\Delta_0(r) = \left(\frac{\zeta_0(r)}{\zeta_0(r)+r\zeta_0'(r)}\right)^2 r^2 \exp\left\{2\int \frac{\zeta_0'(r)}{\zeta_0(r)} \frac{\zeta_0(r)-r\zeta_0'(r)}{\zeta_0(r)+r\zeta_0'(r)} dr\right\}. \tag{1.7}$$

For theorem 2, the $\zeta_0(r)$ is transformed to $Z_0(r)\zeta_0(r).$ The solution after applying the theorem is $\left\{Z_0(r)\zeta_0(r), B_0(r)\right\}.$ The new solution is also a solution of perfect fluid spheres, where the $Z_0(r)$ term should be in the form

$$Z_0(r) = \sigma + \varepsilon \int \frac{r dr}{\zeta_0(r)^2 \sqrt{B_0(r)}}. \tag{1.8}$$

Theorem 3 and theorem 4 are a combination of theorem 1 and theorem 2. Theorem 3 is the application of theorem1 followed by theorem 2. The steps involved in the application of theorem 4 are the reverse of theorem 3.

Besides, a solution generating theorem can also be constructed for the TOV equation with a different approach in reference to ' Solution generating theorems for the TOV equation' by P. Boonserm et. al. (Boonserm et al., 2007). Assuming that we obtain the initial pressure p_0 and the initial density ρ_0, they can be transformed to new solutions in terms of $p_0+\delta p$ and $\rho_0+\delta\rho$.

For theorem (P1), the initial density $\rho_0(r)$ is fixed. The TOV equation is in the form of nonhomogeneous differential equation, referred to as the Riccati equation. The new solution for the TOV equation is $p_0+\delta p,$ where

$$\delta p(r)=\frac{\delta p_c\sqrt{1-2m_0/r}\exp\left\{-2\int_0^r g_0 dr\right\}}{1+4\pi\delta p_c\int_0^r \frac{1}{\sqrt{1-2m_0/r}}\exp\left\{-2\int_0^r g_0 dr\right\}rdr}, \tag{1.9}$$

where δp_c is δp at the center of the sphere, $g_0=\frac{m_0(r)+4\pi p_0(r)r^3}{r^2\left[1-2m_0(r)/r\right]}$, and m_0 is initial mass.

For theorem (P2), the initial pressure p_0 and the initial density ρ_0 are transformed to $p_0+\delta p$ and $\rho_0+\delta\rho$, respectively,

$$\delta p(r)=\frac{\delta p_c}{\left[1+rg_0\right]^2}\frac{1+8\pi p_0 r^2}{1-\frac{2m_0}{r}}\exp\left\{2\int_0^r g_0\frac{1-rg_0}{1+rg_0}dr\right\}, \tag{1.10}$$

$$\delta\rho(r)=-\frac{1}{r^2}\frac{d}{dr}\left(\frac{\delta p(r)r^3}{1+2rg_0(r)}\right). \tag{1.11}$$

Considering the solutions of both theorems, the physical reasonableness of the generated solutions depends on the initial solution. If the initial solution has a finite and positive pressure and density at the center, new solutions are well behaved. Therefore, theorem (P1) and theorem (P2) have a restriction where a black hole cannot be generated by the solution of a star. We can easily consider the physical meaning of the solution in terms of the TOV equation rather than a metric form.

1.5 Charged fluid spheres and an anisotropy of pressure

The perfect fluid sphere is an ideal concept to explain an object. In an approach to real object; however, this concept must be modified. In this thesis, we are particularly interested in anisotropic fluid spheres. In Chapter 3, the charged fluid sphere is presented as an anisotropic pressure system. The property, which is unlike the perfect fluid sphere, where the radial pressure is not equal to the transverse pressure. The energy momentum stress tensor for anisotropic pressure system can be written as follows,

$$T_{\hat{a}\hat{b}} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p_r & 0 & 0 \\ 0 & 0 & p_t & 0 \\ 0 & 0 & 0 & p_t \end{pmatrix}. \tag{1.12}$$

Considering the Einstein field equations, the radial Einstein tensor differs from the transverse,

$$G_{\hat{r}\hat{r}} \neq G_{\hat{\theta}\hat{\theta}}. \tag{1.13}$$

A stellar object always has properties of an anisotropy of pressure. One kind of object which is similar to anisotropic fluid spherical objects is a neutron star. The star is classified as a type of compact star. The existence of electric charge, magnetic field and scalar field in the star are causes of anisotropy.

In this thesis, we construct a solution generating theorem for charged fluid spheres. Considering the solution in metric form, we assume the solution is in the form of Schwarzschild coordinates metric,

$$ds^2 = -\zeta_0(r)^2 dt^2 + \frac{dr^2}{B_0(r)} + r^2 d\Omega^2. \tag{1.14}$$

The solution is denoted by $\{\zeta_0(r), B_0(r)\}$. Following the derived steps of perfect fluid spheres, we have to set up a constraint for charged fluid spheres,

$$G_{\hat{r}\hat{r}} - G_{\hat{\theta}\hat{\theta}} = 8\pi\Delta(r). \tag{1.15}$$

Therefore we can construct the theorems as follows;

For theorem 1, the $B_0(r)$ term is transformed to $B_0(r)+\lambda\Delta_0(r)$. The solution after applying the theorem is $\left\{\zeta_0\left(r\right), B_0\left(r\right)+\lambda\Delta_0\left(r\right)\right\}$. The new solution is also a solution of charged fluid spheres, where the $\Delta_0\left(r\right)$ term should be in the form

$$\Delta_0\left(r\right)=\left(\frac{\zeta_0(r)}{\zeta_0(r)+r\zeta_0'\left(r\right)}\right)^2 r^2\exp\left\{2\int\frac{\zeta_0'\left(r\right)}{\zeta_0(r)}\frac{\zeta_0(r)-r\zeta_0'\left(r\right)}{\zeta_0(r)+r\zeta_0'\left(r\right)}dr\right\}. \tag{1.16}$$

For theorem 2, the $\zeta_0(r)$ term is transformed to $Z_0(r)\zeta_0(r)$. The solution after applying the theorem is $\left\{Z_0(r)\zeta_0(r), B_0\left(r\right)\right\}$. The new solution is also a solution of charged fluid spheres, where the $Z_0\left(r\right)$ term should be in the form

$$Z_0(r)=\sigma+\varepsilon\int\frac{rdr}{\zeta_0(r)^2\sqrt{B_0(r)}}. \tag{1.17}$$

The results are similar to the solution generating theorems for perfect fluid spheres. After that, we use the theorems to generate new solutions of the charged fluid sphere, which is the Tolman-Bayin type with $n=0$,

$$ds^2=-\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)dt^2+\left(1-\frac{2q^2}{r^2}\right)^{-1}dr^2+r^2d\Omega^2. \tag{1.18}$$

For the application of theorem 1, the $B_0(r)$ term is transformed to $B_0(r)+\lambda\Delta_0(r)$, where

$$\Delta_0=4\ln\left(r-q\right)\left[\frac{m}{q}-1\right]-4\ln\left(r+q\right)\left[\frac{m}{q}+1\right]+8\ln\left(r\right). \tag{1.19}$$

For the application of theorem 2, the $\zeta_0(r)$ term is transformed to $Z_0(r)\zeta_0(r)$, where

$$Z_0=\sigma+\varepsilon[\frac{1}{2}r^2+2mr+\frac{q^3\ln\left(r+g\right)}{4m+4q}-\frac{q^3\ln\left(r-g\right)}{4m-4q}+\frac{2\ln\left(-2mr+q^2+r^2\right)}{\left(m+q\right)\left(m-q\right)}\left[m^4-m^2q+\frac{1}{8}\right]$$
$$+\frac{\arctan\left[\frac{1}{2}\frac{2r-2m}{\sqrt{-m^2+q^2}}\right]}{\left[m+q\right]\left[m-q\right]\sqrt{-m^2+q^2}}\left[m^4-m^2q+\frac{1}{8}\right]], \tag{1.20}$$

by assuming $\frac{2q^2}{r^2}\ll 1$.

1.6 The modified TOV equation for charged anisotropic pressure system

The solution of the Einstein field equations in metric form is not enough to understand charged fluid spheres. Research in the area of astrophysics always involves the interior properties. Therefore we should construct a solution generating theorem for the interior solution of charged anisotropic pressure spherical system in terms of pressure and density. The solution is improved from the TOV equation by adding an electromagnetic field and scalar field (Boonserm, Ngampitipan, & Visser, 2015). With the conservation of energy momentum tensor $\nabla_\nu T^{\mu\nu} = 0,$ the modified TOV equation for charged fluid spheres is

$$\begin{gathered} \frac{dp_f}{dr} = -\frac{\left(\rho_f + p_f\right)\left(m + 4\pi p_f r^3\right)}{r^2\left(1-2m/r\right)} - \frac{\sigma_{em} E}{\sqrt{1-2m/r}} - \sigma_s \frac{d\phi}{dr}, \\ \frac{dm(r)}{dr} = 4\pi\left[\rho_f + \sigma_{em} + \sigma_s\right] r^2, \end{gathered} \tag{1.21}$$

where p_f, ρ_f, σ_{em}, and σ_s are perfect fluid pressure, perfect fluid density, electromagnetic density, and scalar field density.

For theorem 1, the initial density $\rho_0(r)$ is fixed. The modified TOV equation is in the form of nonhomogenous differential equation, which is the Riccati equation. Assuming that we obtain the initial solutions p_1 and p_2, the new solution for the TOV equation is p where

$$p(r) = \frac{\lambda \exp\left\{-\int \frac{4\pi r}{(1-2m/r)} p_1(r)\, dr\right\} p_1(r) + (1-\lambda)\exp\left\{-\int \frac{4\pi r}{(1-2m/r)} p_2(r)\, dr\right\} p_2(r)}{\lambda \exp\left\{-\int \frac{4\pi r}{(1-2m/r)} p_1(r)\, dr\right\} + (1-\lambda)\exp\left\{-\int \frac{4\pi r}{(1-2m/r)} p_2(r)\, dr\right\}}, \tag{1.22}$$

where λ is a constant.

For theorem 2, the initial pressure p_0 and the initial density ρ_0 are transformed to $p_0 + \delta p_C + \delta p_P$ and $\rho_0 + \delta\rho$, respectively, where

$$\delta p_C(r) = \frac{\delta p(0)}{\left[1 + r g_0(r)\right]^2} \frac{1 + 8\pi p_0(r) r^2}{1 - 2m_0(r)/r} \exp\left\{2\int_0^r g_0(r) \frac{1 - r g_0(r)}{1 + r g_0(r)} dr\right\}, \tag{1.23}$$

$$\delta\rho_f = -\frac{1}{r^2}\frac{d}{dr}\left(\frac{\delta p_f(r)r^3}{1+2rg_0(r)}\right) - \delta\sigma_{em} - \delta\sigma_s, \tag{1.24}$$

$$\delta p_P = \delta p_C * \int_0^r \left[\frac{\left\{\frac{-\sigma_{EM}E}{\sqrt{1-2m/r}} - \sigma_s \frac{d\phi}{dr}\right\}}{\delta p_C}\right] dr. \tag{1.25}$$

The new solutions from theorem 2 depend on the electromagnetic field and scalar field. In addition, we also consider the effect of charge on pressure. For a special case: when ρ is constant and σ_s is zero, the pressure of fluid spheres rises when the charge increases.

1.7 Structure of the thesis

This thesis starts with an introduction to general relativity in Chapter 2. The general relativity is linked to classical mechanics by special relativity. The involved physics quantities are also explained. An important method in this thesis, which is a solution generating theorem, is presented. After that, Chapter 3 contains the relevant information about our interested system, anisotropic pressure sphere. A dominant characteristic of anisotropy is the difference in the radial pressure and the transverse pressure because of charge. We also construct a solution generating theorem here. In Chapter 4, the theorems are created for the interior solution of charged fluid spheres using the modified TOV equation. Moreover, the effect of electric charge on pressure is also examined. Lastly, the results of the thesis and further works are concluded in Chapter 5.

Chapter 2
Relativity and solution generating theorem

This thesis works on the idea of general relativity which presents a different approach to distinguish physics from classical physics. This chapter describes about background knowledge of the thesis. Moreover, definitions of the relevant physics quantity are explained, such as tensor and the Einstein field equations which is the main idea to construct the new theorems. We will introduce about the solutions of the Einstein field equations in both metric form and interior solution. The important algorithmic technique we use in the thesis is a solution generating theorem. In this chapter we explain the construction of the solution generating theorems for perfect fluid sphere in form of metric solution and interior solution.

2.1 Special relativity

Classical mechanics can describe motion and interaction between objects in daily life by using Newton's laws of motion. But the laws cannot completely explain mechanical phenomena at extreme conditions. For example, classical mechanics gives the wrong description about an object which moving at the speed near the speed of light. In 1905, Albert Einstein presented the theory that gives the concept different from classical mechanics but more precisely, special relativity. The theory makes an impact on physics and is a starting point of modern physics (Carrol, 2014).

Einstein's idea is about setting up an inertial frame and another frame that moves with constant velocity with respect to the inertial frame.

Special relativity was constructed by 2 postulates,

1. Physical laws are the same in all inertial frames.

2. The velocity of light (in vacuum) is constant in all inertial frames.

For the first postulate, a phenomenon that happens in all inertial frames should obey the same laws of physics. The second postulate is supported by Michelson-Morley experiment (Hartle, 2014). Light velocity does not depend on how fast an observer travels. This idea opposes relative speed in Newton's Laws. This postulate allows observers in different frames to be able to observe same event in different time duration.

Because of the two postulates of relativity, time is seen as a coordinate. The time interval of an event as observed in different frames may be different. Therefore, time is coordinate same as spatial coordinates. The transformation from one frame to another frame is Lorentz transformation. This transformation includes both rotation and boost. The rotation transforms spatial coordinates between two frames. The boost leaves the spatial coordinates as it is, while transforms only the time component. As for the 4 dimensions, Lorentz transformation relates two initial frames via both time coordinate and the spatial coordinates. The definition of metric for flat spacetime can be seen from the line element (metric) in the Minkowski coordinates,

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{2.1}$$

The metric can describe geometric property of a spacetime. This Minkowski metric presents a spacetime with no energy or mass. We can write the metric in terms of the infinitesimal coordinate displacement dx^μ and metric tensor $\eta_{\mu\nu}$

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\upsilon. \tag{2.2}$$

The Greek subscripts are run the number 0 to 3, where zero is defined as the time coordinate and the others run for three spatial coordinates. Einstein summation convention is used for summation of all running index without summation sign. The metric tensor $\eta_{\mu\nu}$ stands for metric tensor of a Minkowski metric,

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.3}$$

The metric tensor is related to the matrix Λ that is defined via the Lorentz transformation by

$$\eta = \Lambda^T \eta \Lambda. \tag{2.4}$$

The matrix Λ is constant square matrix and depends on the initial frame and relative frame. The example of Λ, which describes the transformed frame boosts in x direction of the initial frame, is

$$\Lambda^{\mu}{}_{\nu} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2.2 General relativity

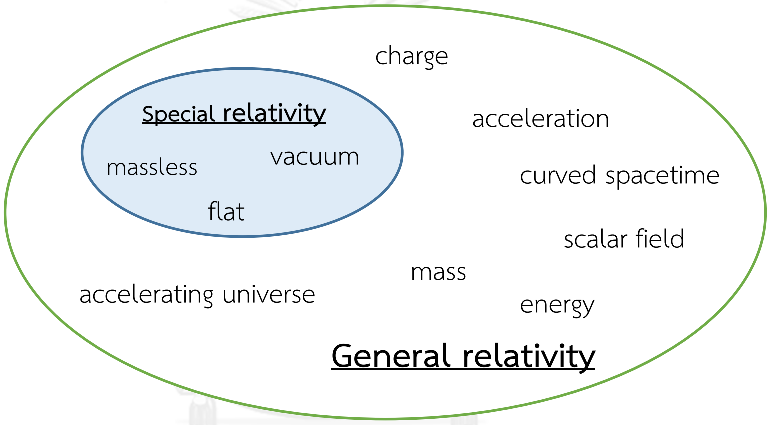


Figure 1: Special relativity as a specific case of general relativity

General theory of relativity was first published in 1915, which introduced a completely new perspective in the understanding of physics. The theory presents new ideas about gravity (Carrol, 2014; Hartle, 2014). This theory is involved with mass and energy. Special relativity is a subsection of general relativity, as shown in Figure 1, and describes the flat spacetime. In classical physics, gravity relates to a force exerted by a mass, which attracts another mass in certain proximity. If the mass disappears, the force is suddenly lost. However, according to general relativity, gravity is not a force. It is a consequence of a mass or energy on spacetime. The spacetime where matter sits will be curved because of the presence of mass. This curvature is recognized as gravity. Any other mass in proximity is also aware of the presence of gravity in spacetime.

