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ORDER-PRESERVING GENERALIZED TRANSFORMATION SEMIGROUPS

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สำหรับเซต X ให้ $P(X), T(X)$ และ $I(X)$ แทนกึ่งกรุปการแปลงบางส่วนบน X กึ่งกรุปการแปลงเต็มบน X และกึ่งกรุปการแปลงบางส่วนหนึ่งต่อหนึ่งบน X ตามลำดับ เราให้นัยทั่วไปของกึ่งกรุปการแปลงเหล่านี้ดังนี้ สำหรับเซต X และ Y ให้ $P(X, Y) = \{ \alpha : A \to Y \mid A \subseteq X \}$, $T(X, Y) = \{ \alpha \in P(X, Y) \mid \mathrm{dom}\, \alpha = X \}$ และ $I(X, Y) = \{ \alpha \in P(X, Y) \mid \alpha$ หนึ่งต่อหนึ่ง $\}$ สำหรับ $\theta \in P(Y, X)$ ให้ $(P(X, Y), \theta)$ แทนกึ่งกรุป $(P(X, Y), *)$ โดย $\alpha * \beta = \alpha\theta\beta$ สำหรับทุก $\alpha, \beta \in P(X, Y)$ เรานิยามกึ่งกรุป $(T(X, Y), \theta)$ โดย $\theta \in T(Y, X)$ และ $(I(X, Y), \theta)$ โดย $\theta \in I(Y, X)$ ในทำนองเดียวกัน

สำหรับโพเซต X ให้ $OP(X), OT(X)$ และ $OI(X)$ แทนกึ่งกรุปการแปลงบางส่วนที่รักษาอันดับบน X กึ่งกรุปการแปลงเต็มที่รักษาอันดับบน X และกึ่งกรุปการแปลงบางส่วนหนึ่งต่อหนึ่งที่รักษาอันดับบน X ตามลำดับ สำหรับโพเซต X และ Y ใดๆ ให้ $OP(X, Y) = \{ \alpha \in P(X, Y) \mid \alpha$ รักษาอันดับ $\}$ สำหรับ $\theta \in OP(Y, X)$ ให้ $(OP(X, Y), \theta)$ แทนกึ่งกรุป $(OP(X, Y), *)$ โดยกำหนดการดำเนินการ $*$ เช่นเดียวกับข้างบน เรานิยามกึ่งกรุป $(OT(X, Y), \theta)$ โดย $\theta \in OT(Y, X)$ และ $(OI(X, Y), \theta)$ โดย $\theta \in OI(Y, X)$ ในทำนองเดียวกัน

ความจริงต่อไปนี้เป็นที่รู้กันแล้ว ถ้า X เป็นเซตอันดับทุกส่วน แล้ว $OP(X)$ และ $OI(X)$ เป็นกึ่งกรุปปรกติ สำหรับสับเซต X ของ $\boldsymbol{Z}$ ที่ไม่ว่างใดๆ $OT(X)$ เป็นกึ่งกรุปปรกติ ยิ่งไปกว่านั้น สำหรับช่วง X ของ $\boldsymbol{I\!R}$ ที่ไม่ว่าง $OT(X)$ เป็นกึ่งกรุปปรกติ ก็ต่อเมื่อ X เป็นเซตปิดและมีขอบเขต

ในการวิจัยนี้ เราให้นำความจริงที่รู้กันอันแรกที่กล่าวไว้แล้วข้างต้นมาใช้ในการบอกลักษณะว่าเมื่อใดกึ่งกรุป $(OP(X,Y), \theta)$ โดย $\theta \in OP(Y, X)$ และ กึ่งกรุป $(OI(X,Y), \theta)$ โดย $\theta \in OI(Y, X)$ เป็นกึ่งกรุปปรกติ โดยที่ X และ Y เป็นเซตอันดับทุกส่วน เราแสดงว่าการเป็นสมสัณฐานของ θ เป็นเงื่อนไขจำเป็นและเพียงพอหลักสำหรับการเป็นปรกติของกึ่งกรุปเหล่านี้ และเรายังให้ลักษณะด้วยว่าเมื่อใดกึ่งกรุป $(OT(X,Y), \theta)$ โดย $\theta \in OT(Y, X)$ เป็นกึ่งกรุปปรกติ โดยที่ X และ Y เป็นเซตอันดับทุกส่วน ในการให้ลักษณะนี้ จะให้ในเทอมของความเป็นกึ่งกรุปปรกติของ $OT(X), |X|, |Y|$ และ θ จากผลที่รู้กันแล้วอันที่สองและที่สามข้างต้นทำให้การให้ลักษณะของความเป็นกึ่งกรุปปรกติของ $(OT(X,Y), \theta)$ โดยที่ทั้ง X และ Y เป็นสับเซตของ $\boldsymbol{Z}$ ที่มีสมาชิกมากกว่าหนึ่งตัว และเมื่อทั้ง X และ Y เป็นช่วงของ $\boldsymbol{I\!R}$ ที่มีสมาชิกมากกว่าหนึ่งตัวสามารถให้ในเทอมของ θ และในเทอมของ X และ θ ตามลำดับ ยิ่งไปกว่านั้นเราให้ทฤษฎีบทสมสัณฐานที่น่าสนใจบางทฤษฎีบท โดยที่ X และ Y เป็นเซตอันดับทุกส่วน เราให้เงื่อนไขที่จำเป็นและเพียงพอเพื่อว่า $(OS(X,Y), \theta) \cong OS(X)$ และเพื่อว่า $(OS(X,Y), \theta) \cong OS(Y)$ โดยที่ $OS(X,Y)$ คือ $OP(X,Y)$, $OT(X,Y)$ หรือ $OI(X,Y)$ และ $\theta \in OS(Y, X)$

ภาควิชา **คณิตศาสตร์** ลายมือชื่อนิสิต.......................................

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For a set X, let $P(X)$, $T(X)$ and $I(X)$ denote respectively the partial transformation semigroup on X, the full transformation semigroup on X and the 1-1 partial transformation semigroup on X. These transformation semigroups are generalized as follows: For sets X and Y, let $P(X, Y) = \{\alpha : A \to Y \mid A \subseteq X\}$, $T(X, Y) = \{\alpha \in P(X, Y) \mid \text{dom}\,\alpha = X\}$ and $I(X, Y) = \{\alpha \in P(X, Y) \mid \alpha \text{ is 1-1}\}$. For $\theta \in P(Y, X)$, let $(P(X, Y), \theta)$ denote the semigroup $(P(X, Y), *)$ where $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in P(X, Y)$. The semigroups $(T(X, Y), \theta)$ with $\theta \in T(Y, X)$ and $(I(X, Y), \theta)$ with $\theta \in I(Y, X)$ are defined similarly.

For a poset X, let $OP(X)$, $OT(X)$ and $OI(X)$ denote the order-preserving partial transformation semigroup on X, the full order-preserving transformation semigroup on X and the order-preserving 1-1 partial transformation semigroup on X, respectively. For any posets X and Y, let $OP(X, Y) = \{\alpha \in P(X, Y) \mid \alpha \text{ is order-preserving}\}$. For $\theta \in OP(Y, X)$, let $(OP(X,Y), \theta)$ denote the semigroup $(OP(X, Y), *)$ where the operation $*$ is defined as above. The semigroups $(OT(X,Y), \theta)$ with $\theta \in OT(Y, X)$ and $(OI(X, Y), \theta)$ with $\theta \in OI(Y, X)$ are defined similarly.

The following facts are known. If X is a chain, then $OP(X)$ and $OI(X)$ are regular semigroups. For any nonempty subsets X of $\boldsymbol{Z}$, $OT(X)$ is regular. Moreover, for a nonempty interval X of $\boldsymbol{IR}$, $OT(X)$ is regular if and only if X is closed and bounded.

In this research, the first known fact mentioned above is used to characterize when the semigroup $(OP(X, Y), \theta)$ with $\theta \in OP(Y, X)$ and the semigroup $(OI(X, Y), \theta)$ with $\theta \in OI(Y, X)$ are regular where X and Y are chains. It is shown that being an order-isomorphism of θ is mainly necessary and sufficient for regularity of these semigroups. We also characterize when the semigroup $(OT(X,Y), \theta)$ with $\theta \in OT(Y, X)$ is regular where X and Y are chains. This characterization is given in terms of regularity of $OT(X)$, $|X|$, $|Y|$ and θ. Due to the above second and third known results, the characterizations of regularity of $(OT(X, Y), \theta)$ when both X and Y are nontrivial subsets of $\boldsymbol{Z}$ and when both X and Y are nontrivial intervals of $\boldsymbol{IR}$ can be given respectively in term of θ and in terms of X and θ. Here, a nontrivial set means a set containing more than one element. Moreover, some interesting isomorphism theorems are provided where X and Y are chains. Necessary and sufficient conditions are given for that $(OS(X, Y), \theta) \cong OS(X)$ and for that $(OS(X, Y), \theta) \cong OS(Y)$ where $OS(X, Y)$ is $OP(X, Y)$, $OT(X,Y)$ or $OI(X, Y)$ and $\theta \in OS(Y, X)$.

Department **Mathematics** Student's signature.......................................

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CONTENTS

page

CHAPTER I

INTRODUCTION AND PRELIMINARIES

For a set X, let $|X|$ denote the cardinality of X. The identity mapping on a nonempty set A is denoted by 1_A. The set of all integers and the set of all real numbers are denoted by $\mathbb{Z}$ and $\mathbb{R}$, respectively.

We call an element a of a semigroup S an *idempotent* of S if $a^2 = a$ and S is said to be an *idempotent semigroup* or a *band* if every element of S is an idempotent.

