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REGRESSIVE GENERALIZED TRANSFORMATION SEMIGROUPS

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ธีรพงษ์ พงษ์พัฒนเจริญ : กึ่งกรุปการแปลงนัยทั่วไปที่ถดถอย (REGRESSIVE GENERALIZED TRANSFORMATION SEMIGROUPS) อ. ที่ปรึกษา : ผศ. ดร. อมร วาสนาวิจิตร์, อ. ที่ปรึกษาร่วม : รศ. ดร. ยุพาภรณ์ เข็มประสิทธิ์ 34 หน้า. ISBN 974-17-3955-9

สำหรับเซต X ให้ $P(X)$, $T(X)$ และ $I(X)$ แทนกึ่งกรุปการแปลงบางส่วนบน X กึ่งกรุปการแปลงเต็มบน X และกึ่งกรุปการแปลงบางส่วนหนึ่งต่อหนึ่งบน X ตามลำดับ เราให้ด้วยว่า $AP(X) = \{ \alpha \in P(X) \mid \alpha$ เกือบเป็นเอกลักษณ์ $\}$ และนิยาม $AT(X)$ และ $AI(X)$ ในทำนองเดียวกัน ดังนั้น $AP(X)$, $AT(X)$ และ $AI(X)$ เป็นกึ่งกรุปย่อยของ $P(X)$, $T(X)$ และ $I(X)$ ตามลำดับ เราให้นัยทั่วไปของกึ่งกรุปการแปลงบน X (กึ่งกรุปย่อยของ $P(X)$) เป็นกึ่งกรุป ($S(X)$, θ) โดยที่ $S(X)$ เป็นกึ่งกรุปการแปลงบน X, $\theta \in S^1(X)$ และ ($S(X)$, θ) = ($S(X)$, $*$) โดย $\alpha * \beta = \alpha\theta\beta$ สำหรับทุก $\alpha, \beta \in S(X)$

สำหรับโพเซต X ให้ $P_{RE}(X) = \{ \alpha \in P(X) \mid \alpha$ เป็นการแปลงบางส่วนที่ถดถอย $\}$ และเรานิยาม $T_{RE}(X)$, $I_{RE}(X)$, $AP_{RE}(X)$, $AT_{RE}(X)$ และ $AI_{RE}(X)$ ในทำนองเดียวกัน ดังนั้น $P_{RE}(X)$, $T_{RE}(X)$, $I_{RE}(X)$, $AP_{RE}(X)$, $AT_{RE}(X)$ และ $AI_{RE}(X)$ เป็นกึ่งกรุปย่อยของ $P(X)$, $T(X)$, $I(X)$, $AP(X)$, $AT(X)$ และ $AI(X)$ ตามลำดับ ความจริงต่อไปนี้เป็นที่รู้กันแล้ว ถ้า $S(X)$ คือ $P_{RE}(X)$, $I_{RE}(X)$, $AP_{RE}(X)$ หรือ $AI_{RE}(X)$ แล้ว $S(X)$ เป็นกึ่งกรุปปรกติ ก็ต่อเมื่อ ทุกจุดใน X เป็นจุดเอกเทศ ถ้า $S(X)$ คือ $T_{RE}(X)$ หรือ $AT_{RE}(X)$ แล้ว $S(X)$ เป็นกึ่งกรุปปรกติ ก็ต่อเมื่อ $|C| \leq 2$ สำหรับทุกเซตย่อยอันดับทุกส่วน C ของ X ถ้า $S(X)$ คือ $P_{RE}(X)$, $T_{RE}(X)$ หรือ $I_{RE}(X)$ แล้ว $S(X)$ เป็นกึ่งกรุปปรกติในที่สุด ก็ต่อเมื่อ มีจำนวนเต็มบวก n ซึ่ง $|C| \leq n$ สำหรับทุกเซตย่อยอันดับทุกส่วน C ของ X ยิ่งไปกว่านั้นทุกกึ่งกรุปการแปลงเกือบเป็นเอกลักษณ์ที่ถดถอยบนX (ทุกกึ่งกรุปย่อยของ $AP_{RE}(X)$) เป็นกึ่งกรุปปรกติในที่สุด

วัตถุประสงค์ของการวิจัยนี้คือให้นัยทั่วไปของผลที่รู้กันแล้วทั้งหมดที่กล่าวไว้ข้างต้นโดยการพิจารณาผลเหล่านั้นบนกึ่งกรุป ($S(X)$, θ) โดย $\theta \in S^1(X)$ และ $S(X)$ เป็นกึ่งกรุปการแปลงที่ถดถอยบน X ตามวัตถุประสงค์ของเรา

นอกจากนี้ เรายังให้ทฤษฎีบทสมสัณฐานบางทฤษฎีบทที่เกี่ยวกับกึ่งกรุปการแปลงนัยทั่วไปที่ถดถอยด้วย

ภาควิชา **คณิตศาสตร์**	ลายมือชื่อนิสิต.......................................
สาขาวิชา **คณิตศาสตร์**	ลายมือชื่ออาจารย์ที่ปรึกษา.......................................
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TEERAPHONG PHONGPATTANACHAROEN: REGRESSIVE GENERALIZED TRANSFORMATION SEMIGROUPS. THESIS ADVISOR : ASSIST. PROF. AMORN WASANAWICHIT, Ph.D., THESIS COADVISOR : ASSOC. PROF. YUPAPORN KEMPRASIT, Ph.D., 34 pp. ISBN 974-17-3955-9

For a set X, let $P(X)$, $T(X)$ and $I(X)$ denote respectively the partial transformation semigroup on X, the full transformation semigroup on X and the one-to-one partial transformation semigroup on X. Also, let $AP(X) = \{ \alpha \in P(X) \mid \alpha$ is almost identical $\}$ and define $AT(X)$ and $AI(X)$ similarly. Then $AP(X)$, $AT(X)$ and $AI(X)$ are subsemigroups of $P(X)$, $T(X)$ and $I(X)$, respectively. We generalize a transformation semigroup on X (a subsemigroup of $P(X)$) to be a semigroup ($S(X), \theta$) where $S(X)$ is a transformation semigroup on X, $\theta \in S^1(X)$ and $(S(X), \theta) = (S(X), *)$ where $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in S(X)$.

For a poset X, let $P_{RE}(X) = \{ \alpha \in P(X) \mid \alpha$ is regressive $\}$, and $T_{RE}(X)$, $I_{RE}(X)$, $AP_{RE}(X)$, $AT_{RE}(X)$ and $AI_{RE}(X)$ are defined similarly. Then $P_{RE}(X)$, $T_{RE}(X)$, $I_{RE}(X)$, $AP_{RE}(X)$, $AT_{RE}(X)$ and $AI_{RE}(X)$ are respectively subsemigroups of $P(X)$, $T(X)$, $I(X)$, $AP(X)$, $AT(X)$ and $AI(X)$. The following facts are known. If $S(X)$ is $P_{RE}(X)$, $I_{RE}(X)$, $AP_{RE}(X)$ or $AI_{RE}(X)$, then $S(X)$ is regular if and only if X is isolated. If $S(X)$ is $T_{RE}(X)$ or $AT_{RE}(X)$, then $S(X)$ is regular if and only if $|C| \leq 2$ for every chain C of X. If $S(X)$ is $P_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$, $S(X)$ is eventually regular if and only if there is a positive integer n such that $|C| \leq n$ for every chain C of X. Moreover, every regressive almost identical transformation semigroup on X (every subsemigroup of $AP_{RE}(X)$) is eventually regular.

The purpose of this research is to generalize all the above known results by considering those on the semigroup ($S(X), \theta$) with $\theta \in S^1(X)$ where $S(X)$ is a regressive transformation semigroup on X of our purpose.

In addition, some isomorphism theorems on regressive generalized transformation semigroups are provided.

Department **Mathematics**	Student's signature......................................
Field of study **Mathematics**	Advisor's signature......................................
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CHAPTER I

INTRODUCTION AND PRELIMINARY

For a set X, let $|X|$ denote the cardinality of X. The set of positive integers, the set of integers and the set of real numbers are denoted by $\mathbb{N}, \mathbb{Z}$ and $\mathbb{R}$, respectively.

An element a of a semigroup S is called an *idempotent* of S if $a^2 = a$. For a semigroup S, let $E(S)$ be the set of all idempotents of S, that is,

$$E(S) = \{ \ a \in S \mid a^2 = a \ \}.$$

If $a \in S$ and $k \in \mathbb{N}$ are such that $a^k = a^{k+1}$, it is clearly seen that $a^k = a^{2k}$ which implies that $a^k \in E(S)$. Hence

$$\{ \ a^k \mid a \in S, k \in \mathbb{N} \text{ and } a^k = a^{k+1} \ \} \subseteq E(S).$$

An element a of a semigroup S is said to be *regular* if $a = aba$ for some $b \in S$, and S is called a *regular semigroup* if every element of S is a regular element of S. The set of all regular elements of a semigroup S will be denoted by $RegS$, that is,

$$RegS = \{ \ a \in S \mid a = aba \text{ for some } b \in S \ \}.$$

Consequently, $E(S) \subseteq RegS$. By an *eventually regular element* of a semigroup S we mean an element a of S such that $a^k \in RegS$ for some $k \in \mathbb{N}$. If every element of S is eventually regular, we call S an *eventually regular semigroup.* Therefore a regular semigroup is eventually regular.

For an element a of a semigroup S, let $<a>$ denote the subsemigroup of S generated by a, that is,

$$<a> = \{ \ a^n \mid n \in \mathbb{N} \ \}.$$

We call S a *periodic semigroup* if $<a>$ is finite for every $a \in S$. It is known that for $a \in S$, if $<a>$ is finite, then $a^k \in E(S)$ for some $k \in \mathbb{N}$. Since $E(S) \subseteq RegS$ for every semigroup S, it follows that every periodic semigroup is eventually regular. In particular, every finite semigroup is eventually regular. Therefore we have that if a semigroup S is regular or periodic, then S is eventually regular. In fact, if $a \in S$ and $a^k \in E(S)$ for some $k \in \mathbb{N}$, then $<a>$ is a finite subsemigroup of S. To see this, let $n \in \mathbb{N}$ be such that $n \geq k$. Then there exist $m \in \mathbb{N}$ and $r \in \{ 0, 1, \ldots, k-1 \}$ such that $n = mk + r$. Thus $a^n = a^{mk+r} = (a^k)^m a^r$. Since $a^k \in E(S)$, $(a^k)^m = a^k$ so $a^n = a^{k+r}$. This implies that $< a > = \{ a, a^2, \ldots, a^k, a^{k+1}, \ldots, a^{2k-1} \}$. We therefore conclude that for $a \in S$, $<a>$ is finite if and only if $a^k \in E(S)$ for some $k \in \mathbb{N}$. Hence we obtain the following proposition.