Anything that travels through spacetime has its speed limit not exceeding the speed of light. If the mass suddenly disappears, the other mass has to take the time to realize the missing gravity. The another concept of gravity is about an equivalent principle. The principle was presented by Einstein. He thought there is no difference for experiment between constant acceleration and uniform gravity. Hence, the object in the constant accelerated frame is equivalent to the objects with free fall moving from the influence of gravity. The key basis of the theory is the Einstein's field equations which express the relationship between mass (or energy) and the geometry of spacetime,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2.5}$$

The right hand side of the equation describes the mass and the energy, while the left hand side is the spacetime curvature part of the equation. The Einstein field equations are nonlinear differential equations with coupling variables. It is hard to solve the equations directly without any assumption or symmetry. The first physicist who solved for the exact solution was Karl Schwarzschild. He applied spherical symmetry with Einstein's theory to explain the gravitational field produced by some objects (Boonserm et al., 2005). In this thesis, we also make use of spherical symmetry. For astronomical objects in the universe, we can see that almost all objects are spherical. By using this symmetry together with model development, we may be able to predict the spacetime around a star.

2.2.1 Metric

In classical mechanics, coordinates are only reference positions of physical phenomenon. An object's move path is influenced by other masses or energy. On the contrary, general relativity explains an action from one mass to another mass is not direct action. The influence is acted on the spacetime and it transfers the action to another mass. Therefore, the spacetime in general relativity are important and have a role in interaction in physics. How to express the geometric property of spacetime, in

4 dimensions, is to use metric. The general form of metric, which is consistent to the spherically symmetric metric, is presented in the form as follows

$$ds^2 = -\zeta(r)^2 dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2, \tag{2.6}$$

which is the type of metric we use in this thesis. The term $\zeta(r)$ shows the scaling of time coordinate and $B(r)$ shows the curve on radial coordinate. This metric is written in spherical coordinate for simply explanation of a spherical symmetry object. The metric tensor $\eta_{\mu\nu}$ in special relativity is turned to be $g_{\mu\nu}$ for a curved spacetime which can be obtained from line elements for a curved spacetime

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

2.2.2 Vector and dual vector

In general relativity, an interaction is observed in 4 dimensions. We need a higher dimensional quantity to explains the phenomena. A tensor is introduced as a type of quantity for a better understanding of the concept of physics. Vector in relativity is a first step for definition of a tensor.

A difference between two coordinates, in general relativity, can be explain by general coordinate transformation. The concept of the transformation is similar to Lorentz transform but done on curved spacetime. From general coordinate transformation, the coordinate $x^{\mu'}$ change from the coordinate dx^ν by

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu} dx^\nu.$$

A vector represents magnitude and direction. The notation we use to define a component of vector is the letter with upper index, V^μ. We can transform the component of a vector V^μ to another frame as follows

$$V^{\nu'} = \frac{\partial x^{\upsilon'}}{\partial x^\mu} V^\mu. \tag{2.7}$$

The basis vector is defined as a letter with lower index, $\hat{e}_{(\mu)}$,

$$V = V^{\mu}\hat{e}_{(\mu)}. \tag{2.8}$$

General coordinate transformation of basis vector differs from the vector,

$$\hat{e}_{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\upsilon}}\hat{e}_{\mu}. \tag{2.9}$$

We can define the quantity that opposes to vector, a dual vector or one-form. It is a mapping of vector in to real space,

$$V(\omega) \equiv \omega(V) = V^{\mu}\omega_{\mu}, \tag{2.10}$$

where V is a scalar. The transformation for a component of dual vector is

$$\omega_{\upsilon'} = \frac{\partial x^{\mu}}{\partial x^{\upsilon'}}\omega_{\mu}. \tag{2.11}$$

For the basis dual vector has a transformation rule likes vector

$$\hat{\theta}^{\nu'} = \frac{\partial x^{\upsilon'}}{\partial x^{\mu}}\hat{\theta}^{\mu}. \tag{2.12}$$

A vector and a dual vector are defined by general coordinate transformation which depends on the properties of spacetime. By this definition, we will explain a tensor quantity that expresses physical property in general relativity.

2.2.3 Tensor

We can construct a tensor as a quantity that has property of both vector and dual vector. The tensor rank $k+l$ is expressed in term of a result of tensor product $\otimes$ as follows

$$T = T^{\mu_1 \ldots \mu_k}{}_{\upsilon_1 \ldots \upsilon_l}\hat{e}_{(\mu_1)} \otimes \ldots \otimes \hat{e}_{(\mu_1)} \otimes \hat{\theta}^{(\upsilon_1)} \otimes \ldots \otimes \hat{\theta}^{(\upsilon_l)}. \tag{2.13}$$

The bases of tensor consist of the tensor product of all coordinate bases

$$T^{\mu_1' \ldots \mu_k'}{}_{\nu_1' \ldots \nu_l'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \ldots \frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} \ldots \frac{\partial x^{\nu_l}}{\partial x^{\nu_l'}} T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}. \tag{2.14}$$

In this thesis, the tensors that we use agree with this general coordinate transformation.

2.2.4 Energy-momentum stress tensor

The metric of spacetime gives a clear understanding about geometric property of spacetime. The curvature of spacetime is a result of mass or energy. Considering in particle and energy, this part can be described by a square matrix, energy momentum stress tensor $T_{\mu\upsilon}$. This quantity is defined by the flux of four-momentum p_μ across a surface x^υ,

$$T_{\mu\nu}=\begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix}. \tag{2.15}$$

From the definition, T_{00} is an energy density, T_{0i} refer to an energy flux through the plane x^i, T_{i0} describe the momentum density in i direction, and T_{ij} are the flux of momentum in i^{th} component flow through the plane x_j. The energy momentum stress tensor is symmetric and should be consistent to the conservation of energy-momentum condition,

$$\nabla_\mu T^{\mu\nu}=0, \tag{2.16}$$

where the symbol ∇_μ is covariant derivative. It is defined by $\nabla_\mu V^\nu=\dfrac{\partial V^\nu}{\partial x_\mu}+\Gamma^\rho_{\mu\sigma}V^\sigma$ for $\Gamma^\rho_{\mu\sigma}$ is a Christoffel connection. The condition is useful to construct the interior solution of anisotropic pressure system in this thesis.

2.2.5 The Einstein field equations

After publishing of special relativity in 1905, Albert Einstein had been developed the more general case and he succeed. The general theory of relativity was published in 1916 and brought a big change to Modern physics. The fundamental idea of general relativity is a relationship between spacetime curvature and mass or energy which is described by the Einstein field equations

$$G_{\mu\nu}=8\pi G T_{\mu\nu}, \tag{2.17}$$

where $G_{\mu\nu}$ is Einstein tensor, defined by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{2.18}$$

where $R_{\mu\nu}$ is a Ricci tensor, R is a Ricci scalar, and $g_{\mu\nu}$ is a metric tensor.

To derive an Einstein tensor $G_{\mu\nu}$ starts from a metric tensor $g_{\mu\nu}$, which can be obtained from line element as follows

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.19}$$

The Christoffel connection can be derived from derivatives of metric tensor,

$$\Gamma^\tau_{\rho\nu} = \frac{1}{2} g^{\tau\mu} \left(\partial_\rho g_{\nu\mu} + \partial_\nu g_{\rho\mu} - \partial_\mu g_{\rho\nu} \right). \tag{2.20}$$

The Christoffel connection can lead to a Ricci tensor and Ricci scalar as foollows

$$R_{\nu\sigma} = \partial_\mu \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\mu} + \Gamma^\mu_{\tau\mu} \Gamma^\tau_{\nu\sigma} - \Gamma^\mu_{\tau\sigma} \Gamma^\tau_{\nu\mu}, \tag{2.21}$$

$$R = R_{\nu\sigma} g^{\nu\sigma}. \tag{2.22}$$

The Einstein field equations show the dynamic of gravity as a result of curved spacetime. This equation is too complicated to solve for an exact solution. The solution of this equation is obtained by supposing some symmetry to a system. In this thesis, the symmetry is anisotropic fluid spheres. However, before we go to this part, we have to consider in first step of approximation, perfect fluid spheres.

2.3 Perfect fluid spheres

The Einstein field equations are composed of Ricci tensor $R_{\mu\nu}$, Ricci scalar R, and metric tensor $g_{\mu\nu}$. These quantities are highly nonlinear differential equation, which are coupled with each other. We cannot solve the equation directly without formulating an assumption to make it simpler. One year after publishing the general theory of relativity, Karl Schwarzschild become the first person to successfully solve for the spherically symmetric solution with the perfect fluid sphere condition (Boonserm et al., 2005). We assume that all matter in a star can be equated to perfect fluid spheres, allowing for symmetry that makes the Einstein field equations simpler. Perfect fluid spheres can represent a spherically symmetric ideal object which is filled by mass or energy. The definitions for perfect fluid in this work differ from fluid mechanics. They are expressed in terms of energy-momentum-stress tensor $T_{\mu\nu}$;

$$T_{\mu\nu} = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix}. \tag{2.23}$$

The quantity $T_{\mu\nu}$ describes four momentum p_{μ} across the surface constant x_{ν} (Carrol, 2014).

In general relativity, the three properties associated with perfect fluid are; no shear stress, no heat (energy) conduction, and isotropic pressure. For presenting an observer's view, we can use the non-coordinate basis known as the Vielbein formalism. It presents the basis vector, which does not depend on any coordinates. In Vielbein formalism, the properties direct the energy-momentum-stress tensor of perfect fluid to the following diagonal matrix (Boonserm et al., 2005),

$$T_{\hat{a}\hat{b}} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{2.24}$$

For isotropic property of interior pressure, the pressure in every direction must be equal. This property allows us to reduce complexity of density distribution inside an object. Using equation (1), we obtain Einstein tensor $G_{\mu\nu}$ for perfect fluid spheres, and perfect fluid constraints

$$G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}}. \tag{2.25}$$

The hat symbols above the coordinates are used to represent the quantities in Veilbein formalism. This form is in non-coordinate basis and agrees with the observer's view.

2.4 The solution generating theorems for perfect fluid sphere metric

Subsequently, many solutions of the Einstein field equations with perfect fluid constraints were explored in various coordinate systems (Bondi, 1947; Buchdahl, 1959;

Thairatana, 2013). Each solution explains different systems in different conditions. Recently, some algorithmic techniques have been developed to generate new solutions from known perfect fluid spheres (Boonserm et al., 2005, 2007). This is very interesting because we no longer need to solve the Einstein field equations directly to obtain the solutions. The idea of a solution generating theorem is presented as in Figure 2. This concept also classifies the type of metric solution and exhibits the association of the solutions in distinct conditions.

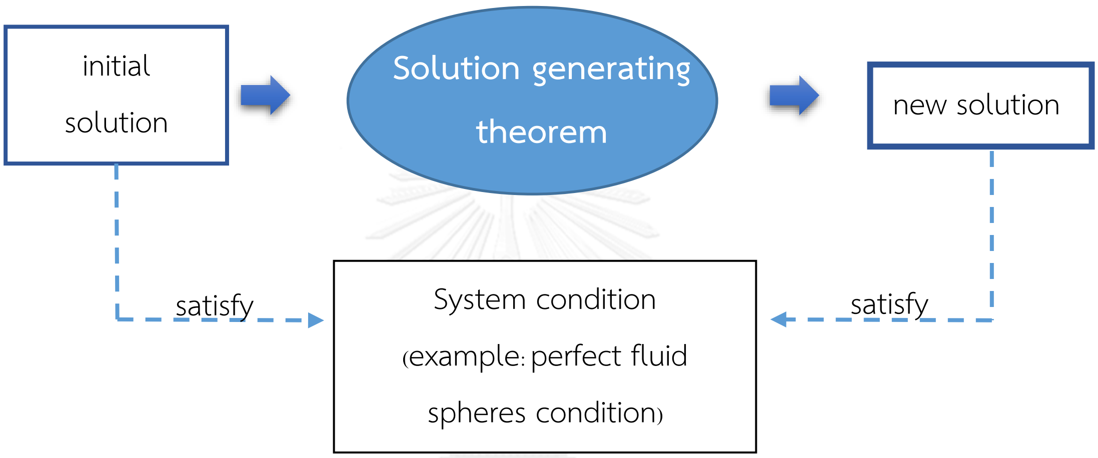


Figure 2: The working of a solution generating theorem under system condition

For constructing the solution generating theorems, a metric of an object was written in suitable coordinates. By definition of the geometric element of the Einstein field equation, $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R,$ Einstein tensor $G_{\mu\nu}$ can be obtained from the metric. We can set perfect fluid constraints to $G_{\mu\nu}$ for ordinary differential equation, which can be used to build the solution generating theorems. These theorems deform perfect fluid spheres in terms of spacetime geometry. In reference to the solution generating theorems for perfect fluid spheres by P. Boonserm (Boonserm et al., 2005), the generating theorems for Schawarszchild metric with perfect fluid sphere conditions were constructed as follows. The coordinates that we want to work is Schwarzschild coordinates,

$$ds^2 = -\zeta_0(r)^2 dt^2 + \frac{dr^2}{B_0(r)} + r^2 d\Omega^2. \tag{2.26}$$

For describing perfect fluid sphere object, the different conditions of any systems influence to $\zeta_0(r)$ and $B_0(r)$ terms. So, the both terms will be transformed to other metric form. Suppose $\{\zeta_0(r), B_0(r)\}$ represents a perfect fluid sphere. According to perfect fluid spheres constraint, $G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}}$, and deriving the Einstein tensor in topic 2.2.5, we can obtain

$$\left[r\left(r\zeta\right)'\right]B' + \left[2r^2\zeta'' - 2\left(r\zeta\right)'\right]B + 2\zeta = 0, \tag{2.27}$$

or in rearranged form of $\zeta(r)$,

$$2r^2B\zeta'' + \left(r^2B' - 2rB\right)\zeta' + \left(rB' - 2B + 2\right)\zeta = 0. \tag{2.28}$$

This equation is under the condition of perfect fluid spheres in Schwarzschild coordinates. The solution that corresponds to equation (2.27) is also a perfect fluid sphere solution of the Einstein field equations.