An element a of a semigroup S is said to be *regular* if $a = aba$ for some $b \in S$ and we call S a *regular semigroup* if every element of S is regular. Therefore every idempotent semigroup is regular.

The domain and the range of any mapping α will be denoted by dom α and ran α, respectively. For an element x in the domain of a mapping α, the image of α at x is written by $x\alpha$. For any mappings α and β, the *composition* $\alpha\beta$ of α and β is defined as follows: $\alpha\beta = 0$ if ran $\alpha \cap$ dom $\beta = \varnothing$, otherwise $\alpha\beta$ is the composition of $\alpha|_{(\text{ran } \alpha \,\cap\, \text{dom } \beta)\alpha^{-1}}$ and $\beta|_{(\text{ran } \alpha \,\cap\, \text{dom } \beta)}$ where 0 is the empty transformation, that is, the mapping with empty domain. Then for mappings α, β and γ, we have

$$\text{dom}(\alpha\beta) = (\text{ran } \alpha \cap \text{dom } \beta)\alpha^{-1} \subseteq \text{dom } \alpha,$$

$$\text{ran}(\alpha\beta) = (\text{ran } \alpha \cap \text{dom } \beta)\beta \subseteq \text{ran } \beta,$$

$$x \in \text{dom}(\alpha\beta) \Leftrightarrow x \in \text{dom } \alpha \text{ and } x\alpha \in \text{dom } \beta,$$

$$(\alpha\beta)\gamma = \alpha(\beta\gamma).$$

For a set X, a *partial transformation* of X is a mapping from a subset of X into X. Then the empty transformation 0 is a partial transformation of X. Let $P(X)$ be the set of all partial transformations of X, that is,

$$P(X) = \{\alpha : A \to X \mid A \subseteq X\}.$$

Then $1_A \in P(X)$ for every nonempty subset A of X. In particular, $1_X \in P(X)$. Therefore under the composition of mappings, $P(X)$ is a semigroup having 0 and 1_X as its zero and identity, respectively. The semigroup $P(X)$ is called the *partial transformation semigroup* on X. By a *transformation semigroup* on X we mean a subsemigroup of $P(X)$.

By a *transformation* of X we mean a mapping of X into itself. Let $T(X)$ be the set of all transformations of X. Then

$$T(X) = \{\alpha \in P(X) \mid \text{dom } \alpha = X\}$$

which is a subsemigroup of $P(X)$ containing 1_X and it is called the *full transformation semigroup* on X.

Let $I(X)$ denote the set of all 1-1 partial transformations of X, that is,

$$I(X) = \{\alpha \in P(X) \mid \alpha \text{ is 1-1}\}.$$

Then $I(X)$ is a subsemigroup of $P(X)$ containing 0 and 1_X and it is called the *1-1 partial transformation semigroup* on X or the *symmetric inverse semigroup* on X.

It is well-known that $P(X)$, $T(X)$ and $I(X)$ are all regular for every set X([2], page 4).

For sets X and Y, let

$$P(X,Y) = \{\alpha : A \to Y \mid A \subseteq X\},$$
$$T(X,Y) = \{\alpha \in P(X,Y) \mid \text{dom } \alpha = X\},$$
$$I(X,Y) = \{\alpha \in P(X,Y) \mid \alpha \text{ is 1-1}\}.$$

Note that $P(X,X) = P(X)$, $T(X,X) = T(X)$ and $I(X,X) = I(X)$. For a nonempty subset A of X and $y \in Y$, let A_y be the element of $P(X,Y)$ with domain A and range $\{y\}$.

Let $S(X,Y)$ be $P(X,Y)$, $T(X,Y)$ or $I(X,Y)$. For $\theta \in S(Y,X)$, let $(S(X,Y),\theta)$ denote the semigroup $(S(X,Y),*)$ where the operation $*$ is defined by

$$\alpha * \beta = \alpha\theta\beta \quad \text{for all } \alpha,\ \beta \in S(X,Y).$$

We observe that $S(X) = (S(X,X), 1_X)$.

Example 1.1. Let X and Y be nonempty sets and $a \in X$. Then $(T(X,Y), Y_a)$ is the semigroup $T(X,Y)$ with the operation $*$ defined as follows:

$$\alpha * \beta = \alpha Y_a \beta = X_{a\beta} \quad \text{for all } \alpha,\ \beta \in T(X,Y).$$

Also, $(P(X,Y), Y_a)$ is the semigroup $P(X,Y)$ with the operation $\circ$ defined by

$$\alpha \circ \beta = \alpha Y_a \beta = \begin{cases} (\text{dom } \alpha)_{a\beta} & \text{if } \alpha \neq 0 \text{ and } a \in \text{ dom } \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for $b \in Y$, the semigroup $(I(X,Y), \{b\}_a)$ is the semigroup $(I(X,Y), \bullet)$ where

$$\alpha \bullet \beta = \alpha \{b\}_a \beta = \begin{cases} \{b\alpha^{-1}\}_{a\beta} & \text{if } b \in \text{ ran } \alpha \text{ and } a \in \text{ dom } \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Let X and Y be partially ordered sets. For $\alpha \in P(X,Y)$, α is said to be *order-preserving* if

$$\text{for } x_1, x_2 \in \text{ dom } \alpha,\ x_1 \leq x_2 \text{ in } X \Rightarrow\ x_1\alpha \leq x_2\alpha \ \text{ in } Y.$$

A bijection $\varphi : X \rightarrow Y$ is called an *order-isomorphism* if φ and φ^{-1} are order-preserving. It is clear that if both X and Y are chains and $\varphi : X \rightarrow Y$ is an order-preserving bijection, then φ is an order-isomorphism from X onto Y. We say that X and Y are *order-isomorphic* if there is an order-isomorphism from X onto Y. Naturally, a bijection $\varphi : X \rightarrow Y$ satisfying the condition

$$\text{for } x_1, x_2 \in\ X,\ x_1 \leq x_2 \text{ in } X \Leftrightarrow\ x_2\varphi \leq x_1\varphi \ \text{ in } Y$$

is called an *anti-order-isomorphism.* We say that X and Y are *anti-order-isomorphic* if there is an anti-order-isomorphism from X onto Y.

A transformation semigroup on a poset X is said to be an *order-preserving transformation semigroup* on X if all of its elements are order-preserving. Define $OP(X)$ by

$$OP(X) = \{\alpha \in\ P(X) \mid \alpha \text{ is order-preserving}\}.$$

Then $OP(X)$ is clearly a subsemigroup of $P(X)$ containing 0 and 1_X. We define $OT(X)$ and $OI(X)$ similarly. Then $OT(X)$ and $OI(X)$ are subsemigoups of $T(X)$ and $I(X)$, respectively. Note that $1_X \in OT(X)$ and $0,\ 1_X \in OI(X)$. The semigroups $OP(X)$, $OT(X)$ and $OI(X)$ are called the *order-preserving partial transformation semigroup* on X, the *full order-preserving transformation semigroup* on X and the *order-preserving 1-1 partial transformation semigroup* on X, respectively.

In this research, the partial order on any subset of $\mathbb{R}$ always means the natural partial order on $\mathbb{R}$.

In [4], Y. Kemprasit and T. Changphas characterized when $OT(X)$ is a regular semigroup where X is a nonempty subset of $\mathbb{Z}$ and X is a nonempty interval of $\mathbb{R}$ as follows:

Theorem 1.2. [4] *For any nonempty subset X of $\mathbb{Z}$, the semigroup $OT(X)$ is regular.*

Theorem 1.3. [4] *For a nonempty interval X of $\mathbb{R}$, $OT(X)$ is a regular semigroup if and only if X is closed and bounded.*

Moreover, they answered similar questions for $OP(X)$ and $OI(X)$ for an arbitrary chain X as follows:

Theorem 1.4. [4] *If X is a chain, then the semigroups $OP(X)$ and $OI(X)$ are regular.*

A significant isomorphism theorem of full order-preserving transformation semigroups is as follows:

Theorem 1.5. [5, page 223] *For posets X and Y, $OT(X) \cong OT(Y)$ if and only if X and Y are order-isomorphic or anti-order-isomorphic.*

Example 1.6. (1) Since $\mathbb{Z}$ is order-isomorphic to $2\mathbb{Z}$ through the map $x \mapsto 2x$, by Theorem 1.5, we have $OT(\mathbb{Z}) \cong OT(2\mathbb{Z})$.

(2) We have that $OT(\mathbb{R}) \cong OT(\mathbb{R}^+)$ where $\mathbb{R}^+$ is the set of positive real numbers because the map $x \mapsto e^x$ is an order-isomorphism of $\mathbb{R}$ onto $\mathbb{R}^+$.

(3) Since $x \mapsto \frac{1}{x}$ is an anti-order-isomorphism from $[1, \infty)$ onto $(0, 1]$, we deduce from Theorem 1.5 that $OT([1, \infty)) \cong OT((0, 1])$.

We generalize the semigroups $OP(X)$, $OT(X)$ and $OI(X)$ where X is a poset as follows: For any posets X and Y, let

$$OP(X, Y) = \{\alpha \in P(X, Y) \mid \alpha \text{ is order-preserving}\}$$

and for $\theta \in OP(Y,X)$, let $(OP(X,Y),\theta)$ denote the semigroup $(OP(X,Y),*)$ where $\alpha * \beta = \alpha\theta\beta$ for all $\alpha,\beta \in OP(X,Y)$. The semigroups $(OT(X,Y),\theta)$ with $\theta \in OT(Y,X)$ and $(OI(X,Y),\theta)$ with $\theta \in OI(Y,X)$ are defined similarly. Note that if $S(X,Y)$ is $P(X,Y)$, $T(X,Y)$ or $I(X,Y)$ and $\theta \in OS(Y,X)$, then $(OS(X,Y),\theta)$ is a subsemigroup of $(S(X,Y),\theta)$. We remark here that $OS(X) = (OS(X,X),1_X)$.

