การขยายทฤษฎีเมเจอร์แรนต์ของซาเกียนไปยังระบบป้อนกลับ ที่มีสัญญาณเข้าสองสัญญาณและสัญญาณออกสองสัญญาณ

นายธัชนนท์ ชุ่มแอ่น

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิศวกรรมศาสตรมหาบัณฑิต
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EXTENSION OF ZAKIAN'S THEORY OF MAJORANTS TO TWO-INPUT TWO-OUTPUT FEEDBACK SYSTEMS

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ธัชนนท์ ชุ่มแอ่น: การขยายทฤษฎีเมเจอร์แรนต์ของซาเกียนไปยังระบบป้อนกลับที่มีสัญญาณเข้าสองสัญญาณและสัญญาณออกสองสัญญาณ (EXTENSION OF ZAKIAN'S THEORY OF MAJORANTS TO TWO-INPUT TWO-OUTPUT FEEDBACK SYSTEMS) อ. ที่ปรึกษาวิทยานิพนธ์หลัก: ผศ. ดร.สุชิน อรุณสวัสดิ์วงศ์, 57 หน้า

วิทยานิพนธ์ฉบับนี้มีเป้าหมายหลักคือขยายทฤษฎีเมเจอร์แรนต์ซึ่งได้ถูกพัฒนาในกรณีของระบบที่มีสัญญาณขาเข้าหนึ่งสัญญาณและสัญญาณขาออกหนึ่งสัญญาณไปยังกรณีที่ระบบมีสัญญาณขาเข้าสองสัญญาณและสัญญาณขาออกสองสัญญาณ ทฤษฎีเมเจอร์แรนต์ได้สร้างอสมการสำหรับออกแบบระบบป้อนกลับที่มีพลานต์ที่มีฟังก์ชันถ่ายโอนแบบอตรรกยะหรือมีพลานต์ที่มีความไม่แน่นอน วัตถุประสงค์ในการออกแบบสำหรับรูปแบบของปัญหาที่พิจารณาในงานชิ้นนี้คือรับประกันว่าสัญญาณคลาดเคลื่อนและสัญญาณขาออกของตัวควบคุมให้อยู่ภายในขอบเขตที่กำหนดตลอดเวลาเมื่อสัญญาณขาเข้าที่เป็นไปได้ทุกสัญญาณสอดคล้องกับเงื่อนไขขอบเขตของขนาดและความชันที่กำหนด วิทยานิพนธ์ฉบับนี้ประกอบด้วยสองส่วนหลัก ส่วนแรกขยายเกณฑ์การประมาณสำหรับระบบที่มีสัญญาณขาเข้าหนึ่งสัญญาณและสัญญาณขาออกหนึ่งสัญญาณไปยังระบบที่มีสัญญาณขาเข้าสองสัญญาณและสัญญาณขาออกสองสัญญาณและไปยังระบบที่มีสัญญาณขาเข้าหลายสัญญาณและสัญญาณขาออกหลายสัญญาณในภายหลังเมื่อเมตริกซ์ถ่ายโอนอตรรกยะถูกประมาณด้วยเมตริกซ์ถ่ายโอนตรรกยะขณะทำการออกแบบ ในส่วนที่สอง จากผลลัพธ์ที่ได้จากเกณฑ์การประมาณที่ได้พัฒนาขึ้นอสมการสำหรับออกแบบระบบที่มีความไม่แน่นอนที่มีสัญญาณขาเข้าสองสัญญาณและสัญญาณขาออกสองสัญญาณได้รับการพัฒนาและตรวจสอบ ตัวอย่างเชิงเลขหลายตัวอย่างถูกนำเสนอเพื่อแสดงให้เห็นถึงประสิทธิผลและประโยชน์ของทฤษฎีที่พัฒนาขึ้นในวิทยานิพนธ์ฉบับนี้

ภาควิชา วิศวกรรมไฟฟ้า **ลายมือชื่อนิสิต** ....

สาขาวิชา วิศวกรรมไฟฟ้า **ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก** ....

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TADCHANON CHUMAN : EXTENSION OF ZAKIAN'S THEORY OF MAJORANTS TO TWO-INPUT TWO-OUTPUT FEEDBACK SYSTEMS.
ADVISOR : ASST. PROF. SUCHIN ARUNSAWATWONG, Ph.D., 57 pp.

The main aim of the thesis is to extend the theory of majorants, which was developed previously for the case of single-input single-output feedback systems, to the case of two-input two-output systems. The theory of majorants provides useful inequalities for designing feedback systems in which the plant model is described by non-rational transfer functions or in which the plant model has uncertainties. For the design formulation considered in this work, the design objective is to ensure that the errors and the controller outputs always stay within prescribed bounds whenever the possible inputs satisfy certain bounding conditions on magnitude and slope. The thesis consists of two main parts. Part 1 extends the criterion of approximation for single-input single-output feedback systems to the case of two-input two-output systems, and later to multi-input multi-output systems, where non-rational transfer matrices are replaced with rational approximants during the design process. In part 2, based on the developed criterion, inequalities for designing two-input two-output vague systems are derived and investigated. Numerical examples are given in order to illustrate the effectiveness and the usefulness of the methods developed here.

Department : Electrical Engineering	Student's Signature ....
Field of Study : Electrical Engineering	Advisor's Signature ....
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CHAPTER I

INTRODUCTION

1.1 Introduction

A common practice in designing a feedback system in which the plant consists of components described by non-rational transfer functions, is to replace the non-rational functions with rational approximants so that well-developed computational tools for rational systems can be utilized. The approach may fail to give satisfactory results if the approximant is not chosen appropriately. Hence, conditions for ensuring the design obtained from using the approximant gives satisfactory results for the original system are required. To this end, a number of researchers have been prompted to investigate and/or develop methods for approximating a non-rational transfer function by a rational one. For example, Gibilaro and Lees [14] and Zakian [27] investigated methods for simplifying transfer functions using moment approximants, whereas Lam [16] reduced models of delay systems using Padé approximants for exponential functions. In addition, Gu et al. [15] used a method based on Fourier transform techniques. A number of references can be found in the literature concerning how to obtain rational approximants (see, for example, [14–16, 27] and the references therein).

The approach of replacing non-rational transfer functions with rational approximants in the design process is useful especially when computational tools for non-rational transfer functions are not readily available. However, it may fail to give satisfactory results for the original system if the approximants are not sufficiently close to the original models. In order to ensure that the design carried out with the approximants is valid for the original system, a criterion of approximation needs to be explicitly taken into account in the formulation of the design problem.

Consider the two-input two-output feedback system shown in Figure 1.1 and described by

$$\left.\begin{aligned} u_i &= k_{i1} \star e_1 + k_{i2} \star e_2 \\ e_i &= f_i - g_{i1} \star u_1 - g_{i2} \star u_2 \end{aligned}\right\}, \quad i = 1, 2 \tag{1.1}$$

where $\mathbf{G}(s) \triangleq [G_{ij}(s)]_{2\times 2}$ is the plant transfer matrix and $\mathbf{K}(s,\mathbf{p}) \triangleq [K_{ij}(s,\mathbf{p})]_{2\times 2}$ is the controller transfer matrix characterized by a design parameter vector $\mathbf{p} \in \mathbb{R}^N$. Let $\mathbf{e} \triangleq [e_1, e_2]^T$ and $\mathbf{u} \triangleq$

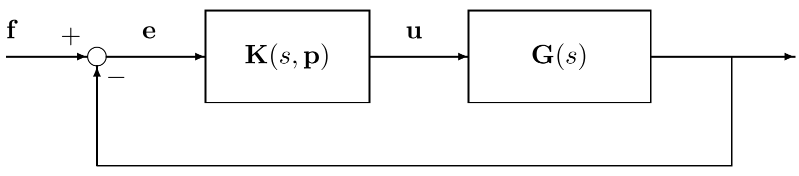

Figure 1.1: The two-input two-output feedback system given in (1.1).

$[u_1, u_2]^T$ be the error and the control vectors of the system, respectively, and $g_j : [0, \infty) \to \mathbb{R}$ and $k_{ij} : [0, \infty) \to \mathbb{R}$ denote the impulse responses of $G_{ij}(s)$ and $K_{ij}(s, \mathbf{p})$, respectively. The symbol $\star$ denotes the convolution; that is, for functions $x : [0, \infty) \to \mathbb{R}$ and $y : [0, \infty) \to \mathbb{R}$,

$$(x \star y)(t) = \int_0^t x(t - \lambda)\, y(\lambda)\, d\lambda, \quad t \geq 0.$$

The vector $\mathbf{f} \triangleq [f_1, f_2]^T$ is the input vector of the system where f_1 and f_2 are known only to the extent that each of them belongs to the sets $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. In this work, $\mathcal{P}_1$ and $\mathcal{P}_2$ are assumed to be the sets of input signals whose magnitude and whose slope satisfy certain bounding conditions.

Define the performance measures $\hat{e}_i$ and $\hat{u}_i$ for $i = 1, 2$.

$$\hat{e}_i \triangleq \sup_{f_1 \in \mathcal{P}_1, f_2 \in \mathcal{P}_2} \|e_i\|_\infty \ \text{ and } \ \hat{u}_i \triangleq \sup_{f_1 \in \mathcal{P}_1, f_2 \in \mathcal{P}_2} \|u_i\|_\infty \tag{1.2}$$

where $\hat{e}_i$ and $\hat{u}_i$ are sometimes called the peak values of e_i and u_i, respectively. The problem investigated here is to determine a design parameter $\mathbf{p} \in \mathbb{R}^N$ such that the following design criteria are satisfied.

$$\hat{e}_1(\mathbf{p}) \leq \mathcal{E}_1 \ \text{ and } \ \hat{e}_2(\mathbf{p}) \leq \mathcal{E}_2. \tag{1.3}$$

$$\hat{u}_1(\mathbf{p}) \leq \mathcal{U}_1 \ \text{ and } \ \hat{u}_2(\mathbf{p}) \leq \mathcal{U}_2. \tag{1.4}$$

where $\mathcal{E}_i$ and $\mathcal{U}_i$ are specified bounds. It should be noted that the design criteria (1.3) and (1.4) are equivalent to the fact that $|e_i(t)|$ and $|u_i(t)|$ do not exceed the bounds $\mathcal{E}_i$ and $\mathcal{U}_i$ for all $t \geq 0$ whenever $f_1 \in \mathcal{P}_1$ and $f_2 \in \mathcal{P}_2$.

Following previous work ([24,29,32–34], and also the references therein), it is readily appreciated that in solving inequalities (1.3) and (1.4) by numerical methods, one needs computational tools for stabilizing and obtaining the time-responses of the systems. Moreover, it is noted that for various cases of the possible set $\mathcal{P}_i$, the peak outputs e_i and u_i are found to be functionals defined implicitly in terms of the system's time-responses. Once the time-responses are obtained, the methods developed in [24, 29, 32–34] can be used to compute the peak outputs. Evidently, for lumped-parameter systems, one can solve inequalities (1.3) and (1.4) easily by using computational tools developed for rational systems, which are readily available and well-established, in conjunction with the methods for computing the peak outputs mentioned above.