Four theorems for perfect fluid sphere

For theorem 1, suppose we know the solution of perfect fluid spheres $\{\zeta_0(r), B_0(r)\}$, we can generate the new solution by fixing the $\zeta_0(r)$ term and extending the $B_0(r)$ term to $B_0 + \lambda\Delta_0(\zeta_0)$. The $\Delta(r)$ can be derived by setting a new solution $\{\zeta_0(r), B_0(r) + \lambda\Delta_0\}$ to satisfy equation (2.27).

Theorem 1: *Suppose $\{\zeta_0(r), B_0(r)\}$ represents a perfect fluid sphere. Define*

$$\Delta_0(r) = \left(\frac{\zeta_0(r)}{\zeta_0(r) + r\zeta_0'(r)}\right)^2 r^2 \exp\left\{2\int \frac{\zeta_0'(r)}{\zeta_0(r)} \frac{\zeta_0(r) - r\zeta_0'(r)}{\zeta_0(r) + r\zeta_0'(r)} dr\right\}. \tag{2.29}$$

Then for all λ, the geometry defined by holding $\zeta_0(r)$ fixed and setting

$$ds^2 = -\zeta_0(r)^2 dt^2 + \frac{dr^2}{B_0(r) + \lambda\Delta_0(r)} + r^2 d\Omega^2$$

is also a perfect fluid sphere. That is, the mapping

$$\{\zeta_0, B_0\} \mapsto \{\zeta_0, B_0 + \lambda\Delta_0(\zeta_0)\} \tag{2.30}$$

takes perfect fluid spheres into perfect fluid spheres.

Theorem 1 transforms a solution to a new solution with the different time scaling but the geometric property about spatial parts remain the same.

For theorem 2, suppose we know the solution of perfect fluid spheres $\{\zeta_0(r), B_0(r)\}$, we can generate the new solution by transforming the $\zeta_0(r)$ term to $\zeta_0 Z_0(\zeta_0, B_0)$ and holding the $B_0(r)$ term. The $Z_0(\zeta_0, B_0)$ term can be derived by setting a new solution $\{\zeta_0 Z_0(\zeta_0, B_0), B_0\}$ satisfy equation (2.28).

Theorem 2: *Let* $\{\zeta_0(r), B_0(r)\}$ *describe a perfect fluid sphere. Define*

$$Z_0(r) = \sigma + \varepsilon \int \frac{r dr}{\zeta_0(r)^2 \sqrt{B_0(r)}}, \tag{2.31}$$

then for all σ *and* ε, *the geometry defined by holding* $B_0(r)$ *fixed and setting*

$$ds^2 = -\zeta_0(r)^2 Z_0(r)^2 dt^2 + \frac{dr^2}{B_0(r)} + r^2 d\Omega^2$$

is also a perfect fluid sphere. That is, the mapping

$$\{\zeta_0, B_0\} \mapsto \{\zeta_0 Z_0(\zeta_0, B_0), B_0\} \tag{2.32}$$

takes perfect fluid spheres into perfect fluid spheres.

Theorem 2 transforms a solution to a new solution with the different radial explanation but the description about time part remains the same.

Theorem 3: *If* $\{\zeta_0(r), B_0(r)\}$ *represents a perfect fluid sphere, then for all* σ, ε *and* λ, *the three parameter geometry defined by*

$$ds^2 = -\zeta_0(r)^2 \left\{ \sigma + \varepsilon \int \frac{r dr}{\zeta_0(r)^2 \sqrt{B_0(r) + \lambda \Delta_0(r)}} \right\}^2 dt^2 + \frac{dr^2}{B_0(r) + \lambda \Delta_0(r)} + r^2 d\Omega^2$$

is also a perfect fluid sphere, where $\Delta_0(r)$ *is*

$$\Delta_0(r) = \left(\frac{\zeta_0(r)}{\zeta_0(r) + r\zeta_0'(r)} \right)^2 r^2 \exp\left\{ 2\int \frac{\zeta_0'(r)}{\zeta_0(r)} \frac{\zeta_0(r) - r\zeta_0'(r)}{\zeta_0(r) + r\zeta_0'(r)} dr \right\}. \tag{2.33}$$

That is

$$T_3 = T_2 \circ T_1 : \{\zeta_0, B_0\} \mapsto \{\zeta_0, B_0 + \lambda\Delta_0(\zeta_0)\} \mapsto \{\zeta_0 Z_0(\zeta_0, B_0 + \lambda\Delta_0(\zeta_0)), B_0 + \lambda\Delta_0(\zeta_0)\}. \tag{2.34}$$

For constructing the theorems, suppose we know the solution of perfect fluid spheres $\{\zeta_0(r), B_0(r)\}$, we can generate a new solution using the following 2 steps. First, applying theorem 1, then the $B_0(r)$ term is extended to $B_1 = B_0 + \lambda\Delta_0(\zeta_0)$ while $\zeta_0(r)$ is held. After that, by applying theorem 2, the $\zeta_0(r)$ term is transformed to

$\zeta_0 Z_0\left(\zeta_0, B_0+\lambda\Delta_0\right)$ while holding the B_1 term. The $Z_0\left(\zeta_0, B_1\right)$ term and $\Delta_0\left(\zeta_0\right)$ term can be derived by setting new solutions $\left\{\zeta_0, B_1\right\}$ and $\left\{\zeta_0 Z_0\left(\zeta_0, B_1\right), B_1\right\}$ satisfy equation (2.27) and (2.28), respectively. Theorem 3 transforms a solution to a new solution with the different geometric property of radial and new time scaling.

Theorem 4: *If* $\{\zeta_0(r), B_0(r)\}$ *represents a perfect fluid sphere, then for all* σ, ε, *and* λ, *the three-parameter geometry defined by*

$$ds^2 = -\zeta_0\left(r\right)^2\left\{\sigma+\varepsilon\int\frac{rdr}{\zeta_0\left(r\right)^2\sqrt{B_0\left(r\right)}}\right\}^2 dt^2+\frac{dr^2}{B_0\left(r\right)+\lambda\Delta_0\left(\zeta_1, r\right)}+r^2 d\Omega^2 \tag{2.35}$$

is also a perfect fluid sphere, where $\Delta_0\left(\zeta_1, r\right)$ is defined as

$$\Delta_0\left(\zeta_1, r\right)=\left(\frac{\zeta_1(r)}{\zeta_1(r)+r\zeta_1^{'}\left(r\right)}\right)^2 r^2\exp\left\{2\int\frac{\zeta_1^{'}\left(r\right)}{\zeta_1(r)}\frac{\zeta_1(r)-r\zeta_1^{'}\left(r\right)}{\zeta_1(r)+r\zeta_1^{'}\left(r\right)}dr\right\}, \tag{2.36}$$

depending on $\zeta_1=\zeta_0 Z_0$, whereas before

$$Z_0\left(r\right)=\sigma+\varepsilon\int\frac{rdr}{\zeta_0\left(r\right)^2\sqrt{B_0\left(r\right)}}. \tag{2.37}$$

That is

$$T_4 = T_1\circ T_2:\left\{\zeta_0, B_0\right\}\mapsto\left\{\zeta_0 Z_0\left(\zeta_0, B_0\right), B_0\right\}\mapsto\left\{\zeta_0 Z_0\left(\zeta_0, B_0\right), B_0+\lambda\Delta_0\left(\zeta_0 Z_0\left(\zeta_0, B_0\right)\right)\right\} \tag{2.38}$$

For constructing the theorems, suppose we know the solution of perfect fluid spheres $\{\zeta_0(r), B_0(r)\}$, we can generate a new solution by reversing the steps of theorem 3. First we apply theorem 2 to the initial solution. Therefore, the $\zeta_0(r)$ term is extended to $\zeta_1=\zeta_0 Z_0\left(\zeta_0, B_0\right)$ while $B_0(r)$ is held. After that, by applying theorem 1, the $B_0(r)$ term is transformed to $B_0+\lambda\Delta_0\left(\zeta_1\right)$, while holding the $\zeta_1(r)$ term. The $Z_0\left(\zeta_0, B_0\right)$ term and $\Delta_0(\zeta_1)$ term can be derived by setting new solutions $\left\{\zeta_1, B_0\right\}$ and $\left\{\zeta_1, B_0+\lambda\Delta_0\left(\zeta_1\right)\right\}$ satisfy equation (2.28) and (2.27), respectively. Theorem 4 transforms a solution to a new solution with the different geometric property of radial and new time scaling.

Four theorems transform a solution in metric terms. The theorems can generate new solutions from known solution but sometimes the generated solutions have the same mathematical form as the initial solution of generation. We can classify the type of solutions from generation of four theorems to be seed and non- seed solutions (Boonserm et al., 2005).

If a solution gives a new solution from both theorem 1 and theorem 2, the solution is a seed solution or a seed metric.

If a solution gives a new solution from either theorem 1 or theorem 2, the solution is a non-seed solution or a non-seed metric

	Seed solution	Non-seed solution	
Theorem 1	New solution	New solution	initial solution
Theorem 2	New solution	initial solution	New solution

Table 1: Types of the solution, divided from the solution generating theorems

2.5 Interior solution of perfect fluid sphere and the TOV equation

For the solution generating theorems, the theorem transformations are based on spacetime geometry. But for us, a star can more easily be observed in terms of mass and energy rather than the geometry of spacetime around it. Because the star matter can be expressed by energy-momentum-stress tensor in terms of pressure and density, the solution generating theorems should be applied to physical observables: pressure and density.

The relationship between pressure and density profile is given in the Tolman-Oppenheimer-Volkov (TOV) equation;

$$\frac{dp(r)}{dr}=-\frac{[\rho(r)+p(r)][m(r)+4\pi p(r)r^3]}{r^2[1-2m(r)/r]}, \tag{2.39}$$

$$\frac{dm(r)}{dr}=4\pi\rho(r)r^2. \tag{2.40}$$

2.6 The solution generating theorems for the pressure and density terms

The Tolman-Oppenheimer-Volkov (TOV) equation describes the interior properties of spherical static perfect fluid object as a relationship between two physical observables, pressure p and density ρ. For a fluid sphere object, which contains electric charge, magnetic field, and scalar field, the pressure becomes anisotropic. We deform the TOV equation in terms of $\delta\rho$ and $\delta p.$

Before considering interior part of a perfect fluid sphere system, we first cite to the solution generating theorems on the TOV equation in perfect fluid system. Referring to the article 'Solution generating theorems for the TOV equation' by P. Boonserm et al. (Boonserm et al., 2007), the solution generating theorems for the TOV equation were developed as follows.

Theorem (P1): *Let $p_0(r)$ and $\rho_0(r)$ solve the TOV equation, and hold $m_0(r)=4\pi\int\rho_0(r)r^2dr$ as fixed. Define an auxiliary function $g_0(r)$ by*

$$g_0=\frac{m_0(r)+4\pi p_0(r)r^3}{r^2\left[1-2m_0(r)/r\right]}. \tag{2.41}$$

Then the general solution to the TOV equation is $p(r)=p_0(r)+\delta p(r)$ where

$$\delta p(r)=\frac{\delta p_c\sqrt{1-2m_0/r}\,\exp\left\{-2\int_0^r g_0dr\right\}}{1+4\pi\delta p_c\int_0^r\frac{1}{\sqrt{1-2m_0/r}}\exp\left\{-2\int_0^r g_0dr\right\}rdr}, \tag{2.42}$$

and where δp_c is the shift in the central pressure.

Theorem (P1) gives a new solution in terms of the variation of pressure. The mass is fixed allows us to solve for $\delta p(r)$ by perturbing solution of Riccati equation. The physical reasonableness at the center of a new solution, in terms of pressure, depend on the well-behaved properties of the initial pressure $p_0(r)$, the initial density $\rho_0(r)$, the central pressure and the central density.

Theorem (P2): *Let $p_0(r)$ and $\rho_0(r)$ solve the TOV equation, and hold g_0 fixed, such that*

$$g_0=\frac{m_0(r)+4\pi p_0(r)r^3}{r^2\left[1-2m_0(r)/r\right]}=\frac{m(r)+4\pi p(r)r^3}{r^2\left[1-2m(r)/r\right]}. \tag{2.43}$$

Then the general solution to the TOV equation is given by $p(r) = p_0(r) + \delta p(r)$ *and* $\rho(r) = \rho_0(r) + \delta\rho(r)$ *where*

$$\delta m(r) = \frac{4\pi r^3 \delta\rho_c}{3\left[1+rg_0\right]^2} \exp\left\{2\int g_0 \frac{1-rg_0}{1+rg_0} dr\right\}, \tag{2.44}$$

and

$$\delta p(r) = -\frac{\delta m}{4\pi r^3} \frac{1+8\pi p_0 r^2}{1-2m_0/r}. \tag{2.45}$$

Here δp_c *is the shift in the central density. By explicitly combining these formulae we have*

$$\delta p(r) = \frac{\delta p_c}{\left[1+rg_0\right]^2} \frac{1+8\pi p_0 r^2}{1-\frac{2m_0}{r}} \exp\left\{2\int_0^r g_0 \frac{1-rg_0}{1+rg_0} dr\right\}, \tag{2.46}$$

and

$$\delta\rho(r) = -\frac{1}{r^2}\frac{d}{dr}\left(\frac{\delta p(r) r^3}{1+2rg_0(r)}\right). \tag{2.47}$$

Theorem (P2) transforms pressure and mass terms with holding the function g_0. We obtain a shift in pressure and mass by transforming of theorem (P2) through relating between mass and density in the TOV equation.

2.7 Conclusions

The theory of relativity gives an idea in understanding the physical phenomenon using the concept of spacetime curvature After that, the physicist found this approach can explain and predict an interaction more precisely and correctly than Newtonian mechanics. Special relativity is a special case of general relativity. In this thesis, our work is based on general theory of relativity. The work will perform on both matter and energy part and spacetime part.

In this chapter, new solutions of perfect fluid spheres can be generated by applying theorems to initial solution. We can classify solutions of perfect fluid spheres to seed and non-seed metrics by considering results after applying theorem 1 and theorem 2.

We have established several relationships among the generating theorem. The transformation theorems provide the unexpected structure of perfect fluid spheres solutions and yield a new way of viewing the interrelationships between different static fluid spheres.

Moreover, the solution generating theorems for the TOV equation in terms of pressure and density can be generated. The deformed solutions are parameterized in terms of $\delta\rho_c$ and δp_c.

Chapter 3
Charged fluid sphere and solution generating theorem

The main work of this thesis is to consider anisotropic pressure spherical system. The system is interesting because it is the next step which try to explain a realistic object in our universe after perfect fluid spheres. The important method, that we use, is also a solution generating theorem. Therefore, building steps of the theorems for charged fluid spheres are similar to the algorithm in Chapter 2. The application of the solution generating theorems for charged fluid spheres is also shown in this chapter.