Proposition 1.1. *Let S be a semigroup. If $Reg\,S = E(S)$, then S is eventually regular if and only if S is periodic.*

For semigroup S, let $S^1 = S$ if S has an identity, otherwise, let S^1 be the semigroup S with the identity 1 adjoined. For the later case, $S^1 = S \cup \{1\}$ and $1 \notin S$ and extend the operation in S to 1 in $S \cup \{1\}$ by defining $1a = a1 = a$ for every $a \in S \cup \{1\}$.

A *partial transformation* of a set X is a map from a subset of X into X. The *empty transformation* 0 is the partial transformation with empty domain. Let $P(X)$ be the set of all partial transformations of X, that is,

$$P(X) = \{ \alpha\colon A \to X \mid A \subseteq X \}.$$

Then $0 \in P(X)$. The identity map on a nonempty set A is denoted by 1_A. Then $1_A \in P(X)$ for every nonempty subset A of X. In particular, $1_X \in P(X)$. We denote the domain and the image of $\alpha \in P(X)$ by $\operatorname{dom}\alpha$ and $\operatorname{im}\alpha$, respectively. Also, for $\alpha \in P(X)$ and $x \in \operatorname{dom}\alpha$, the image of x under α is written by $x\alpha$. The composition $\alpha\beta$ of α, $\beta \in P(X)$, is defined as follows: $\alpha\beta = 0$ if $\operatorname{im}\alpha \cap \operatorname{dom}\beta = \varnothing$, otherwise $\alpha\beta$ is the usual composition of the functions $\alpha\,|_{(\operatorname{im}\alpha\cap\operatorname{dom}\beta)\alpha^{-1}}$ and $\beta\,|_{(\operatorname{im}\alpha\cap\operatorname{dom}\beta)}$. Then under this composition, $P(X)$ is a semigroup having 0 and 1_X as its zero and identity, respectively. Observe that for $\alpha, \beta \in P(X)$,

$$\begin{aligned}
\operatorname{dom}(\alpha\beta) \quad &= \quad (\operatorname{im}\alpha \cap \operatorname{dom}\beta)\alpha^{-1} \subseteq \operatorname{dom}\alpha, \\
\operatorname{im}(\alpha\beta) \quad &= \quad (\operatorname{im}\alpha \cap \operatorname{dom}\beta)\beta \subseteq \operatorname{im}\beta, \\
x \in \operatorname{dom}(\alpha\beta) \quad &\Longleftrightarrow \quad x \in \operatorname{dom}\alpha \text{ and } x\alpha \in \operatorname{dom}\beta.
\end{aligned}$$

The semigroup $P(X)$ is called the *partial transformation semigroup* on X. By a *transformation semigroup* on X we mean a subsemigroup of $P(X)$.

By a *transformation* of X we mean a map of X into itself. Let $T(X)$ be the set of all transformations of X. Then

$$T(X) = \{\ \alpha \in P(X) \mid \operatorname{dom}\alpha = X\ \}$$

which is a subsemigroup of $P(X)$ containing 1_X and it is called the *full transformation semigroup* on X.

Let $I(X)$ denote the set of all 1-1 partial transformations of X, that is,

$$I(X) = \{\ \alpha \in P(X) \mid \alpha \text{ is 1-1 }\}.$$

Then $I(X)$ is a subsemigroup of $P(X)$ containing 0 and 1_X and it is called the *1-1 partial transformation semigroup* of X or the *symmetric inverse semigroup* on X.

It is well-known that all $P(X)$, $T(X)$ and $I(X)$ are regular ([1], page 4) and for $\alpha \in P(X)$, $\alpha^2 = \alpha$ ($\alpha \in E(P(X))$) if and only if $\operatorname{im}\alpha \subseteq \operatorname{dom}\alpha$ and $x\alpha = x$ for all $x \in \operatorname{im}\alpha$.

For a nonempty subset A of X and $x \in X$, let A_x be the element of $P(X)$ with domain A and image $\{x\}$.

The *shift* of $\alpha \in P(X)$ is defined to be the set

$$S(\alpha) = \{\ x \in \operatorname{dom}\alpha \mid x\alpha \neq x\ \},$$

and we call α *almost identical* if $S(\alpha)$ is finite ($|S(\alpha)| < \infty$). Next, let

$$\begin{aligned} AP(X) &= \{\, \alpha \in P(X) \mid \alpha \text{ is almost identical } \}, \\ AT(X) &= \{\, \alpha \in T(X) \mid \alpha \text{ is almost identical } \}, \\ AI(X) &= \{\, \alpha \in I(X) \mid \alpha \text{ is almost identical } \}. \end{aligned}$$

Then 0 and 1_X belong to $AP(X)$ and $AI(X)$ and $1_X \in AT(X)$. Let $\alpha, \beta \in P(X)$ and $x \in S(\alpha\beta)$. Then $x \in \operatorname{dom}(\alpha\beta) \subseteq \operatorname{dom}\alpha$ and $x\alpha\beta \neq x$. If $x \notin S(\alpha)$, then $x\alpha = x \in \operatorname{dom}\beta$, so $x\beta = x\alpha\beta \neq x$ which implies that $x \in S(\beta)$. This shows that $S(\alpha\beta) \subseteq S(\alpha) \cup S(\beta)$. It follows that if $|S(\alpha)| < \infty$ and $|S(\beta)| < \infty$ then $|S(\alpha\beta)| < \infty$. Hence $AP(X), AT(X)$ and $AI(X)$ are respectively subsemigroups of $P(X), T(X)$ and $I(X)$. The proofs of regularity of $P(X), T(X)$ and $I(X)$ show that $AP(X), AT(X)$ and $AI(X)$ are also regular. Note that if X is finite, then $AP(X) = P(X), AT(X) = T(X)$ and $AI(X) = I(X)$.

If $S(X)$ is a transformation semigroup on X and $\theta \in S^1(X)$, let $(S(X), \theta)$ denote the semigroup $(S(X), *)$ where the operation $*$ is defined by

$$\alpha * \beta = \alpha\theta\beta \text{ for all } \alpha, \beta \in S(X).$$

We call such a semigroup $(S(X), \theta)$ a *generalized transformation semigroup* on X. Note that $S(X) = (S(X), 1)$ where 1 is the identity of $S^1(X)$. To distinguish between α^n in the semigroup $S(X)$ and the product $\alpha * \alpha \ldots * \alpha$ (n times) in the semigroup $(S(X), *) = (S(X), \theta)$ where $\alpha \in S(X)$ and n is a positive integer, we shall use $(\alpha, \theta)^n$ to denote the later product. For examples, $(\alpha, \theta)^2$ and $(\alpha, \theta)^4$ denote $\alpha\theta\alpha$ and $\alpha\theta\alpha\theta\alpha\theta\alpha$, respectively. Observe that $(\alpha, \theta)^n = (\alpha\theta)^{n-1}\alpha$ if $n > 1$.

Example 1.2. Let X be a nonempty set and $a \in X$. Then $(T(X), X_a)$ is the semigroup $T(X)$ with the operation $*$ defined as follows:

$$\alpha * \beta = \alpha X_a \beta = X_{a\beta} \text{ for all } \alpha, \beta \in T(X).$$

Also, $(P(X), X_a)$ is the semigroup $P(X)$ with the operation $\circ$ defined by

$$\alpha \circ \beta = \alpha X_a \beta = \begin{cases} (dom\,\alpha)_{a\beta} & \text{if } \alpha \neq 0 \text{ and } a \in dom\,\beta\,, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $(I(X), \{a\}_a) = (I(X), \bullet)$ where

$$\alpha \bullet \beta = \alpha \{a\}_a \beta = \begin{cases} \{a\alpha^{-1}\}_{a\beta} & \text{if } a \in im\,\alpha \cap dom\,\beta\ , \\ 0 & \text{otherwise.} \end{cases}$$

Next, let X be a partially ordered set (a poset). An element $a \in X$ is called an *isolated point* if

$$\text{for } x \in X,\ x \leq a \text{ or } x \geq a \Longrightarrow x = a,$$

and we call X *isolated* if every element of X is an isolated point of X. By a *chain* of X we mean a *chain* Y such that $Y \subseteq X$ and the partial order of Y is the partial order of X restricted to Y.

For $\alpha \in P(X)$, α is said to be *regressive* if

$$x\alpha \leq x \text{ for all } x \in \operatorname{dom}\alpha.$$

A transformation semigroup on X is said to be *regressive* if all of its elements are regressive. Let

$$\begin{aligned} P_{RE}(X) &= \{\ \alpha \in P(X) \mid \alpha \text{ is regressive }\}, \\ AP_{RE}(X) &= \{\ \alpha \in AP(X) \mid \alpha \text{ is regressive }\}. \end{aligned}$$

Then 0 and 1_X belong to both $P_{RE}(X)$ and $AP_{RE}(X)$ and both $P_{RE}(X)$ and $AP_{RE}(X)$ are subsemigroups of $P(X)$ and $AP_{RE}(X) \subseteq P_{RE}(X)$. The notations $T_{RE}(X)$, $AT_{RE}(X)$, $I_{RE}(X)$ and $AI_{RE}(X)$ are defined analogously. Thus $1_X \in AT_{RE}(X) \subseteq T_{RE}(X), 0, 1_X \in AI_{RE}(X) \subseteq I_{RE}(X)$, $T_{RE}(X)$ and $AT_{RE}(X)$ are subsemigroups of $T(X)$ and $I_{RE}(X)$ and $AI_{RE}(X)$ are subsemigroups of $I(X)$. Observe that

$$\begin{aligned} AP_{RE}(X) &= \{\ \alpha \in P(X) \mid \alpha \text{ is regressive and } |S(\alpha)| < \infty\ \}, \\ AT_{RE}(X) &= \{\ \alpha \in T(X) \mid \alpha \text{ is regressive and } |S(\alpha)| < \infty\ \}, \\ AI_{RE}(X) &= \{\ \alpha \in I(X) \mid \alpha \text{ is regressive and } |S(\alpha)| < \infty\ \}. \end{aligned}$$

By a *regressive transformation semigroup* on X and a *regressive almost identical transformation semigroup* on X we mean a subsemigroup of $P_{RE}(X)$ and a subsemigroup of $AP_{RE}(X)$, respectively.