Example 1.7. From Example 1.1, if X and Y are posets, $a \in X$ and $b \in Y$, then $Y_a \in OT(Y,X) \subseteq OP(Y,X)$ and $\{b\}_a \in OI(Y,X)$, then $(OT(X,Y),Y_a)$, $(OP(X,Y),Y_a)$ and $(OI(X,Y),\{b\}_a)$ are subsemigroups of $(T(X,Y),Y_a)$, $(P(X,Y),Y_a)$ and $(I(X,Y),\{b\}_a)$, respectively.

Example 1.8. Let $\theta : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$n\theta = (n+1)\theta = n \text{ for every n } \in 2\mathbb{Z}.$$

Then $\theta \in OT(\mathbb{Z})$ and ran $\theta = 2\mathbb{Z}$. Suppose that $(OT(\mathbb{Z}),\theta)$ has an identity, say η. Thus

$$\alpha\theta\eta = \eta\theta\alpha = \alpha \text{ for every } \alpha \in OT(\mathbb{Z}),$$

in particular, $\eta\theta 1_{\mathbb{Z}} = \eta\theta = 1_{\mathbb{Z}}$. This implies that ran $\theta = \mathbb{Z}$, a contradiction. Hence $(OT(\mathbb{Z}),\theta)$ does not have an identity. But by Example 1.6(1), $OT(\mathbb{Z}) \cong OT(2\mathbb{Z})$ and both have an identity, so we conclude that

$$(OT(\mathbb{Z}),\theta) \ncong OT(\mathbb{Z}) \text{ and } (OT(\mathbb{Z}),\theta) \ncong OT(2\mathbb{Z}).$$

In Chapter II, we are concerned with regularity of the order-preserving generalized transformation semigroups $(OP(X,Y),\theta)$ with $\theta \in OP(Y,X)$ and $(OI(X,Y),\theta)$ with $\theta \in OI(Y,X)$ where X and Y are any chains. We give necessary and sufficient conditions for θ and $|X|$ so that the semigroup $(OP(X,Y),\theta)$

is regular and for θ so that $(OI(X,Y),\theta)$ is a regular semigroup. The main tool for this chapter is Theorem 1.4.

The main purpose of Chapter III is to characterize when the semigroup $(OT(X,Y),\theta)$ with $\theta \in OT(Y,X)$ is regular where X and Y are chains. The characterization is given in terms of regularity of $OT(X)$, $|X|$, $|Y|$ and θ.

Some interesting isomorphism theorems are provided in Chapter IV. We characterize when the following statements hold where X and Y are chains.

$$(OP(X,Y),\theta) \cong OP(X) \quad \text{where} \quad \theta \in OP(Y,X),$$

$$(OP(X,Y),\theta) \cong OP(Y) \quad \text{where} \quad \theta \in OP(Y,X),$$

$$(OI(X,Y),\theta) \cong OI(X) \quad \text{where} \quad \theta \in OI(Y,X),$$

$$(OI(X,Y),\theta) \cong OI(Y) \quad \text{where} \quad \theta \in OI(Y,X),$$

$$(OT(X,Y),\theta) \cong OT(X) \quad \text{where} \quad \theta \in OT(Y,X),$$

$$(OT(X,Y),\theta) \cong OT(Y) \quad \text{where} \quad \theta \in OT(Y,X).$$

We can see from our purpose that we confine our attention when posets X and Y are chains. However, some required lemmas for our main results can be given in terms of any posets X and Y.

CHAPTER II

REGULAR ORDER-PRESERVING GENERALIZED PARTIAL TRANSFORMATION SEMIGROUPS

We know from Theorem 1.4 that for any chain X, the semigroups $OP(X)$ and $OI(X)$ are always regular. The purpose of this chapter is to extend this result by considering when the semigroup $(OP(X,Y),\theta)$ with $\theta \in OP(Y,X)$ and the semigroup $(OI(X,Y),\theta)$ with $\theta \in OI(Y,X)$ are regular.

To obtain the main two theorems of this chapter, Theorem 1.4 and the following two lemmas are required.

Lemma 2.1. *Let X and Y be posets and let $OS(X,Y)$ be $OP(X,Y)$ or $OI(X,Y)$ and $\theta \in OS(Y,X)$. If the semigroup $(OS(X,Y),\theta)$ is regular, then dom $\theta = Y$ and ran $\theta = X$.*

Proof. We prove the lemma by contrapositive. Assume that dom $\theta \neq Y$ or ran $\theta \neq X$.

Case 1: dom $\theta \neq Y$. Let $y \in Y\backslash$dom θ and $x \in X$. Then $\{x\}_y \in OS(X,Y)$ and $\{x\}_y\theta = 0$. This implies that $\{x\}_y\theta\alpha\theta\{x\}_y = 0 \neq \{x\}_y$ for every $\alpha \in OS(X,Y)$. Thus $\{x\}_y$ is not a regular element of $(OS(X,Y),\theta)$.

Case 2: ran $\theta \neq X$. Let $x \in X\backslash$ran θ and $y \in Y$. Then $\{x\}_y \in OS(X,Y)$ and $\theta\{x\}_y = 0$ which implies that $\{x\}_y\theta\alpha\theta\{x\}_y = 0 \neq \{x\}_y$ for every $\alpha \in OS(X,Y)$, and so $\{x\}_y$ is not a regular element of $(OS(X,Y),\theta)$.

Therefore $(OS(X,Y),\theta)$ is not a regular semigroup, and hence the lemma is proved. □

Lemma 2.2. *Let X and Y be posets and let $OS(X,Y)$ be $OP(X,Y)$ or $OI(X,Y)$ and $\theta \in OS(Y,X)$. If θ is an order-isomorphism from Y onto X, then the following statements hold.*

(i) The map $\alpha \mapsto \alpha\theta$ is an isomorphism of $(OS(X,Y),\theta)$ onto $OS(X)$.

(ii) The map $\alpha \mapsto \theta\alpha$ is an isomorphism of $(OS(X,Y),\theta)$ onto $OS(Y)$.

Proof. It is clear that $\alpha\theta \in OS(X)$ and $\theta\alpha \in OS(Y)$ for all $\alpha \in OS(X,Y)$. Define $\varphi : OS(X,Y) \to OS(X)$ and $\varphi' : OS(X,Y) \to OS(Y)$ by $\alpha\varphi = \alpha\theta$ and $\alpha\varphi' = \theta\alpha$ for all $\alpha \in OS(X,Y)$. Then for $\alpha,\ \beta \in OS(X,Y)$,

$$(\alpha\theta\beta)\varphi = \alpha\theta\beta\theta = (\alpha\theta)(\beta\theta) = (\alpha\varphi)(\beta\varphi),$$
$$(\alpha\theta\beta)\varphi' = \theta\alpha\theta\beta = (\theta\alpha)(\theta\beta) = (\alpha\varphi')(\beta\varphi'),$$

so φ and φ' are homomorphisms. Next, we will show that φ and φ' are bijections. For $\alpha,\ \beta \in OS(X,Y)$, then

$$\alpha\varphi = \beta\varphi \Rightarrow \alpha = \alpha 1_Y = \alpha\theta\theta^{-1} = (\alpha\varphi)\theta^{-1} = (\beta\varphi)\theta^{-1} = \beta\theta\theta^{-1} = \beta 1_Y = \beta,$$
$$\alpha\varphi' = \beta\varphi' \Rightarrow \alpha = 1_X\alpha = \theta^{-1}\theta\alpha = \theta^{-1}(\alpha\varphi') = \theta^{-1}(\beta\varphi') = \theta^{-1}\theta\beta = 1_X\beta = \beta.$$

Thus φ and φ' are 1-1. Also, for $\gamma \in OS(X)$ and $\lambda \in OS(Y)$, we have $\gamma\theta^{-1},\ \theta^{-1}\lambda \in OS(X,Y)$ and $(\gamma\theta^{-1})\varphi = (\gamma\theta^{-1})\theta = \gamma(\theta^{-1}\theta) = \gamma 1_X = \gamma$ and $(\theta^{-1}\lambda)\varphi' = \theta(\theta^{-1}\lambda) = (\theta\theta^{-1})\lambda = 1_Y\lambda = \lambda$, so φ and φ' are onto.

Hence φ is an isomorphism of $(OS(X,Y),\theta)$ onto $OS(X)$ and φ' is an isomorphism of $(OS(X,Y),\theta)$ onto $OS(Y)$. Therefore (i) and (ii) are proved. □

Theorem 2.3. *Let X and Y be chains. For $\theta \in OI(Y,X)$, the semigroup $(OI(X,Y),\theta)$ is regular if and only if θ is an order-isomorphism from Y onto X.*

Proof. Assume that $(OI(X,Y),\theta)$ is regular. By Lemma 2.1, we have dom $\theta = Y$ and ran $\theta = X$. Since $\theta \in OI(Y,X)$, θ is order-preserving and 1-1. It therefore follows that θ is an order-isomorphism from Y onto X.