Zakian [30] derives a criterion of approximation for the case of single-input single output (SISO) feedback systems (see Section 2.1 for further details) in which the plant transfer function $G(s)$ is replaced by a rational approximant $G^*(s)$ during the design process. The criterion provides simple sufficient conditions to ensure that the controller obtained through the use of the approximant $G^*(s)$ still gives satisfactory results for the original system in the sense that the original design criteria (2.6) are satisfied. Zakian's criterion of approximation has been employed in the design of delayed control systems [4, 5] for SISO systems. Based on the criterion, the theory of majorants were derive

([31, 33, 34]). and were employed to design robust control systems in [6, 23]. The derivation of the criterion of approximation and the theory of majorants for multi-input multi-output (MIMO) feedback systems is still an open question.

So far, the design problem defined inequalities of the form (1.3) and (1.4) can be solved by numerical method for the case of MIMO lumped-parameter feedback systems or, by using Zakian's criterion of approximation, SISO distributed-parameter feedback systems (See [24, 29] and the references therein for details on this). As suggested by Arunsawatwong [3], the design problem for the case of general MIMO distributed-parameter feedback systems is still an open question.

In this regard, it is the intention of this thesis to extend the criterion of approximation to the case of MIMO systems that can be used to design general distributed-parameter MIMO feedback systems. Then base on the criterion for MIMO systems, the theory of majorants will be investigated for the case of two-input two-output systems.

1.2 Objectives

1. To study Zakian's criterion of approximation and extend it to 2×2 feedback systems.

2. Based on the obtained results, to develop a practical method for designing a controller for 2×2 feedback systems where the non-rational plant transfer matrix is replaced by rational approximants during the design process.

3. To develop inequalities for designing 2×2 feedback systems where the plant has parametric uncertainties.

4. To illustrate the effectiveness of the developed methods by carrying out numerical examples.

1.3 Scope of Thesis

1. To extend Zakian's theory of majorants to 2×2 feedback systems.

2. The design requirement is to ensure that all the errors and all the controller outputs lie within prescribed bounds for all time in the presence of any input whose magnitude and whose slope do not exceed respective bounds.

3. To develop a practical method for designing a controller for 2×2 feedback systems subject to inputs satisfying bounding conditions so as to ensure that the design criteria (1.3) and (1.4) are fulfilled.

4. To develop inequalities for designing a robust controller for 2×2 feedback systems subject to inputs satisfying bounding conditions so as to ensure that the design criteria (1.3) and (1.4) are fulfilled.

5. To design robust controllers for 2×2 feedback systems where the plant transfer matrix has parametric uncertainties.

1.4 Methodology

Sufficient conditions for ensuring (1.3) and (1.4) in terms of inequalities are developed, thereby providing surrogate design criteria that are in keeping with the method of inequalities. That is to say, the obtained criteria are inequalities that can be solved in practice.

1.5 Expected Outcomes

1. Readily computable inequalities for designing a 2×2 feedback system described by a non-rational transfer matrix so that the criteria (1.3) and (1.4) are satisfied.

2. Readily computable inequalities for designing a robust controller for 2×2 feedback systems with uncertainties so that the criteria (1.3) and (1.4) are satisfied.

3. Numerical examples demonstrating the effectiveness of the developed methods.

1.6 Achievements

The contributions of this thesis are as follows:

- First and foremost, we develop a practical method for designing a controller for non-rational MIMO feedback systems subject to inputs satisfying bounding conditions on magnitude and slope. Zakian's criterion of approximation for SISO systems is extended not only to the case of 2×2 systems but also to the case of MIMO systems where the non-rational functions in the plant transfer matrix are replaced by rational approximants throughout the design process. Accordingly, the obtained criterion enables ones to solve the design problems (1.3) and (1.4) for non-rational systems by using only computational tools for rational systems.

- Second, based on the criterion of approximation for MIMO systems, the inequalities for designing a robust controller for 2×2 feedback systems subject to inputs satisfying bounding conditions are developed.

1.7 Thesis Outline

The structure of the thesis is as follows. Chapter 2 reviews the criterion of approximation and the theory of majorants for SISO systems. Chapter 3 presents the application of the theory of majorants for SISO systems. Chapter 4 extends the criterion of approximation to the case of MIMO systems. To illustrate the usefulness of the criterion, a numerical design of a binary distillation column is carried out. Chapter 5 extends the theory of majorants to the case of 2×2 systems. Finally, the thesis is concluded in Chapter 6.

CHAPTER II

RECAP OF THE THEORY OF MAJORANTS FOR SISO SYSTEMS

2.1 Zakian's Criterion of Approximation

Consider the scalar feedback system described by

$$\left.\begin{aligned} u &= e \star k \\ e &= f - g \star u \end{aligned}\right\} \tag{2.1}$$

where $G(s)$ denotes the plant transfer function and $K(s, \mathbf{p})$ denotes the controller transfer function with the design parameter $\mathbf{p} \in \mathbb{R}^N$ (see Figure 2.1). The responses e and u are the error and the control of the system, respectively, and g and k denote the impulse responses of $G(s)$ and $K(s, \mathbf{p})$, respectively.

Suppose that f is a *possible input* (*i.e.*, input that can happen or is likely to happen in practice) and is known only to the extent that it belongs to a set $\mathcal{P}$, to be called a *possible set*. Accordingly, $\mathcal{P}$ contains all possible inputs. In this work, the set $\mathcal{P}$ are subset of $\mathbb{L}_\infty$, which denoted the set of all bounded functions defined on $[0, \infty)$.

Note, in passing, that there are different models of the possible set $\mathcal{P}$ which have been investigated by many researchers. For example, the set $\mathcal{P}$ given by

$$\mathcal{P} = \{f : \|f\|_\infty \le M_\infty \text{ and } \|\dot{f}\|_\infty \le D_\infty\} \tag{2.2}$$

was considered by [8, 29, 34], while the set $\mathcal{P}$ given by

$$\mathcal{P} = \{f : \|f\|_2 \le M_2 \text{ and } \|\dot{f}\|_2 \le D_2\} \tag{2.3}$$

was considered in [2, 17]. Recently, the set $\mathcal{P}$ given by

$$\mathcal{P} = \{f : \|f\|_2 \le M_2,\ \|\dot{f}\|_2 \le D_2,\ \|f\|_\infty \le M_\infty\} \tag{2.4}$$

has been considered in [24]. For the characterization of the above sets $\mathcal{P}$ and their implications, see [17, 24, 34] and the references therein.

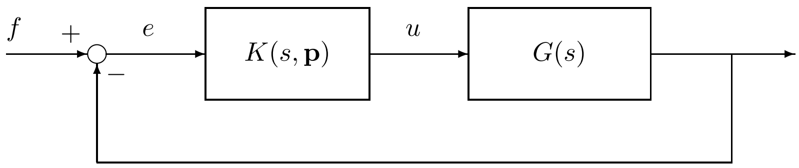

Figure 2.1: The scalar feedback system given in (2.1).

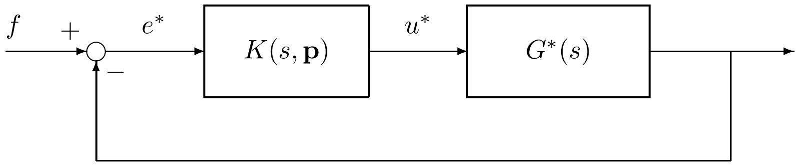

Figure 2.2: The nominal system for (2.1).

In connection with the possible set $\mathcal{P}$, define the performance measures $\hat{e}$ and $\hat{u}$ as follows.

$$\hat{e} \triangleq \sup_{f\in\mathcal{P}} \|e\|_\infty \quad \text{and} \quad \hat{u} \triangleq \sup_{f\in\mathcal{P}} \|u\|_\infty. \tag{2.5}$$

Note that for the possible set $\mathcal{P}$, $\hat{e}$ and $\hat{u}$ are the peak error and the peak control of the system (2.1). Methods for computing such peak values in association with the possible sets described by (2.2), (2.3) and (2.4) are readily available. See [17, 24, 34] and the references therein for the details of the methods. It is worth noting at this point that in computing the peak values $\hat{e}$ and $\hat{u}$, one needs to compute the time-responses of the system.

Assume that the design problem for the system (2.1) is to determine a design parameter $\mathbf{p}$ such that the following design criteria are satisfied.

$$\hat{e}(\mathbf{p}) \le E_{\max} \quad \text{and} \quad \hat{u}(\mathbf{p}) \le U_{\max}, \tag{2.6}$$

where the bounds $E_{\max}$ and $U_{\max}$ are specified.

Let $G^*(s)$ be an approximant of the original plant transfer function $G(s)$. In connection with the system (2.1), the nominal system (see Figure 2.2) is described by

$$\begin{aligned} u^* &= e^* \star k \\ e^* &= f - g^* \star u^* \end{aligned} \tag{2.7}$$

where e^* and u^* are the error and the control of the nominal system, respectively, and g^* be the impulse response of $G^*(s)$. Let $\hat{e}^*$ and $\hat{u}^*$ denote the peak values of e^* and u^*, respectively, for the possible set $\mathcal{P}$. That is to say,

$$\hat{e}^* \triangleq \sup_{f\in\mathcal{P}} \|e^*\|_\infty \quad \text{and} \quad \hat{u}^* \triangleq \sup_{f\in\mathcal{P}} \|u^*\|_\infty. \tag{2.8}$$

Let μ denote the approximation index and be defined by

$$\mu \triangleq \|w\|_1$$

where $w : [0,\infty) \to \mathbb{R}$ is the inverse Laplace transform of $W(s)$ given by

$$W(s,\mathbf{p}) = \frac{K(s,\mathbf{p})}{1 + K(s,\mathbf{p})G^*(s)} \left[G(s) - G^*(s)\right].$$

Now it is ready to state the main theorem on the criterion of approximation.

Theorem 2.1. *[30] Suppose that the nominal system* (2.7) *is stable in the sense that* $\hat{e}^* < \infty$ *and* $\hat{u}^* < \infty$. *Let* $\mu < 1$. *Then the original design criteria* (2.6) *for the system* (2.1) *are satisfied if the following inequalities hold.*

$$\frac{\hat{e}^*}{1-\mu} \leq E_{\max} \quad \textit{and} \quad \frac{\hat{u}^*}{1-\mu} \leq U_{\max}. \tag{2.9}$$

Furthermore,

$$\frac{\hat{e}^*}{1+\mu} \leq \hat{e} \leq \frac{\hat{e}^*}{1-\mu} \quad \textit{and} \quad \frac{\hat{u}^*}{1+\mu} \leq \hat{u} \leq \frac{\hat{u}^*}{1-\mu}.$$

Since the approximant $G^*(s)$ is a rational function, the nominal system (2.7) is finite-dimensional. In this case, the computation of $\hat{e}^*$ and $\hat{u}^*$ can be readily carried out and the solution of inequalities (2.9) is easily obtainable (see, for example, [24,34]). When tools for stability analysis and computing time-responses for non-rational systems are not available, inequalities (2.9) becomes more computationally tractable than the original design criteria (2.6).

Following the method of inequalities [30, 32–35], it is readily appreciated that in solving inequalities (2.9) by numerical methods, it is necessary that a search algorithm should start from a point $\mathbf{p} \in \mathbb{R}^N$ such that $\mu(\mathbf{p}) < \infty$, $\hat{e}^*(\mathbf{p}) < \infty$ and $\hat{u}^*(\mathbf{p}) < \infty$. In this connection, the following theorem provides a practical and useful sufficient condition that enables the algorithm to start from an arbitrary point in $\mathbb{R}^N$.

Define Λ as the set of all of the finite poles of the transfer function

$$\frac{F(s)}{U^*(s)} = \frac{K(s,\mathbf{p})}{1+K(s,\mathbf{p})G^*(s)}$$

where $F(s)$ and $U^*(s)$ are the Laplace transforms of f and u^*, respectively.