3.1 Introduction to anisotropic pressure spheres

For describing the interior of a spherical object in the general relativistic frame, some objects can be considered using the concept of perfect fluid spheres for simplicity. The absence of heat conduction and shear stress, and the presence of isotropic pressure are the characteristics of perfect fluid spheres. In this thesis, we are interested in generating a solution for charged fluid spheres (Bayin, 1982; Herrera, Ospino, & Di Prisco, 2008; Patel & Mehta, 1995). The particular property of anisotropic pressure, which differs from the property of perfect fluid spheres, is that the radial pressure and the transverse pressure are not equal. One cause of anisotropy is the presence of charge inside an object. Charged fluid spheres with anisotropic pressure are models for describing a charged star such as a neutron star (Boonserm et al., 2015; Heintzmann & Hillebrandt, 1975; Sulaksono, 2015). An important tool in studying fluid sphere solutions is a solution generating algorithm. This technique can be used to generate a new solution from known solutions without having to solve the Einstein field equations directly. A solution generating theorem for charged fluid spheres are constructed in terms of the metric of spacetime.

Many solutions of the Einstein field equations with perfect fluid constraints were explored in various coordinate systems (Bondi, 1947; Buchdahl, 1959). Each solution explains different systems in different conditions. Recently, some algorithmic techniques have been developed to generate new solutions from known perfect fluid spheres (Boonserm et al., 2005, 2007; Thairatana, 2013). This concept also classifies the type of metric solution and exhibits the association of the solutions in distinct conditions.

Perfect fluid spheres are the first approximation of solution for many objects. But there are also many other spherical objects that do not fit the properties of perfect fluid spheres. One kind of such object is a neutron star. The radial pressure of a neutron star may differ from its tangential pressure (Boonserm et al., 2015; Heintzmann & Hillebrandt, 1975; Sulaksono, 2015). The applied idea for these stars is referred to as charged fluid spheres with anisotropic pressure.

For finding a solution generating theorem, a metric (or line element) of an object was written in a suitable coordinate. By definition of the geometric element of the Einstein field equations, $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R,$ the Einstein tensor $G_{\mu\nu}$ can be obtained from the metric. We can set the fluid constraints to $G_{\mu\nu}$ for ordinary differential equation, which can be used to build the solution generating theorems. These theorems deform fluid spheres in terms of spacetime geometry.

In case of spherical objects with a charge on the inside, the electromagnetic charge makes the pressures of the objects become anisotropic. The radial pressure and the transverse pressure in these kinds of objects are unequal. The energy-momentum-stress tensor for anisotropic pressure spheres is

$$T_{\hat{a}\hat{b}} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p_r & 0 & 0 \\ 0 & 0 & p_t & 0 \\ 0 & 0 & 0 & p_t \end{pmatrix}. \tag{3.1}$$

The constraint of perfect fluid spheres cannot be used in constructing the solution generating theorems. Therefore, we have to set a new ordinary differential equation that satisfies charged fluid spheres from the Einstein field equations. This equation gives an Einstein tensor that agrees with the anisotropic energy momentum stress tensor. This model is a primary model that can be applied to a real object in our universe; the neutron stars (Bayin, 1982; Boonserm et al., 2015; Patel & Mehta, 1995; Sulaksono, 2015).

3.2 Anisotropic pressure system: a neutron star

The type of compact star that a star will eventually become, in the end point, is up to the mass of the star. A neutron star is one type of an end point star. It is the result of gravitational collapse of a star into a very small size compared to the mass. The matter of the star is condensed and any chemical potential becomes broken up. Finally, there are only neutrons and elementary particles that remain in the star (Herrera et al., 2008; Ray & Das, 2004).

In studying a neutron star, we can observe the star as pulsars which are created from the rapid rotation of the neutron star. Pulsars are the key evidence of magnetic field in the stars. Additionally, the large magnetic field that can be observed indicate that the magnetic field in a star are anisotropic (Sulaksono, 2015). Moreover, we also found that the matter within a neutron star constitutes several particle types with different charges. The lightest charged particles, electrons, pass into the boundary and create an unbalanced electric field on the surface of the star. The electric field affects the energy-momentum-stress tensor, the tensor that was filled with four-momentum across the surface, which gives us the anisotropic pressure. A neutron star is an example of an anisotropic pressure object that we are particularly interested in this study.

3.3 The solution generating theorems for charged anisotropic pressure spheres and classifying steps

Suppose the solution of charged fluid spheres in Schwarzschild coordinates is defined as a specific metric (Boonserm et al., 2005)

$$ds^2 = -\zeta_0(r)^2 dt^2 + \frac{dr^2}{B_0(r)} + r^2 d\Omega^2, \tag{3.2}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, or with the notation

$$\{\zeta_0, B_0\}. \tag{3.3}$$

From the definition of Einstein tensor, we obtain its components (Boonserm et al., 2015)

$$G_{\hat{r}\hat{r}} = \frac{4B\zeta' r - 2\zeta + 2\zeta B}{2r^2\zeta}, \tag{3.4}$$

$$G_{\hat{\theta}\hat{\theta}} = \frac{B'\zeta r + 2B\zeta' r + 2r^2\zeta'' B + r^2\zeta' B'}{2r^2\zeta}. \tag{3.5}$$

Then setting $G_{\hat{r}\hat{r}} - G_{\hat{\theta}\hat{\theta}} = 8\pi\Delta(r)$, we obtain a second order homogeneous linear ODE for $\zeta(r)$ as

$$2r^2B\zeta'' + \left(r^2B' - 2Br\right)\zeta' + \left(2 - 2B + B'r + 16\pi\Delta r^2\right)\zeta = 0, \tag{3.6}$$

which can be rearranged to a first order nonhomogeneous linear ODE for $B(r)$ as

$$\left(r^2\zeta' + r\zeta\right)B' + \left(2r^2\zeta'' - 2r\zeta' - 2\zeta\right)B + \left(2 + 16\pi\Delta r^2\right)\zeta = 0. \tag{3.7}$$

3.3.1 The solution generating theorems for charged anisotropic pressure spheres

Theorem 1: Suppose $\{\zeta_0(r), B_0(r)\}$ represents a charged fluid sphere. Define

$$\Delta_0 = \left(\frac{\zeta_0(r)}{\zeta_0(r) + r\zeta_0'(r)}\right)^2 r^2 \exp\left\{2\int \frac{\zeta_0'(r)}{\zeta_0(r)} \frac{\zeta_0(r) - r\zeta_0'(r)}{\zeta_0(r) + r\zeta_0'(r)} dr\right\}. \tag{3.8}$$

We can define a new metric solution for charged fluid sphere with fixed $\zeta_0(r)$ *as follows*

$$ds^2 = -\zeta_0(r)^2 dt^2 + \frac{dr^2}{B_0(r) + \lambda\Delta_0(r)} + r^2 d\Omega^2, \tag{3.9}$$

where λ *is a constant. The mapping for theorem 1 is*

$$T_1(\lambda): \{\zeta_0, B_0\} \mapsto \{\zeta_0, B_0 + \lambda\Delta_0(\zeta_0)\}. \tag{3.10}$$

If we apply theorem 1 at the second time, we get the same form of the solution as applying theorem 1 first time. Therefore, T_1 *is "idempotent", in the sense that*

$$T_1(\lambda_n)\circ\ldots\circ T_1(\lambda_2)\circ T_1(\lambda_1):\{\zeta_0, B_0\}\mapsto\left\{\zeta_0, B_0+\left(\sum_{i=1}^{n}\lambda_i\right)\Delta_0(\zeta_0)\right\}. \quad (3.11)$$

For a general form

$$\prod_{i=1}^{n} T_1 \triangleq T_1, \quad (3.12)$$

the symbol ≜ *represents "equality up to relabeling of the parameters"*(Boonserm *et al.,* 2005).

Proof for theorem 1: Assume the metric represents a charged fluid sphere

$$ds^2 = -\zeta_0(r)^2 dt^2 + \frac{dr^2}{B_0(r)} + r^2 d\Omega^2. \quad (3.13)$$

We know ζ_0 and B_0 are solutions of charged fluid sphere. Therefore, they satisfy ODE (3.7)

$$\left(r^2\zeta'+r\zeta\right)B'+\left(2r^2\zeta''-2r\zeta'-2\zeta\right)B+\left(2+16\pi\Delta r^2\right)\zeta_0=0.$$

If we modified the metric as

$$ds^2 = -\zeta_0(r)^2 dt^2 + \frac{dr^2}{B_1(r)} + r^2 d\Omega^2, \quad (3.14)$$

where $B_1(r)=B_0(r)+\lambda\Delta_0(r)$. For this metric, ζ_0 and B_1 also satisfy charged fluid spheres

$$\left(r^2\zeta_0'+r\zeta_0\right)B_1'+\left(2r^2\zeta_0''-2r\zeta_0'-2\zeta_0\right)B_1+\left(2+16\pi\Delta r^2\right)\zeta_0=0. \quad (3.15)$$

Substituting $B_1(r)=B_0(r)+\lambda\Delta_0(r)$ into equation (3.15)

$$\left(r^2\zeta_0'+r\zeta_0\right)B_0'+\left(r^2\zeta_0'+r\zeta_0\right)\lambda\Delta_0'+\left(2r^2\zeta_0''-2r\zeta_0'-2\zeta_0\right)B_0$$
$$+\left(2r^2\zeta_0''-2r\zeta_0'-2\zeta_0\right)\lambda\Delta_0+\left(2+16\pi\Delta r^2\right)\zeta_0=0. \quad (3.16)$$

Using equation (3.7) gives

$$\left(r^2\zeta_0'+r\zeta_0\right)\lambda\Delta_0'+\left(2r^2\zeta_0''-2r\zeta_0'-2\zeta_0\right)\lambda\Delta_0=0 \quad (3.17)$$

This first order homogeneous linear differential equation in $\boldsymbol{\Delta_0}$ leads to

$$\Delta_0 = \left(\frac{\zeta_0(r)}{\zeta_0(r) + r\zeta_0{}'(r)} \right)^2 r^2 \exp\left\{ 2\int \frac{\zeta_0{}'(r)}{\zeta_0(r)} \frac{\zeta_0(r) - r\zeta_0{}'(r)}{\zeta_0(r) + r\zeta_0{}'(r)} dr \right\}. \qquad (3.18)$$

Proof for idempotent property: For a second application of theorem 1, the mapping is

$$T_1(\lambda_2) \circ T_1(\lambda_1) : \{\zeta_0, B_0\} \mapsto \{\zeta_0, B_2\}, \qquad (3.19)$$

where $B_2(r) = B_1(r) + \lambda_2 \Delta_0(r)$ and $B_1(r) = B_0(r) + \lambda_1 \Delta_0(r)$. So,

$$T_1(\lambda_2) \circ T_1(\lambda_1) : \{\zeta_0, B_0\} \mapsto \{\zeta_0, B_0 + (\lambda_2 + \lambda_1)\Delta_0\}. \qquad (3.20)$$

For n applications of theorem 1, the mapping is

$$T_1(\lambda_n) \circ \ldots \circ T_1(\lambda_2) \circ T_1(\lambda_1) : \{\zeta_0, B_0\} \mapsto \left\{ \zeta_0, B_0 + \left(\sum_{i=1}^{n} \lambda_i \right) \Delta_0(\zeta_0) \right\}. \qquad (3.21)$$

Corollary 1: Let $\{\zeta_0, B_a\}$ and $\{\zeta_0.B_b\}$ both represent charged fluid spheres, then for all p, $\{\zeta_0, pB_a + (1-p)B_b\}$ is also a charged fluid sphere. Furthermore, all fluid spheres for a fixed ζ_0 can also be written in this form.

Proof for corollary 1: Suppose $\{\zeta_0, B_a\}$ and $\{\zeta_0.B_b\}$ represent charged fluid spheres, the solutions could satisfy (3.7)

$$\left(r^2\zeta' + r\zeta\right)(B_a)' + \left(2r^2\zeta'' - 2r\zeta' - 2\zeta\right)B_a + \left(2 + 16\pi\Delta r^2\right)\zeta = 0, \qquad (3.22)$$

$$\left(r^2\zeta' + r\zeta\right)(B_b)' + \left(2r^2\zeta'' - 2r\zeta' - 2\zeta\right)B_b + \left(2 + 16\pi\Delta r^2\right)\zeta = 0. \qquad (3.23)$$

$\{\zeta_0, pB_a + (1-p)B_b\}$ also represents charged fluid spheres,

$$\left(r^2\zeta' + r\zeta\right)\left(pB_a + (1-p)B_b\right)' + \left(2r^2\zeta'' - 2r\zeta' - 2\zeta\right)\left(pB_a + (1-p)B_b\right) + \left(2 + 16\pi\Delta r^2\right)\zeta = 0$$

$$\left(r^2\zeta'+r\zeta\right)pB_a'+\left(r^2\zeta'+r\zeta\right)(1-p)B_b'$$
$$+\left(2r^2\zeta''-2r\zeta'-2\zeta\right)\left(pB_a+(1-p)B_b\right)+\left(2+16\pi\Delta r^2\right)\zeta=0. \tag{3.24}$$

From (3.24) – [p *(3.22) + $(1-p)$*(3.23)];

$$-p\left(2+16\pi\Delta r^2\right)\zeta+(1-p)\left(2+16\pi\Delta r^2\right)\zeta+\left(2+16\pi\Delta r^2\right)\zeta=0. \tag{3.25}$$

Therefore, $\left\{\zeta_0, pB_a+(1-p)B_b\right\}$ represent a charged fluid sphere.