Conversely, assume that θ is an order-isomorphism from Y onto X. It then deduces from Lemma 2.2(i) that $(OI(X,Y),\theta) \cong OI(X)$. Since X is a chain, $OI(X)$ is a regular semigroup by Theorem 1.4. Therefore the semigroup $(OI(X,Y),\theta)$ is regular, as required. □

We observe here from the proof of Theorem 2.3 that the following fact is true. For posets X and Y, if the semigroup $(OI(X,Y),\theta)$ with $\theta \in OI(Y,X)$ is regular, then θ is an order-isomorphism from Y onto X.

Theorem 2.4. *Let X and Y be chains. For $\theta \in OP(Y,X)$, the semigroup $(OP(X,Y),\theta)$ is regular if and only if*

(i) θ is an order-isomorphism from Y onto X or

(ii) dom $\theta = Y$, ran $\theta = X$ and $|X| = 1$.

Proof. To prove necessity, assume that $(OP(X,Y),\theta)$ is a regular semigroup. We have by Lemma 2.1 that dom $\theta = Y$ and ran $\theta = X$. If $|X| = 1$, then (ii) holds, that is, dom $\theta = Y$, ran $\theta = X$ and $|X| = 1$. Assume that $|X| > 1$. We will show that θ is an order-isomorphism from Y onto X. It remains to show that θ is 1-1. Suppose in the contrary that θ is not 1-1. Then there exist $a \in X$, $e, f \in Y$ such that $e < f$ and $e\theta = f\theta = a$. Since $|X| > 1$, there is $b \in X \setminus \{a\}$. Then $b < a$ or $a < b$ because X is a chain.

Case 1: $b < a$. Define $\alpha : \{a,b\} \to Y$ by $a\alpha = f$ and $b\alpha = e$. Then $\alpha \in OI(X,Y) \subseteq OP(X,Y)$. Since $e\theta = f\theta = a$, we have that $a\alpha\theta = a = b\alpha\theta$. But dom$(\alpha\theta) \subseteq$ dom α, thus dom$(\alpha\theta) = \{a,b\}$ and ran$(\alpha\theta) = \{a\}$. Consequently, for

$\beta \in OP(X, Y)$,

$$|\text{ran}(\alpha\theta\beta\theta\alpha)| \leq |\text{ran}(\alpha\theta)| = 1.$$

Therefore $\alpha \neq \alpha\theta\beta\theta\alpha$ for every $\beta \in OP(X, Y)$ since $|\text{ran } \alpha| = |\{e, f\}| = 2$. Thus α is not a regular element of $(OP(X, Y), \theta)$ which is a contradiction.

Case 2: $a < b$. Define $\lambda : \{a, b\} \to Y$ by $b\lambda = f$ and $a\lambda = e$. Then $\lambda \in OI(X, Y) \subseteq OP(X, Y)$. We can show similarly to Case 1 that λ is not a regular element of $(OP(X, Y), \theta)$. This is contrary to the assumption.

Hence we deduce that θ is 1-1, so (i) holds if $|X| > 1$.

To prove sufficiency, assume that (i) or (ii) holds. If (i) is true, then we have that $(OP(X, Y), \theta) \cong OP(X)$ by Lemma 2.2(i). Since $OP(X)$ is regular from Theorem 1.4, it follows that $(OP(X, Y), \theta)$ is a regular semigroup.

Next, assume that (ii) holds, that is, dom $\theta = Y$, ran $\theta = X$ and $|X| = 1$. Let $X = \{x\}$. Then $Y\theta = \{x\}$. If $\alpha \in OP(X, Y)\setminus\{0\}$, then dom $\alpha = \{x\}$ and ran $\alpha = \{x\alpha\}$. Since $x\alpha \in Y =$ dom θ, $x\alpha\theta = x$, and so $x\alpha\theta\alpha = x\alpha$. Thus $\alpha\theta\alpha = \alpha$. This proves that $(OP(X, Y), \theta)$ is an idempotent semigroup, and therefore $(OP(X, Y), \theta)$ is a regular semigroup.

Hence the theorem is completely proved. □

Example 2.5. Define $\theta_1, \theta_2 : \mathbb{Z} \to \mathbb{Z}$ by

$$x\theta_1 = x + 1 \text{ and } x\theta_2 = 2x \text{ for all } x \in \mathbb{Z}.$$

Then the mappings θ_1 and θ_2 are order-preserving. Moreover, θ_1 is an order-isomorphism from $\mathbb{Z}$ onto $\mathbb{Z}$ and θ_2 is an order-isomorphism from $\mathbb{Z}$ onto $2\mathbb{Z}$. We then deduce from Theorem 2.3 and Theorem 2.4 that the semigroups $(OI(\mathbb{Z}, \mathbb{Z}), \theta_1)$, $(OI(2\mathbb{Z}, \mathbb{Z}), \theta_2)$, $(OP(\mathbb{Z}, \mathbb{Z}), \theta_1)$ and $(OP(2\mathbb{Z}, \mathbb{Z}), \theta_2)$ are all regular but the semigroups $(OI(\mathbb{Z}, \mathbb{Z}), \theta_2)$ and $(OP(\mathbb{Z}, \mathbb{Z}), \theta_2)$ are not regular. For the later conclusion, we can see directly from the fact that $\{1\}_0 \in OI(\mathbb{Z},\mathbb{Z}) \subseteq OP(\mathbb{Z},\mathbb{Z})$ and $\theta_2\{1\}_0 = 0$ (since $1 \notin \text{ran } \theta_2$) which implies that $\{1\}_0\theta_2\beta\theta_2\{1\}_0 = 0 \neq \{1\}_0$ for all $\beta \in OP(\mathbb{Z},\mathbb{Z})$.

CHAPTER III

REGULAR FULL ORDER-PRESERVING GENERALIZED TRANSFORMATION SEMIGROUPS

In this chapter, we consider the semigroup $(OT(X,Y),\theta)$ with $\theta \in OT(Y,X)$ where X and Y are chains. The main purpose is to characterize when $(OT(X,Y),\theta)$ is a regular semigroup. This characterization is given in terms of regularity of $OT(X)$, $|X|$, $|Y|$ and θ. This characterization with Theorem 1.2 and Theorem 1.3 will tell us when the semigroup $(OT(X,Y),\theta)$ is regular where both X and Y are nontrivial subsets of $\mathbb{Z}$ and both X and Y are nontrivial intervals of $\mathbb{R}$. By a nontrivial set we mean a set containing more than one element.

Throughout this chapter, let X and Y be any chains and θ any element of $OT(Y,X)$, unless otherwise mentioned.

The following sequence of lemmas is desired to obtain our main result of this chapter.

Lemma 3.1. *Let $a,\ b \in X$ and $c,\ d \in Y$ be such that $a < b$, $c < d$ and $c\theta = d\theta$. If $\alpha : X \to Y$ is defined by*

$$x\alpha = \begin{cases} c & \text{if } x < b, \\ d & \text{if } x \geq b, \end{cases}$$

then $\alpha \in OT(X,Y), |ran\ \alpha| = 2$ and $|ran(\alpha\theta)| = 1$.

Proof. Since $a \in \{x \in X \mid x < b\}$, we have that $\{x \in X \mid x < b\} \neq \varnothing$ and so $\{x \in X \mid x < b\}\alpha = \{c\}$. Also, $\{x \in X \mid x \geq b\}\alpha = \{d\}$. But $c < d$, thus $\alpha \in OT(X,Y)$ and ran $\alpha = \{c,d\}$. Consequently, $\text{ran}(\alpha\theta) = (\text{ran }\alpha)\theta = \{c,d\}\theta = \{c\theta, d\theta\} = \{c\theta\}$ because $c\theta = d\theta$. Hence $|\text{ran }\alpha| = 2$ and $|\text{ran}(\alpha\theta)| = 1$. □

Lemma 3.2. *Let $|X| > 1$. If the semigroup $(OT(X,Y),\theta)$ is regular, then θ is 1-1.*

Proof. We will prove the lemma by contrapositive. Assume that θ is not 1-1. Then there are $a, b \in X$ and $c, d \in Y$ such that $a < b, c < d$ and $c\theta = d\theta$. Define $\alpha : X \to Y$ as in Lemma 3.1. By Lemma 3.1, $\alpha \in OT(X,Y)$, $|\text{ran }\alpha| = 2$ and $|\text{ran}(\alpha\theta)| = 1$. Since for each $\beta \in OT(X,Y)$, $|\text{ran}(\alpha\theta\beta\theta\alpha)| \leq |\text{ran}(\alpha\theta)| = 1$, so we have that $|\text{ran}(\alpha\theta\beta\theta\alpha)| = 1 \neq |\text{ran }\alpha|$. Thus $\alpha\theta\beta\theta\alpha \neq \alpha$ for every $\beta \in OT(X,Y)$. Hence α is not a regular element of $(OT(X,Y),\theta)$. Therefore, $(OT(X,Y),\theta)$ is not a regular semigroup. □

Lemma 3.3. *Let $e, f \in Y$ be such that $e < f$ and $a \in X$.*

(i) If $x < a$ for all $x \in$ ran θ and $\alpha : X \to Y$ is defined by

$$x\alpha = \begin{cases} e & \text{if } x < a, \\ f & \text{if } x \geq a, \end{cases}$$

then $\alpha \in OT(X,Y), |ran\ \alpha| = 2$ and $|ran(\theta\alpha)| = 1$.

(ii) If $x > a$ for all $x \in$ ran θ and $\beta : X \to Y$ is defined by

$$x\beta = \begin{cases} e & \text{if } x \leq a, \\ f & \text{if } x > a, \end{cases}$$

then $\beta \in OT(X,Y), |ran\ \beta| = 2$ and $|ran(\theta\beta)| = 1$.