A.Umar [4] proved that if X is a finite chain, then the subsemigroup

$$S = \{\ \alpha \in T_{RE}(X) \mid |\mathrm{im}\,\alpha| < |X|\ \}$$

of $T_{RE}(X)$ is generated by $E(S)$ and S is not regular if $|X| \geq 3$.

It was shown in [2] that $Reg(S(X))$ and $E(S(X))$ coincide for every regressive transformation semigroup $S(X)$ on any poset X.

Proposition 1.3. *If X is a poset and $S(X)$ is a regressive transformation semigroup on X, then $Reg(S(X)) = E(S(X))$.*

Using Proposition 1.3 as a lemma, the above six regressive transformation semigroups on a poset were considered when they are regular as follows:

Theorem 1.4. [2] *Let X be a poset and let $S(X)$ be $P_{RE}(X)$, $I_{RE}(X)$, $AP_{RE}(X)$ or $AI_{RE}(X)$. Then the semigroup $S(X)$ is regular if and only if X is isolated.*

Theorem 1.5. [2] *Let X be a poset and let $S(X)$ be $T_{RE}(X)$ or $AT_{RE}(X)$. Then the semigroup $S(X)$ is regular if and only if $|C| \leq 2$ for every chain C of X.*

In [2], some interesting results relating to eventually regular regressive transformation semigroups were provided as follows:

Theorem 1.6. [2] *Let X be a poset and let $S(X)$ be $P_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$. Then the semigroup $S(X)$ is eventually regular if and only if there exists a positive integer n such that $|C| \leq n$ for every chain C of X.*

Theorem 1.7. [2] *Every regressive almost identical transformation semigroup on any poset is eventually regular.*

A significant isomorphism theorem on full regressive transformation semigroups was given by A.Umar in [5] as follows: For chains X and Y, $T_{RE}(X)$ and $T_{RE}(Y)$ are isomorphic if and only if X and Y are order-isomorphic. Moreover, in [3], T. Saitô, K. Aoki and K. Kajitori gave necessary and sufficient conditions for posets X and Y so that $T_{RE}(X)$ and $T_{RE}(Y)$ are isomorphic, and A. Umar's theorem mentioned above becomes their special case.

By a *regressive generalized transformation semigroup* on a poset X we mean a semigroup $(S(X), \theta)$ where $S(X)$ is a subsemigroup of $P_{RE}(X)$ and $\theta \in S^1(X)$. A *regressive almost identical generalized transformation semigroup* on a poset is a semigroup $(S(X), \theta)$ where $S(X)$ is a subsemigroup of $AP_{RE}(X)$ and $\theta \in S^1(X)$.

In Chapter II, we show that Proposition 1.3 holds for any regressive generalized transformation semigroups. Moreover, we generalize Theorem 1.4 and Theorem 1.5 to their regressive generalized transformation semigroups.

In Chapter III, we generalize Theorem 1.6 and Theorem 1.7 by considering regressive generalized transformation semigroups. Beside Theorem 1.6 and Theorem 1.7, some interesting consequences of our results are also provided.

The purpose of Chapter IV is to give some isomorphism theorems on regressive generalized transformation semigroups for some certain poset X and θ.

CHAPTER II

REGULAR REGRESSIVE GENERALIZED TRANSFORMATION SEMIGROUPS

The purpose of this chapter is to determine when $(S(X), \theta)$ is a regular semigroup where X is any poset, $\theta \in S(X)$ and $S(X)$ is any of $P_{RE}(X)$, $T_{RE}(X)$, $I_{RE}(X)$, $AP_{RE}(X)$, $AT_{RE}(X)$ and $AI_{RE}(X)$.

We begin this chapter by showing that every regular element of a regressive generalized transformation semigroup $(S(X), \theta)$ with $\theta \in S^1(X)$ must be an idempotent of $(S(X), \theta)$.

Proposition 2.1. *If $S(X)$ is a regressive transformation semigroup on a poset X, then $Reg(S(X), \theta) = E(S(X), \theta)$ for every $\theta \in S^1(X)$.*

Proof. Let $\theta \in S^1(X)$ and $\alpha \in Reg(S(X), \theta)$. Then $\alpha = \alpha\theta\beta\theta\alpha$ for some $\beta \in S(X)$. Thus

$$\begin{aligned} \text{for } x \in \operatorname{dom}\alpha, \quad x\alpha &= x\alpha\theta\beta\theta\alpha = (x\alpha\theta\beta)\theta\alpha \\ &\leq x\alpha\theta\beta = (x\alpha)\theta\beta \leq x\alpha \end{aligned}$$

which implies that

$$x\alpha = x\alpha\theta\beta \quad \text{for every } x \in \operatorname{dom}\alpha. \tag{1}$$

But $\alpha = \alpha\theta\beta\theta\alpha$, so from (1),

$$\begin{aligned} \text{for } x \in \operatorname{dom}\alpha, \quad x\alpha &= x\alpha\theta\beta\theta\alpha = (x\alpha\theta\beta)\theta\alpha \\ &= x\alpha\theta\alpha = x(\alpha, \theta)^2. \end{aligned} \tag{2}$$

It then follows from (2) that

$$\operatorname{dom}\alpha \subseteq \operatorname{dom}(\alpha,\theta)^2 \text{ and } x\alpha = x(\alpha,\theta)^2 \text{ for every } x \in \operatorname{dom}\alpha. \tag{3}$$

But $\operatorname{dom}(\alpha,\theta)^2 = \operatorname{dom}(\alpha\theta\alpha) \subseteq \operatorname{dom}\alpha$, so (3) yields $\alpha = (\alpha,\theta)^2 \in E(S(X),\theta)$. Consequently, $Reg(S(X),\theta) = E(S(X),\theta)$, as required. □

In Proposition 2.1, $S(X) = (S(X),1)$ where 1 is the identity of S^1(X). Hence Proposition 1.3 becomes a consequence of Proposition 2.1.

Corollary 2.2. *If $S(X)$ is a regressive transformation semigroup on a poset X, then $Reg(S(X)) = E(S(X))$.*

The regressive transformation semigroup $S(X)$ on a poset X may not contain an identity. It is natural to ask whether there is a regressive transformation semigroup $S(X)$ on a poset X such that $S(X)$ is not isomorphic to $(S(X),\theta)$ for every $\theta \in S(X)$. An existence of such a semigroup is given by the following example.

Example 2.3. For each $n \in \mathbb{N}$, let $\alpha_n \colon \mathbb{Z} \to \mathbb{Z}$ be defined by

$$x\alpha_n = x - n \text{ for all } x \in \mathbb{Z}.$$

Then with the natural order on $\mathbb{Z}$, $\alpha_n \in T_{RE}(\mathbb{Z})$ for every $n \in \mathbb{N}$ and $\alpha_i \neq \alpha_j$ for distinct $i \neq j$ in $\mathbb{N}$. Let

$$S(\mathbb{Z}) = \{\ \alpha_n \mid n \in \mathbb{N}\ \}.$$

Clearly, $\alpha_n\alpha_m = \alpha_{n+m}$ for all $n, m \in \mathbb{N}$. We therefore conclude that $S(\mathbb{Z})$ is a subsemigroup of $T_{RE}(\mathbb{Z})$ and $S(\mathbb{Z})$ has no identity. We claim that $S(\mathbb{Z}) \ncong (S(\mathbb{Z}),\alpha_n)$ for every $n \in \mathbb{N}$. Suppose that $S(\mathbb{Z}) \cong (S(\mathbb{Z}),\alpha_k)$ for some $k \in \mathbb{N}$. Let φ be an isomorphism from $S(\mathbb{Z})$ onto $(S(\mathbb{Z}),\alpha_k)$. Then there exists $t \in \mathbb{N}$ such that $\alpha_t\varphi = \alpha_1$.

Case 1 : $t > 1$. Then $t-1 \in \mathbb{N}$, so

$$\begin{aligned}
\alpha_1 &= \alpha_t\varphi \\
&= (\alpha_{t-1}\alpha_1)\varphi \\
&= (\alpha_{t-1}\varphi)\alpha_k(\alpha_1\varphi) \\
&= \alpha_s\alpha_k\alpha_r \quad \text{for some } s, r \in \mathbb{N}. \\
&= \alpha_{s+k+r}
\end{aligned}$$

which is a contradiction since $s+k+r > 1$.

Case 2 : $t = 1$. Then $\alpha_1\varphi = \alpha_1$, so

$$\alpha_2\varphi \;=\; (\alpha_1\alpha_1)\varphi \;=\; (\alpha_1\varphi)\alpha_k(\alpha_1\varphi) \;=\; \alpha_1\alpha_k\alpha_1 \;=\; \alpha_{k+2}.$$

To show that $\alpha_n\varphi = \alpha_{(n-1)k+n}$ for all $n \in \mathbb{N}$ with $n > 1$, suppose that $l \in \mathbb{N}, l > 1$ and $\alpha_l\varphi = \alpha_{(l-1)k+l}$. Thus

$$\begin{aligned}
\alpha_{l+1}\varphi &= (\alpha_l\alpha_1)\varphi \\
&= (\alpha_l\varphi)\alpha_k(\alpha_1\varphi) \\
&= (\alpha_{(l-1)k+l})\alpha_k\alpha_1 \\
&= \alpha_{lk-k+l+k+1} \\
&= \alpha_{lk+(l+1)}.
\end{aligned}$$

This proves that $\operatorname{im}\varphi = \{\, \alpha_{(n-1)k+n} \mid n \in \mathbb{N} \,\}$, so $\operatorname{im}\varphi = \{\, \alpha_1, \alpha_{k+2}, \alpha_{2k+3}, \alpha_{3k+4}, \ldots \}$ which does not contain α_2. This is contrary to that φ is onto.

Theorem 1.4 and Theorem 1.5 provide respectively the next two theorems easily.