Theorem 2: Suppose $\{\zeta_0(r), B_0(r)\}$ represents a charged fluid sphere. Define

$$Z_0(r)=\sigma+\varepsilon\int\frac{rdr}{\zeta_0(r)^2\sqrt{B_0(r)}}. \tag{3.26}$$

Then for all σ and ε, we can define a new metric solution for charged fluid sphere with fixed $\zeta_0(r)$ as follows

$$ds^2=-\zeta_0(r)^2 Z_0(r)^2 dt^2+\frac{dr^2}{B_0(r)}+r^2 d\Omega^2. \tag{3.27}$$

The mapping of theorem 2 is

$$T_2(\lambda):\{\zeta_0,B_0\}\mapsto\{\zeta_0 Z_0(\zeta_0,B_0),B_0\}. \tag{3.28}$$

If we apply theorem 2 at a second time, we get the same form of the solution as applying theorem 2 first time. Therefore, T_2 is "idempotent", in the sense that [1, 11, 22]

$$T_2(\sigma_n,\varepsilon_n)\circ\ldots\circ T_2(\sigma_2,\varepsilon_2)\circ T_2(\sigma_1,\varepsilon_1)=T_2(\sigma_n\ldots\sigma_2\sigma_1,\varepsilon_{n\ldots321}), \tag{3.29}$$

where

$$\varepsilon_{n\ldots321}=\left(\varepsilon_1\sigma_2\sigma_3\ldots\sigma_n\right)+\left(\sigma_1^{-1}\varepsilon_2\sigma_3\ldots\sigma_n\right)+\left(\sigma_1^{-1}\sigma_2^{-1}\varepsilon_3\ldots\sigma_n\right)+\ldots+\left(\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\ldots\varepsilon_n\right). \tag{3.30}$$

Proof for theorem 2: Assume the metric represents a charged fluid sphere

$$ds^2=-\zeta_0(r)^2 dt^2+\frac{dr^2}{B_0(r)}+r^2 d\Omega^2. \tag{3.31}$$

We know ζ_0 and B_0 are solutions of charged fluid sphere. Therefore, they satisfy ODE (3.6)

$$2r^2B_0\zeta_0''+\left(r^2B_0'-2B_0r\right)\zeta_0'+\left(2-2B_0+B_0'r+16\pi\Delta r^2\right)\zeta_0=0.$$

The generated metric should be in the form as follows

$$ds^2=-\zeta_0(r)^2Z_0\left(r\right)^2dt^2+\frac{dr^2}{B_0(r)}+r^2d\Omega^2. \tag{3.32}$$

Then $\zeta_1(r)=\zeta_0(r)Z_0(r).$ For this metric, ζ_1 and B_0 also satisfy charged fluid spheres

$$2r^2B_0\zeta_1''+\left(r^2B_0'-2B_0r\right)\zeta_1'+\left(2-2B_0+B_0'r+16\pi\Delta r^2\right)\zeta_1=0. \tag{3.33}$$

Substituting $\zeta_1(r)=\zeta_0(r)Z_0(r)$ into equation (3.6)

$$\begin{aligned}&\left[2r^2B_0\right]\zeta_0''Z_0+2\left[2r^2B_0\right]\zeta_0'Z_0'+\left[2r^2B_0\right]\zeta_0Z_0''+\left[r^2B_0'-2B_0r\right]\zeta_0'Z_0\\&\qquad+\left[r^2B_0'-2B_0r\right]\zeta_0Z_0'+\left[2-2B_0+B_0'r+16\pi\Delta r^2\right]\zeta_0Z_0=0.\end{aligned} \tag{3.34}$$

Substituting $\zeta_0\left(r\right)$ into equation (3.6), equation (3.34) reduces to

$$\left[r^2B_0'-2B_0r\right]\zeta_0Z_0'+2\left[2r^2B_0\right]\zeta_0'Z_0'+\left[2r^2B_0\right]\zeta_0Z_0''=0. \tag{3.35}$$

Therefore,

$$\begin{aligned}\frac{Z_0''}{Z_0'}&=-\frac{4r^2B_0\zeta_0'+r^2B_0'\zeta_0-2rB_0\zeta_0}{\left[2r^2B_0\right]\zeta_0}\\&=-\frac{2\zeta_0'}{\zeta_0}-\frac{B_0'}{2B_0}+\frac{1}{r}.\end{aligned} \tag{3.36}$$

Integrating over r,

$$\int\frac{Z_0''}{Z_0'}dr=-\int\frac{2\zeta_0'}{\zeta_0}dr-\int\frac{B_0'}{2B_0}dr+\int\frac{1}{r}dr, \tag{3.37}$$

we obtain

$$\ln\left|Z_0'\right|=-2\ln\left|\zeta_0\right|-\frac{1}{2}\ln\left|B_0\right|+\ln\left|r\right|+c. \tag{3.38}$$

Then,

$$Z_0'=\varepsilon\frac{r}{\zeta_0^2\sqrt{B_0}}. \tag{3.39}$$

for all $Z_0' > 0, \zeta_0 > 0$ and $B_0 > 0.$

Finally,

$$Z_0(r) = \sigma + \varepsilon \int \frac{rdr}{\zeta_0(r)^2 \sqrt{B_0(r)}}. \tag{3.40}$$

Proof for idempotent property: For a second application of theorem 2, the mapping is

$$T_2(\sigma_2, \varepsilon_2) \circ T_2(\sigma_1, \varepsilon_1) : \{\zeta_0, B_0\} \mapsto \{\zeta_2, B_0\}, \tag{3.41}$$

where $\zeta_2 = \zeta_1 Z_1(\zeta_1, B_0)$, $\zeta_1 = \zeta_0 Z_0(\zeta_0, B_0)$ and $dZ_0 = \varepsilon_1 \dfrac{rdr}{\zeta_0^2 \sqrt{B_0}}.$ We can substitute ζ_1 in ζ_2,

$$\begin{aligned}
\zeta_2 &= \zeta_0 Z_0(\zeta_0, B_0) Z_1(\zeta_1, B_0) \\
&= \zeta_0 Z_0 \left(\sigma_2 + \varepsilon_2 \int \frac{rdr}{\zeta_1^2 \sqrt{B_0(r)}} \right) \\
&= \zeta_0 Z_0 \left(\sigma_2 + \frac{\varepsilon_2}{\varepsilon_1} \int \frac{rdr}{(\zeta_0 Z_0)^2 \sqrt{B_0(r)}} \right) \\
&= \zeta_0 Z_0 \left(\sigma_2 + \frac{\varepsilon_2}{\varepsilon_2} \int \frac{dZ_0}{Z_0^2} \right) \\
&= \zeta_0 Z_0 \left(\sigma_2 - \frac{\varepsilon_2}{\varepsilon_1} \left[\frac{1}{Z_0} - \frac{1}{\sigma_1} \right] \right) \\
&= \zeta_0 \left(-\frac{\varepsilon_2}{\varepsilon_1} + Z_0 \left[\sigma_2 + \frac{\varepsilon_2}{\varepsilon_1} \frac{1}{\sigma_1} \right] \right),
\end{aligned} \tag{3.42}$$

then

$$Z_1 = -\frac{\varepsilon_2}{\varepsilon_1} + Z_0 \left[\sigma_2 + \frac{\varepsilon_2}{\varepsilon_1} \frac{1}{\sigma_1} \right]. \tag{3.43}$$

As above, the law of composition for theorem 2 is

$$T_2(\sigma_2, \varepsilon_2) \circ T_2(\sigma_1, \varepsilon_1) = T_2\left(\sigma_2 \sigma_1, \varepsilon_1 \sigma_2 + \frac{\varepsilon_2}{\sigma_1} \right). \tag{3.44}$$

Theorem 2 has idempotent property only in special case ($\sigma = 1$ for all σ_n)

$$\prod_{i=1}^{n} T_2(1, \varepsilon_i) = T_2\left(1, \sum_{i=1}^{n} \varepsilon_i\right), \tag{3.45}$$

and for n applications of theorem 2 with the same σ and ε

$$T_2(\sigma, \varepsilon)^n = T_2\left(\sigma^n, \varepsilon\left[\sigma^{n-1} + \sigma^{n-3} + \ldots + \sigma^{-(n-1)} + \sigma^{-(n-3)}\right]\right). \tag{3.46}$$

For large n

$$T_2(\sigma, \varepsilon)^n \approx T_2(\sigma^n, \varepsilon^n) = \sigma^{n-1} T_2(\sigma, \varepsilon) \triangleq T_2(\sigma, \varepsilon). \tag{3.47}$$

Corollary 2: Let $\{\zeta_a, B_0\}$ and $\{\zeta_b, B_0\}$ both represent charged fluid spheres, then for all p, q $\{p\zeta_a + q\zeta_b, B_0\}$ is also a charged fluid sphere. Furthermore, all charged fluid spheres for a fixed B_0 can also be written in this form.

Proof for corollary 2: Suppose $\{\zeta_0, B_a\}$ and $\{\zeta_0, B_b\}$ represent charged fluid spheres. The solutions could satisfy (3.6)

$$2r^2 B_0 (\zeta_a)'' + \left(r^2 B_0' - 2B_0 r\right)(\zeta_a)' + \left(2 - 2B_0 + B_0' r + 16\pi\Delta r^2\right)(\zeta_a) = 0, \tag{3.48}$$

and

$$2r^2 B_0 (\zeta_b)'' + \left(r^2 B_0' - 2B_0 r\right)(\zeta_b)' + \left(2 - 2B_0 + B_0' r + 16\pi\Delta r^2\right)(\zeta_b) = 0. \tag{3.49}$$

Multiplying (3.48) with p and multiplying (3.49) with q and adding together give

$$2r^2 B_0 (p\zeta_a'' + q\zeta_b'') + \left(r^2 B_0' - 2B_0 r\right)(p\zeta_a' + q\zeta_b') + \left(2 - 2B_0 + B_0' r + 16\pi\Delta r^2\right)(p\zeta_a + q\zeta_b) = 0. \tag{3.50}$$

That is the solution $\{p\zeta_a + q\zeta_b, B_0\}$ also represents charged fluid sphere.

3.3.2 Application of the solution generating theorems on Tolman-Bayin solution

The Tolman-Bayin solution is the applied solution from Tolman VI and Bayin solution by S. Ray and B. Das (Ray & Das, 2004). It is a model for electromagnetic mass model. The total charge on the sphere can be calculated using radius, $q = \mathrm{K} r^n$ where n is an integer. With the condition n=0, the given form of The Tolman-Bayin solution metric with n=0 is

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)dt^2 + \left(1 - \frac{2q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{3.51}$$

The Tolman-Bayin solution represents Schwarzschild coordinates metric with charge.

Application of theorem 1 to the Tolman-Bayin solution

By comparing the Tolman-Bayin metric with the Schwarzschild coordinates metric, the ζ_0 can be written as follows

$$\zeta_0 = \sqrt{1 - \frac{2m}{r} + \frac{q^2}{r^2}}. \tag{3.52}$$

In applying theorem 1, use Δ_0 in the form below

$$\Delta_0 = \frac{r^2}{\left[(r\zeta_0)'\right]^2} \exp\left\{\int \frac{4\zeta_0'}{(r\zeta_0)'} dr\right\}, \tag{3.53}$$

This equation is rearranged from equation *(3.8)*

$$\Delta_0 = \left(\frac{\zeta_0(r)}{\zeta_0(r) + r\zeta_0'(r)}\right)^2 r^2 \exp\left\{2\int \frac{\zeta_0'(r)}{\zeta_0(r)} \frac{\zeta_0(r) - r\zeta_0'(r)}{\zeta_0(r) + r\zeta_0'(r)} dr\right\}.$$

From calculations using the Maple programme, we obtain

$$\Delta_0 = -\frac{4q^2}{mr} + \left(4\ln\left[\frac{m}{r} - 1\right]\right)\left(1 - \frac{q^2}{m^2}\right). \tag{3.54}$$

Then for all λ, the new metric is

$$ds^2 = -\zeta_0(r)^2 dt^2 + \frac{dr^2}{B_0(r) + \lambda\Delta_0(r)} + r^2 d\Omega^2. \tag{3.55}$$

With this Δ_0, the metric also satisfies charged fluid spheres.

Application of theorem 2 to the Tolman-Bayin solution

The given form of The Tolman-Bayin solution for n=0 is

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)dt^2 + \left(1 - \frac{2q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{3.56}$$

then

$$\zeta_0 = \sqrt{1 - \frac{2m}{r} + \frac{q^2}{r^2}}, \tag{3.57}$$

and

$$B_0 = \left(1 - \frac{2q^2}{r^2}\right). \tag{3.58}$$

Theorem 2 can be applied to the Tolman-Bayin metric using equation (3.26)

$$Z_0(r) = \sigma + \varepsilon \int \frac{rdr}{\zeta_0(r)^2 \sqrt{B_0(r)}}. \tag{3.59}$$

$$Z_0(r) = \sigma + \varepsilon \int \frac{rdr}{\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)\left(\sqrt{1 - \frac{2q^2}{r^2}}\right)}, \tag{3.60}$$

We can approximate $1 - \frac{2q^2}{r^2}$ term by considering the velocity of light and the gravitational constant. The term, together with the constants, is written as $1 - \frac{2q^2 G}{r^2 c^4}$. Therefore, to reduce the complexity of calculation, this term can be approximated to 1 under the condition $\frac{2q^2 G}{r^2 c^4} \ll 1.$ By using the Maple programme, we obtain

$$Z_0 = \sigma + \varepsilon[\frac{1}{2}r^2 + 2mr + \frac{q^3 \ln(r+g)}{4m+4q} - \frac{q^3 \ln(r-g)}{4m-4q} + \frac{2\ln(-2mr+q^2+r^2)}{(m+q)(m-q)}\left[m^4 - m^2 q + \frac{1}{8}\right]$$

$$+ \frac{\arctan\left[\frac{1}{2}\frac{2r-2m}{\sqrt{-m^2+q^2}}\right]}{[m+q][m-q]\sqrt{-m^2+q^2}}\left[m^4 - m^2 q + \frac{1}{8}\right]], \tag{3.61}$$

then for all σ and ε, the new metric is

$$ds^2 = -\zeta_0(r)^2 Z_0(r)^2 dt^2 + \frac{dr^2}{B_0(r)} + r^2 d\Omega^2. \tag{3.62}$$

With this Z_0, the metric also satisfies anisotropic pressure fluid spheres.

3.3.3 Classifying charged fluid spheres

In referrence to P. Boonserm et. al. (Boonserm et al., 2005), we use a similar concept to classify charged fluid spheres into seed and non-seed metrics based on the following definitions.

Definition (seed metric): take a metric g and apply theorem 1 or theorem 2 to it. Two different cases are possible: each of the applications supplies us with a new solution, $\left[T_1(g) \ncong g \ncong T_2(g)\right]$. We define a metric with this pattern as a seed metric. For example, we apply theorem 1 to the Tolman-Bayin metric to derive a new solution (see equation (3.54)). In addition, when we apply theorem 2 to the Tolman-Bayin metric, we derive a new solution (see equation (3.61)).

Definition (non-seed metric): take a metric g and apply theorem 1 or theorem 2 to it. Two different cases are possible: only one of the applications supplies us with a new solution, while the other one gives us the same metric we started with $\left[T_1(g) \triangleq g\right]$ or $\left[T_2(g) \triangleq g\right]$. These metrics are non-seed metrics.

In the classifying step, the generated solutions are classified either as seed or non-seed solutions. If the generated solution can give rise to a new solution after reapplying it with every generating theorem, the solution is regarded as a seed type. On the other hand, if the reapplication of the generated solution with the generating theorems only results in a new solution from just one generating theorem, the solution is classified as a non-seed type.

3.4 Conclusions

The solution generating theorems for anisotropic pressure spheres can be constructed in terms of the geometric solution of the Einstein field equations, in Schwarzschild coordinates, for a charged fluid sphere. Theorem 1 can generate a new solution in terms of variation of the component of radial coordinate by transforming radial component and fixing time component. Theorem 2 transforms time scaling component to another solution which still satisfy charged fluid sphere system while the radius component is fixed. Both theorems can generate solutions that satisfy anisotropic pressure spheres. The idempotent property and corollary of the theorems are presented in this thesis. The application of the solution generating theorems has been tested with the Tolman-Bayin metric which is one of the static charged type for anisotropic pressure model. We use the maple programme for calculating a new solution from the theorem 1 and theorem 2. As a result, the new solution is obtained from theorem 1 and theorem 2. Finally, we define classification of the types of solutions into seed and non-seed by the solution generating theorems.

Chapter 4
Interior solution for charged fluid spheres

In this chapter, we introduce an interior solution of charged fluid spheres by generalizing the TOV equation. The interior solution of the Einstein field equations, with perfect fluid sphere constrain, satisfies the TOV equation. The electromagnetic field and scalar field inside a sphere are considered in this case (Bayin, 1982; Herrera et al., 2008; Patel & Mehta, 1995; Sulaksono, 2015). In addition, we will construct a solution generating theorem for an interior solution of anisotropic pressure spheres in pressure and density profiles. The interior solution is important to link general relativity to astrophysics.