Theorem 2.2. ***[30,32,33]*** *Assume that* $G^*(s)$ *and* $K(s)$ *are rational transfer functions. Then* $\mu < \infty$ *if the two conditions hold.*

(a) $\|z\|_1 < \infty$, $z \triangleq g - g^*$.

(b) $\operatorname{Re}\lambda(\mathbf{p}) < 0$ *for all* $\lambda(\mathbf{p}) \in \Lambda$.

It is clear from Theorem 2.2 that with appropriate approximant $G^*(s)$, condition (a) is always satisfied. Consequently, condition (b) provides a useful inequality for computing a point $\mathbf{p}$ satisfying $\mu(\mathbf{p}) < \infty$ that is always soluble by numerical methods. This is because $\operatorname{Re}\lambda(\mathbf{p}) < \infty$ for every $\mathbf{p} \in \mathbb{R}^N$.

From Theorems 2.1 and 2.2, it readily follows that the solution of inequalities (2.6) involves three phases of computation as follows.

- Phase I : With a starting point, find $\mathbf{p}_0$ satisfying

$$\max_{\lambda \in \Lambda} \operatorname{Re}\lambda(\mathbf{p}_0) \leq -\epsilon \tag{2.10}$$

  where $0 < \epsilon \ll 1$ is given.

- Phase II: By starting from $\mathbf{p}_0$, find $\mathbf{p}_1$ satisfying

$$\left.\begin{aligned} &\max_{\lambda\in\Lambda}\mathrm{Re}\,\lambda(\mathbf{p}_1) \le -\epsilon \\ &\mu(\mathbf{p}_1) < 1. \end{aligned}\right\}. \tag{2.11}$$

- Phase III: By starting from $\mathbf{p}_1$, find $\mathbf{p}$ satisfying both the design criteria (2.9) and the inequality $\mu(\mathbf{p}) < 1$.

Since the plant transfer function $G(s)$ in Figure 2.1 is uncertain, which is known only to the extent that is belongs to a set $\mathcal{G}$, a difficulty arises because μ depends on $G \in \mathcal{G}$. Zakian [31, 33, 34] suggests replacing μ by its upper bound that is easier to compute. This upper bound can be called a *majorant*.

2.2 Majorants for Vague Systems

Suppose that the plant transfer function $G(s)$ in Section 2.1 is uncertain and is known only to the extent that is belongs to a set $\mathcal{G}$. Then the design criteria (2.6) become the design criteria given by

$$\sup_{G\in\mathcal{G}} \hat{e} \le E_{\max} \quad \text{and} \quad \sup_{G\in\mathcal{G}} \hat{u} \le U_{\max}. \tag{2.12}$$

From Theorem 2.1, it easy to see that

$$\sup_{G\in\mathcal{G}} \hat{e} \;\le\; \frac{\hat{e}^*}{1-\sup\limits_{G\in\mathcal{G}}\mu} \quad \text{and} \quad \sup_{G\in\mathcal{G}} \hat{u} \;\le\; \frac{\hat{u}^*}{1-\sup\limits_{G\in\mathcal{G}}\mu}.$$

Upon noting the computational difficulty, Zakian [31, 33, 34] proposes to replace $\sup\limits_{G(s)\in\mathcal{G}} \mu$ by a majorant μ_a given by

$$\mu_a \triangleq A|\sigma^*_{ss}| + B\,\|\sigma^* - \sigma^*_{ss}\|_1 \tag{2.13}$$

where σ^* is the unit-step response of the control u^*, σ^*_{ss} is the steady-state value of σ^*, and the constants A and B are given by

$$\left.\begin{aligned} A &\triangleq \sup_{G\in\mathcal{G}} \|z\|_1 \\ B &\triangleq \sup_{G\in\mathcal{G}}\{|z(0)| + \|\dot{z}\|_1\} \end{aligned}\right\}, \quad z \triangleq g - g^*. \tag{2.14}$$

Now, in applying Theorem 2.1, it is ready to state the main theorem on the majorants.

Theorem 2.3. *Suppose that the nominal system* (2.7) *is stable in the sense that* $\hat{e}^* < \infty$ *and* $\hat{u}^* < \infty$. *Let* $\mu_a < 1$. *Then the design criteria* (2.12) *for the system* (2.1) *are satisfied if the following inequalities hold.*

$$\frac{\hat{e}^*}{1-\mu_a} \le E_{\max} \quad \textit{and} \quad \frac{\hat{u}^*}{1-\mu_a} \le U_{\max}. \tag{2.15}$$

Proof. See [31, 33] for details. □

Following Theorem 2.2 in Section 2.1, the following theorem provides a sufficient condition to ensure that $\mu_a(\mathbf{p}) < \infty$.

Theorem 2.4 ([30, 32, 33])**.** *Assume that* $G^*(s)$ *is a rational transfer function. Then* $\mu_a < \infty$ *if the two conditions hold.*

(a) $\|z\|_1 < \infty$*,*

(b) $\operatorname{Re}\lambda(\mathbf{p}) < 0$ *for all* $\lambda(\mathbf{p}) \in \Lambda$*,*

where Λ *is the set of all of the finite poles of the transfer function*

$$\frac{U^*(s)}{F(s)} = \frac{K(s, \mathbf{p})}{1 + F(s, \mathbf{p})G^*(s)}.$$

Proof. See [31, 33] for details. □

In the same way, from Theorems 2.3 and 2.4, it readily follows that the solution of inequalities (2.9) is the three phases of computation in Section 2.1 if μ is replaced by μ_a.

CHAPTER III

APPLICATION OF ZAKIAN'S MAJORANTS TO ROBUST CONTROLLER FOR HYDRAULIC FORCE CONTROL SYSTEMS

3.1 Introduction

In modern industry, the precision of mechanical positioning systems is important in automation process. Such positioning systems are usually driven by electric, hydraulic or pneumatic actuators. For heavy load application, hydraulic actuators are more attractive because they possess a high force-to-weight ratio and fast response time and also because they are able to maintain their loading capacity indefinitely. However, uncertainties in hydraulic actuators limit their use in high precision application with a simple closed-loop controller. Under different operating conditions, the flow and the pressure coefficients, which characterize fluid flow into and out of the value, can vary. The uncertainty in the valve characteristic causes the variation of the valve dynamics. Furthermore, the value of the bulk modulus can vary significantly owing to changes of the oil temperature, the pressure and the air inside the cylinder. See, e.g., [11, 20] and the references therein for details on this.

The control design for hydraulic force control systems, especially in high precision application, is a challenging problem and has been investigated by many authors. For example, Niksefat and Sepehri [20] applied the quantitative feedback theory (QFT) to the design of robust force control of hydraulic actuators. Marusak and Kuntanapreeda [19] designed an analytical model predictive controller for force control of an electrohydraulic actuator.

This chapter presents the design of a robust controller for a hydraulic force control system, in which the parametric uncertainties are due to the variations of environmental stiffness and pressure sensitivity gain of the value. The principal design objective is to ensure that, for all possible inputs (that is, inputs that happen or are likely to happen in practice), the error and the control signals always stay within their prescribed bounds despite all uncertainties. When the objective is fulfilled, one can fully ensure that the control system will operate with high precision as required and the components of the system will not be damaged.

The design problem is formulated using Zakian's theory of majorants stated in Chapter 2 and other theories developed by Zakian and his group ([24, 33–35] and the references therein). It may be noted that the theory of majorants can be used effectively to formulate the design problem when the principal design objective of the chapter is taken into account. As a result, the problem is expressed explicitly as a set of inequalities that can readily be solved in practice.

3.2 Hydraulic Actuator Model

In a hydraulic actuator, the control input signal controls the spool displacement of valve that controls the flow of fluid from pump to the actuator. This flow builds a pressure difference that is proportional to the sensed force. For the detail on this, see [20].

Since hydraulic systems are highly nonlinear, a linear model of a hydraulic actuator that is obtained from linearization about an operating point will be used in the subsequent design. Following [20], the transfer function of the hydraulic actuator is given by

$$\frac{F(s)}{U(s)} = \frac{k_{sp}}{(\tau s+1)}\left[\frac{K_s k_e(A_i+A_o)}{(K_p+Cs)(m_a s^2+ds+k_e)+(A_i^2 s+A_o^2 s)}\right] \tag{3.1}$$

where $F(s)$ and $U(s)$ are the Laplace transforms of the sensed force f and the input control voltage u, respectively.

The meanings of the parameters in the transfer function of the actuator are as follows. Parameters k_e and d represent damping and stiffness of the environment, K_s and K_p are the flow and the pressure sensitivity gains of the valve, respectively, C is the approximant of the volume of fluid to the effective bulk modulus of the fluid, m_a represents the mass of the hydraulic piston, A_i and A_o are the piston effective areas, τ and k_{sp} are gains describing the valve dynamics. The nominal values of the parameters used in the subsequent design are given in Table 3.1.

Parameter	**Nominal Value**	**Parameter**	**Nominal Value**
k_e	75 (kN/m)	m_a	20 (kg)
K_s	$0.375\ (\mathrm{m}^3/\mathrm{pa.s})$	A_i	$0.00203\ (\mathrm{m}^2)$
K_p	$2.5\times10^{-12}\ (\mathrm{m/s}^2)$	A_o	$0.00152\ (\mathrm{m}^2)$
C	$1.5\times10^{-11}\ (\mathrm{m}^3/\mathrm{pa})$	k_{sp}	$0.0012\ (\mathrm{m/V})$
d	700 (N/m/s)	τ	35 (ms)

Table 3.1: Nominal values of parameters of the transfer function ([20]).

Parameter k_e depends on the variation in the environmental stiffness of the system, whereas K_p depends on the supply pressure of hydraulic actuator and the orifice area gradient ([20]). During the operation of the hydraulic actuator, the parameters k_e and K_p may change. Therefore, in the following, assume that k_e and K_p are parametric uncertainties within the ranges $[50, 100]$ kN/m and $[0, 5\times10^{-12}]$ m^2/s, respectively.

The frequency responses of the plant for different environmental stiffness and pressure sensitivity gains of the valve are shown in Figure 3.1 for $k_e \in [50, 100]$ kN/m and for $K_p \in [0, 5\times10^{-12}]$ m^2/s.

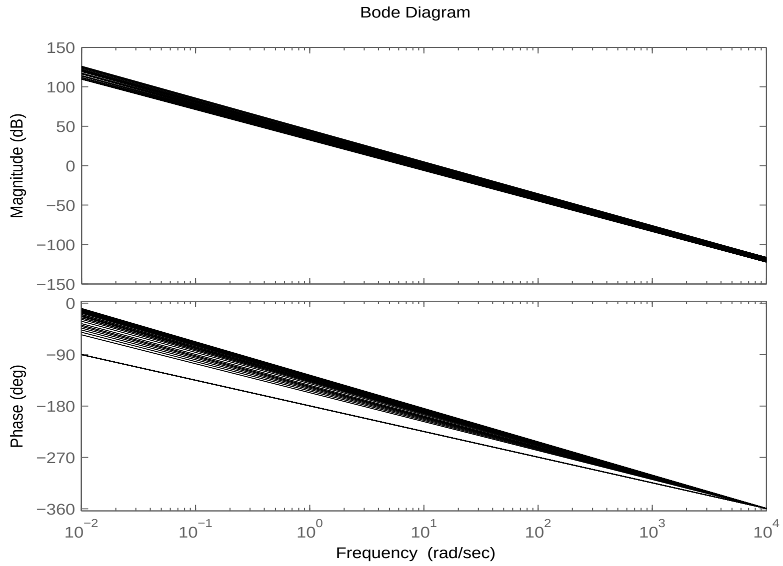

Figure 3.1: The frequency responses of $G_p(s)$ with uncertainties in k_e and K_p.