Theorem 2.4. *Let X be a poset and let $S(X)$ be $P_{RE}(X)$, $I_{RE}(X)$, $AP_{RE}(X)$ or $AI_{RE}(X)$ and $\theta \in S(X)$. Then the semigroup $(S(X), \theta)$ is regular if and only if $\theta = 1_X$ and X is isolated.*

Proof. Assume that $(S(X), \theta)$ is regular. By Proposition 2.1, we know that $Reg(S(X), \theta) = E(S(X), \theta)$. Then by assumption, we get $(S(X), \theta) = E(S(X), \theta)$. Since $1_X \in S(X)$, 1_X is an idempotent of $(S(X), \theta)$, so $1_X = 1_X\theta 1_X = \theta$. This shows that $(S(X), \theta) = S(X)$. It follows from Theorem 1.4 that X is isolated.

The converse follows directly from Theorem 1.4. □

Theorem 2.5. *Let X be a poset and let $S(X)$ be $T_{RE}(X)$ or $AT_{RE}(X)$ and $\theta \in S(X)$. Then the semigroup $(S(X), \theta)$ is regular if and only if $\theta = 1_X$ and $|C| \leq 2$ for every chain C of X.*

Proof. Using Proposition 2.1 and Theorem 1.5, the proof of the theorem can be given similarly to that of Theorem 2.4. □

Example 2.6. Let $S(\mathbb{N})$ be one of $P_{RE}(\mathbb{N}), T_{RE}(\mathbb{N}), I_{RE}(\mathbb{N}), AP_{RE}(\mathbb{N}), AT_{RE}(\mathbb{N})$ or $AI_{RE}(\mathbb{N})$ under the natural order on $\mathbb{N}$. We then have by Theorem 2.4 amd 2.5 that the semigroup $(S(\mathbb{N}), \theta)$ is not regular for every $\theta \in S(\mathbb{N})$.

Next, let

$$C(\mathbb{N}) \ = \ \{ \ A_1 \mid 1 \in A \subseteq \mathbb{N} \ \}.$$

Recall that A_1 is an element of $P(\mathbb{N})$ with domain A and image $\{1\}$. Then $C(\mathbb{N})$ is an infinite subset of $P_{RE}(\mathbb{N})$ and for $1 \in A \subseteq \mathbb{N}$ and $1 \in B \subseteq \mathbb{N}, A_1B_1 = A_1$. This implies that $C(\mathbb{N})$ is an infinite regular subsemigroup of $P_{RE}(\mathbb{N})$. Also, for any $\theta \in C^1(\mathbb{N}), (C(\mathbb{N}), \theta)$ is an infinite regular subsemigroup of $(P_{RE}(\mathbb{N}), \theta)$.

CHAPTER III

EVENTUALLY REGULAR REGRESSIVE GENERALIZED TRANSFORMATION SEMIGROUPS

Our purpose of this chapter is to generalize Theorem 1.6 and 1.7. We first provide necessary and sufficient conditions for X and θ so that the semigroup $(S(X), \theta)$ is eventually regular where X is any poset, $S(X)$ is $P_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$ and $\theta \in S(X)$. Next, we show that for every subsemigroup $S(X)$ of $AP_{RE}(X)$, $(S(X), \theta)$ with $\theta \in S^1(X)$ is an eventually regular semigroup.

To obtain the first main result, the following series of lemmas is required.

Lemma 3.1. *Let X be any poset and let $\theta \in P_{RE}(X)$. If there exists a positive integer n such that every chain of X of the form*

$$x_1 \geq x_1\theta \geq x_2 \geq x_2\theta \geq x_3 \geq x_3\theta \geq \ldots$$

with $x_i \in dom\theta$ has length at most n, then for every $\alpha \in P_{RE}(X)$, $(\alpha, \theta)^{n+2} \in E(P_{RE}(X), \theta)$.

Proof. Let $\alpha \in P_{RE}(X)$ and $x \in \mathrm{dom}(\alpha\theta)^{n+1}$. By assumption, the chain

$$x\alpha \geq x\alpha\theta \geq x(\alpha\theta)\alpha \geq x(\alpha\theta)^2 \geq \ldots \geq x(\alpha\theta)^n\alpha \geq x(\alpha\theta)^{n+1}$$

has length at most n, so its subchain

$$x\alpha\theta \geq x(\alpha\theta)^2 \geq \ldots \geq x(\alpha\theta)^{n+1} \tag{1}$$

has length at most n. If $x(\alpha\theta)^i > x(\alpha\theta)^{i+1}$ for every $i \in \{1, 2, \ldots, n\}$, then the chain (1) has length $n + 1$, a contradiction. Thus $x(\alpha\theta)^i = x(\alpha\theta)^{i+1}$ for some

$i \in \{1, 2, \ldots, n\}$. Since $x \in \mathrm{dom}\,(\alpha\theta)^{n+1}$, $x(\alpha\theta)^i \in \mathrm{dom}\,(\alpha\theta)^{n+1-i}$. We then deduce that

$$x(\alpha\theta)^{n+1} = x(\alpha\theta)^i(\alpha\theta)^{n+1-i} = x(\alpha\theta)^{i+1}(\alpha\theta)^{n+1-i} = x(\alpha\theta)^{n+2}. \qquad (2)$$

It follows from (2) that

$$\mathrm{dom}\,(\alpha\theta)^{n+1} \subseteq \mathrm{dom}\,(\alpha\theta)^{n+2} \text{ and } x(\alpha\theta)^{n+1} = x(\alpha\theta)^{n+2} \qquad (3)$$
$$\text{for every } x \in \mathrm{dom}(\alpha\theta)^{n+1}.$$

But $\mathrm{dom}\,(\alpha\theta)^{n+2} \subseteq \mathrm{dom}\,(\alpha\theta)^{n+1}$, so (3) yields $(\alpha\theta)^{n+1} = (\alpha\theta)^{n+2}$. This implies that

$$(\alpha, \theta)^{n+2} = (\alpha\theta)^{n+1}\alpha = (\alpha\theta)^{n+2}\alpha = (\alpha, \theta)^{n+3}.$$

Consequently, $(\alpha, \theta)^{n+2} \in E(P_{RE}(X), \theta)$ (see Chapter I, page 1). □

By the definition of regressive generalized transformation semigroups, the following lemma is a direct consequence of Lemma 3.1.

Lemma 3.2. *Let $S(X)$ be a regressive transformation semigroup on any poset X and $\theta \in S^1(X)$. If there exists a positive integer n such that every chain of X of the form*

$$x_1 \geq x_1\theta \geq x_2 \geq x_2\theta \geq x_3 \geq x_3\theta \geq \ldots$$

with $x_i \in dom\,\theta$ has length at most n, then the semigroup $(S(X), \theta)$ is eventually regular.

The following corollary follows directly from Lemma 3.2.

Corollary 3.3. *If $S(X)$ is a regressive transformation semigroup on a poset X and $\theta \in S^1(X)$ with $|im\,\theta| < \infty$, then the semigroup $(S(X), \theta)$ is eventually regular.*

Moreover, Proposition 1.1, Proposition 2.1, Lemma 3.2 and Corollary 3.3 yield the following fact.

Corollary 3.4. *Let $S(X)$ be a regressive transformation semigroup on a poset X and $\theta \in S^1(X)$. If there exists a positive integer n such that every chain of X of the form*

$$x_1 \geq x_1\theta \geq x_2 \geq x_2\theta \geq x_3 \geq x_3\theta \geq \dots$$

with $x_i \in dom\,\theta$ has length at most n, then the semigroup $(S(X), \theta)$ is periodic. In particular, if $|im\,\theta| < \infty$, then the semigroup $(S(X), \theta)$ is a periodic semigroup.

Lemma 3.5. *Let X be a poset and let $S(X)$ be $P_{RE}(X)$, $T_{RE}(X)$, or $I_{RE}(X)$ and $\theta \in S(X)$. If X contains a sequence of pairewise disjoint finite chains $C_1, C_2, C_3, \dots$ such that each C_i is of the form*

$$x_1 \geq x_1\theta > x_2 \geq x_2\theta > \dots > x_{k_i} \geq x_{k_i}\theta$$

and $k_1 < k_2 < k_3 < \dots$, then the semigroup $(S(X), \theta)$ is not eventually regular.

Proof. For each $i \in \mathbb{N}$, let

$$\begin{aligned} C_i \;&=\; \{\, x_{i1}, x_{i1}\theta, x_{i2}, x_{i2}\theta, \dots, x_{ik_i}, x_{ik_i}\theta \,\} \\ &\qquad \text{where } x_{i1} \geq x_{i1}\theta > x_{i2} \geq x_{i2}\theta > \dots > x_{ik_i} \geq x_{ik_i}\theta. \end{aligned}$$

We may assume that $k_1 \geq 2$, otherwise we consider the sequence $C_2, C_3, C_4, \dots$ instead. To show that $(S(X), \theta)$ is not eventually regular, define α from the set $\bigcup_{i=1}^{\infty}\{\, x_{i1}\theta, x_{i2}\theta, \dots, x_{i,k_i-1}\theta \,\}$ onto the set $\bigcup_{i=1}^{\infty}\{\, x_{i2}, x_{i3}, \dots, x_{ik_i} \,\}$ by

$$(x_{ij}\theta)\alpha = x_{i,j+1} \;\text{ for } i \in \mathbb{N} \text{ and } j \in \{\, 1, 2, \dots, k_i - 1 \,\}.$$

The map α is well-defined because $C_1, C_2, C_3, \dots$ are pairwise disjoint. Because of the form of each C_i, we have that α is 1-1 and regressive. Then $\alpha \in I_{RE}(X) \subseteq P_{RE}(X)$. Let $n \in \mathbb{N}$. Since the sequence $k_1, k_2, k_3, \dots$ of positive integers is strictly increasing, there exists $m \in \mathbb{N}$ such that $k_m > 2n$. We then deduce that

$$\begin{aligned}(x_{m1}\theta)(\alpha,\theta)^n &= (x_{m1}\theta)(\alpha\theta)^{n-1}\alpha \\ &= x_{m,n+1} \\ &> x_{m,2n+1} \\ &= (x_{m1}\theta)(\alpha,\theta)^{2n}.\end{aligned}$$

This proves that $(\alpha,\theta)^n \neq (\alpha,\theta)^{2n}$ for every $n \in \mathbb{N}$. Thus $(\alpha,\theta)^n \notin E(P_{RE}(X),\theta)$ for every $n \in \mathbb{N}$. By Proposition 2.1, α is not an eventually regular element of the semigroup $(S(X),\theta)$ if $S(X)$ is $P_{RE}(X)$ or $I_{RE}(X)$.