4.1 Charged fluid spheres

In astrophysics, the interior properties of a stellar structure are often presented in pressure and density profiles. The interior information of an astrophysical objects can determine the type of an object, e.g. white dwarf star, neutron star. Almost all of the stars are anisotropic pressure objects because they always contain charged particles, magnetic field or scalar field. The interior solution of the Einstein field equations in general relativity for ideal objects, perfect fluid spheres, is described by the TOV equation. For improving the solution to get close the realistic objects, we need to consider the anisotropic property of pressures. As in chapter 3, the simple expression of anisotropic pressure spheres can be written in terms of energy momentum stress tensor,

$$T_{\hat{a}\hat{b}} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p_r & 0 & 0 \\ 0 & 0 & p_t & 0 \\ 0 & 0 & 0 & p_t \end{pmatrix}. \tag{4.1}$$

In this chapter, the pressure profiles are emphasized more than in previous chapter. The TOV equation will be modified to depend on the electromagnetic charged and scalar field.

4.2 The modified TOV equation

In this work, we cannot use the perfect fluid constrains because of the electromagnetic field and the massless scalar field within this object. The TOV equation was thus generalized to involve the electromagnetic and the scalar field. This model is close to the realistic objects in our universe such as a Neutron star.

Because of the presence of electromagnetic field and scalar field, the interior properties of an object differ from properties of perfect fluid sphere. In reference to Mimicking static charged fluid spheres in general relativity by P. Boonserm (Boonserm et al., 2007), we can obtain interior solution of anisotropic pressured system via modifying the TOV equation. The energy momentum stress tensor of anisotropic pressured fluid sphere is linear combination of stress tensor of electric charge, magnetic field, and scalar field

$$T_f^{ab} = \left(\rho_f + p_f\right)V^a V^b + p_f g^{ab}, \tag{4.2}$$

$$T_{em}^{ab} = F^{ac} g_{cd} F^{bd} - \frac{1}{4} g^{ab}\left(F_{cd} F^{cd}\right), \tag{4.3}$$

$$T_s^{ab} = \phi^{;a}\phi^{;b} - \frac{1}{2} g^{ab}\left(g^{cd}\phi_{;c}\phi_{;d}\right), \tag{4.4}$$

where F_{ab} is a field strength tensor, V^a is a four velocity, ϕ is a scalar field, and the notation $A_{;b}$ expresses $\nabla_b A$. By using the covariant conservation of the total energy momentum stress tensor $\nabla_b T^{ab} = 0$ and considering the unit vector in the radial direction, we obtain a modified TOV equation

$$\left(\rho_f + p_f\right)V^a_{;b}V^b + g^{ab}\left(\left[p_f\right]_{;b} + \sigma_s \phi_{;b}\right) - F^{ab}\left(\sigma_{em} V_b\right) = 0. \tag{4.5}$$

A four-velocity V^a is defined by $\frac{dx^a}{d\tau}$, where $d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$. We can construct the modified TOV equation with refers the form of the TOV equation as follows

$$\frac{dp_f}{dr} = -\frac{\left(\rho_f + p_f\right)\left(m + 4\pi p_f r^3\right)}{r^2\left(1 - 2m/r\right)} - \frac{\sigma_{em}E}{\sqrt{1 - 2m/r}} - \sigma_s \frac{d\phi}{dr}, \tag{4.6}$$

$$\frac{dm(r)}{dr} = 4\pi\left[\rho_f + \sigma_{em} + \sigma_s\right]r^2. \tag{4.7}$$

Equation (4.6) shows the relationship of a fluid pressure p_f, a fluid density ρ_f, electromagnetic density σ_{em}, and scalar field density σ_s. The other equation is a definition of mass for charged fluid spheres.

4.3 The solution generating theorems and the modified TOV equation

The solution generating theorems for the modified TOV equation can be constructed in terms of pressure and density by Riccati equation solution for theorem 1, and definition of mass on the modified TOV equation in theorem 2.

Theorem 1: *Suppose we know two specific solutions p_1 and p_2 and fix $m_0(r) = \int \rho_0(r) r^2 dr$, the generated general solution for the modified TOV equation is*

$$p(r) = \frac{\lambda \exp\left\{-\int \frac{4\pi r}{\left(1-2m_0/r\right)} p_1(r)dr\right\} p_1(r) + (1-\lambda)\exp\left\{-\int \frac{4\pi r}{\left(1-2m_0/r\right)} p_2(r)dr\right\} p_2(r)}{\lambda \exp\left\{-\int \frac{4\pi r}{\left(1-2m_0/r\right)} p_1(r)dr\right\} + (1-\lambda)\exp\left\{-\int \frac{4\pi r}{\left(1-2m_0/r\right)} p_2(r)dr\right\}}, \tag{4.8}$$

where λ is constant.

Proof of theorem 1: The form of Riccati equation is

$$\frac{dp(r)}{dr} = \alpha(r) + \beta(r)p(r) + \gamma(r)p^2(r). \tag{4.9}$$

The modified TOV equation is

$$\frac{dp_f}{dr} = -\frac{\left(\rho_f + p_f\right)\left(m + 4\pi p_f r^3\right)}{r^2\left(1 - 2m/r\right)} - \frac{\sigma_{em}E}{\sqrt{1 - 2m/r}} - \sigma_s \frac{d\phi}{dr}, \tag{4.10}$$

where ρ_f, p_f, σ_{EM} and σ_s are fluid density, fluid pressure, electromagnetic charge density and scalar field density, respectively.

$$\frac{dp_f}{dr} = -\frac{\rho_f m}{r^2\left(1-2m/r\right)} - \frac{\sigma_{em}E}{\sqrt{1-2m/r}} - \sigma_s\frac{d\phi}{dr} - \frac{\left(m+\rho_f 4\pi r^3\right)}{r^2\left(1-2m/r\right)}p_f$$

$$-\frac{4\pi r^3}{r^2\left(1-2m/r\right)}{p_f}^2. \tag{4.11}$$

We can treat the modified TOV equation as a Riccati equation where

$$\alpha(r) = -\frac{\rho_f m}{r^2\left(1-2m/r\right)} - \frac{\sigma_{em}E}{\sqrt{1-2m/r}} - \sigma_s\frac{d\phi}{dr},$$

$$\beta(r) = -\frac{\left(m+\rho_f 4\pi r^3\right)}{r^2\left(1-2m/r\right)},$$

$$\gamma(r) = -\frac{4\pi r^2}{\left(1-2m/r\right)}.$$

The one-parameter general solution may be written as

$$p(r) = \frac{\lambda\exp\left\{-\int\gamma(r)p_1(r)dr\right\}p_1(r) + (1-\lambda)\exp\left\{-\int\gamma(r)p_2(r)dr\right\}p_2(r)}{\lambda\exp\left\{-\int\gamma(r)p_1(r)dr\right\} + (1-\lambda)\exp\left\{-\int\gamma(r)p_2(r)dr\right\}}. \tag{4.12}$$

Theorem 2: *Suppose we know two specific solutions* p_0 *and* ρ_0 *and fix* g_0*, where the auxiliary function* g_0 *is*

$$g_0(r) = \frac{m_0(r) + 4\pi(r)r^3}{r^2\left[1-2m_0(r)/r\right]}. \tag{4.13}$$

The generated general solution for the modified TOV equation is given by $p(r) = p_0(r) + \delta p_P(r) + \delta p_C(r)$ *and* $m(r) = m_0(r) + \delta m(r),$ *where*

$$\delta m(r) = -\frac{4\pi\left[\delta p_C + \delta p_P\right]r^3}{1+2rg_0(r)}, \tag{4.14}$$

$$\delta p_C(r) = \frac{\delta p(0)}{\left[1+rg_0(r)\right]^2}\frac{1+8\pi p_0(r)r^2}{1-2m_0(r)/r}\exp\left\{2\int_0^r g_0(r)\frac{1-rg_0(r)}{1+rg_0(r)}dr\right\}, \tag{4.15}$$

$$\delta p_P = \delta p_C\int_0^r\left[\left\{-\left[\delta\sigma_{em}\left\{\frac{E}{\sqrt{1-2m/r}} - g_0(r)\right\} + \delta\sigma_s\left\{\frac{d\phi}{dr} - g_0(r)\right\}\right]\right\}\left[\frac{1+2rg_0(r)}{1+rg_0(r)}\delta p_C\right]dr. \tag{4.16}$$

Proof of theorem 2: Function $g_0(r)$ is defined as $g_0\left(r\right)=\frac{m_0\left(r\right)+4\pi p_f\left(r\right)r^3}{r^2\left[1-2m_0\left(r\right)/r\right]}$ can be rearranged in term of $m_0(r)$ as

$$m_0(r)=\frac{g_0(r)r^2-4\pi p_f(r)r^3}{1+2rg_0(r)}. \tag{4.17}$$

We hold $g_0(r)$ fixed for this transformation,

$$\delta m_0(r)=-\frac{4\pi\delta p_f(r)r^3}{1+2rg_0(r)}. \tag{4.18}$$

From the definition of $m_0(r)$

$$\frac{dm_0(r)}{dr}=4\pi\left[\rho_f+\sigma_{em}+\sigma_s\right]r^2, \tag{4.19}$$

$$\frac{d\delta m_0(r)}{dr}=4\pi\left[\delta\rho_f+\delta\sigma_{em}+\delta\sigma_s\right]r^2, \tag{4.20}$$

$$4\pi\left[\delta\rho_f+\delta\sigma_{em}+\delta\sigma_s\right]r^2=\frac{d}{dr}\left(-\frac{4\pi\delta p_f\left(r\right)r^3}{1+2rg_0\left(r\right)}\right),$$

$$\delta\rho_f=\frac{1}{r^2}\frac{d}{dr}\left(-\frac{\delta p_f\left(r\right)r^3}{1+2rg_0\left(r\right)}\right)-\delta\sigma_{em}-\delta\sigma_s. \tag{4.21}$$

We substitute equation (4.21) into the modified TOV equation and simplify it as follows

$$\frac{d\delta p_f}{dr}=-\left(\delta\rho_f+\delta p_f\right)g_0-\frac{\delta\sigma_{em}E}{\sqrt{1-2m/r}}-\delta\sigma_s\frac{d\phi}{dr}, \tag{4.22}$$

$$\frac{d\delta p_f}{dr}=-\left(-\frac{1}{r^2}\frac{d}{dr}\left(\frac{\delta p_f(r)r^3}{1+2rg_0(r)}\right)-\delta\sigma_{em}-\delta\sigma_s+\delta p_f\right)g_0-\frac{\delta\sigma_{em}E}{\sqrt{1-2m/r}}-\delta\sigma_s\frac{d\phi}{dr}, \tag{4.23}$$

$$\left[\frac{1+rg_0(r)}{1+2rg_0(r)}\right]\frac{d\delta p_f}{dr}+g_0\left[1-\frac{d}{dr}\left\{\frac{r^3}{1+2rg_0(r)}\right\}\right]\delta p_f$$
$$=-\delta\sigma_{em}\left[\frac{E}{\sqrt{1-2m/r}}-g_0(r)\right]-\delta\sigma_s\left[\frac{d\phi}{dr}-g_0(r)\right]. \tag{4.24}$$

We obtain a first order linear nonhomogeneous differential equation for $\delta p_f(r)$. We can solve the equation by using integrating factor. The general form of a first order linear nonhomogeneous ordinary differential equation is

$$y'+P(r)y=Q(r). \tag{4.25}$$

This equation gives a solution in terms of the combination of a particular solution and the complementary solution

$$y(r)=y_c(r)+y_p(r), \tag{4.26}$$

where

$$y_C(r)=Ce^{-\int_{r0}^{r}P(r)dr}, \tag{4.27}$$

$$y_P(r)=e^{-\int_{r0}^{r}P(r)dr}\int Q(r)e^{-\int_{r0}^{r}P(r)dr}dr. \tag{4.28}$$

From equation (4.24)

$$P(r)=\left[\frac{1+2rg_0(r)}{1+rg(r)}\right]g_0(r)\left[1-\frac{d}{dr}\left\{\frac{r^3}{1+2rg_0}\right\}\right], \tag{4.29}$$

$$Q(r)=-\left[\delta\sigma_{EM}\left\{\frac{E}{\sqrt{1-2m/r}}-g_0(r)\right\}+\delta\sigma_s\left\{\frac{d\phi}{dr}-g_0(r)\right\}\right]\left[\frac{1+2rg_0(r)}{1+rg_0(r)}\right]. \tag{4.30}$$

Therefore, the solution is $\delta p_f=\delta p_C+\delta p_P$, where

$$\delta p_C=Ce^{-\int_{x0}^{x}\left[\frac{1+2rg_0(r)}{1+rg(r)}\right]g_0(r)\left[1-\frac{d}{dr}\left\{\frac{r^3}{1+2rg_0(r)}\right\}\right]dr}=-\frac{\delta m}{4\pi r^3}\frac{1+8p_0(r)r^2}{1-2m_0(r)/r}, \tag{4.31}$$

$$\delta p_P=\delta p_C\int_0^r\left[\left\{-\left[\delta\sigma_{em}\left\{\frac{E}{\sqrt{1-2m/r}}-g_0(r)\right\}+\delta\sigma_s\left\{\frac{d\phi}{dr}-g_0(r)\right\}\right]\right\}\left[\frac{1+2rg_0(r)}{1+rg_0(r)}\right]\delta p_C\right]dr. \tag{4.32}$$

The new solution in terms of mass is shifted from the initial $m_0(r)$ with the variation

$$\delta m(r)=-\frac{4\pi\delta p_f r^3}{1+2rg_0(r)}=-\frac{4\pi[\delta p_C+\delta p_P]r^3}{1+2rg_0(r)}. \tag{4.32}$$

Theorem 2 can generate new solution for mass and pressure in forms $m(r) = m_0 + \delta m(r)$, and $p_f(r) = p_0(r) + \delta p_P + \delta p_C$.

4.4 The effect of charge on pressure

From the modified TOV equation, we consider an effect of electric charge on the pressure profile (Boonserm et al., 2016).

4.4.1. Special case: when ρ is constant and σ_s is zero

In this case, the generalized TOV equation becomes

$$\frac{dp_f}{dr} = -\frac{\left(\rho_f + p_f\right)\left(m + 4\pi p_f r^3\right)}{r^2\left(1 - 2m/r\right)} - \frac{\sigma_{EM} E}{\sqrt{1 - 2m/r}}, \tag{4.33}$$

supplemented with

$$\frac{dm(r)}{dr} = 4\pi\rho(r)r^2 = 4\pi\left(\rho_f + \rho_{em}\right)r^2. \tag{4.34}$$

Integrating the above equation, we obtain

$$m(r) = \frac{4}{3}\pi\left(\rho_f + \rho_{em}\right)r^3. \tag{4.35}$$

Substituting $m(r)$ into the generalized TOV equation, we can numerically solve for p_f as shown in Figure 3.