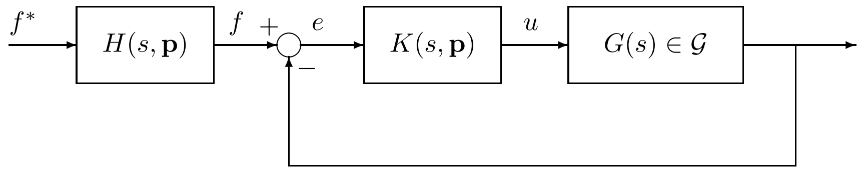

Figure 3.2: Two–degree-of–freedom feedback control system.

3.3 Design Formulation

In this section, the two-degree-of-freedom control configuration shown in Figure 3.2 is used where $H(s,\mathbf{p})$ is the prefilter transfer function, $K(s,\mathbf{p})$ is the controller transfer function, $\mathbf{p} \in \mathbb{R}^N$ is the design parameter to be determined. To this end, let $K(s)$ and $H(s)$ be characterized by

$$\left.\begin{aligned} K(s,\mathbf{p}) &= \frac{p_4 s^2 + p_5 s + p_6}{(s+p_1)(s^2+p_2 s+p_3)} \\ H(s,\mathbf{p}) &= \frac{p_{10}(p_7 s+1)}{(p_8 s+1)(p_9 s+1)} \end{aligned}\right\},$$

where $\mathbf{p} = [p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}]^T \in \mathbb{R}^{10}$ is the design parameter vector to be determined.

In our early study, it has been found that the one-degree-of-freedom control configuration is not appropriate. This is because a sufficiently high loop gain is required in order to make the error fulfill the control objective. However, such a loop gain makes the control system not robust to be against the uncertainties in k_e and K_p.

The main control objective is to ensure that during the operation, the absolute values of the error e and the control u do not exceed 105 N and 0.1 V, respectively, in the presence of all input

$f^* \in \mathcal{P}_\infty$ given by

$$\mathcal{P}_\infty = \left\{ f^* \mid ||f^*||_\infty \leq 1000 \quad \text{and} \quad ||\dot{f}^*||_\infty \leq 1000 \right\}. \tag{3.2}$$

Accordingly, the design problem is to find a value of $\mathbf{p}$ that satisfies

$$\left.\begin{array}{rcl} \mu_a(\mathbf{p}) & \leq & 0.99 \\ \hat{e}^*(\mathbf{p})/(1-\mu_a(\mathbf{p})) & \leq & 105 \text{ N} \\ \hat{u}^*(\mathbf{p})/(1-\mu_a(\mathbf{p})) & \leq & 0.1 \text{ V} \end{array}\right\}. \tag{3.3}$$

3.4 Numerical Results

A design solution is obtained from simultaneously solving inequalities (3.3) by a numerical search algorithm called the moving-boundaries-process (MBP). The detail of the algorithm can be found in [34], [35].

By conducting numerical searches in the range $[50, 100]$ kN/m for k_e and the range $[0, 5 \times 10^{-12}]$ m^2/s for K_p, we find that

$$A = 9.431 \times 10^6 \ \text{ at } k_e = 100 \text{ kN/m and } K_p = 0 \text{ m}^2/\text{s}$$

and

$$B = 2.2976 \times 10^4 \ \text{ at } k_e = 50 \text{ kN/m and } K_p = 0 \text{ m}^2/\text{s}.$$

The plots of $\|z\|_1$ and $|z(0)| + \|\dot{z}\|_1$ versus k_e and K_p are given in Figures 3.3 and 3.4, which confirm the obtained numerical results.

After a number of iterations, the MBP algorithm locates a design solution

$$\begin{aligned} \mathbf{p} \ = \ & [99.0580,\ 4.2871 \times 10^3,\ 9.6540 \times 10^4,\ 0.2580, 58713, \\ & 1.8897 \times 10^{-6},\ 1.5712 \times 10^{-3},\ 0.2268,\ 0.2000,\ 0.1000]^T. \end{aligned}$$

The corresponding performance measures are

$$\left.\begin{array}{rcl} \mu_a & = & 0.1376 \\ \hat{e}^*/(1-\mu_a) & = & 100.4894 \text{ N} \\ \hat{u}^*/(1-\mu_a) & = & 8.1271 \times 10^{-5} \text{ V} \end{array}\right\}.$$

To verify the design, a simulation is carried out for the case in which the control system is subject to a test input $\hat{f}$, which is generated randomly so that their magnitude and slope satisfy (3.2). The waveform of $\hat{f}$ and the corresponding responses e and u are displayed in Figure 3.5 for $k_e \in [50, 100]$ kN/m and $K_p \in [0, 5 \times 10^{-12}]$ m^2/s. Clearly, the design objectives are satisfied.

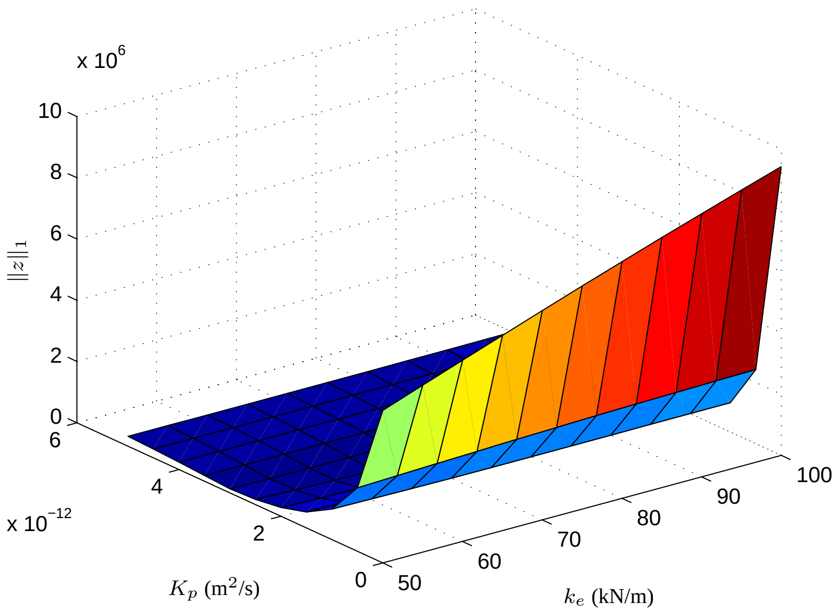

Figure 3.3: The plot of $\|z\|_1$ versus k_e and K_p.

3.5 Conclusions

This chapter presents the design of a robust controller for a hydraulic force control system, in which the uncertainties are in the parameters k_e and K_p. The design objective is to ensure that the magnitudes of the error and the control output signal always stay within their prescribed bounds for all time and for any input in the set $\mathcal{P}_\infty$ in spite of all uncertainties. The design problem is formulated using Zakian's majorants [31] in conjunction with other theories in Zakian's framework [24, 33–35], and consequently is expressed explicitly as a set of inequalities that can be solved in practice by numerical methods.

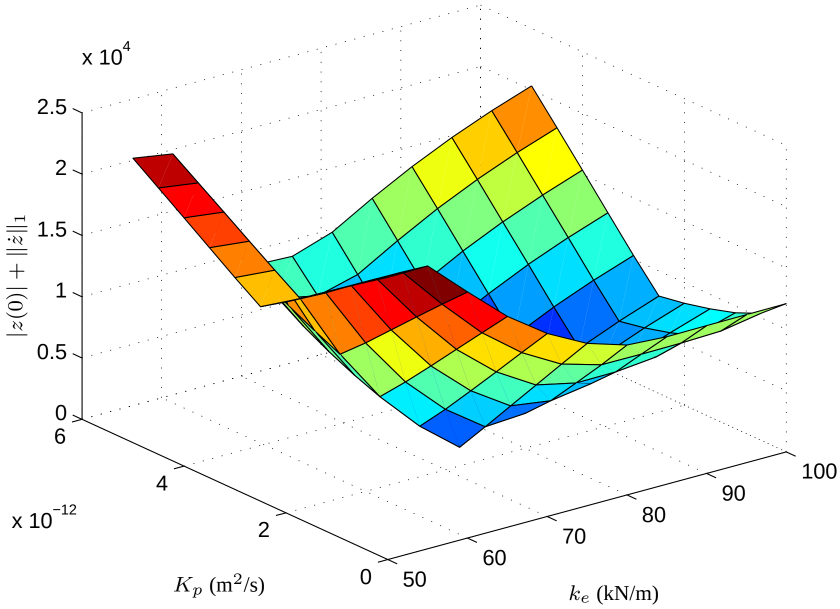

Figure 3.4: The plot of $|z(0)| + \|\dot{z}\|_1$ versus k_e and K_p.

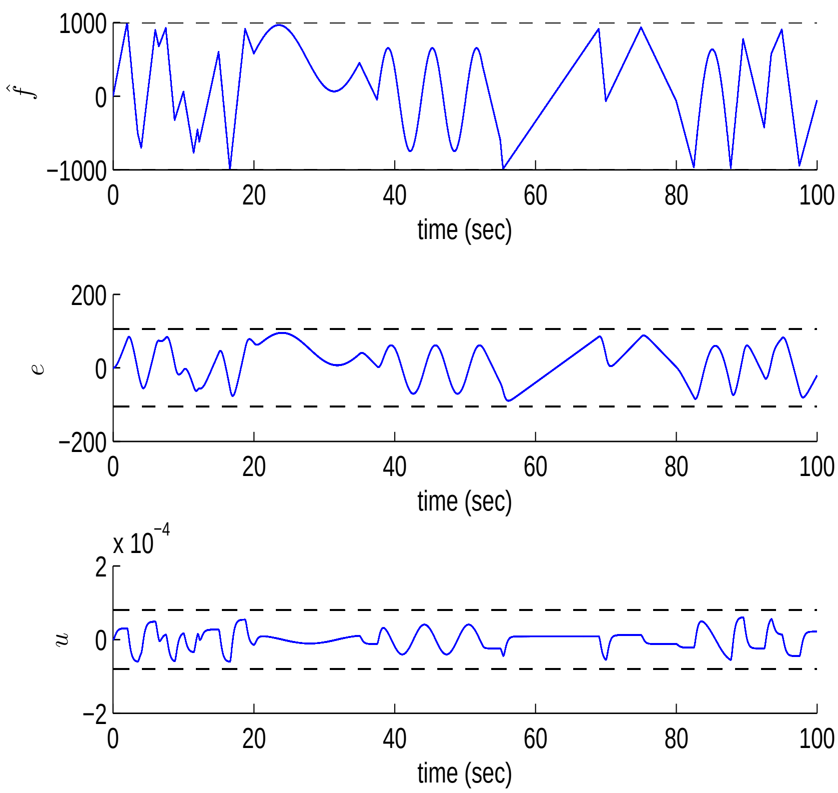

Figure 3.5: Responses e and u to f^* for $k_e \in [50, 100]$ kN/m and $K_p \in [0, 5 \times 10^{-12}]$ m^2/s.

CHAPTER IV

CRITERION OF APPROXIMATION FOR MIMO FEEDBACK SYSTEMS

This chapter presents an extension of Zakian's criterion of approximation [30] in which the plant is a non-rational MIMO feedback system.