Next, assume that $S(X) = T_{RE}(X)$. Then $\theta \in T_{RE}(X) \subseteq P_{RE}(X)$. Let $\beta\colon X \to X$ be defined by

$$x\beta = \begin{cases} x\alpha & \text{if } x \in \operatorname{dom}\alpha, \\ x & \text{if } x \in X \smallsetminus \operatorname{dom}\alpha. \end{cases}$$

Then $\beta \in T_{RE}(X)$. If $n \in \mathbb{N}$, from the above proof, there exists an element $y \in \operatorname{dom}\alpha$ such that $y(\alpha,\theta)^n > y(\alpha,\theta)^{2n}$, that is, $y(\alpha\theta)^{n-1}\alpha > y(\alpha\theta)^{2n-1}\alpha$. Consequently,

$$y(\beta\theta)^{n-1}\beta = y(\alpha\theta)^{n-1}\alpha > y(\alpha\theta)^{2n-1}\alpha = y(\beta\theta)^{2n-1}\beta$$

which implies that $(\beta,\theta)^n \neq (\beta,\theta)^{2n}$. We therefore have from Proposition 2.1 that β is not an eventually regular element of the semigroup $(T_{RE}(X),\theta)$.

Hence the lemma is completely proved. □

An interesting consequence of Lemma 3.5 is as follow:

Corollary 3.6. *Let X be any poset and let $S(X)$ be $P_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$. If X does not have a minimal element, then the semigroup $(S(X),\theta)$ is not eventually regular for every $\theta \in S(X)$ with dom$\,\theta = X$.*

Proof. Let $x_1 \in X$. Thus $x_1 \geq x_1\theta$. By assumption, $x_1\theta$ is not a minimal element, so $x_1\theta > x_2$ for some $x_2 \in X$. Then $x_1 \geq x_1\theta > x_2 \geq x_2\theta$. By this process, we obtain a sequence

$$x_1 \geq x_1\theta > x_2 \geq x_2\theta > x_3 \geq x_3\theta > \dots \quad .$$

Let (k_n) be a strictly increasing sequence of positive integers such that $k_1 > 1$ and let $l_i = k_1 + k_2 + \cdots + k_i$ for all $i \in \mathbb{N}$. Define the chains C_i for $i \in \mathbb{N}$ as follows:

$$C_1 = \{x_1, x_1\theta, \dots, x_{l_1}, x_{l_1}\theta\}$$

$$C_2 = \{x_{l_1+1}, x_{l_1+1}\theta, \dots, x_{l_2}, x_{l_2}\theta\}$$

$$C_3 = \{x_{l_2+1}, x_{l_2+1}\theta, \dots, x_{l_3}, x_{l_3}\theta\}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Then each C_i is a finite chain of X, $C_i \cap C_j = \varnothing$ if $i \neq j$ and each C_i is of the form $y_1 \geq y_1\theta > y_2 \geq y_2\theta > \dots > y_{k_i} \geq y_{k_i}\theta$. Also, $k_1 < k_2 < k_3 < \dots$. Therefore we have from Lemma 3.5 that $(S(X), \theta)$ is not an eventually regular semigroup. $\square$

As was mentioned previously, we have that eventual regularity and periodicity of regressive generalized transformation semigroups are identical. Then we have

Corollary 3.7. *Let X be a poset and let $S(X)$ be $P_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$. If X does not have a minimal element, then the semigroup $(S(X), \theta)$ is not a periodic semigroup for every $\theta \in S(X)$ with $dom\,\theta = X$.*

Now we are ready to give the first main result.

Theorem 3.8. *Let X be a poset and let $S(X)$ be $P_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$ and $\theta \in S(X)$. Then the semigroup $(S(X), \theta)$ is eventually regular if and only if there exists a positive integer n such that every chain of X of the form*

$$x_1 \geq x_1\theta \geq x_2 \geq x_2\theta \geq x_3 \geq x_3\theta \geq \dots$$

with $x_i \in dom\,\theta$ has length at most n.

Proof. If there is an element $n \in \mathbb{N}$ such that every chain of X of the form

$$x_1 \geq x_1\theta \geq x_2 \geq x_2\theta \geq x_3 \geq x_3\theta \geq \dots \tag{1}$$

with $x_i \in \operatorname{dom}\theta$ has length at most n, then by Lemma 3.2, $(S(X), \theta)$ is an eventually regular semigroup.

To prove necessity by contrapositive, assume that for every positive integer n, there exists a chain of X of the form (1) of length greater than n.

For better understanding in counting, the chain (1) can be revised as follows: If there is $i \in \mathbb{N}$ such that $x_i = x_i\theta = x_{i+1}$ in (1), then we can replace $x_i = x_i\theta = x_{i+1}$ by x_{i+1} and the revised chain is still of the form (1). Also, if there is $i \in \mathbb{N}$ such that $x_i\theta = x_{i+1} = x_{i+1}\theta$, then this can be replaced by $x_i\theta$ and the revised chain is still of the form (1). Because of these facts, (1) can be considered as

$$x_1 \geq x_1\theta \geq x_2 \geq x_2\theta \geq x_3 \geq x_3\theta \geq \dots \tag{2}$$

with $x_i \in \operatorname{dom}\theta$ and any three consecutive

terms not identical.

In the remainder of this proof, elements x_i, x_i', x_{ij} which we use always belong to $\operatorname{dom}\theta$. Then by the above assumption, we have that every positive integer n, there exists a chain of X of the form (2) of length greater than n. If there is no chain of X of the form

$$x_1 \geq x_1\theta > x_2 \geq x_2\theta, \tag{3}$$

then any chain of X of the form (2) has length not more than the chain

$$x_1 > x_1\theta = x_1' > x_1'\theta$$

(because if we can add $\geq y \geq y\theta$ with $y \in \operatorname{dom}\theta$ after $x_1'\theta$, we obtain a chain of the form (3)), so every chain of X of the form (2) has length at most 3 which is contrary to the assumption. Then there is a chain

$$C_1 = \{\ x_{11}, x_{11}\theta, x_{12}, x_{12}\theta\ \} \text{ with}$$
$$x_{11} \geq x_{11}\theta > x_{12} \geq x_{12}\theta.$$

If there is no chain of the form (2) of the subposet $X \smallsetminus C_1$ of the form

$$x_1 \geq x_1\theta > x_2 \geq x_2\theta > x_3 \geq x_3\theta, \tag{4}$$

then any chain of $X \smallsetminus C_1$ of the form (2) has length not more than the length of the chain

$$x_1 > x_1\theta = x_1' > x_1'\theta > x_2 > x_2\theta = x_2' > x_2'\theta$$

(because if we can add $\geq y \geq y\theta$ with $y \in \operatorname{dom}\theta$ after $x_i'\theta$, we obtain a chain of the form (4)), so every chain of X of the form (2) has length at most $|C_1| + 6$, a contradiction. Let C_2 be a chain of $X \smallsetminus C_1$ such that

$$C_2 = \{\ x_{21}, x_{21}\theta, x_{22}, x_{22}\theta, x_{23}, x_{23}\theta\ \} \text{ with}$$
$$x_{21} \geq x_{21}\theta > x_{22} \geq x_{22}\theta > x_{23} \geq x_{23}\theta.$$

Thus $C_1 \cap C_2 = \varnothing$. Again, if the subposet $X \smallsetminus (C_1 \cup C_2)$ of X does not contain a chain of the form

$$x_1 \geq x_1\theta > x_2 \geq x_2\theta > x_3 \geq x_3\theta > x_4 \geq x_4\theta, \tag{5}$$

then any chain of $X \smallsetminus (C_1 \cup C_2)$ of the form (2) has length not more than the length of the chain

$$x_1 > x_1\theta = x_1' > x_1'\theta > x_2 > x_2\theta = x_2' > x_2'\theta > x_3 > x_3\theta = x_3' > x_3'\theta$$

(because we will obtain a chain of the form (5) if we can add $\geq y \geq y\theta$ with $y \in \operatorname{dom}\theta$ after $x_i'\theta$), which implies that every chain of X of the form (2) has length at most $|C_1|+|C_2|+9$, a contradiction. Let C_3 be a chain of $X \smallsetminus (C_1 \cup C_2)$ such that

$$C_3 = \{\ x_{31}, x_{31}\theta, x_{32}, x_{32}\theta, x_{33}, x_{33}\theta, x_{34}, x_{34}\theta\ \} \text{ with}$$

$$x_{31} \geq x_{31}\theta > x_{32} \geq x_{32}\theta > x_{33} \geq x_{33}\theta > x_{34} \geq x_{34}\theta.$$

Then $C_3 \cap C_1 = \varnothing$ and $C_3 \cap C_2 = \varnothing$. By this process, we obtain a sequence of pairwise disjoint finite chains $C_1, C_2, C_3, \ldots$ of X such that each $i \in \mathbb{N}$,

$$C_i = \{\ x_{i1}, x_{i1}\theta, x_{i2}, x_{i2}\theta, \ldots, x_{i,i+1}, x_{i,i+1}\theta\ \} \text{ with}$$

$$x_{i1} \geq x_{i1}\theta > x_{i2} \geq x_{i2}\theta > \ldots > x_{i,i+1} \geq x_{i,i+1}\theta.$$

We therefore deduce from Lemma 3.5 that the semigroup $(S(X), \theta)$ is not eventually regular.

Hence the theorem is completely proved. $\square$

Also, from Proposition 1.1, Proposition 2.1 and Theorem 3.8, the following corollary is directly obtained.

Corollary 3.9. *Let X be a poset and let $S(X)$ be $P_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$ and $\theta \in S(X)$. Then the semigroup $(S(X), \theta)$ is periodic if and only if there exists a positive integer n such that every chain of X of the form*

$$x_1 \geq x_1\theta \geq x_2 \geq x_2\theta \geq x_3 \geq x_3\theta \geq \ldots$$

with $x_i \in dom\theta$ has length at most n.

It is easily seen that the following two statements on X are equivalent.

(1) There is a positive integer n such that every chain of the form $x_1 \geq x_2 \geq x_3 \geq \ldots$ has length at most n.

(2) There is a positive integer n such that $|C| \leq n$ for every chain C of X.