Proof. (i) Since ran $\theta \subseteq \{x \in X \mid x < a\}$, $\{x \in X \mid x < a\} \neq \varnothing$, so $\{x \in X \mid x < a\}\alpha = \{e\}$. We also have $\{x \in X \mid x \geq a\}\alpha = \{f\}$. It then follows that $\alpha \in OT(X,Y)$ since $e < f$, ran $\alpha = \{e,f\}$ and $\text{ran}(\theta\alpha) = (\text{ran } \theta)\alpha = \{e\}$. Therefore $|\text{ran } \alpha| = 2$ and $|\text{ran}(\theta\alpha)| = 1$.

(ii) Because ran $\theta \subseteq \{x \in X \mid x > a\}$, we have $\{x \in X \mid x > a\}\beta = \{f\}$. But $\{x \in X \mid x \leq a\}\beta = \{e\}$ and $e < f$, so we have $\beta \in OT(X,Y)$, ran $\beta = \{e,f\}$ and $\text{ran}(\theta\beta) = (\text{ran } \theta)\beta = \{f\}$. Therefore $|\text{ran } \beta| = 2$ and $|\text{ran}(\theta\beta)| = 1$. □

Lemma 3.4. *Let $|Y| > 1$. If the semigroup $(OT(X,Y),\theta)$ is regular, then for every $x \in X$, $y \leq x \leq z$ for some y, $z \in$ ran θ.*

Proof. We prove the lemma by contrapositive. Assume that it is not true that for every $x \in X$, $y \leq x \leq z$ for some y, $z \in$ ran θ. Then there is an element $a \in X$ such that $x < a$ for all $x \in$ ran θ or $x > a$ for all $x \in$ ran θ. Let $e, f \in Y$ be such that $e < f$.

Case 1: $x < a$ for all $x \in$ ran θ. Define $\alpha : X \to Y$ as in Lemma 3.3 (i). Then $\alpha \in OT(X,Y), |\text{ran } \alpha| = 2$ and $|\text{ran}(\theta\alpha)| = 1$. But for each $\lambda \in OT(X,Y)$, $|\text{ran}(\alpha\theta\lambda\theta\alpha)| \leq |\text{ran}(\theta\alpha)| = 1$ for all $\lambda \in OT(X,Y)$, so it follows that $|\text{ran}(\alpha\theta\lambda\theta\alpha)| = 1$ for every $\lambda \in OT(X,Y)$. Thus $\alpha\theta\lambda\theta\alpha \neq \alpha$ for all $\lambda \in OT(X,Y)$. Hence α is not a regular element of the semigroup $(OT(X,Y),\theta)$.

Case 2: $x > a$ for all $x \in$ ran θ. Define $\beta : X \to Y$ as in Lemma 3.3 (ii). Then $\beta \in OT(X,Y), |\text{ran } \beta| = 2$ and $|\text{ran}(\theta\beta)| = 1$. Since for each $\lambda \in OT(X,Y)$ $|\text{ran}(\beta\theta\lambda\theta\beta)| \leq |\text{ran}(\theta\beta)| = 1$, we deduce that $|\text{ran}(\beta\theta\lambda\theta\beta)| = 1$ for every $\lambda \in OT(X,Y)$. Thus $\beta\theta\lambda\theta\beta \neq \beta$ for all $\lambda \in OT(X,Y)$. Hence β is not a regular element of the semigroup $(OT(X,Y),\theta)$.

From Case 1 and Case 2, we have that $(OT(X,Y),\theta)$ is not a regular semigroup, and hence the lemma is proved. □

Lemma 3.5. *Let $a \in X \backslash ran\ \theta$ be such that $b < a < c$ for some $b,\ c \in ran\ \theta$ and $e,\ f\ ,\ g \in Y$ such that $e < f < g$. If $\alpha : X \to Y$ is defined by*

$$x\alpha = \begin{cases} e & \text{if } x < a, \\ f & \text{if } x = a, \\ g & \text{if } x > a. \end{cases}$$

Then $\alpha \in OT(X, Y)$, $|ran\ \alpha| = 3$ and $|ran(\theta\alpha)| = 2$.

Proof. Since $b \in \{x \in X \mid x < a\}$, $c \in \{x \in X \mid x > a\}$, it follows that

$$\{x \in X \mid x < a\}\alpha = \{e\},\ a\alpha = f,\ \{x \in X \mid x > a\}\alpha = g,$$

and hence ran $\alpha = \{e, f, g\}$. But $e < f < g$, so $\alpha \in OT(X, Y)$. Moreover,

$$\begin{aligned} \text{ran}(\theta\alpha) &= (\text{ran }\theta)\alpha \\ &= \{x \in \text{ran }\theta \mid x < a\}\alpha \ \cup\ \{x \in \text{ran }\theta \mid x > a\}\alpha \quad \text{since} \quad a \notin \text{ran }\theta \\ &= \{e\} \ \cup\ \{f\} \quad \text{since} \quad b \in \{x \in \text{ran }\theta \mid x < a\} \text{ and} \\ &\qquad\qquad\qquad\qquad c \in \{x \in \text{ran }\theta \mid x > a\} \\ &= \{e, f\}. \end{aligned}$$

Hence $|\text{ran } \alpha| = 3$ and $|\text{ran}(\theta\alpha)| = 2$, as required. □

Lemma 3.6. *Let $|Y| > 2$. If the semigroup $(OT(X, Y), \theta)$ is regular, then ran $\theta = X$.*

Proof. This lemma is proved by contrapositive. Since $|Y| > 2$, there are $e, f, g \in Y$ be such that $e < f < g$. Assume that ran $\theta \neq X$. Then there is $a \in X \backslash \text{ran }\theta$ satisfying one of three following conditions.

(1) $x < a$ for all $x \in \text{ran } \theta$.

(2) $x > a$ for all $x \in \text{ran } \theta$.

(3) $b < a < c$ for some $b, c \in \text{ran } \theta$.

If (1) or (2) holds, then by Lemma 3.4, $(OT(X, Y), \theta)$ is not regular. Assume that (3) holds, define $\alpha : X \to Y$ as in Lemma 3.5. By Lemma 3.5, $\alpha \in OT(X, Y)$, $|\text{ran } \alpha| = 3$ and $|\text{ran}(\theta\alpha)| = 2$. Hence for every $\lambda \in OT(X, Y)$, $|\text{ran}(\alpha\theta\lambda\theta\alpha)| \leq |\text{ran}(\theta\alpha)| = 2$, so $\alpha \neq \alpha\theta\lambda\theta\alpha$ for every $\lambda \in OT(X, Y)$. Thus α is not a regular element of $(OT(X, Y), \theta)$. Therefore $(OT(X, Y), \theta)$ is not a regular semigroup if (3) is true.

Hence the lemma is proved. □

Lemma 3.7. *Let $|Y| = 2$. If ran $\theta = \{minX, maxX\}$, then $(OT(X, Y), \theta)$ is an idempotent semigroup.*

Proof. Let $\alpha \in OT(X, Y)$. Then either $|\text{ran } \alpha| = 1$ or $|\text{ran } \alpha| = 2$ because $|Y| = 2$. Since $\text{ran}(\alpha\theta\alpha) \subseteq \text{ran } \alpha$, it follows that $\alpha\theta\alpha = \alpha$ if $|\text{ran } \alpha| = 1$. Next, assume that $|\text{ran } \alpha| = 2$. Then ran $\alpha = Y$. Let $Y = \{e, f\}$ with $e < f$. Thus $X = e\alpha^{-1} \cup f\alpha^{-1}$ which is a disjoint union. Then $\text{min}X \in e\alpha^{-1}$ and $\text{max}X \in f\alpha^{-1}$ because $e < f$ and α is order-preserving. Since θ is order-preserving, ran $\theta = \{e, f\}\theta = \{\text{min}X, \text{max}X\}$ and $e < f$, it follows that $e\theta = \text{min}X$ and $f\theta = \text{max}X$. Consequently,

$$(e\alpha^{-1})\alpha\theta\alpha = \{e\theta\}\alpha = \{\text{min}X\}\alpha = \{e\} = (e\alpha^{-1})\alpha,$$

$$(f\alpha^{-1})\alpha\theta\alpha = \{f\theta\}\alpha = \{\text{max}X\}\alpha = \{f\} = (f\alpha^{-1})\alpha,$$

which implies that $\alpha = \alpha\theta\alpha$, so α is an idempotent of $(OT(X, Y), \theta)$.

This proves that $(OT(X, Y), \theta)$ is an idempotent semigroup, as desired. □

Lemma 3.8. *Let θ be an order-isomorphism from Y onto X. Then the following statements hold.*

(i) The map $\alpha \mapsto \alpha\theta$ is an isomorphism of $(OT(X,Y),\theta)$ onto $OT(X)$.

(ii) The map $\alpha \mapsto \theta\alpha$ is an isomorphism of $(OT(X,Y),\theta)$ onto $OT(Y)$.

Proof. It is clear that for any $\alpha \in OT(X,Y)$, $\alpha\theta \in OT(X)$ and $\theta\alpha \in OT(Y)$. Define $\varphi : (OT(X,Y),\theta) \to OT(X)$ by $\alpha\varphi = \alpha\theta$ for all $\alpha \in OT(X,Y)$ and define $\varphi' : (OT(X,Y),\theta) \to OT(Y)$ by $\alpha\varphi' = \theta\alpha$ for all $\alpha \in OT(X,Y)$. We can show similarly to the proof of Lemma 2.2 that φ is an isomorphism of $(OT(X,Y),\theta)$ onto $OT(X)$ and φ' is an isomorphism of $(OT(X,Y),\theta)$ onto $OT(Y)$. □

Now we are ready to provide our main theorem of this chapter.