4.1 Introduction

Consider the multi-input multi-output (MIMO) feedback system described by

$$\left.\begin{aligned} u_i &= \sum_{j=1}^{n} k_{ij} \star e_j \\ e_i &= f_i - \sum_{j=1}^{n} g_{ij} \star u_j \end{aligned}\right\}, \quad i = 1, 2, \ldots, n \tag{4.1}$$

where $\mathbf{G}(s) \triangleq [G_{ij}(s)]_{n \times n}$ is the plant transfer matrix and $\mathbf{K}(s, \mathbf{p}) \triangleq [K_{ij}(s, \mathbf{p})]_{n \times n}$ is the controller transfer matrix characterized by a design parameter $\mathbf{p} \in \mathbb{R}^N$ (see Figure 4.1). Let $\mathbf{e} \triangleq [e_1, e_2, \ldots, e_n]^T$ and $\mathbf{u} \triangleq [u_1, u_2, \ldots, u_n]^T$ be the error vector and the control vector of the system (4.1), respectively. Also let $k_{ij} : [0, \infty) \to \mathbb{R}$ and $g_{ij} : [0, \infty) \to \mathbb{R}$ denote the inverse Laplace transforms of $G_{ij}(s)$ and $K_{ij}(s, \mathbf{p})$, respectively.

Suppose that the input vector $\mathbf{f} \triangleq [f_1, f_2, \ldots, f_n]^T$ is known only to the extent that each element $f_i : [0, \infty) \to \mathbb{R}$ belongs to the possible set $\mathcal{P}_i$ where $\mathcal{P}_i \subseteq \mathbb{L}_\infty$ for all $i = 1, 2 \ldots, n$ and $\mathbb{L}_\infty$ is the set of all bounded functions. In this connection, define the Cartesian product of the possible sets as

$$\boldsymbol{\mathcal{P}} \triangleq \mathcal{P}_1 \times \mathcal{P}_2 \times \ldots \times \mathcal{P}_n.$$

Define, for the system (4.1), the vectors of performance measures

$$\hat{\mathbf{e}} \triangleq [\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n]^T \quad \text{and} \quad \hat{\mathbf{u}} \triangleq [\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n]^T \tag{4.2}$$

where $\hat{e}_i$ and $\hat{u}_i$ denote the peak values of e_i and u_i, respectively, associated with the space $\boldsymbol{\mathcal{P}}$ given

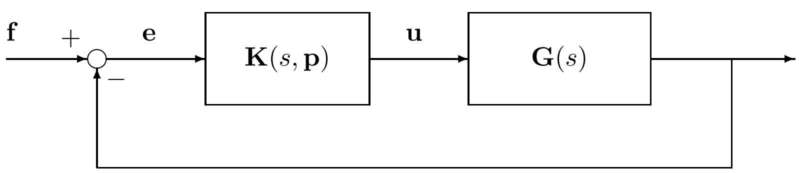

Figure 4.1: The MIMO feedback system given in (4.1).

by

$$\hat{e}_i \triangleq \sup_{\mathbf{f}\in\boldsymbol{\mathcal{P}}} \|e_i\|_\infty \;\text{ and }\; \hat{u}_i \triangleq \sup_{\mathbf{f}\in\boldsymbol{\mathcal{P}}} \|u_i\|_\infty. \tag{4.3}$$

Let $\mathbf{x} \preceq \mathbf{y}$ denote a componentwise inequality between vectors $\mathbf{x}$ and $\mathbf{y}$. The design problem is to find $\mathbf{p}$ such that the following design criteria are satisfied.

$$\hat{\mathbf{e}} \preceq \mathcal{E} \tag{4.4}$$

$$\hat{\mathbf{u}} \preceq \mathcal{U} \tag{4.5}$$

where the bound vectors $\mathcal{E} \triangleq [\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n]^T$ and $\mathcal{U} \triangleq [\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n]^T$ are specified. It should be noted that the criteria (4.4) and (4.5) are equivalent to the fact that, for every i, $|e_i(t)|$ and $|u_i(t)|$ are within the bounds $\mathcal{E}_i$ and $\mathcal{U}_i$, respectively, for all time t whenever $\mathbf{f} \in \boldsymbol{\mathcal{P}}$.

4.2 Main Results

This section derives the criterion of approximation for the system (4.1). To this end, let $\mathbf{G}^*(s) \triangleq [G^*_{ij}(s)]_{n\times n}$ be a rational approximant matrix of $\mathbf{G}(s)$ Then replacing $\mathbf{G}(s)$ with $\mathbf{G}^*(s)$ yields the resulting system which is called a nominal system (see Figure 4.2) and described by

$$\left.\begin{aligned} u_i^* &= \sum_{j=1}^{n} k_{ij} \star e_j^* \\ e_i^* &= f_i - \sum_{j=1}^{n} g^*_{ij} \star u_j^* \end{aligned}\right\}, \quad i = 1, 2, \ldots, n \tag{4.6}$$

where $\mathbf{e}^* \triangleq [e_1^*, e_2^*, \ldots, e_n^*]^T$ and $\mathbf{u}^* \triangleq [u_1^*, u_2^*, \ldots, u_n^*]^T$ are the error vector and the control vector of the nominal system (4.6), respectively, and $g^*_{ij} : [0, \infty) \to \mathbb{R}$ denotes the inverse Laplace transform of $G^*_{ij}(s)$.

For the nominal system (4.6), define

$$\hat{\mathbf{e}}^* \triangleq [\hat{e}_1^*, \hat{e}_2^*, \ldots, \hat{e}_n^*]^T \;\text{ and }\; \hat{\mathbf{u}}^* \triangleq [\hat{u}_1^*, \hat{u}_2^*, \ldots, \hat{u}_n^*]^T$$

where $\hat{e}_i^*$ and $\hat{u}_i^*$ denote the peak value of e_i^* and u_i^*, respectively, given by

$$\hat{e}_i^* \triangleq \sup_{\mathbf{f}\in\boldsymbol{\mathcal{P}}} \|e_i^*\|_\infty \;\text{ and }\; \hat{u}_i^* \triangleq \sup_{\mathbf{f}\in\boldsymbol{\mathcal{P}}} \|u_i^*\|_\infty. \tag{4.7}$$

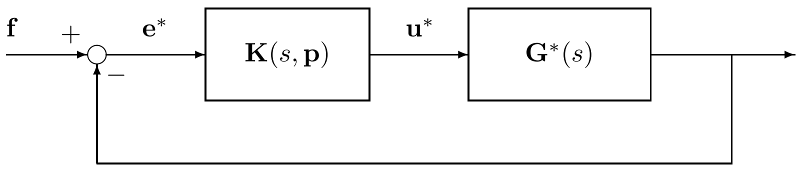

Figure 4.2: The nominal system for (4.1).

Define

$$\mathbf{X}(s) = [X_{ij}(s)]_{n\times n} \triangleq [I + \mathbf{G}^*(s)\mathbf{K}(s)]^{-1} \quad (4.8)$$

$$\mathbf{Z}(s) = [Z_{ij}(s)]_{n\times n} \triangleq \mathbf{G}(s) - \mathbf{G}^*(s) \quad (4.9)$$

$$\mathbf{W}(s) = [W_{ij}(s)]_{n\times n} \triangleq \mathbf{X}(s)\mathbf{Z}(s)\mathbf{K}(s) \quad (4.10)$$

$$\mathbf{V}(s) = [V_{ij}(s)]_{n\times n} \triangleq \mathbf{K}(s)\mathbf{X}(s)\mathbf{Z}(s) \quad (4.11)$$

$$\mathbf{M} = [\mu_{ij}]_{n\times n}, \quad \mu_{ij} \triangleq \|w_{ij}\|_1 \quad (4.12)$$

$$\mathbf{N} = [\nu_{ij}]_{n\times n}, \quad \nu_{ij} \triangleq \|v_{ij}\|_1 \quad (4.13)$$

where w_{ij} and v_{ij} are the inverse Laplace transforms of $W_{ij}(s)$ and $V_{ij}(s)$, respectively. The following results provide useful mathematical expressions for upper bounds and lower bounds of $\hat{e}_i$ and $\hat{u}_i$.

Lemma 4.1. *Suppose that* $\hat{e}_i^* < \infty$ *for* $i = 1, \ldots, n$. *Let* $(I - \mathbf{M})^{-1}$ *exists and all of its elements are positive. Then it follows that*

$$(I + \mathbf{M})^{-1}\hat{\mathbf{e}}^* \preceq \hat{\mathbf{e}} \preceq (I - \mathbf{M})^{-1}\hat{\mathbf{e}}^*. \quad (4.14)$$

Proof. Using equations (4.1), (4.6), (4.8), (4.9) and (4.10), one can verify that

$$E_i(s) = E_i^*(s) - \sum_{j=1}^{n} W_{ij}(s)E_j(s) \quad (4.15)$$

where $E_i(s)$ and $E_i^*(s)$ are the Laplace transforms of e_i and e_i^*, respectively, due to the input $\mathbf{f}$. Consequently,

$$e_i(t) = e_i^*(t) - \sum_{j=1}^{n}(w_{ij} \star e_j)(t) \quad (4.16)$$

and then

$$\begin{aligned} \|e_i\|_\infty &\leq \|e_i^*\|_\infty + \sum_{j=1}^{n}\|w_{ij} \star e_j\|_\infty \\ &\leq \|e_i^*\|_\infty + \sum_{j=1}^{n}\|w_{ij}\|_1\|e_j\|_\infty. \end{aligned}$$

Thus, by using (4.12), one obtains

$$\|e_i\|_\infty \leq \|e_i^*\|_\infty + \sum_{j=1}^{n}\mu_{ij}\|e_j\|_\infty. \quad (4.17)$$

From equations (4.3), (4.7) and (4.17), it follows that

$$\hat{e}_i \leq \hat{e}_i^* + \sum_{j=1}^{n}\mu_{ij}\hat{e}_j. \quad (4.18)$$

It is easy to see from (4.2), (4.12) and (4.18) that

$$\hat{\mathbf{e}} \preceq \hat{\mathbf{e}}^* + \mathbf{M}\hat{\mathbf{e}}. \quad (4.19)$$

By using (4.12) and (4.16), we have

$$\|e_i^* - e_i\|_\infty \leq \sum_{j=1}^{n} \mu_{ij} \|e_j\|_\infty. \tag{4.20}$$

From the identity

$$e_i^*(t) = [e_i^*(t) - e_i(t)] + e_i(t),$$

one obtains

$$\|e_i^*\|_\infty \leq \|e_i^* - e_i\|_\infty + \|e_i\|_\infty$$

and (4.20) implies that

$$\|e_i^*\|_\infty \leq \sum_{j=1}^{n} \mu_{ij} \|e_j\|_\infty + \|e_i\|_\infty.$$

Hence,

$$\hat{\mathbf{e}}^* \preceq \mathbf{M}\hat{\mathbf{e}} + \hat{\mathbf{e}}. \tag{4.21}$$

Now, if $(I - \mathbf{M})^{-1}$ exists and all of its elements are positive, then (4.19) and (4.21) imply that

$$(I + \mathbf{M})^{-1}\hat{\mathbf{e}}^* \preceq \hat{\mathbf{e}} \preceq (I - \mathbf{M})^{-1}\hat{\mathbf{e}}^*.$$

□

Lemma 4.2. *Suppose that $\hat{u}_i^* < \infty$ for $i = 1, \ldots, n$. Let $(I - \mathbf{N})^{-1}$ exists and all of its elements are positive. Then it follows that*

$$(I + \mathbf{N})^{-1}\hat{\mathbf{u}}^* \preceq \hat{\mathbf{u}} \preceq (I - \mathbf{N})^{-1}\hat{\mathbf{u}}^*. \tag{4.22}$$

Proof. From the definition of ν_{ij} in (4.13), the proof can be completed by the technique used in Lemma 4.1. □

Now, it is ready to state the main theorems in this section.