Hence the following result given in [2] becomes our special case.

Corollary 3.10. *Let X be any poset and let $S(X)$ be $P_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$. Then the semigroup $S(X)$ is eventually regular if and only if there is a positive integer n such that $|C| \leq n$ for every chain C of X.*

Example 3.11. Under the natural order on $\mathbb{R}$, let $S(\mathbb{R})$ be $P_{RE}(\mathbb{R}), T_{RE}(\mathbb{R})$ or $I_{RE}(\mathbb{R})$. If $\theta \in S(\mathbb{R})$ with $\operatorname{dom}\theta = \mathbb{R}$, then by Corollary 3.6, $(S(\mathbb{R}), \theta)$ is not an eventually regular semigroup.

Let $n \in \mathbb{N}$ and define $\theta_n \in P(\mathbb{R})$ and $\theta'_n \in I(\mathbb{R})$ by

$$\operatorname{dom}\theta_n = [0, \infty),$$

$$[n, \infty)\theta_n = n,\ [n-1, n)\theta_n = n-1, \ldots,\ [0,1)\theta_n = 0,$$

$$\operatorname{dom}\theta'_n = \{\ n+1, n, n-1, \ldots, 1\ \},$$

$$(n+1)\theta'_n = n, n\theta'_n = n-1, \ldots, 1\theta'_n = 0.$$

Then $\theta_n \in P_{RE}(\mathbb{R})$, $\theta'_n \in I_{RE}(\mathbb{R})$, $\operatorname{im}\theta_n = \{\ n, n-1, \ldots, 1, 0\ \} = \operatorname{im}\theta'_n$. We therefore deduce from Corollary 3.3 that $(P_{RE}(\mathbb{R}), \theta_n)$ and $(I_{RE}(\mathbb{R}), \theta'_n)$ are both eventually regular.

Example 3.12. Define $\theta_1 \colon \mathbb{N} \to \mathbb{N}$ by

$$x\theta_1 = \begin{cases} x-1 & \text{if } \ x \in \mathbb{N} \smallsetminus \{1\}, \\ 1 & \text{if } \ x = 1 \end{cases} \quad \textit{and} \quad \theta_2 = \theta_1|_{\mathbb{N} \smallsetminus \{1\}}.$$

Then $\theta_1 \in T_{RE}(\mathbb{N}) \subseteq P_{RE}(\mathbb{N})$ and $\theta_2 \in I_{RE}(\mathbb{N}) \subseteq P_{RE}(\mathbb{N})$. Moreover, for every $n \in \mathbb{N}$,

$$2n > (2n)\theta_1 > (2n)\theta_1^2 > \ldots > (2n)\theta_1^{2n-1} \ \text{ and}$$

$$2n > (2n)\theta_2 > (2n)\theta_2^2 > \ldots > (2n)\theta_2^{2n-1}$$

and both chains have length $2n > n$. Then by Theorem 3.8, all of the semigroups $(P_{RE}(\mathbb{N}), \theta_1)$, $(T_{RE}(\mathbb{N}), \theta_1)$, $(P_{RE}(\mathbb{N}), \theta_2)$, and $(I_{RE}(\mathbb{N}), \theta_2)$ are not eventually regular.

In fact, for $n \in \mathbb{N}$ and for $i \in \{ 1, 2 \}$, we have

$$4n(\theta_i, \theta_i)^{2n} = 4n\theta_i^{4n-1} = 1$$

$$< 2n + 1 = 4n\theta_i^{2n-1} = 4n(\theta_i, \theta_i)^n,$$

Hence $(\theta_1, \theta_1)^n$ is not an idempotent of $(P_{RE}(\mathbb{N}), \theta_1)$ and $(T_{RE}(\mathbb{N}), \theta_1)$ and $(\theta_2, \theta_2)^n$ is not an idempotent of $(P_{RE}(\mathbb{N}), \theta_2)$ and $(I_{RE}(\mathbb{N}), \theta_2)$ for every $n \in \mathbb{N}$. By Proposition 2.1, θ_1 is not an eventually regular element of $(P_{RE}(\mathbb{N}), \theta_1)$ and $(T_{RE}(\mathbb{N}), \theta_1)$ and θ_2 is not an eventually regular element of $(P_{RE}(\mathbb{N}), \theta_2)$ and $(I_{RE}(\mathbb{N}), \theta_2)$.

The last theorem is the second main result of this chapter.

Theorem 3.13. *For any poset X, if $S(X)$ is a regressive almost identical transformation semigroup on X, then the semigroup $(S(X), \theta)$ is eventually regular for every $\theta \in S^1(X)$.*

Proof. Let $\theta \in S^1(X)$ and $\alpha \in S(X)$. Then $\alpha\theta \in S(X)$, so $S(\alpha\theta)$ is finite, say $|S(\alpha\theta)| = n$. Let $x \in \operatorname{dom}(\alpha\theta)^{n+2}$. Then

$$x(\alpha\theta) \geq x(\alpha\theta)^2 \geq \ldots \geq x(\alpha\theta)^{n+2}.$$

If $x(\alpha\theta) > x(\alpha\theta)^2 > \ldots > x(\alpha\theta)^{n+2}$, then $\{ x(\alpha\theta), x(\alpha\theta)^2, \ldots, x(\alpha\theta)^{n+1} \} \subseteq S(\alpha\theta)$ and $| \{ x(\alpha\theta), x(\alpha\theta)^2, \ldots, x(\alpha\theta)^{n+1} \} | = n + 1$, a contradiction. Thus $x(\alpha\theta)^i = x(\alpha\theta)^{i+1}$ for some $i \in \{ 1, 2, \ldots, n+1 \}$. Since $x \in \operatorname{dom}(\alpha\theta)^{n+2}$, $x(\alpha\theta)^i \in \operatorname{dom}(\alpha\theta)^{n+2-i}$. Then we deduce that

$$x(\alpha\theta)^{n+2} = x(\alpha\theta)^i(\alpha\theta)^{n+2-i} = x(\alpha\theta)^{i+1}(\alpha\theta)^{n+2-i} = x(\alpha\theta)^{n+3}.$$

This shows that

$$\text{for each } x \in \operatorname{dom}(\alpha\theta)^{n+2},\ x(\alpha\theta)^{n+2} = x(\alpha\theta)^{n+3}.$$

It then follows from this fact that $\operatorname{dom}(\alpha\theta)^{n+2} \subseteq \operatorname{dom}(\alpha\theta)^{n+3}$ and $x(\alpha\theta)^{n+2} = x(\alpha\theta)^{n+3}$ for every $x \in \operatorname{dom}(\alpha\theta)^{n+2}$. But $\operatorname{dom}(\alpha\theta)^{n+3} \subseteq \operatorname{dom}(\alpha\theta)^{n+2}$, so we have

$(\alpha\theta)^{n+2} = (\alpha\theta)^{n+3}$. Hence $(\alpha, \theta)^{n+3} = (\alpha\theta)^{n+2}\alpha = (\alpha\theta)^{n+3}\alpha = (\alpha, \theta)^{n+4}$, and thus $(\alpha, \theta)^{n+3} \in E(S(X), \theta)$.

Therefore the proof is complete. $\square$

CHAPTER IV

ISOMORPHISM THEOREMS

In this chapter, some isomorphism theorems on regressive generalized transformation semigroups are provided. The purpose is to show that for some posets X, some regressive transformation semigroups $S(X)$ on X and certain $\theta_1, \theta_2 \in S(X)$, $(S(X), \theta_1) \cong (S(X), \theta_2)$ if and only if $\theta_1 = \theta_2$.

Theorem 4.1. *Let $S(X)$ be a regressive transformation semigroup on a poset X containing an identity η. Then for $\theta \in S(X)$,*

$$(S(X), \theta) \cong S(X) \iff \theta = \eta.$$

Proof. Note that $S(X) = (S(X), \eta)$. Assume that $(S(X), \theta) \cong S(X)$. Since $S(X)$ has an identity, $(S(X), \theta)$ has an identity, say μ, that is,

$$\alpha\theta\mu = \mu\theta\alpha = \alpha \text{ for every } \alpha \in S(X).$$

In particular, $\eta\theta\mu = \mu\theta\eta = \eta$ since $\eta \in S(X)$. But since $\theta \in S(X)$ and η is the identity of $S(X)$, we have $\theta\eta = \eta\theta = \theta$. Now, we have

$$\theta\mu = \mu\theta = \eta \text{ and } \theta\eta = \eta\theta = \theta.$$

Then $\operatorname{dom}\eta = \operatorname{dom}(\theta\mu) \subseteq \operatorname{dom}\theta$ and $\operatorname{dom}\theta = \operatorname{dom}(\eta\theta) \subseteq \operatorname{dom}\eta$, so $\operatorname{dom}\theta = \operatorname{dom}\eta$. Also, for $x \in \operatorname{dom}\theta (= dom\,\eta)$

$$x\theta = x\eta\theta = (x\eta)\theta \leq x\eta,$$

$$x\eta = x\theta\mu = (x\theta)\mu \leq x\theta.$$

Consequently, $\operatorname{dom}\theta = \operatorname{dom}\eta$ and $x\theta = x\eta$ for every $x \in \operatorname{dom}\theta$, so $\theta = \eta$.

The converse is trivial. $\square$

The following is an immediate consequence of Theorem 4.1.

Corollary 4.2. *Let X be a poset and let $S(X)$ be $P_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$. Then for $\theta \in S(X)$,*

$$(S(X), \theta) \cong S(X) \iff \theta = 1_X.$$

Theorem 4.3. *Let $S(X)$ be a transformation semigroup on a set X containing an identity and a zero ξ. Then for $\theta \in S(X)$,*

$$(S(X), \theta) \cong (S(X), \xi) \iff \theta = \xi.$$

Proof. Let η be the identity of $S(X)$. First, assume that $(S(X), \theta) \cong (S(X), \xi)$. Since ξ is the zero of $S(X)$, it follows that $\alpha\xi\beta = \xi$ for all $\alpha, \beta \in S(X)$, that is, $(S(X), \xi)$ is a zero semigroup. Consequently, $(S(X), \theta)$ is a zero semigroup. But $\alpha\theta\xi = \xi\theta\alpha = \xi$ for all $\alpha \in S(X)$, so ξ is the zero of $(S(X), \theta)$. In particular, $\eta\theta\eta = \xi$. Since η is the identity of $S(X)$, we have that $\theta = \xi$.