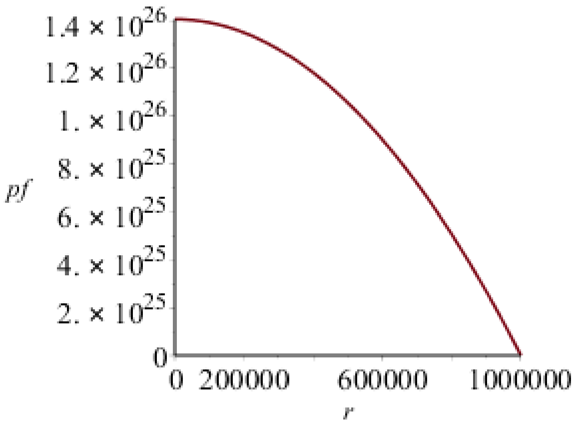


Figure 3: The fluid pressure as a function of radius

4.4.2. The effect between charge and pressure

We consider the relationship between electric charge and pressure inside a sphere. We assume fluid density, and electric field are fixed at all radius. Consider with neutron star model, we suppose the total electric charge inside the star depends on a mass of the star (Chamel, Haensel, Zdunik, & Fantina, 2013; Malheiro & Ray, 2004). The charge can be calculated by comparing to the mass of the sun

$$Q \simeq 10^{20}\frac{M}{M_{sun}} \simeq 2\times 10^{20}\,\mathrm{Coulomb}.$$

The charge density is considered to be constant for all radius of the object. The relationship between charge density and the total for this case is $\sigma_{em} = \frac{4}{3}\pi r^3 Q.$ Therefore, the range of the charged density can be approximated in order of $10^{11} - 10^{12}\,\mathrm{C/m^3}$. For a star with maximum radius $R = 10^4\,\mathrm{m},$ the electric charge affects the perfect fluid pressure at radius $r = 5\times 10^3\,\mathrm{m},$ which can be calculated using the maple programme. The results of the numerical calculation are presented in table 2.

Charge density σ_{em} (C/m^3)	Pressure p_f (N/m^2)
4×10^{11}	1.090×10^{24}
6×10^{11}	1.111×10^{24}
8×10^{11}	1.131×10^{24}
1×10^{12}	1.152×10^{24}
1.2×10^{12}	1.173×10^{24}

Table 2: The effect of charged density on the pressure for the modified TOV equation at radius $5\times 10^3\,\mathrm{m}$ in special case $\rho_f = 10^{13}\,\mathrm{kg/m^3}$ with $R = 10^4\,\mathrm{m}$

From table 2, we can see that when the charge density increases, the pressure also increases. Changing charge affects the fluid pressure. For example, we compare the case $\sigma_{em} = 6\times10^{11}\mathrm{C/m^3}$, with the case $\sigma_{em} = 8\times10^{11}\mathrm{C/m^3}$ as shown in Figure 4 and 5, respectively.

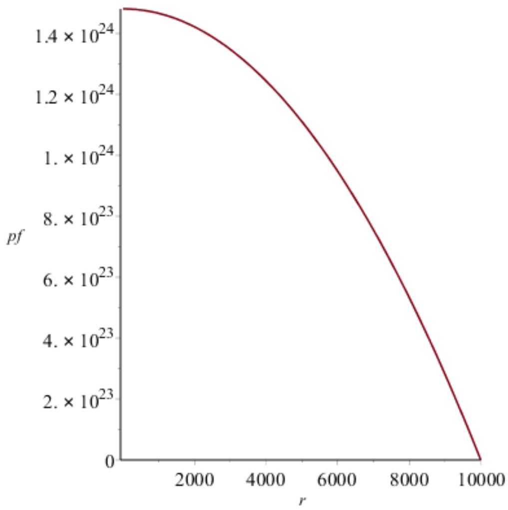


Figure 4: The fluid pressure for $\sigma_{em} = 6\times10^{11}\mathrm{C/m^3}$ with the fluid pressure at the center $p_f\left(r=0\right)=1.408\times10^{24}\,\mathrm{N/m^2}$

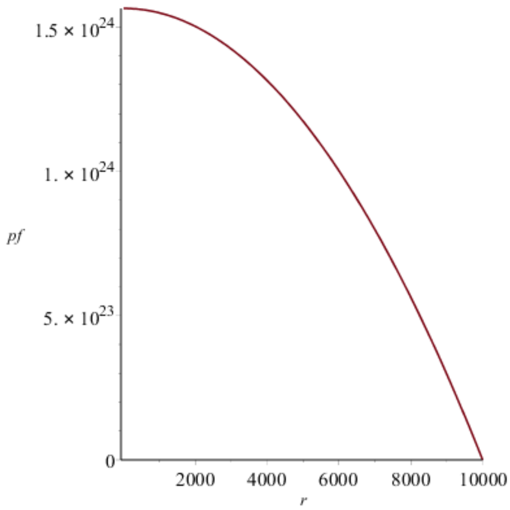


Figure 5: The fluid pressure for $\sigma_{em} = 1.2\times10^{12}\,\mathrm{kg/m^3}$ with the fluid pressure at the center $p_f\left(r=0\right)=1.564\times10^{24}\,\mathrm{N/m^2}$

We found that the higher electric charge gives a higher fluid pressure. Consider at the center of the sphere where the fluid pressure is the highest, the electric charge increase by 2 times, the pressure increase by approximately 1.07 times. Additionally, the star with same maximum radius 10^4 has the effect of electric charge to the fluid pressure as shown in table 4.

Fluid density ρ_f (kg/m^3)	Pressure p_f (N/m^2)
2×10^{13}	4.359×10^{24}
4×10^{13}	1.710×10^{25}
6×10^{13}	3.822×10^{25}
8×10^{13}	6.773×10^{25}
1×10^{14}	1.056×10^{26}

Table 3: The effect of fluid density on the pressure for the generalized TOV equation at radius $5\times10^3\,\mathrm{m}$ in special case $\sigma_{em}=8\times10^{11}\,\mathrm{C/m^3}$ with $R=10^4\,\mathrm{m}$

We can see that when the fluid density increases, the pressure also increases rapidly. Therefore only the electromagnetic charge affects the fluid pressure are less than fluid density.

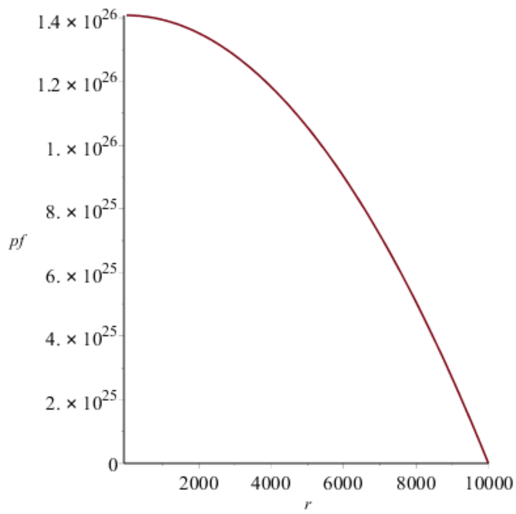


Figure 6: The fluid pressure for $\sigma_{em}=8\times10^{11}\,\mathrm{C/m^3}$ and $\rho_f=10^{14}\,\mathrm{kg/m^3}$ with the fluid pressure at the center $p_f(r=0)=1.408\times10^{26}\,\mathrm{N/m^2}$

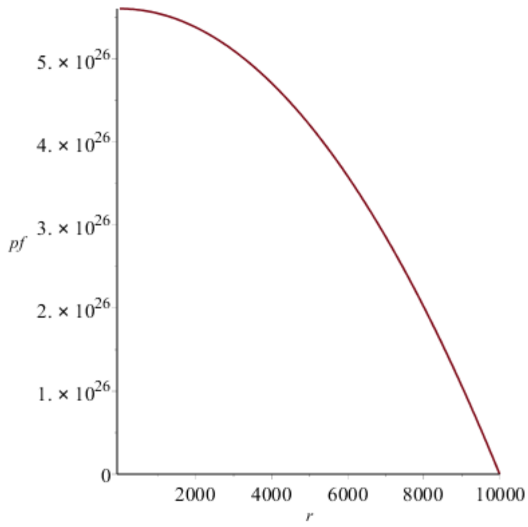


Figure 7: The fluid pressure for $\sigma_{em} = 8\times10^{11}\,\mathrm{C/m^3}$ and $\rho_f = 2\times10^{14}\,\mathrm{kg/m^3}$ with the fluid pressure at the center $p_f\left(r=0\right) = 5.611\times10^{26}\,\mathrm{N/m^2}$

Figure 6 and 7 show the effect of the fluid density on fluid pressure. We can see that when the fluid density increases by 2 times, the pressure can increase by approximately 2.98 times.

4.5 Conclusions

The modified TOV equation is derived to get close a real object such as charged star or neutron star by setting a combination of energy momentum stress tensor of perfect fluid sphere, electromagnetic charge, and scalar field. The solution generating theorems for the modified TOV equation can be created. Theorem 1 gives a general form of a new solution in terms of two known pressures. For Theorem 2, the derived forms of a solution are a new pressure and a shifted term of mass. Both forms are written in terms of charge and infinitesimal variation of pressure at the center. Moreover, we can consider for the effects of charge on the modified TOV equation. The pressure of fluid spheres grows when the charge increases. The electric charge density affects the pressure less than fluid density.

Chapter 5
Conclusions

The thesis began with the overall review of this work. The thesis focuses on a problem of the anisotropic pressure spherical system. General theory of relativity has been described for the understanding of the relevant background information. The theory provides a new idea in the explanations of a system in the form of spacetime geometry. The field is interesting because it can explain certain physical phenomena more precisely than classical mechanics. In this thesis, general relativity is briefly reviewed, particularly focusing on the Einstein field equations and the TOV equation. The anisotropy of pressure is a property of charged fluid spheres, which is one of the solutions of the Einstein field equations. The perfect fluid spheres are important in our work. This system is an ideal approximation for presenting an object in general relativity. However, we modified the system to involve a charged fluid sphere, which is more suitable for studying a realistic stellar object.

A solution generating theorem is a technique we use in studying the anisotropic pressure system. The theorems are constructed for the generation of both a spacetime solution and an interior solution. For a spacetime solution, we have constructed two theorems that can generate a solution. Applications of the theorems are performed on Tolman-Bayin type solution. The new solutions are obtained via the application of the theorems.

Considering metric solution, we focus on the solution as a type of Schwarzschild coordinates. Static and spherically symmetric are the characteristics of the coordinates. The solution is denoted as $\{\zeta_0(r), B_0(r)\}$, where $\zeta_0(r)$ represents a scaling in time coordinate and $B_0(r)$ indicates a geometric property along the radial coordinate. The electromagnetic field and scalar field are added into the constraint of charged fluid spheres. The solution generating theorems for an anisotropic pressure sphere is created in a similar way to the method used in the construction of the theorems for a perfect fluid sphere. The theorems we obtain are as follows;

In deriving theorem 1, the $\zeta_0(r)$ term is held fixed, while the $B_0(r)$ term is transformed to $B_0 + \lambda\Delta_0(\zeta_0)$. The new solution is $\{\zeta_0, B_0 + \lambda\Delta_0(\zeta_0)\}$, where

$$\Delta_0 = \left(\frac{\zeta_0(r)}{\zeta_0(r) + r\zeta_0{}'(r)}\right)^2 r^2 \exp\left\{2\int \frac{\zeta_0{}'(r)}{\zeta_0(r)} \frac{\zeta_0(r) - r\zeta_0{}'(r)}{\zeta_0(r) + r\zeta_0{}'(r)} dr\right\}. \tag{5.1}$$

We can see that the new solution has a different appearance to the initial solution, only in term of the radial coordinate. The shifted term is a function of time scaling of the initial solution.

In deriving theorem 2, from theorem 1, the $B_0(r)$ term is held fixed, while the $\zeta_0(r)$ term is transformed to $\zeta_0 Z_0(\zeta_0, B_0)$, with the new solution being $\{\zeta_0 Z_0(\zeta_0, B_0), B_0\}$, where

$$Z_0(r) = \sigma + \varepsilon \int \frac{rdr}{\zeta_0(r)^2 \sqrt{B_0(r)}}. \tag{5.2}$$

We can see that the new solution has a different appearance to the initial solution, only in time coordinate. The shift of the component of time coordinate has been shown to depend on its own scaling factor $\zeta_0(r)$ and $B_0(r)$.

For the interior solution, the modified TOV describes the internal structure of a charged fluid sphere. It is derived from the TOV equation combined with the electromagnetic field and the scalar field. The solution of the modified TOV equation can be written in terms of the initial pressure p_0 and the initial density ρ_0. A mass is defined by density, thus m_0 can be a solution of the modified TOV equation.

In constructing theorem 1, the theorem can be simplified in the form of Riccati equation in term of perfect fluid pressure p_f. Hence we can use the general solution of the Riccati equation as a solution of the modified TOV equation. Fixing m_0 leads to the constant ρ_0 in the transformation. For the starting solutions p_1 and p_2, a generated solution by theorem 1 is in the form of

$$p(r) = \frac{\lambda \exp\left\{-\int \gamma(r) p_1(r) dr\right\} p_1(r) + (1-\lambda) \exp\left\{-\int \gamma(r) p_2(r) dr\right\} p_2(r)}{\lambda \exp\left\{-\int \gamma(r) p_1(r) dr\right\} + (1-\lambda) \exp\left\{-\int \gamma(r) p_2(r) dr\right\}}. \tag{5.3}$$

In building theorem 2, the definition of mass, for charged fluid spheres, is substituted into the modified TOV equation. With integration by-parts, a form of the new solution can be written as $p_f\left(r\right)=p_0\left(r\right)+\delta p_P+\delta p_C$ and $m(r)=m_0+\delta m\left(r\right)$, where

$$\delta p_C=Ce^{-\int_{x_0}^{x}\left[\frac{1+2rg_0(r)}{1+rg(r)}\right]g_0(r)\left[1-\frac{d}{dr}\left\{\frac{r^3}{1+2rg_0(r)}\right\}\right]dx}=-\frac{\delta m}{4\pi r^3}\frac{1+8p_0(r)r^2}{1-2m_0(r)/r}, \tag{5.4}$$

$$\delta p_C(r)=\frac{\delta p(0)}{\left[1+rg_0(r)\right]^2}\frac{1+8\pi p_0(r)r^2}{1-2m_0(r)/r}\exp\left\{2\int_0^r g_0(r)\frac{1-rg_0(r)}{1+rg_0(r)}dr\right\}, \tag{5.5}$$

$$\delta m\left(r\right)=-\frac{4\pi\delta p_f r^3}{1+2rg_0(r)}=-\frac{4\pi\left[\delta p_C+\delta p_P\right]r^3}{1+2rg_0(r)}. \tag{5.6}$$

For the first advantage of the solution generating theorems, we can obtain a solution without solving through the complicated the Einstein field equations. As for the other advantage, the relationship between the distinct solutions is apparent through the theorem. However, the solution generating theorems have a limitation. An initial solution is needed to make the theorem. This is especially true for theorem 1, where the theorem, derived from the modified TOV equation, requires 2 initial solutions in determining a new solution. Because of this, the solution generating theorems cannot generate a complete set of solutions for charged fluid spheres. It can only build solutions under the condition of a starting solution.

Considering the effect of charge on pressure, a relationship between electric charge and fluid pressure can be considered using the modified TOV equation. The absence of the scalar field and the constant electric field are the conditions necessary for our special case. First, we investigate the pressure by varying the radius of an object at a constant scalar field. The fluid pressure reduces when radius increases. The charge density has an effect on the fluid pressure. When the charge density increases, the pressure also rises.