Theorem 4.1. *Suppose that $\hat{e}_i^* < \infty$ for $i = 1, \ldots, n$. Let $(I - \mathbf{M})^{-1}$ exists and let all of its elements be positive. Then the original design criteria* (4.4) *for the system* (4.1) *are satisfied if*

$$(I - \mathbf{M})^{-1}\hat{\mathbf{e}}^* \preceq \mathcal{E}. \tag{4.23}$$

Proof. The theorem readily follows from Lemma 4.1. □

Theorem 4.2. *Suppose that $\hat{u}_i^* < \infty$ for $i = 1, \ldots, n$. Let $(I - \mathbf{N})^{-1}$ exists and let all of its elements be positive. Then the original design criteria* (4.5) *for the system* (4.1) *are satisfied if*

$$(I - \mathbf{N})^{-1}\hat{\mathbf{u}}^* \preceq \mathcal{U}. \tag{4.24}$$

Proof. The theorem readily follows from Lemma 4.2. □

From Theorems 4.1 and 4.2, we can see that the upper bounds of $\hat{e}_i$ and $\hat{u}_i$ are expressed in terms of the matrices $\mathbf{M}$, $\mathbf{N}$ and the peak values $\hat{e}_i^*$ and $\hat{u}_i^*$ of the nominal system (4.6). It is important to note that when the system (4.1) becomes the scalar system (2.1), the conditions in Theorems 4.1 and 4.2 turn out to be identical to those in Theorem 2.1 in Chapter 2.

Since $\mathbf{G}^*(s)$ and $\mathbf{K}(s)$ are rational matrices, tools for stability analysis and computing time-responses are readily available for the case of the nominal system (4.6). Hence, when such tools for the non-rational system (4.1) are not available, inequalities (4.23) and (4.24) are more computationally tractable than the original design criteria (4.4) and (4.5). For this reason, (4.23) and (4.24) are called the *surrogate* design criteria.

4.3 Finiteness of $\hat{\mathrm{e}}$ and $\hat{\mathrm{u}}$

Following the method of inequalities [29,30,32,33,35], it is readily appreciated that in solving inequalities (4.23) and (4.24) by numerical methods, a search algorithm needs to start from a point $\mathbf{p} \in \mathbb{R}^N$ such that $\hat{e}_i(\mathbf{p}) < \infty$ and $\hat{u}_i(\mathbf{p}) < \infty$ for all i. In this regard, the following proposition provide useful sufficient conditions for ensuring that $\hat{e}_i < \infty$ and $\hat{u}_i < \infty$ for all i.

Proposition 4.1. *Consider the original system* (4.1) *and the nominal system* (4.6). *Suppose that* $\{A_{G^*}, B_{G^*}, C_{G^*}, 0\}$ *and* $\{A_K, B_K, C_K, D_K\}$ *are state-space realizations of* $\mathbf{G}^*(s)$ *and* $\mathbf{K}(s)$, *respectively. Let* Λ_{cl} *denotes the set of all the eigenvalues of the closed-loop state transfer matrix*

$$A_{cl} = \left[\begin{array}{c|c} A_K & -B_K C_{G^*} \\ \hline B_G C_K & A_{G^*} - B_{G^*} D_K C_{G^*} \end{array}\right].$$

Then $\hat{e}_i < \infty$ *and* $\hat{u}_i < \infty$ *for all* $i = 1, 2, \ldots, n$ *if the following conditions hold.*

(a) $\operatorname{Re} \lambda(\mathbf{p}) < 0$ *for all* $\lambda(\mathbf{p}) \in \Lambda_{cl}$.

(b) $(I - \mathbf{M})^{-1}$ *and* $(I - \mathbf{N})^{-1}$ *exists and all of their elements are positive.*

Proof. Condition (a) is a well-known result of bounded-input bounded-output (BIBO) stability for rational systems. It implies that $\hat{e}_i^* < \infty$ and $\hat{u}_i^* < \infty$ for any bounded input. Therefore, it follows from (4.14) and (4.22) that if conditions (a) and (b) hold, then $\hat{e}_i < \infty$ and $\hat{u}_i < \infty$ for all $i = 1, 2, \ldots, n$. □

Consider condition (b) in Proposition 4.1. The finiteness of $\mathbf{M}$ and $\mathbf{N}$ is necessary for existence of $(I - \mathbf{M})^{-1}$ and $(I - \mathbf{M})^{-1}$. Therefore, a search algorithm should find a point $\mathbf{p}$ such that $\mathbf{M}$ and $\mathbf{N}$ are finite before finding $\mathbf{p}$ that satisfying condition (b).

The following lemmas provide sufficient conditions for ensuring that $\mathbf{M}$ and $\mathbf{N}$ are finite.

The following notation will be used in Lemmas 4.3, 4.4 and Theorem 4.3. For given integers i, j, k and l, define

$$\alpha_{ij}^{kl}(\mathbf{p}) \triangleq \max_{\lambda \in \Lambda_{ij}^{kl}} \operatorname{Re} \lambda(\mathbf{p})$$

where Λ_{ij}^{kl} denotes the set of all the poles of the function $K_{ij}(s)X_{kl}(s)$. Let z_{kl} be the impulse response of $Z_{kl}(s)$.

Lemma 4.3. *Assume that* $\mathbf{G}^*(s)$ *and* $\mathbf{K}(s)$ *are rational matrices. Then* $\mu_{ij} < \infty$ *if the two conditions hold.*

(a) $\|z_{kl}\|_1 < \infty,\ \forall\, k, l = 1, 2, \ldots, n.$

(b) $\alpha_{lj}^{ik}(\mathbf{p}) < 0,\ \forall\, k, l = 1, 2, \ldots, n.$

Proof. It is easy to see from (4.8), (4.9) and (4.10) that

$$W_{ij}(s) = \sum_{k=1}^{n}\sum_{l=1}^{n} K_{lj}(s)X_{ik}(s)Z_{kl}(s).$$

From the definition of μ_{ij} in (4.12), one can easily see that

$$\mu_{ij} = \left\| \sum_{k=1}^{n}\sum_{l=1}^{n} \mathcal{L}^{-1}\big\{K_{lj}(s)X_{ik}(s)Z_{kl}(s)\big\} \right\|_1 . \tag{4.25}$$

Evidently, $K_{lj}X_{im}$ is a proper rational function of s for any $i, j, k, l = 1, 2, \ldots, n$. Thus,

$$K_{lj}(s)X_{ik}(s) = a_{lj}^{ik} + R_{lj}^{ik}(s) \tag{4.26}$$

where a_{lj}^{ik} is a constant and R_{lj}^{ik} is a strictly proper function of s. Then it follows from (4.25) and (4.26) that

$$\mu_{ij} \leq \sum_{k=1}^{n}\sum_{l=1}^{n} |a_{lj}^{ik}| \cdot ||z_{kl}||_1 + ||r_{lj}^{ik} \star z_{kl}||_1$$

where r_{lj}^{ik} denotes the impulse response of $R_{lj}^{ik}(s)$. By a well-known result (see, for example, [13]), it follows that if $||r_{lj}^{ik}||_1 < \infty$ and $||z_{kl}||_1 < \infty$, then $||r_{lj}^{ik} \star z_{kl}||_1 < \infty$. Condition (b) is necessary and sufficient for $||r_{lj}^{ik}||_1 < \infty$ for all $k, l = 1, 2, \ldots, n$. □

Lemma 4.4. *Assume that* $\mathbf{G}^*(s)$ *and* $\mathbf{K}(s)$ *are rational matrices. Then* $\nu_{ij} < \infty$ *if the two conditions hold.*

(a) $\|z_{lj}\|_1 < \infty,\ \forall\, l = 1, 2, \ldots, n.$

(b) $\alpha_{ik}^{kl}(\mathbf{p}) < 0,\ \forall\, k, l = 1, 2, \ldots, n.$

Proof. It is easy to see from (4.8), (4.9) and (4.11) that

$$V_{ij}(s) = \sum_{k=1}^{n}\sum_{l=1}^{n} K_{ik}(s)X_{kl}(s)Z_{lj}(s).$$

From the definition of ν_{ij} in (4.13), the proof can be completed by the technique used in Lemma 4.3. □

Now it is ready to state the main theorem for the finiteness of the matrices $\mathbf{M}$ and $\mathbf{N}$.

Theorem 4.3. *Assume that* $\mathbf{G}^*(s)$ *and* $\mathbf{K}(s)$ *are rational matrices. Then all elements of* $\mathbf{M}$ *and* $\mathbf{N}$ *are finite if the two conditions hold.*

(a) $\|z_{kl}\|_1 < \infty, \ \forall\, k, l = 1, 2, \ldots, n.$

(b) $\alpha_{kl}^{pq}(\mathbf{p}) < 0, \ \forall\, k, l, p, q = 1, 2, \ldots, n.$

Proof. The theorem readily follows from Lemmas 4.3 and 4.4. □

From above, the solution of inequalities (4.23) and (4.24) involves three phases of computation as follows. Define

$$\left.\begin{array}{rcl} [\mu_{ij}^{\dagger}] & \triangleq & (I - \mathbf{M})^{-1} \\ [\nu_{ij}^{\dagger}] & \triangleq & (I - \mathbf{N})^{-1} \end{array}\right\}, \ i, j = 1, 2, \ldots, n.$$

- Phase I : With an arbitrary starting point, find $\mathbf{p}_0$ satisfying

$$\alpha_{kl}^{pq}(\mathbf{p}_0) \leq -\varepsilon, \ \forall\, k, l, p, q = 1, 2, \ldots, n.$$

  where $0 < \varepsilon \ll 1$ is given.

- Phase II : By starting from $\mathbf{p}_0$, find $\mathbf{p}_1$ satisfying

$$\left.\begin{array}{rcl} \alpha_{kl}^{pq}(\mathbf{p}_1) & \leq & -\varepsilon, \ \forall\, k, l, p, q = 1, 2, \ldots, n, \\ \mu_{ij}^{\dagger}(\mathbf{p}_1) & > & 0, \ \ \forall\, i, j = 1, 2, \ldots, n, \\ \nu_{ij}^{\dagger}(\mathbf{p}_1) & > & 0, \ \ \forall\, i, j = 1, 2, \ldots, n. \end{array}\right\}. \tag{4.27}$$

- Phase III : By starting from $\mathbf{p}_1$, find $\mathbf{p}$ satisfying the design criteria (4.23), (4.24), and inequalities (5.28).

4.4 Majorants for MIMO Feedback Systems

The original definition of a majorant for the case of SISO feedback systems is defined as follows.