The converse is immediate. $\square$

Corollary 4.4. *Let X be a poset and $S(X)$ be $P_{RE}(X)$ or $I_{RE}(X)$. Then for $\theta \in S(X)$,*

$$(S(X), \theta) \cong (S(X), 0) \iff \theta = 0.$$

Proof. It follows directly from Theorem 4.3 since $1_X, 0 \in S(X)$. $\square$

If a is the minimum element of a poset X, then X_a is the zero of $T_{RE}(X)$. That is because $a\alpha = a$ for all $\alpha \in T_{RE}(X)$, which implies that

$$\alpha X_a = X_a = X_a\alpha \quad \text{for every } \alpha \in T_{RE}(X).$$

Hence the following theorem is obtained directly from Theorem 4.3 and that $1_X \in T_{RE}(X)$ for every poset X.

Corollary 4.5. *Let X be a poset containing a minimum element a. Then for $\theta \in T_{RE}(X)$,*

$$(T_{RE}(X), \theta) \cong (T_{RE}(X), X_a) \iff \theta = X_a.$$

For a poset X containing a minimum element a, $X_a \in P_{RE}(X)$ but not a zero of $P_{RE}(X)$ because 0 is the zero of $P_{RE}(X)$. However, Corollary 4.5 remains valid if $T_{RE}(X)$ is replaced by $P_{RE}(X)$.

Theorem 4.6. *Let X be a poset containing a minimum element a. Then for $\theta \in P_{RE}(X)$,*

$$(P_{RE}(X), \theta) \cong (P_{RE}(X), X_a) \iff \theta = X_a.$$

Proof. First we note that $\{x\}_y \in P_{RE}(X)$ for all $x, y \in X$ with $y \leq x$. Assume that $(P_{RE}(X), \theta) \cong (P_{RE}(X), X_a)$. Since X_a is not the zero of $(P_{RE}(X), X_a)$, it follows from Corollary 4.4 that $\theta \neq 0$. Let φ be an isomorphism of $(P_{RE}(X), \theta)$ onto $(P_{RE}(X), X_a)$. Since 0 is the zero of both $(P_{RE}(X), \theta)$ and $(P_{RE}(X), X_a)$, for $\alpha \in P_{RE}(X), \alpha\varphi = 0$ if and only if $\alpha = 0$. Since a is the minimum element of X, we deduce that

$$\text{for } \alpha \in P_{RE}(X), a \in \operatorname{dom}\alpha \implies a\alpha = a. \tag{1}$$

Pick $b \in \operatorname{dom}\theta$. Then $\{b\}_b\theta 1_X = \{b\}_b\theta \neq 0$ which implies that

$$(\{b\}_b\theta 1_X)\varphi = (\{b\}_b)\varphi X_a(1_X\varphi) \neq 0,$$

and hence

$$a \in \operatorname{dom}(1_X\varphi). \tag{2}$$

To show that $\operatorname{dom}\theta = X$, let $x \in X$. Then $\{x\}_x\varphi \neq 0$, so $(\{x\}_x\varphi)X_a = (\operatorname{dom}(\{x\}_x\varphi))_a$. Hence by (2), $(\{x\}_x)\varphi X_a(1_X\varphi) \neq 0$. Consequently,

$$0 \neq (\{x\}_x\varphi)X_a(1_X\varphi) = (\{x\}_x\theta 1_X)\varphi = (\{x\}_x\theta)\varphi$$

which implies that $x \in \operatorname{dom}\theta$. This proves that $\operatorname{dom}\theta = X$. By (1), $a \in \operatorname{im}\theta$. To show that $\operatorname{im}\theta = \{a\}$, suppose not. Then there is $c \in \operatorname{im}\theta \smallsetminus \{a\}$. Then $c > a$ and $d\theta = c$ for some $d \in X$. But θ is regressive, so $d \geq c$. Hence

$$\begin{aligned} 0 \neq \{d\}_c\varphi &= (\{d\}_d\theta\{c\}_c)\varphi && \text{since } d\theta = c \\ &= (\{d\}_d\varphi)X_a(\{c\}_c\varphi) \end{aligned}$$

which implies that

$$a \in \operatorname{dom}(\{c\}_c\varphi). \tag{3}$$

It then follows that

$$0 \neq (\{c\}_c\varphi)X_a(\{c\}_c\varphi) = (\{c\}_c\theta\{c\}_c)\varphi.$$

Thus $c \in \operatorname{dom}\theta$ and $c\theta = c$. But

$$\begin{aligned} 0 \neq \{c\}_a\varphi &= (\{c\}_c\theta\{c\}_a)\varphi && \text{since } c\theta = c \\ &= (\{c\}_c\varphi)X_a(\{c\}_a\varphi) \\ &= (\operatorname{dom}(\{c\}_c\varphi))_a && \text{because of (1) and (3),} \end{aligned}$$

so

$$\begin{aligned} 0 \neq (\operatorname{dom}(\{c\}_c\varphi))_a &= (\operatorname{dom}(\{c\}_c\varphi))_a X_a (\operatorname{dom}(\{c\}_c\varphi))_a && \text{from (1) and (3)} \\ &= \{c\}_a\varphi X_a \{c\}_a\varphi \\ &= (\{c\}_a\theta\{c\}_a)\varphi \\ &= 0\varphi && \text{since } a\theta = a < c \\ &= 0. \end{aligned}$$

This is a contradiction. Thus $\theta = X_a$, as required.

The converse is trivial. $\square$

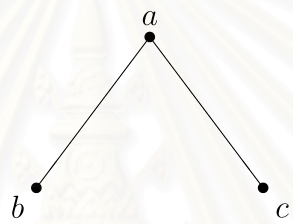
From Corollary 4.2, Corollary 4.4, Corollary 4.5 and Theorem 4.6, it is natural to ask whether the following question is true. If X is a poset and $S(X)$ is $P_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$, and $\theta_1, \theta_2 \in S(X)$, is it true that

$$(S(X), \theta_1) \cong (S(X), \theta_2) \iff \theta_1 = \theta_2 \text{ ?}$$

The following example gives a negative answer.

Example 4.7. Let $X = \{$a, b, c$\}$ with a partial order defined by the following Hasse diagram.

X :



(1) Define $\theta_1, \theta_2 \colon X \to X$ by

$$a\theta_1 = b,\ b\theta_1 = b,\ c\theta_1 = c,$$

$$a\theta_2 = c,\ b\theta_2 = b,\ c\theta_2 = c.$$

Then $T_{RE}(X) = \{\ 1_X, \theta_1, \theta_2\ \}$ and $\theta_1 \neq \theta_2$. It is clearly seen that $\theta_1\theta_2 = \theta_1 = \theta_1\theta_1$ and $\theta_2\theta_1 = \theta_2 = \theta_2\theta_2$. From this fact, it is easy to see that the map $\varphi \colon T_{RE}(X) \to T_{RE}(X)$ defined by

$$1_X\varphi = 1_X, \theta_1\varphi = \theta_2 \text{ and } \theta_2\varphi = \theta_1$$

is an isomorphism of $(T_{RE}(X), \theta_1)$ onto $(T_{RE}(X), \theta_2)$.

(2) We have that $\{b\}_b$ and $\{c\}_c$ are distinct elements of $I_{RE}(X) (\subseteq P_{RE}(X))$. We claim that

$$(S(X), \{b\}_b) \cong (S(X), \{c\}_c) \text{ where } S(X) \text{ is } P_{RE}(X) \text{ or } I_{RE}(X).$$

For convenience, let (b,c) denote the element of $T(X)$ defined by

$$x(b,c) = \begin{cases} c & \text{if } \ x = b, \\ b & \text{if } \ x = c, \\ x & \text{otherwise.} \end{cases}$$

Note that $(b,c) \in I(X)$. First, we shall show that for $\alpha \in S(X), (b,c)\alpha(b,c) \in S(X)$. Let $\alpha \in S(X)$ and $x \in \operatorname{dom}(b,c)\alpha(b,c)$. Since b and c are minimal elements of X and α is regressive, we have $b\alpha = b$ if $b \in \operatorname{dom}\alpha$ and $c\alpha = c$ if $c \in \operatorname{dom}\alpha$.

Case 1 : $x = a$. Then $a(b,c)\alpha(b,c) \leq a$ since a is the maximum element of X.

Case 2 : $x = b$. Then $b(b,c)\alpha(b,c) = c\alpha(b,c) = c(b,c) = b$.

Case 3 : $x = c$. Then $c(b,c)\alpha(b,c) = b\alpha(b,c) = b(b,c) = c$.

Thus $(b,c)\alpha(b,c)$ is regressive. If $\alpha \in I(X)$, then so is $(b,c)\alpha(b,c)$ since $(b,c) \in I(X)$. This shows that $(b,c)\alpha(b,c) \in S(X)$. Define $\Psi\colon S(X) \to S(X)$ by $\alpha\Psi = (b,c)\alpha(b,c)$ for all $\alpha \in S(X)$. If $\alpha, \beta \in S(X)$ are such that $\alpha\Psi = \beta\Psi$, then $(b,c)\alpha(b,c) = (b,c)\beta(b,c)$, so $\alpha = \beta$ since $(b,c)(b,c) = 1_X$. Hence Ψ is 1-1. But $S(X)$ is finite, so Ψ is onto. If $\alpha, \beta \in S(X)$, then

$$\begin{aligned} (\alpha\{b\}_b\beta)\Psi &= (b,c)(\alpha\{b\}_b\beta)(b,c) \\ &= ((b,c)\alpha(b,c))((b,c)\{b\}_b(b,c))((b,c)\beta(b,c)) \\ &\qquad \text{since } \ (b,c)(b,c) = 1_X \\ &= (\alpha\Psi)\{c\}_c(\beta\Psi), \end{aligned}$$

so,Ψ is a homomorphism of $(S(X), \{b\}_b)$ into $(S(X), \{c\}_c)$. This proves that Ψ is an isomorphism of $(S(X), \{b\}_b)$ onto $(S(X), \{c\}_c)$.

Example 4.7(2) shows that if X is a poset and $S(X)$ is $P_{RE}(X)$ or $I_{RE}(X)$, it is not generally true that for $a, b \in X, (S(X), \{a\}_a) \cong (S(X), \{b\}_b)$ implies a = b. However, the next theorem shows that this is true if X is a finite chain.