Theorem 3.9. *The semigroup $(OT(X,Y),\theta)$ is regular if and only if one of the following statements holds.*

(i) The semigroup $OT(X)$ is regular and θ is an order-isomorphism from Y onto X.

(ii) $|X| = 1$.

(iii) $|Y| = 1$.

(iv) $|Y| = 2$ and ran $\theta = \{minX, maxX\}$.

Proof. To prove necessity, assume that the semigroup $(OT(X,Y),\theta)$ is regular and suppose that (ii),(iii) and (iv) are false. Then

$$|X| > 1,\ |Y| > 1 \text{ and } (|Y| \neq 2 \text{ or ran } \theta \neq \{\min X, \max X\}).$$

Therefore we have $|X| > 1$ and either $|Y| > 2$ or $|Y| = 2$ and ran $\theta \neq \{\min X$, $\max X\}$. Note that $\min X$ or $\max X$ may not exist. We will show that (i) is true, that is, $OT(X)$ is regular and θ is an order-isomorphism from Y onto X. From that $|X| > 1$, we have by Lemma 3.2 that θ is 1-1. We claim that the case $|Y| = 2$ and ran $\theta \neq \{\min X,\ \max X\}$ cannot occur. Suppose that $|Y| = 2$ and ran $\theta \neq \{\min X,\ \max X\}$. Since $|Y| = 2$ and θ is 1-1, $|\text{ran } \theta| = 2$. Let ran $\theta = \{b, c\}$ with $b < c$. Then $\{b, c\} \neq \{\min X,\ \max X\}$.

Case 1: min$X$ does not exist. Then there exists $a \in X$ such that $a < b$, so $a < b < c$.

Case 2: max$X$ does not exist. Then $a > c$ for some $a \in X$, so $a > c > b$.

Case 3: min$X$ and max$X$ exist. But $\{b, c\} \neq \{\text{min}X, \text{max}X\}$, so min$X < b$ or max$X > c$. Then either min$X < b < c$ or max$X > c > b$.

From Case 1 - Case 3, we conclude that there exists an element $a \in X$ such that $x < a$ for all $x \in \text{ran } \theta$ or $x > a$ for all $x \in \text{ran } \theta$. It therefore follows from Lemma 3.4 that $(OT(X,Y), \theta)$ is not a regular semigroup which contradicts the assumption. Hence we prove the claim. Thus $|Y| > 2$, and so by Lemma 3.6, we have ran $\theta = X$. Consequently, θ is an order-isomorphism from Y onto X. We then deduce from Lemma 3.8(i) that $(OT(X,Y), \theta) \cong OT(X)$. But $(OT(X,Y), \theta)$ is regular, so $OT(X)$ is regular. Hence (i) holds.

To prove sufficiency, assume that one of (i)-(iv) holds.

Case 1: (i) is true. By Lemma 3.8(i), we have $(OT(X,Y), \theta) \cong OT(X)$. Since the semigroup $OT(X)$ is regular, $(OT(X,Y), \theta)$ is a regular semigroup.

Case 2: $|X| = 1$. For $\alpha \in OT(X,Y)$, $|\text{ran } \alpha| = 1$, so $\alpha = \alpha\theta\alpha$ since $\text{ran}(\alpha\theta\alpha) \subseteq \text{ran } \alpha$. Thus α is an idempotent element of $(OT(X,Y), \theta)$. For this case, $(OT(X,Y), \theta)$ is an idempotent semigroup, so it is regular.

Case 3: $|Y| = 1$. Then $|OT(X,Y)| = 1$, and thus the semigroup $(OT(X,Y), \theta)$ is trivially regular.

Case 4: (iv) is true. Then by Lemma 3.7, $(OT(X,Y), \theta)$ is an idempotent semigroup, so it is regular.

Hence the theorem is completely proved. $\square$

We know from Theorem 1.2 that $OT(X)$ is a regular semigroup for any nonempty subset of $\mathbb{Z}$. Then this fact and Theorem 3.9 yield the following two corollaries directly.

Corollary 3.10. *If X is a nonempty subset of $\mathbb{Z}$, then the semigroup $(OT(X,Y),\theta)$ is regular if and only if one of the following statements holds.*

(i) *θ is an order-isomorphism from Y onto X.*

(ii) $|X| = 1$.

(iii) $|Y| = 1$.

(iv) $|Y| = 2$ *and ran* $\theta = \{minX, maxX\}$.

Corollary 3.11. *Let X and Y be nontrivial subsets of $\mathbb{Z}$. Then the semigroup $(OT(X,Y),\theta)$ is regular if and only if*

(i) *θ is an order-isomorphism from Y onto X or*

(ii) $|Y| = 2$ *and ran* $\theta = \{minX, maxX\}$.

We note that if (ii) of Corollary 3.11 holds, then X must be finite.

It is known from Theorem 1.3 that for a nonempty interval X of $\mathbb{R}$, then $OT(X)$ is regular if and only if X is closed and bounded. We also know that for a nonempty interval X of $\mathbb{R}$, either $|X| = 1$ or X is (uncountably) infinite. Then following three corollaries are directly obtained from these facts and Theorem 3.9.

Corollary 3.12. *Let X be a nonempty interval of $\mathbb{R}$. Then the semigroup $(OT(X,Y),\theta)$ is regular if and only if one of the following statements holds.*

(i) *X is closed and bounded and θ is an order-isomorphism from Y onto X.*

(ii) $|X| = 1$.

(iii) $|Y| = 1$.

(iv) $|Y| = 2$ *and ran* $\theta = \{minX, maxX\}$.

Corollary 3.13. *Let X and Y be nonempty intervals of $\mathbb{R}$. Then the semigroup $(OT(X,Y),\theta)$ is regular if and only if one of the following statements holds.*

(i) X *is closed and bounded and* θ *is an order-isomorphism from* Y *onto* X*.*

(ii) $|X| = 1$.

(iii) $|Y| = 1$.

Corollary 3.14. *Let X and Y be nontrivial intervals of $\mathbb{R}$. Then the semigroup $(OT(X,Y),\theta)$ is regular if and only if X is closed and bounded and θ is an order-isomorphism from Y onto X.*

Example 3.15. Define $\theta_1, \theta_2 : \mathbb{Z} \to \mathbb{Z}$ as in Example 2.5, that is,

$$x\theta_1 = x + 1 \text{ and } x\theta_2 = 2x \text{ for all } x \in \mathbb{Z}.$$

Since θ_1 is an order-isomorphism from $\mathbb{Z}$ onto $\mathbb{Z}$ and θ_2 is an order-isomorphism from $\mathbb{Z}$ onto $2\mathbb{Z}$, by Corollary 3.10, $(OT(\mathbb{Z},\mathbb{Z}),\theta_1)$ and $(OT(2\mathbb{Z},\mathbb{Z}),\theta_2)$ are regular semigroups but $(OT(\mathbb{Z},\mathbb{Z}),\theta_2)$ is not a regular semigroup. For the later inclusion, we can show directly as follows: Since $1_{\mathbb{Z}} \in OT(\mathbb{Z},\mathbb{Z})$ and for any $\alpha \in OT(\mathbb{Z},\mathbb{Z})$,

$$\text{ran}(1_{\mathbb{Z}}\theta_2\alpha\theta_2 1_{\mathbb{Z}}) = \text{ran}(\theta_2\alpha\theta_2) \subseteq \text{ran}(\theta_2) = 2\mathbb{Z} \subsetneq \mathbb{Z},$$

so $1_{\mathbb{Z}}\theta_2\alpha\theta_2 1_{\mathbb{Z}} \neq 1_{\mathbb{Z}}$ for all $\alpha \in OT(\mathbb{Z},\mathbb{Z})$, so $1_{\mathbb{Z}}$ is not a regular element of $(OT(\mathbb{Z},\mathbb{Z}),\theta_2)$.

Next, let $\theta_3 = \theta_1|_{\{0,1\}}$. Then ran $\theta_3 = \{1,2\}$. If $X = \{0,1,2\}$, then ran $\theta_3 = \{1,2\} \neq \{\min X, \max X\} = \{0,2\} \neq X$. Therefore from Corollary 3.11, $(OT(\{0,1,2\},\{0,1\}),\theta_3)$ is not a regular semigroup. If $\theta_4 = \theta_2|_{\{0,1\}}$. Then ran $\theta_4 = \{0,2\}$. If X is as above, that is, $X = \{0,1,2\}$, then ran $\theta_4 = \{0,2\} = \{\min X, \max X\}$, so by Corollary 3.11, the semigroup $(OT(\{0,1,2\},\{0,1\}),\theta_4)$ is a regular semigroup.

Example 3.16. Let $\theta : \mathbb{R} \to \mathbb{R}^+$ and $\theta' : \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$x\theta = 10^x \text{ for all } x \in \mathbb{R} \text{ and } x\theta' = log_{10}x \text{ for all } x \in \mathbb{R}^+.$$

Then θ is an order-isomorphism from $\mathbb{R}$ onto $\mathbb{R}^+$ and θ' is an order-isomorphism from $\mathbb{R}^+$ onto $\mathbb{R}$. Let $\theta_1 = \theta|_{[0,1]}$ and $\theta_2 = \theta'|_{[10,100]}$. Then θ_1 is an order-isomorphism from [0,1] onto [1,10] and θ_2 is an order-isomorphism from [10,100] onto [1,2]. It therefore follows from Corollary 3.14 that $(OT([1,10],[0,1]),\theta_1)$ and $(OT([1,2],[10,100]),\theta_2)$ are both regular semigroups.