In this thesis, the solution generating theorems for charged fluid spheres are created in terms of metric and interior solution. They can be beneficial in solving for the solution of charged fluid spheres. The modified TOV equation is analyzed for the influence of charge on the pressure. This is a little step in the field of astronomy in trying to understand a realistic spherical object. In the future, if unknown solutions are observed, the solution generating theorems can prove to be one of the several techniques that can be helpful in understanding the solutions even more.

As for further work, the other forms of metric in Schwarzschild coordinates should also be applied with the solution generating theorems for the generation of a new solution. With a higher number of generated solutions, the relationship between charged fluid sphere solutions will become clearer. Moreover, a solution generating theorem can also be constructed in other coordinates of metric. The solution from the modified TOV equation can be investigated for other physical quantities, e.g. central red shift and surface shift. Almost all types of fluid spheres have a non-constant fluid density along its radius. Hence, the electric charge effects can be considered in comparing the fluid pressure with variation of fluid density in terms of the radius. Additionally, the electric field, in terms of electric charge density, may also have an effect on the fluid pressure.

REFERENCES

Bayin, S. S. (1982). Anisotropic fluid spheres in general relativity. *Physical Review D, 26*, 1262.

Bondi, H. (1947). Spherically symmetrical models in general relativity. *Monthly Notices of the Royal Astronomical Society, 107*, 410.

Boonserm, P., Jongjittanon, N., & Ngampitipan, T. (2016). The Modified Tolman-Oppenheimer-Volkov (TOV) Equation and the Effect of Charge on Pressure in Charge Anisotropy. *American Journal of Physics and Applications, 4*, 57.

Boonserm, P., Ngampitipan, T., & Visser, M. (2015). Mimicking static anisotropic fluid spheres in general relativity. *International Journal of Modern Physics D, 25*, 1650019.

Boonserm, P., Visser, M., & Weinfurtner, S. (2005). Generating perfect fluid spheres in general relativity. *Physical Review D, 71*, 124037.

Boonserm, P., Visser, M., & Weinfurtner, S. (2007). Solution generating theorems for the TOV equation. *Physical Review D, 76*, 044024.

Buchdahl, H. A. (1959). General relativistic fluid spheres. *Physical Review, 116*, 1027.

Carrol, S. M. (2014). *Spacetime and geometry: an introduction to general relativity*. U.S.A.: Pearson Education Limited.

Chamel, N., Haensel, P., Zdunik, J. L., & Fantina, A. F. (2013). On the maximum mass of neutron stars. *International Journal of Modern Physics E, 22*, 1330018.

Delgaty, M. S. R., & Lake, K. (1998). Physical acceptability of isolated, static, spherically symmetric, perfect fluid solutions of Einstein's equations. *Computer Physics Communications, 115*, 395.

Hartle, J. B. (2014). *Gravity: An Introduction to Einstein's General Relativity*. U.S.A.: Pearson Education Limited.

Heintzmann, H., & Hillebrandt, W. (1975). Neutron stars with an anisotropic equation of state ---- Mass, redshift and stability. *Astronomy and Astrophysics, 38*, 51.

Herrera, L., Ospino, J., & Di Prisco, A. (2008). All static spherically symmetric anisotropic solutions of Einstein's equations. *Physical Review D, 77*, 027502.

Lake, K. (2003). All static spherically symmetric perfect-fluid solutions of Einstein's equations. *Physical Review D, 67*, 104015.

Malheiro, M., & Ray, S. (2004). Electrically charged nutron stars. *Workshop on strong magnetic field and neutron stars, Ed. World Scientific*, 253.

Patel, L. K., & Mehta, N. P. (1995). An exact model of an anisotropic relativistic sphere. *Australian Journal of Physics, 48*, 635.

Rahman, S., & Visser, M. (2002). Spacetime geometry of static fluid spheres. *Classical and Quantum Gravity, 19*, 935.

Ray, S., & Das, B. (2004). Tolman–Bayin type static charged fluid spheres in general relativity. *Monthly Notices of the Royal Astronomical Society, 349*, 1331.

Sulaksono, A. (2015). Anisotropic pressure and hyperons in neutron stars. *International Journal of Modern Physics E, 24*, 1550007.

Thairatana, K. (2013). *Transformation for perfect fluid spheres in isotropic coordinates.* (Master of Science), Chulalongkorn University.

APPENDICES

Appendix A1

Example of some starting charged fluid sphere metrics

Solution of the Einstein field equations, which provides descriptions of spheres with electromagnetic charge, is one of the models for realistic astronomical objects. The solution can be a starting metric for the solution generating theorems. Some of the anisotropic pressured fluid spheres in metric form are presented in the table below.

Name	Metric form $\left(ds^2\right)$
Reissner-Nordstrom (for a non-rotating charged spherical object)	$-\left(1-\frac{R_s}{r}+\frac{R_Q^2}{r^2}\right)dt^2+\left(1-\frac{R_s}{r}+\frac{R_Q^2}{r^2}\right)^{-1}dr^2+r^2d\Omega^2.$
Tikekar (for positive pressure and negative gradient pressure)	$-\left[A\left(-\frac{11}{24}-\frac{7r^2}{12R^2}+\frac{49r^4}{24R^4}\right)+B\sqrt{1-\frac{r^2}{R^2}}\left(\frac{1}{8}+\frac{7r^2}{8R^2}\right)^{3/2}\right]^2dt^2$ $+\frac{1}{B_0}dr^2+r^2d\Omega^2.$
Tolman-Bayin (specific choice $n=0$ and the total gravitational mass m, and radius a	$-\left(1-\frac{2m}{a}+\frac{q^2}{a^2}\right)dt^2+\left(1-\frac{2q^2}{a^2}\right)^{-1}dr^2+r^2d\Omega^2.$
Psuedo-spheroidal (where A and B are constants, and $u^2=\left\{k/\left(k-1\right)\right\}\left(1+r^2/R^2\right)$	$-\left(Au+B\left\{u\log\left(u+\sqrt{u^2-1}\right)-\sqrt{u^2-1}\right\}\right)^2dt^2+\frac{1+\frac{kr^2}{R^2}}{1+\frac{r^2}{R^2}}dr^2$ $+r^2d\Omega^2.$

In this thesis, the solutions from applying the theorems are calculated in Maple programme. The code of calculation for the Tolman-Bayin solution is shown below.

Calculation for the application of theorem 1 on the Tolman-Bayin metric

The form of Tolman-Bayin metric with $n=0$

$$ds^2=-\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)dt^2+\left(1-\frac{2q^2}{r^2}\right)^{-1}dr^2+r^2d\Omega^2.$$

> *restart*

> $f(r) := \sqrt{1-\frac{2\cdot m}{r}+\frac{q^2}{r^2}}$

$$f := r \to \sqrt{1-\frac{2\,m}{r}+\frac{q^2}{r^2}}$$

>

> $\frac{d}{dr} f(r)$

$$\frac{1}{2}\,\frac{\frac{2\,m}{r^2}-\frac{2\,q^2}{r^3}}{\sqrt{1-\frac{2\,m}{r}+\frac{q^2}{r^2}}}$$

> $\frac{4\cdot\frac{d}{dr}f(r)}{r\cdot\frac{d}{dr}f(r)+f(r)}$

$$\left(2\left(\frac{2\,m}{r^2}-\frac{2\,q^2}{r^3}\right)\right) \Bigg/ \left(\sqrt{1-\frac{2\,m}{r}+\frac{q^2}{r^2}}\left(\frac{1}{2}\,\frac{r\left(\frac{2\,m}{r^2}-\frac{2\,q^2}{r^3}\right)}{\sqrt{1-\frac{2\,m}{r}+\frac{q^2}{r^2}}}+\sqrt{1-\frac{2\,m}{r}+\frac{q^2}{r^2}}\right)\right)$$

> $\int \frac{4\cdot\frac{d}{dr}f(r)}{r\cdot\frac{d}{dr}f(r)+f(r)}\,dr$

$$-\frac{4\,q^2}{m\,r} + 4\ln\left(2\,m^2 - 2\,q^2 - \frac{2\,(m^2-q^2)\,r}{m}\right)$$
$$-\frac{4\ln\left(2\,m^2 - 2\,q^2 - \frac{2\,(m^2-q^2)\,r}{m}\right)q^2}{m^2}$$
$$-4\ln\left(\frac{2\,(m^2-q^2)\,r}{m}\right)$$
$$+\frac{4\ln\left(\frac{2\,(m^2-q^2)\,r}{m}\right)q^2}{m^2}$$

>

Calculation for the application of theorem 2 on the Tolman-Bayin metric

> *restart*

> $\displaystyle\int \frac{r}{1-\frac{2\cdot m}{r}+\frac{2\cdot m\cdot q^2}{r^3}-\frac{q^4}{r^4}}\,\mathrm{d}r$

$$\frac{1}{2}\,r^2 + 2\,m\,r + \frac{q^3\ln(r+q)}{4\,m+4\,q}$$
$$+\frac{2\ln(-2\,m\,r+q^2+r^2)\,m^4}{(m+q)\,(m-q)}$$
$$-\frac{2\ln(-2\,m\,r+q^2+r^2)\,m^2\,q^2}{(m+q)\,(m-q)}$$
$$+\frac{1}{4}\,\frac{\ln(-2\,m\,r+q^2+r^2)\,q^4}{(m+q)\,(m-q)}$$
$$+\frac{4\arctan\left(\frac{1}{2}\,\frac{2\,r-2\,m}{\sqrt{-m^2+q^2}}\right)m^5}{(m+q)\,(m-q)\,\sqrt{-m^2+q^2}}$$
$$-\frac{6\arctan\left(\frac{1}{2}\,\frac{2\,r-2\,m}{\sqrt{-m^2+q^2}}\right)m^3\,q^2}{(m+q)\,(m-q)\,\sqrt{-m^2+q^2}}$$
$$+\frac{2\arctan\left(\frac{1}{2}\,\frac{2\,r-2\,m}{\sqrt{-m^2+q^2}}\right)m\,q^4}{(m+q)\,(m-q)\,\sqrt{-m^2+q^2}}$$
$$-\frac{q^3\ln(r-q)}{4\,m-4\,q}$$

>

Considering the effects of charge on fluid pressure, the calculated pressures and graphs in each specific case are created using codes as follows;

Calculation for the effect of charge on pressure

> $restart :$

> $with(plots) :$

> $G := 6.67 \cdot 10^{-11}$

$$G := 6.670000000\ 10^{-11}$$

> $c := 3 \cdot 10^{8}$

$$c := 300000000$$

> $rhof := 10^{13}$

$$rhof := 10000000000000$$

> $sigmaem := 8 \cdot 10^{11}$

$$sigmaem := 800000000000$$

> $\mathrm{rho} := rhof + sigmaem$

$$\rho := 10800000000000$$

> $m := \frac{4}{3} \cdot \mathrm{Pi} \cdot \mathrm{rho} \cdot r^{3}$

$$m := 14400000000000\ \pi\ r^{3}$$

> $E := 1$

$$E := 1$$

>

$$TOV := \frac{\mathrm{d}}{\mathrm{d}\,r}\, pf(r) = -\frac{G \cdot \left(rhof + \frac{pf(r)}{c^{2}}\right) \cdot \left(m + \frac{4 \cdot \mathrm{Pi} \cdot r^{3} \cdot pf(r)}{c^{2}}\right)}{r^{2} \cdot \left(1 - \frac{2 \cdot G \cdot m}{c^{2} \cdot r}\right)} - \frac{sigmaem \cdot E}{\sqrt{1 - \frac{2 \cdot G \cdot m}{c^{2} \cdot r}}}$$

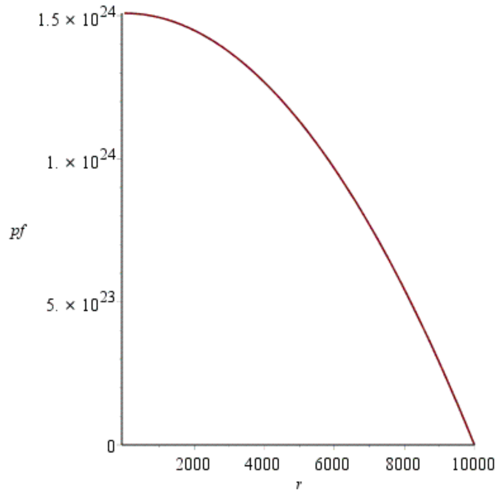
$$TOV := \frac{d}{dr}\, pf(r) = -\Big(6.670000000\ 10^{-11}\ \Big(10000000000000 + \frac{1}{9000000000000000}\, pf(r)\Big) \Big(14400000000000\ \pi\, r^3 + \frac{1}{22500000000000000}\, \pi\, r^3\, pf(r)\Big)\Big) \Big/ \big(r^2\ \big(1 - 6.705415360\ 10^{-14}\, r^2\big)\big) - \frac{800000000000}{\sqrt{1 - 6.705415360\ 10^{-14}\, r^2}}$$

> $TOV1 := eval\big(TOV, sigmaem = 8 \cdot 10^{11}\big)$

$$TOV1 := \frac{d}{dr}\, pf(r) = -\Big(6.670000000\ 10^{-11}\ \Big(10000000000000 + \frac{1}{9000000000000000}\, pf(r)\Big) \Big(14400000000000\ \pi\, r^3 + \frac{1}{22500000000000000}\, \pi\, r^3\, pf(r)\Big)\Big) \Big/ \big(r^2\ \big(1 - 6.705415360\ 10^{-14}\, r^2\big)\big) - \frac{800000000000}{\sqrt{1 - 6.705415360\ 10^{-14}\, r^2}}$$

> $p1 := dsolve\big(\{TOV1, pf(10^4) = 0\}, type = numeric, range = 1\,..10^4\big):$

> $odeplot(p1)$



> $p1\left(5\cdot 10^{3}\right)$

$$\left[r = 5000.,\ pf(r) = 1.13154627538709\ 10^{24}\right]$$

>

Appendix A2

Electromagnetic tensor

In classical electrodynamics, Maxwell's equations cover almost all descriptions about the static and dynamic of the electromagnetic phenomena. The electric field is introduced as a result of the existence of electric charge. Meanwhile, an origin of the magnetic field is an electric current. They have self-inducing effect, which is a property of the electromagnetic waveท. Because an electric field and a magnetic field can be detected using different methods, they are separated as different quantities. However, in general relativity, these two quantities can be defined by the same tensor: which is the field strength tensor $F_{\mu\nu}$ represented as follows,

$$E_i = F_{0i,} \tag{A2.1}$$

$$B_i = -\frac{1}{2}\varepsilon_{ijk}F^{jk}, \tag{A2.2}$$

where ε_{ijk} is Levi-Civita symbol. The contravariant form of Field strength tensor in Cartesian coordinates in four dimension is

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_Z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}, \tag{A2.3}$$

for $c = 1.$ The energy momentum stress tensor for the electromagnetic field can be written in terms of field strength tensor

$$T^{ab}_{em\,;b} = F^{ac}g_{cd}F^{bd} - \frac{1}{4}g^{ab}\left(F_{cd}F^{cd}\right). \tag{A2.4}$$

This tensor is used in the construction of the energy momentum stress tensor for a charge fluid sphere.

VITA

Miss Napasorn Jongjittanon was born on 27 April 1992. She received her Bachelor's degree in physics with the first class honors from Chiang Mai University in 2014. She has studied general relativity for her Master degree. Her research interest is the solution generating theorems for charged anisotropic fluid sphere.