Lemma 4.8. *Let X be a chain and $a \in X$. For $\alpha \in P_{RE}(X)$, $\alpha \in E(P_{RE}(X), \{a\}_a)$ if and only if either $\alpha = 0$ or $\alpha = A_a$ for some $A \subseteq \{ x \in X \mid x \geq a \}$ with $a \in A$. In particular, $E(I_{RE}(X), \{a\}_a) = \{ 0, \{a\}_a \}$.*

Proof. Assume that $\alpha \in E(P_{RE}(X), \{a\}_a)$ and $\alpha \neq 0$. Since $0 \neq \alpha = \alpha\{a\}_a\alpha$, we have $a \in \operatorname{dom}\alpha$ and so $a\alpha = a\alpha\{a\}_a\alpha$ which implies that $a\alpha = a$. If $x \in \operatorname{dom}\alpha$, then $x\alpha = x\alpha\{a\}_a\alpha$, so $a = x\alpha \leq x$. Hence $\alpha = A_a$ for some $A \subseteq \{ x \in X \mid x \geq a \}$ with $a \in A$. If α is also 1-1, then $\alpha = \{a\}_a$.

The converse is obvious. Hence the lemma is proved. □

Theorem 4.9. *Let X be a finite chain and $a, b \in X$. If $S(X)$ is $P_{RE}(X)$ or $I_{RE}(X)$, then*

$$(S(X), \{a\}_a) \cong (S(X), \{b\}_b) \iff a = b.$$

Proof. We may assume that $a \leq b$ since X is a chain. By Lemma 4.8,

$$\left.\begin{aligned} E(P_{RE}(X), \{a\}_a) &= \{ (A \cup \{a\})_a \mid A \subseteq \{x \in X \mid x > a\} \} \cup \{0\}, \\ E(P_{RE}(X), \{b\}_b) &= \{ (A \cup \{b\})_b \mid A \subseteq \{x \in X \mid x > b\} \} \cup \{0\}, \\ E(I_{RE}(X), \{a\}_a) &= \{ 0, \{a\}_a \}, \\ E(I_{RE}(X), \{b\}_b) &= \{ 0, \{b\}_b \}. \end{aligned}\right\} \quad (1)$$

Suppose that $(S(X), \{a\}_a) \cong (S(X), \{b\}_b)$. Let φ be an isomorphism from $(S(X), \{a\}_a)$ onto $(S(X), \{b\}_b)$, so $|E(S(X), \{a\}_a)| = |E(S(X), \{b\}_b)|$.

Case 1: $S(X) = P_{RE}(X)$. Since $|E(P_{RE}(X), \{a\}_a)| = |E(P_{RE}(X), \{b\}_b)|$, from (1),

$$| \{ A \mid A \subseteq \{x \in X \mid x > a\} \} | = | \{ A \mid A \subseteq \{x \in X \mid x > b\} \} |.$$

But X is a finite chain, so $a = b$.

Case 2: $S(X) = I_{RE}(X)$. Let φ be an isomorphism of $(I_{RE}(X), \{a\}_a)$ onto

$(I_{RE}(X), \{b\}_b)$. By (1), $\{a\}_a\varphi = \{b\}_b$. Let $c \in X$ be such that $c \geq a$. Then

$$\begin{aligned}
0 \neq \{c\}_a\varphi &= (\{c\}_a\{a\}_a\{a\}_a)\varphi \\
&= \{c\}_a\varphi\{b\}_b\{b\}_b && \text{since } \{a\}_a\varphi = \{b\}_b \\
&= (\{c\}_a\varphi)\{b\}_b \\
&= \{d\}_b && \text{for some } d \in X \text{ with } d \geq b \\
& && \text{since } S(X) = I_{RE}(X).
\end{aligned}$$

This shows that

$$\{\ \{c\}_a\varphi \mid c \in X \text{ and } c \geq a\ \} \subseteq \{\ \{d\}_b\varphi \mid d \in X \text{ and } d \geq b\ \}$$

Hence

$$\begin{aligned}
\mid \{c \in X \mid c \geq a\} \mid\ &= \mid \{\ \{c\}_a\varphi \mid c \in X \text{ and } c \geq a\ \} \mid && \text{since } \varphi \text{ is } 1-1 \\
&\leq \mid \{\ \{d\}_b \mid d \in X \text{ and } d \geq b\} \mid \\
&= \mid \{d \in X \mid d \geq b\} \mid \\
&\leq \mid \{c \in X \mid c \geq a\} \mid && \text{since } a \leq b,
\end{aligned}$$

so $\mid \{c \in X \mid c \geq a\} \mid = \mid \{d \in X \mid d \geq b\} \mid$. Since X is a finite chain, we deduce that $a = b$.

Hence the theorem is completely proved. □

Theorem 4.9 need not be true if X is an infinite chain. It is shown by the next theorem. Moreover, this theorem also shows that if X is an infinite chain order-isomorphic to $\mathbb{Z}$ and $S(X)$ is $P_{RE}(X)$ or $I_{RE}(X)$, then for all $a, b \in X$, $(S(X), \{a\}_a) \cong (S(X), \{b\}_b)$.

Theorem 4.10. *Let $S(\mathbb{Z})$ be $P_{RE}(\mathbb{Z})$ or $I_{RE}(\mathbb{Z})$ under the natural order. Then for all $a, b \in \mathbb{Z}$,*

$$(S(\mathbb{Z}), \{a\}_a) \cong (S(\mathbb{Z}), \{b\}_b).$$

Proof. For a subset A of $\mathbb{Z}$ and $k \in \mathbb{Z}$, let

$$A + k = \{ \ x + k \mid x \in A \ \}.$$

Let $a, b \in \mathbb{Z}$ be such that $a < b$. For each $\alpha \in P(\mathbb{Z})$, define $\alpha' \in P(\mathbb{Z})$ by

$$\operatorname{dom}\alpha' = \operatorname{dom}\alpha + (b-a) \quad \text{and}$$
$$x\alpha' = (x-(b-a))\alpha + (b-a) \quad \text{for every } x \in \operatorname{dom}\alpha'.$$

It is clear that if α is regressive, then so is α', and if α is 1-1, then so is α'. Hence

$$\{ \ \alpha' \mid \alpha \in P_{RE}(\mathbb{Z}) \ \} \subseteq P_{RE}(\mathbb{Z}),$$
$$\{ \ \alpha' \mid \alpha \in I_{RE}(\mathbb{Z}) \ \} \subseteq I_{RE}(\mathbb{Z}).$$

Define $\varphi\colon S(\mathbb{Z}) \to S(\mathbb{Z})$ by $\alpha\varphi = \alpha'$ for all $\alpha \in S(\mathbb{Z})$. First, to show φ is 1-1, let $\alpha, \beta \in S(\mathbb{Z})$ be such that $\alpha' = \beta'$. Then

$$\operatorname{dom}\alpha + (b-a) = \ \operatorname{dom}\alpha' = \ \operatorname{dom}\beta' = \ \operatorname{dom}\beta + (b-a) \quad \text{and}$$
$$x\alpha' = x\beta' \quad \text{for all } x \in \operatorname{dom}\alpha' (= \operatorname{dom}\beta').$$

This implies that $\operatorname{dom}\alpha = \operatorname{dom}\beta$ and for every $x \in \operatorname{dom}\alpha$,

$$x\alpha = (x+(b-a))\alpha' - (b-a) = (x+(b-a))\beta' - (b-a) = x\beta.$$

Hence $x\alpha = x\beta$.

To show φ is onto, let $\beta \in S(\mathbb{Z})$. Define $\alpha \in P(\mathbb{Z})$ by

$$\operatorname{dom}\alpha = \operatorname{dom}\beta - (b-a) \quad \text{and}$$
$$x\alpha = (x+(b-a))\beta - (b-a) \quad \text{for all } x \in \ \operatorname{dom}\alpha.$$

Then we have similarly that $\alpha \in S(\mathbb{Z})$ and also

$$\begin{aligned}\operatorname{dom}\alpha' &= \operatorname{dom}\alpha + (b-a)\\ &= \operatorname{dom}\beta - (b-a) + (b-a)\\ &= \operatorname{dom}\beta \quad \text{and}\\ x\alpha' &= (x-(b-a))\alpha + (b-a)\\ &= (x-(b-a)+(b-a))\beta - (b-a) + (b-a)\\ &= x\beta.\end{aligned}$$

Hence $\alpha\varphi = \alpha' = \beta$.

To show that φ is a homomorphism, it is equivalent to show $(\alpha\{a\}_a\beta)' = \alpha'\{b\}_b\beta'$ for all $\alpha, \beta \in S(\mathbb{Z})$. Let $\alpha, \beta \in S(\mathbb{Z})$ be arbitrary fixed. Then for $x \in \mathbb{Z}$,

$$\begin{aligned}x \in \operatorname{dom}(\alpha\{a\}_a\beta)' &\iff x-(b-a) \in \operatorname{dom}(\alpha\{a\}_a\beta)\\ &\iff x-(b-a) \in \operatorname{dom}\alpha, (x-(b-a))\alpha = a\\ &\qquad \text{and } a \in \operatorname{dom}\beta\\ &\iff x \in \operatorname{dom}\alpha' \text{ and } x\alpha'(= (x-(b-a))\alpha + (b-a)\\ &\qquad = a+b-a) = b \text{ and } b = a+(b-a) \in \operatorname{dom}\beta'\\ &\iff x \in \operatorname{dom}(\alpha'\{b\}_b\beta').\end{aligned}$$

Therefore $\operatorname{dom}(\alpha\{a\}_a\beta)' = \operatorname{dom}(\alpha'\{b\}_b\beta')$ and for $x \in \operatorname{dom}(\alpha\{a\}_a\beta)'$,

$$\begin{aligned}x(\alpha\{a\}_a\beta)' &= (x-(b-a))(\alpha\{a\}_a\beta) + (b-a)\\ &= a\beta + (b-a) && \text{since } (x-(b-a))\alpha = a\\ &= (b-(b-a))\beta + (b-a)\\ &= b\beta'\\ &= x(\alpha'\{b\}_b\beta') && \text{since } x\alpha' = (x-(b-a))\alpha + (b-a) = b.\end{aligned}$$

Hence the theorem is proved, as required. $\square$

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