Definition 4.1 ([31]). *Suppose that* $\Pi \subseteq \mathbb{R}^N$ *and there is a function* $\phi : \mathbb{R}^N \to [0, \infty]$ *such that*

$$\phi(\mathbf{p}) < \infty \quad \forall \mathbf{p} \in \Pi.$$

A function $\hat{\phi} : \hat{\Pi} \to [0, \infty]$ *is said to be a majorant of* ϕ *if the following conditions are satisfied.*

(A1) $\phi(\mathbf{p}) \leq \hat{\phi}(\mathbf{p}) \quad \forall \mathbf{p} \in \hat{\Pi}$

(A2) $\hat{\Pi} \subseteq \Pi$

(A3) Suppose that there is a sequence S *of points in* $\hat{\Pi}$ *such that* $\phi(\mathbf{p})$ *converges to zero. Then, for the same sequence* S, $\hat{\phi}(\mathbf{p})$ *also converges to zero.*

(A4) The function $\hat{\phi}$ *can be calculated more easily than* ϕ.

Now the definition of a majorant for the case of vector-valued functions is defined as follows.

Definition 4.2. *Suppose that* $\Pi \subseteq \mathbb{R}^N$ *and there are functions* $\phi_i : \mathbb{R}^N \to [0, \infty]$, $i = 1, 2, \ldots, m$, *such that*

$$\phi_i(\mathbf{p}) < \infty, \text{ for } i = 1, 2, \ldots, m, \ \forall \mathbf{p} \in \Pi.$$

A function $\hat{\boldsymbol{\phi}} \triangleq [\hat{\phi}_1, \hat{\phi}_2, \ldots, \hat{\phi}_m]^T$ *is said to be a majorant vector of* $\boldsymbol{\phi} \triangleq [\phi_1, \phi_2, \ldots, \phi_m]^T$ *if the following conditions are satisfied.*

(B1) $\phi_i(\mathbf{p}) \le \hat{\phi}_i(\mathbf{p})$, *for* $i = 1, 2, \ldots, n \ \forall \mathbf{p} \in \hat{\Pi}$

(B2) $\hat{\Pi} \subseteq \Pi$

(B3) Suppose that there is a sequence S *of points in* $\hat{\Pi}$ *such that* $\phi_i(\mathbf{p})$ *converges to zero for all* i. *Then, for the same sequence* S, $\hat{\phi}_i(\mathbf{p})$ *also converges to zero for all* i.

(B4) The functions $\hat{\phi}_i$ *can be calculated more easily than* ϕ_i *for all* i.

The following propositions show that the upper bounds derived in Lemmas 4.1 and 4.2 are majorants of $\hat{\mathbf{e}}$ and $\hat{\mathbf{u}}$, respectively.

Proposition 4.2. *The function* $(I - \mathbf{M})^{-1}\hat{\mathbf{e}}^*$ *is a majorant vector of* $\hat{\mathbf{e}}$.

Proof. From the result in Proposition 4.1, if $\hat{e}_i^* < \infty$ for all i and $(I - \mathbf{M})^{-1}$ exists and all of its elements are positive, then condition (B1) in Definition 4.2 is satisfied.

Since $\hat{e}_i^* < \infty$ for all i and $(I - \mathbf{M})^{-1}$ exists and all of its elements are positive, then $\hat{\mathbf{e}}$ is always not greater than $(I - \mathbf{M})^{-1}\hat{\mathbf{e}}^*$. Hence, these conditions are sufficient for ensuring that $\hat{\mathbf{e}}$ is finite. Then condition (B2) in Definition 4.2 is satisfied.

Consider inequalities (4.14) in Lemma 4.1. If $\hat{\mathbf{e}}$ converges to zero, then $\hat{\mathbf{e}}^*$ will converge to zero and then, if $(I - \mathbf{M})^{-1} > 0$, the upper bound $(I - \mathbf{M})^{-1}\hat{\mathbf{e}}^*$ also converges to zero. Hence, condition (B3) in Definition 4.2 is satisfied.

Since the approximant $\mathbf{G}^*(s)$ and the controller $\mathbf{K}(s, \mathbf{p})$ are rational, $\hat{\mathbf{e}}^*$ are easily obtainable. When tools for computing time-responses for non-rational systems are not available, the upper bound $(I - \mathbf{M})^{-1}\hat{\mathbf{e}}^*$ can be calculated more easily than $\hat{\mathbf{e}}$. Therefore, condition (B4) in Definition 4.2 is satisfied. □

Proposition 4.3. *The function* $(I - \mathbf{N})^{-1}\hat{\mathbf{u}}^*$ *is a majorant vector of* $\hat{\mathbf{u}}$.

Proof. The proof can be completed by the technique used in Proposition 4.2. □

4.5 The Criterion of Approximation for 2×2 Feedback Systems

This section states the criterion of approximation for 2×2 feedback systems. This was published in [10] and can be seen as a special case of Theorems 4.1 and 4.2.

Theorem 4.4. *[10] Suppose that* $\hat{e}_1^* < \infty$ *and* $\hat{e}_2^* < \infty$. *Let* $\mu_{11} < 1$, $\mu_{22} < 1$ *and* $\det(I - \mathbf{M}) > 0$. *Then the original design criteria* (4.4) *for the system* (4.1) *are satisfied if the following hold.*

$$\frac{(1 - \mu_{22})\hat{e}_1^* + \mu_{12}\hat{e}_2^*}{\det(I - \mathbf{M})} \le \mathcal{E}_1 \ \text{ and } \ \frac{(1 - \mu_{11})\hat{e}_2^* + \mu_{21}\hat{e}_1^*}{\det(I - \mathbf{M})} \le \mathcal{E}_2. \tag{4.28}$$

Theorem 4.5. *[10] Suppose that $\hat{u}_1^* < \infty$ and $\hat{u}_2^* < \infty$. Let $\nu_{11} < 1$, $\nu_{22} < 1$ and $\det(I - \mathbf{N}) > 0$. Then the original design criteria* (4.5) *for the system* (4.1) *are satisfied if the following hold.*

$$\frac{(1-\nu_{22})\hat{u}_1^* + \nu_{12}\hat{u}_2^*}{\det(I-\mathbf{N})} \leq \mathcal{U}_1 \;\; \textit{and} \;\; \frac{(1-\nu_{11})\hat{u}_2^* + \nu_{21}\hat{u}_1^*}{\det(I-\mathbf{N})} \leq \mathcal{U}_2. \tag{4.29}$$

Theorems 4.4 and 4.5 can be seen as a special case of Theorems 4.1 and 4.2 in Section 4.2. From Lemmas 4.1 and 4.2, one obtains

$$\hat{\mathbf{e}} \leq (I - \mathbf{M})^{-1}\hat{\mathbf{e}}^* \;\; \text{and} \;\; \hat{\mathbf{u}} \leq (I - \mathbf{N})^{-1}\hat{\mathbf{u}}^*.$$

For 2×2 systems, it is easy to see that

$$(I-\mathbf{M})^{-1}\hat{\mathbf{e}}^* = \frac{1}{\det(I-\mathbf{M})}\begin{bmatrix} 1-\mu_{22} & \mu_{12} \\ \mu_{21} & 1-\mu_{11} \end{bmatrix}\begin{bmatrix} \hat{e}_1^* \\ \hat{e}_2^* \end{bmatrix}.$$

Consequently,

$$\hat{e}_1 \leq \frac{(1-\mu_{22})\hat{e}_1^* + \mu_{12}\hat{e}_2^*}{\det(I-\mathbf{M})} \;\; \text{and} \;\; \hat{e}_2 \leq \frac{(1-\mu_{11})\hat{e}_2^* + \mu_{21}\hat{e}_1^*}{\det(I-\mathbf{M})}.$$

Similarity, for $\hat{u}_1$ and $\hat{u}_2$, it is easy to verify that

$$\hat{u}_1 \leq \frac{(1-\nu_{22})\hat{u}_1^* + \nu_{12}\hat{u}_2^*}{\det(I-\mathbf{N})} \;\; \text{and} \;\; \hat{u}_2 \leq \frac{(1-\nu_{11})\hat{u}_2^* + \nu_{21}\hat{u}_1^*}{\det(I-\mathbf{N})}.$$

Hence, it can be readily seen that Theorems 4.4 and 4.5 are equivalent to Theorems 4.1 and 4.2, respectively, for the cases 2×2 systems.

The following theorem provides useful sufficient conditions for ensuring that the matrices $\mathbf{M}$ and $\mathbf{N}$ are finite.

Theorem 4.6. *[10] Assume that $\mathbf{G}^*(s)$ and $\mathbf{K}(s)$ are rational matrices. Then all elements of $\mathbf{M}$ and $\mathbf{N}$ are finite if the two conditions hold.*

(a) $\|z_{kl}\|_1 < \infty, \; \forall\, k, l = 1, 2.$

(b) $\alpha_{kl}^{pq}(\mathbf{p}) < 0, \; \forall\, k, l, p, q = 1, 2.$

From Theorems 4.4, 4.5 and 4.6, it readily follows that the solution of inequalities (4.28) and (4.29) involves three phases of computation as follows.

- Phase I : With an arbitrary starting point, find $\mathbf{p}_0$ satisfying

$$\alpha_{kl}^{pq}(\mathbf{p}_0) \leq -\varepsilon, \; \forall\, k, l, p, q = 1, 2.$$

where $0 < \varepsilon \ll 1$ is given.

- Phase II : By starting from $\mathbf{p}_0$, find $\mathbf{p}_1$ satisfying

$$\left.\begin{aligned} \alpha_{kl}^{pq}(\mathbf{p}_0) \leq -\varepsilon, \; \forall\, k, l, p, q = 1, 2. \\ \mu_{11}(\mathbf{p}_1) < 1 \;\; \text{and} \;\; \mu_{22}(\mathbf{p}_1) < 1 \\ \nu_{11}(\mathbf{p}_1) < 1 \;\; \text{and} \;\; \nu_{22}(\mathbf{p}_1) < 1 \\ \det(I-\mathbf{M}) > 0 \;\; \text{and} \;\; \det(I-\mathbf{N}) > 0 \end{aligned}\right\}. \tag{4.30}$$

- Phase III : By starting from $\mathbf{p}_1$, find $\mathbf{p}$ satisfying the design criteria (4.28), (4.29), and inequalities (4.30).

4.6 Numerical Examples

In this section, the linearized model of a binary distillation column is obtained from linearization about an operating point [25]. The plant transfer matrix $\mathbf{G}(s)$ is given by

$$\mathbf{G}(s) = \begin{bmatrix} \frac{12.8e^{-s}}{16.7s+1} & \frac{-18.9e^{-3s}}{21.0s+1} \\ \frac{6.6e^{-7s}}{10.9s+1} & \frac{-19.4e^{-2s}}{14.4s+1} \end{bmatrix}. \tag{4.31}$$

Let $\mathbf{K}(s,p)$ take the form

$$\mathbf{K}(s,p) = \begin{bmatrix} \frac{p_1(p_2s+1)(p_3s+1)}{s(p_4s+1)} & p_9 \\ p_{10} & \frac{p_5(p_6s+1)(p_7s+1)}{s(p_8s+1)} \end{bmatrix}.$$

where $\mathbf{p} \triangleq [p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}]^T$ is a design parameter. Assume the inputs f_1 belongs to the set $\mathcal{P}$ where

$$\mathcal{P} \triangleq \{f_1 : \|f_1\|_\infty \leq 0.2 \;\text{ and }\; \|\dot{f}_1\|_\infty \leq 0.2\}. \tag{4.32}$$

For simplicity, let $f_2 = 0$. It may be noted that if f_2 is not zero, one can use the principle of superposition to compute the peaks due to both inputs f_1 and f_2.