Remark 3.17. In fact for $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, there is an order-isomorphism θ from [a,b] onto [c,d]. To show this, define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$x\varphi = (\frac{b-a}{d-c})(x-c) + a \quad \text{for all } x \in \mathbb{R}.$$

Then the slope of the line φ is $\frac{b-a}{d-c} > 0$, so φ is a strictly increasing continuous function. But $c\varphi = a$ and $d\varphi = b$, so $\varphi|_{[c,d]}$ is an order-isomorphism from [c,d] onto [a,b]. Let $\theta = \varphi|_{[c,d]}$. Then θ is an order-isomorphism from [c,d] onto [a,b]. This implies by Corollary 3.14 that $(OT([a,b],[c,d]),\theta)$ is a regular semigroup.

Note that if $\theta' = \varphi|_{(c,d)}$, then θ' is an order-isomorphism from (c,d) onto (a,b). However, the semigroup $(OT((a,b),(c,d)),\theta')$ is not regular by Corollary 3.14.

CHAPTER IV

SOME ISOMORPHISM THEOREMS

In the last chapter, we provide some isomorphism theorems of order-preserving generalized transformation semigroups. The purpose is to characterize when the semigroup $(OS(X,Y),\theta)$ is isomorphic to $OS(X)$ and when it is isomorphic to $OS(Y)$ where X and Y are chains, $OS(X,Y)$ is $OP(X,Y)$, $OT(X,Y)$ or $OI(X,Y)$ and $\theta \in OS(Y,X)$. We obtain some interesting isomorphism theorems as follows: $(OS(X,Y),\theta) \cong OS(X)[OS(Y)]$ if and only if θ is an order-isomorphism from Y onto X where $OS(X,Y)$ is $OP(X,Y)$ or $OI(X,Y)$ and $\theta \in OS(Y,X)$. Also, $(OT(X,Y),\theta) \cong OT(X)$ if and only if θ is an order-isomorphism from Y onto X, but $(OT(X,Y),\theta) \cong OT(Y)$ if and only if $|Y| = 1$ or θ is an order-isomorphism from Y onto X. To obtain these results, Theorem 1.4, Lemma 2.2, Theorem 2.3 and Theorem 2.4 will be referred.

Throughout this chapter, let X and Y be chains.

Theorem 4.1. *For $\theta \in OI(Y,X)$, $(OI(X,Y),\theta) \cong OI(X)$ if and only if θ is an order-isomorphism from Y onto X.*

Proof. First, assume that $(OI(X,Y),\theta) \cong OI(X)$. We know from Theorem 1.4 that $OI(X)$ is a regular semigroup. We then have that the semigroup $(OI(X,Y),\theta)$ is regular. It therefore follows from Theorem 2.3 that θ is an order-isomorphism from Y onto X.

Conversely, assume that θ is an order-isomorphism from Y onto X. We have from Lemma 2.2(i) that $(OI(X,Y),\theta) \cong OI(X)$, as required. □

Theorem 4.2. *For $\theta \in OI(Y,X)$, $(OI(X,Y),\theta) \cong OI(Y)$ if and only if θ is an order-isomorphism from Y onto X.*

Proof. Assume that $(OI(X,Y),\theta) \cong OI(Y)$. Since the semigroup $OI(Y)$ is regular by Theorem 1.4, we deduce that the semigroup $(OI(X,Y),\theta)$ is regular. Therefore by Theorem 2.3, θ is an order-isomorphism from Y onto X.

Conversely, assume that θ is an order-isomorphism from Y onto X. It therefore follows from Lemma 2.2(ii) that $(OI(X,Y),\theta) \cong OI(Y)$, as desired. □

As a consequence of Theorem 4.1 and Theorem 4.2, we have

Corollary 4.3. *For $\theta \in OI(Y,X)$, the following statements are equivalent.*

(i) $(OI(X,Y),\theta) \cong OI(X)$.

(ii) $(OI(X,Y),\theta) \cong OI(Y)$.

(iii) *θ is an order-isomorphism from Y onto X.*

The following lemma gives necessary conditions for the semigroup $(OS(X,Y),\theta)$ to have an identity where $OS(X,Y)$ is $OP(X,Y)$ or $OT(X,Y)$ and $\theta \in OS(Y,X)$.

Lemma 4.4. *Let $OS(X,Y)$ be $OP(X,Y)$ or $OT(X,Y)$ and $\theta \in OS(Y,X)$. If the semigroup $(OS(X,Y),\theta)$ has an identity η, then $\theta\eta = 1_Y$, and hence θ is 1-1 and ran $\eta = Y$.*

Proof. We have that for any $y \in Y, X_y \in OS(X,Y)$. Since η is the identity of $(OS(X,Y),\theta)$, we have

$$\eta\theta\alpha = \alpha\theta\eta = \alpha \text{ for every } \alpha \in OS(X,Y),$$

in particular,

$$X_y\theta\eta = X_y \text{ for every } y \in Y.$$

Therefore for $x \in X$,

$$y\theta\eta = xX_y\theta\eta = xX_y = y \text{ for every } y \in Y.$$

This shows that $\theta\eta = 1_Y$ which implies that θ is 1-1 and ran $\eta = Y$. □

We remark here from the proof of Lemma 4.4 that Lemma 4.4 is true for any posets X and Y.

Theorem 4.5. *For $\theta \in OP(Y,X)$, $(OP(X,Y),\theta) \cong OP(X)$ if and only if θ is an order-isomorphism from Y onto X.*

Proof. First, assume that $(OP(X,Y),\theta) \cong OP(X)$. By Theorem 1.4, the semigroup $OP(X)$ is regular, and therefore $(OP(X,Y),\theta)$ is a regular semigroup. From Theorem 2.4, one of the following statements holds.

(1) θ is an order-isomorphism from Y onto X.

(2) dom $\theta = Y$, ran $\theta = X$ and $|X| = 1$.

Since $(OP(X,Y),\theta) \cong OP(X)$ and $OP(X)$ has an identity, we deduce from Lemma 4.4 that θ is 1-1. Hence if (2) holds, then $|Y| = 1$. Therefore we conclude that θ must be an order-isomorphism from Y onto X.

For the converse, assume that θ is an order-isomorphism from Y onto X. Then $(OP(X,Y),\theta) \cong OP(X)$ by Lemma 2.2(i). □

Theorem 4.6. *For $\theta \in OP(Y,X)$, $(OP(X,Y),\theta) \cong OP(Y)$ if and only if θ is an order-isomorphism from Y onto X.*

Proof. By Theorem 1.4, $OP(Y)$ is a regular semigroup.

If $(OP(X,Y),\theta) \cong OP(Y)$, then the semigroup $(OP(X,Y),\theta)$ is regular, so by Theorem 2.4,

(1) θ is an order-isomorphism from Y onto X or

(2) dom $\theta = Y$, ran $\theta = X$ and $|X| = 1$.

Since $OP(Y)$ has an identity, $(OP(X,Y),\theta)$ has an identity. Thus θ is 1-1 by Lemma 4.4, so (2) implies $|Y| = 1$. Hence θ is an order-isomorphism from Y onto

X.

Conversely, if θ is an order-isomorphism from Y onto X, then $(OP(X,Y),\theta) \cong OP(Y)$ by Lemma 2.2(ii). $\square$

The following corollary is an immediate consequence of Theorem 4.5 and Theorem 4.6.

Corollary 4.7. *For $\theta \in OP(Y,X)$, the following statements are equivalent.*

(i) $(OP(X,Y),\theta) \cong OP(X)$.

(ii) $(OP(X,Y),\theta) \cong OP(Y)$.

(iii) *θ is an order-isomorphism from Y onto X.*

Beside Lemma 4.4, the following series of lemmas are required to determine when $(OT(X,Y),\theta) \cong OT(X)$ and when $(OT(X,Y),\theta) \cong OT(Y)$ where $\theta \in OT(Y,X)$.

Lemma 4.8. *For $\theta \in OT(Y,X)$, if $|Y| > 1$ and the semigroup $(OT(X,Y),\theta)$ has an identity, then for every $x \in X$, $y \le x \le z$ for some $y,z \in$ ran θ.*

Proof. Let $e,f \in Y$ be such that $e < f$. Suppose that the conclusion is false. Then there is an element $a \in X$ such that

(1) $x < a$ for all $x \in$ ran θ or

(2) $x > a$ for all $x \in$ ran θ.

Case 1: (1) holds. Define $\alpha : X \to Y$ as in Lemma 3.3(i), Then by Lemma 3.3(i), $\alpha \in OT(X,Y)$, $|\text{ran } \alpha| = 2$ and $|\text{ran}(\theta\alpha)| = 1$. Thus for any $\eta \in OT(X,Y)$, $\text{ran}(\eta\theta\alpha) \subseteq \text{ran}(\theta\alpha)$, so $|\text{ran}(\eta\theta\alpha)| = 1$. Hence

$$\eta\theta\alpha \neq \alpha \text{ for every } \eta \in OT(X,Y)$$

which implies that $(OT(X,Y),\theta)$ has no identity.

Case 2: (2) holds. Let $\beta : X \to Y$ be defined as in Lemma 3.3(ii). By Lemma 3.3(ii), $\beta \in OT(X,Y)$, $|\text{ran } \beta| = 2$ and $|\text{ran}(\theta\beta)| = 1$. We then have similarly to Case 1 that

$$\eta\theta\beta \neq \beta \text{ for every } \eta \in OT(X,Y)$$

and hence $(OT(X,Y),\theta)$ has no identity.

Therefore the lemma is proved. □

Lemma 4.9. *For $\theta \in OT(Y,X)$, if $|Y| > 2$ and the semigroup $(OT(X,Y),\theta)$ has an identity, then ran $\theta = X$.*

Proof. Let $e,f,g \in Y$ be such that $e < f < g$. Suppose that ran $\theta \neq X$. Then there is an element $a \in X \backslash \text{ran } \theta$. Then one of the following three cases must occur.

(1) $x < a$ for all $x \in$ ran θ.

(2) $x > a$ for all $x \in$ ran θ.

(3) $b < a < c$ for some $b, c \in$ ran θ.

Case 1: (1) or (2) holds. By Lemma 4.8, the semigroup $(OT(X,Y),\theta)$ has no identity.

Case 2: (3) holds. Let $\alpha : X \to Y$ be defined as in Lemma 3.5. Then by this lemma, $\alpha \in OT(X,Y)$, $|\text{ran } \alpha| = 3$ and $|\text{ran}(\theta\alpha)| = 2$. But $|\text{ran}(\eta\theta\alpha)| \leq |\text{ran}(\theta\alpha)|$ for any $\eta \in OT(X,Y)$, so $|\text{ran}(\eta\theta\alpha)| \leq 2$ for all $\eta \in OT(X,Y)$. Hence

$$\eta\theta\alpha \neq \alpha \text{ for every } \eta \in OT(X,Y)$$

which implies that the semigroup $(OT(X,Y),\theta)$ has no identity.

Therefore the lemma is proved. □

Lemma 4.10. *For $\theta \in OT(Y, X)$, if $|Y| = 2$, ran $\theta = \{minX, maxX\}$ and the semigroup $(OT(X, Y), \theta)$ has an identity, then $|X| = 2$.*

Proof. Let $Y = \{e, f\}$ with $e < f$ and η the identity of the semigroup $(OT(X, Y), \theta)$. From Lemma 4.4, θ is 1-1. But $|Y| = 2$ and $\theta : Y = \{e, f\} \to \text{ran } \theta = \{\text{min}X, \text{max}X\}$ is order-preserving, so $e\theta = \text{min}X < \text{max}X = f\theta$. To show that $|X| = 2$, suppose not. Then $|X| > 2$ and so $\text{min}X < a < \text{max}X$ for some $a \in X$. Since $\eta : X \to Y = \{e, f\}$, $a\eta = e$ or $a\eta = f$. Define $\alpha, \beta : X \to Y$ by

$$x\alpha = \begin{cases} e & \textit{if } x < a, \\ f & \textit{if } x \geq a, \end{cases} \quad \text{and} \quad x\beta = \begin{cases} e & \textit{if } x \leq a, \\ f & \textit{if } x > a. \end{cases}$$

Since $e < f$ and $\text{min}X < a < \text{max}X$, we have $\alpha, \beta \in OT(X, Y)$, $(\text{min}X)\alpha = e$ and $(\text{max}X)\beta = f$.

Case 1: $a\eta = e$. Then $a\eta\theta\alpha = e\theta\alpha = (\text{min}X)\alpha = e < f = a\alpha$.

Case 2: $a\eta = f$. Then $a\eta\theta\beta = f\theta\beta = (\text{max}X)\beta = f > e = a\beta$.

From Case 1 and Case 2, we have $\eta\theta\alpha \neq \alpha$ and $\eta\theta\beta \neq \beta$, respectively. This is contrary to that η is the identity of the semigroup $(OT(X, Y), \theta)$. This proves that $|X| = 2$, as required. □

Lemma 4.11. *For $\theta \in OT(Y, X)$, the semigroup $(OT(X, Y), \theta)$ has an identity if and only if $|Y| = 1$ or θ is an order-isomorphism from Y onto X.*

Proof. To prove necessity, assume that the semigroup $(OT(X, Y), \theta)$ has an identity and $|Y| > 1$. From Lemma 4.4, θ is 1-1. We will show that ran $\theta = X$.

Case 1: $|Y| = 2$. Let $Y = \{e, f\}$ with $e < f$. Then ran $\theta = \{e\theta, f\theta\}$ and $e\theta < f\theta$ since θ is 1-1 and order-preserving. It then follows from Lemma 4.8, $e\theta \leq x \leq f\theta$ for all $x \in X$. This implies that $e\theta = \text{min}X$ and $f\theta = \text{max}X$.

Hence ran $\theta = \{\min X, \max X\}$. It therefore follows from Lemma 4.10 that $|X| = 2$. Consequently, ran $\theta = X$

Case 2: $|Y| > 2$. Therefore that ran $\theta = X$ is directly obtained from Lemma 4.9.

Therefore θ is an order-isomorphism from Y onto X.

To prove sufficiently, assume that $|Y| = 1$ or θ is an order-isomorphism from Y onto X. If $|Y| = 1$, then $|OT(X,Y)| = 1$, so $(OT(X,Y), \theta)$ has an identity. If θ is an order-isomorphism from Y onto X, then by Lemma 3.8(i), we have that $(OT(X,Y), \theta) \cong OT(X)$. But $OT(X)$ has an identity, thus $(OT(X,Y), \theta)$ has an identity. □

Theorem 4.12. *For $\theta \in OT(Y,X)$, $(OT(X,Y), \theta) \cong OT(X)$ if and only if θ is an order-isomorphism from Y onto X.*

Proof. First, assume that $(OT(X,Y), \theta) \cong OT(X)$. Then the semigroup $(OT(X,Y), \theta)$ has an identity since the semigroup $OT(X)$ does. By Lemma 4.11, $|Y| = 1$ or θ is an order-isomorphism from Y onto X. Assume that $|Y| = 1$. Then $|OT(X,Y)| = 1$, so $|OT(X)| = 1$ since $(OT(X,Y), \theta) \cong OT(X)$. Since $|OT(X)| = 1$ and $X_x \in OT(X)$ for every $x \in X$, we deduce that $|X| = 1$. This shows that θ is an order-isomorphism from Y onto X.

The converse is obtained directly from Lemma 3.8(i). □

Theorem 4.13. *For $\theta \in OT(Y,X)$, $(OT(X,Y), \theta) \cong OT(Y)$ if and only if $|Y| = 1$ or θ is an order-isomorphism from Y onto X.*

Proof. Assume that $(OT(X,Y), \theta) \cong OT(Y)$. Then $(OT(X,Y), \theta)$ has an identity. Then from Lemma 4.11, we have $|Y| = 1$ or θ is an order-isomorphism from Y onto X.

If $|Y| = 1$, then $|OT(X,Y)| = 1 = |OT(Y)|$, so $(OT(X,Y), \theta) \cong OT(Y)$. If θ is an order-isomorphism from Y onto X, then by Lemma 3.8(ii), we have that

$(OT(X,Y),\theta) \cong OT(Y)$.

Hence the theorem is proved, as required. □

We can see in this chapter that having an identity and being isomorphic are closely related. We combine this relationship to be a theorem as follows:

Theorem 4.14. *For $\theta \in OT(Y,X)$ and $|Y| > 1$, the following statements are equivalent.*

(i) $(OT(X,Y),\theta)$ *has an identity.*

(ii) $(OT(X,Y),\theta) \cong OT(X)$.

(iii) $(OT(X,Y),\theta) \cong OT(Y)$.

(iv) θ *is an order-isomorphism from Y onto X.*

Proof. Since $|Y| > 1$, by Lemma 4.11, (i)⇔(iv). That (ii)⇔(iv) follows from Theorem 4.12. Because $|Y| > 1$, we obtain that (iii)⇔(iv) from Theorem 4.13. □

Example 4.15. Let $\theta_2 : \mathbb{Z} \to \mathbb{Z}$ be defined as in Example 3.15, that is,

$$x\theta_2 = 2x \ \text{ for all } x \in \mathbb{Z}.$$

By Theorem 4.1-4.2, Theorem 4.4-4.5 and Theorem 4.12-4.13, we have $(OI(2\mathbb{Z},\mathbb{Z}),\theta_2) \cong OI(2\mathbb{Z}) \cong OI(\mathbb{Z})$, $(OP(2\mathbb{Z},\mathbb{Z}),\theta_2) \cong OP(2\mathbb{Z}) \cong OP(\mathbb{Z})$ and $(OT(2\mathbb{Z},\mathbb{Z}),\theta_2) \cong OT(2\mathbb{Z}) \cong OT(\mathbb{Z})$, respectively.

Remark 4.16. Let $a,b,c,d \in \mathbb{R}$ be such that $a < b$ and $c < d$, then from Remark 3.16, there are order-isomorphisms $\theta : [a,b] \to [c,d]$ and $\theta' : (a,b) \to (c,d)$. By Theorem 4.1-4.2, Theorem 4.4-4.5 and Theorem 4.12-4.13, we have respectively that

(1) $(OI([a,b],[c,d]),\theta) \cong OI([a,b]) \cong OI([c,d])$,

$(OI((a,b),(c,d)),\theta') \cong OI((a,b)) \cong OI((c,d))$,

(2) $(OP([a,b],[c,d]),\theta) \cong OP([a,b]) \cong OP([c,d])$,

$(OP((a,b),(c,d)),\theta') \cong OP((a,b)) \cong OP((c,d))$,

(3) $(OT([a,b],[c,d]),\theta) \cong OT([a,b]) \cong OT([c,d])$,

$(OT((a,b),(c,d)),\theta') \cong OT((a,b)) \cong OT((c,d))$.

Note that all the above semigroups except those on the last line are regular semigroups.

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