Suppose that the plant transfer matrix $\mathbf{G}(s)$ is replaced by a strictly proper rational approximant matrix $\mathbf{G}^*(s)$. Here, we replace $e^{-\tau s}$ with its $[1/2]$ Padé approximant[1]; that is

$$e^{-\tau s} \approx \frac{1 - \tau s/3}{(\tau s)^2/6 + 2\tau s/3 + 1}.$$

Then the resultant approximant matrix $\mathbf{G}^*(s)$ is given by

$$\begin{aligned} G^*_{11}(s) &= \frac{-25.6(s-3)}{(16.7s+1)(s^2+4s+6)} \\ G^*_{12}(s) &= \frac{37.8(s-1)}{(21.0s+1)(3s^2+4s+2)} \\ G^*_{21}(s) &= \frac{-13.2(7s-3)}{(10.9s+1)(49s^2+28s+6)} \\ G^*_{22}(s) &= \frac{19.4(2s-3)}{(14.4s+1)(2s^2+4s+3)}. \end{aligned}$$

The impulse responses of the original system and the nominal system are shown in Figure 4.3.

The main control objective is to ensure that, during the operation,

[1]The $[M/N]$ Padé approximant to a function $h(s)$ is defined as a rational function $P(s)/Q(s)$ where P and Q are polynomials of degree M and N, respectively. See, e.g., [5] for the details.

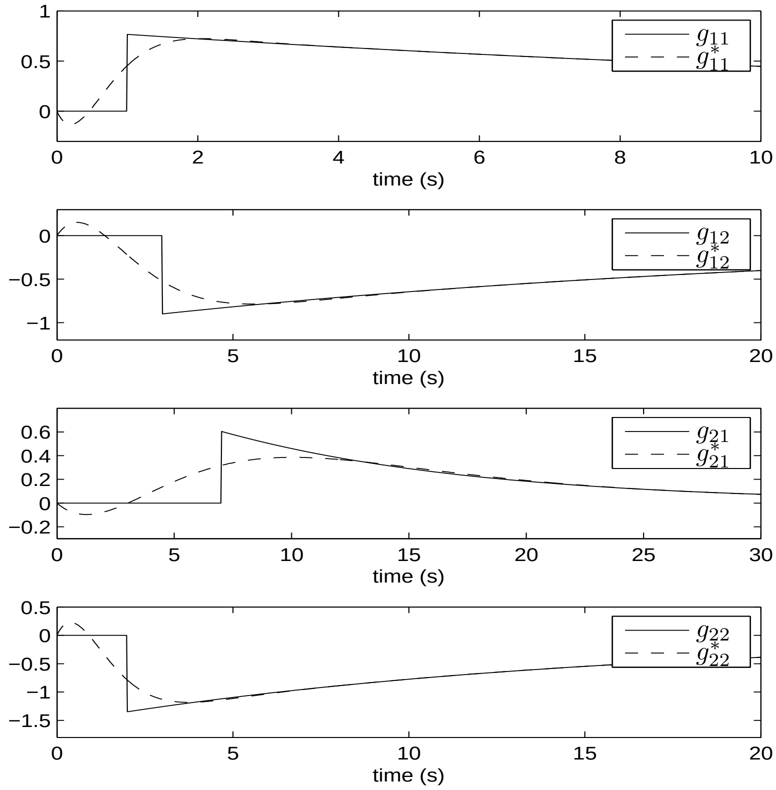

Figure 4.3: Comparison of the impulse responses of $\mathbf{G}(s)$ and $\mathbf{G}^*(s)$.

- the top product deviation e_1 stays within ± 0.38 mol%,
- the bottom product deviation e_2 stays within ± 0.20 mol%,
- the deviation of the reflux rate u_1 stays within ± 0.10 lb/min,
- the deviation of the reboiler rate u_2 stays within ± 0.05 lb/min.

Accordingly, the principal design criteria can be expressed as

$$\hat{e}_1 \leq 0.38, \quad \hat{e}_2 \leq 0.20, \quad \hat{u}_1 \leq 0.10, \quad \hat{u}_2 \leq 0.05. \tag{4.33}$$

In this work, inequalities (4.33) are solved by using the MBP algorithm (See [34, 35] for the detail of the MBP algorithm). Alternatively, other algorithms for solving a set of inequalities may be used (see [22] and the references therein). In addition, the nominal peaks $\hat{e}_i^*$ and $\hat{u}_i^*$ associated with the possible set (4.32) are computed by the method developed in [14].

To verify the design, a simulation is carried out for the case in which the control system is subject to a test input $f = [\hat{f}_1, 0]^T$, which is generated randomly so that its magnitude and slope satisfy (4.32). The waveform of the test input $\hat{f}_1$ is shown in Figure 4.4.

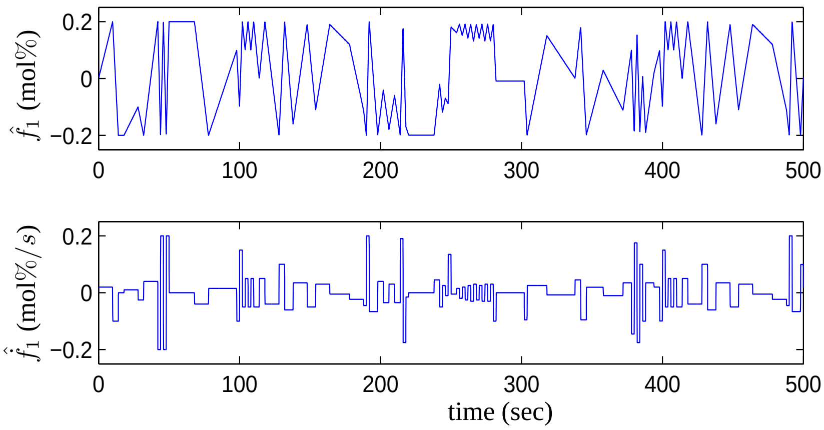

Figure 4.4: The waveforms of the test input $\hat{f}_1$ and its derivative.

4.6.1 Design without the criterion of approximation

In this part, the nominal system (4.6) is designed without the criterion of approximation. A controller is obtained by solving the design criteria

$$\begin{aligned} \hat{e}_1^* &\leq 0.38 \text{ mol\%}, & \hat{e}_2^* &\leq 0.20 \text{ mol\%}, \\ \hat{u}_1^* &\leq 0.10 \text{ lb/min}, & \hat{u}_2^* &\leq 0.05 \text{ lb/min}. \end{aligned}$$

After a number of iterations, the MBP algorithm locates a design solution resulting in

$$\begin{aligned} K_{11}(s,\mathbf{p}) &= \frac{0.213s^2 + 0.1892s + 0.04204}{s(0.1015s+1)} \\ K_{12}(s,\mathbf{p}) &= 0.08298 \\ K_{21}(s,\mathbf{p}) &= -0.01224 \\ K_{22}(s,\mathbf{p}) &= \frac{-0.724s^2 - 0.4234s - 0.05974}{s(0.05371s+1)} \end{aligned} \tag{4.34}$$

and the corresponding performance measures are

$$\begin{aligned} \hat{e}_1^* &= 0.3726 \text{ mol\%}, & \hat{e}_2^* &= 0.0626 \text{ mol\%}, \\ \hat{u}_1^* &= 0.0860 \text{ lb/min}, & \hat{u}_2^* &= 0.0352 \text{ lb/min}. \end{aligned}$$

The responses e_1, e_2, u_1 and u_2 of the nominal system and the original system are displayed in Figures 4.5 and 4.6, respectively.

The simulation results in Figure 4.6 show that the responses of the original system oscillate; that is, the system is unstable. From this example, it is seen that the design can give a failure when the system is designed without the criterion of approximation.

4.6.2 Design by using the criterion of approximation

Now the criterion of approximation is used. Following Theorems 4.4, 4.5 and 4.6, a controller is obtained by solving the design criteria

$$\max_{\lambda \in \Lambda_{kl}^{pq}} \operatorname{Re} \lambda(\mathbf{p}_0) \leq -10^{-6} \quad \forall k,l,p,q = 1,2 \tag{4.35}$$

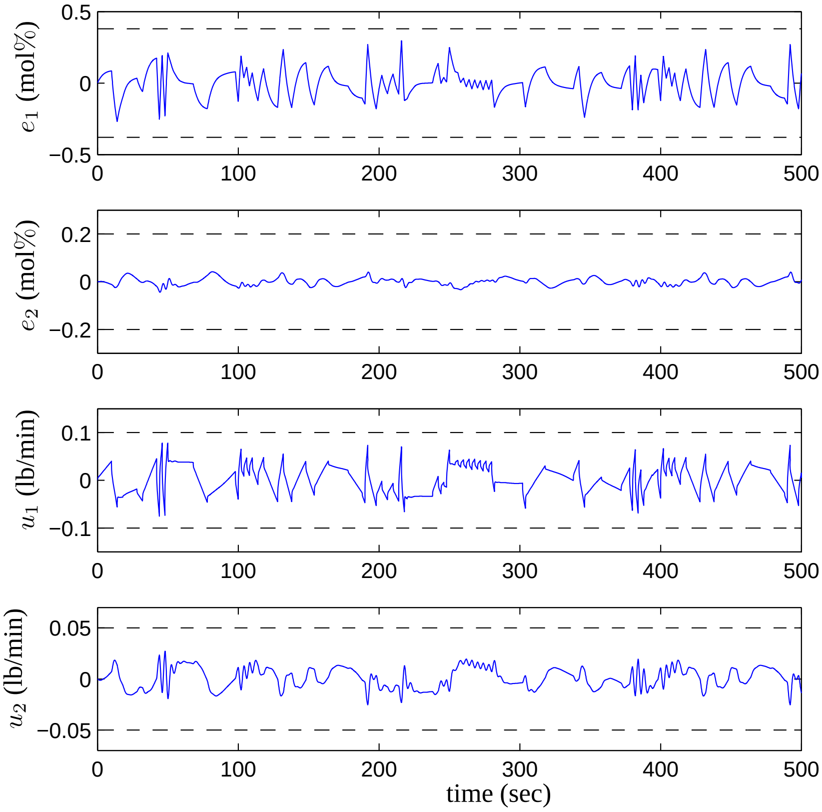

Figure 4.5: Responses of the nominal system using controller (4.34).

$$\left.\begin{array}{cc} \mu_{11}(\mathbf{p}) \leq 0.5, & \mu_{22}(\mathbf{p}) \leq 0.5, \\ \nu_{11}(\mathbf{p}) \leq 0.5, & \nu_{22}(\mathbf{p}) \leq 0.5, \\ -\det(I-\mathbf{M}) \leq -0.5, & -\det(I-\mathbf{N}) \leq -0.5 \end{array}\right\} \tag{4.36}$$

$$\left.\begin{array}{rcl} \dfrac{(1-\mu_{22})\hat{e}_1^* + \mu_{12}\hat{e}_2^*}{\det(I-\mathbf{M})} & \leq & 0.38 \text{ mol\%} \\ \dfrac{(1-\mu_{11})\hat{e}_2^* + \mu_{21}\hat{e}_1^*}{\det(I-\mathbf{M})} & \leq & 0.20 \text{ mol\%} \end{array}\right\} \tag{4.37}$$

$$\left.\begin{array}{rcl} \dfrac{(1-\nu_{22})\hat{u}_1^* + \nu_{12}\hat{u}_2^*}{\det(I-\mathbf{N})} & \leq & 0.10 \text{ lb/min} \\ \dfrac{(1-\nu_{11})\hat{u}_2^* + \nu_{21}\hat{u}_1^*}{\det(I-\mathbf{N})} & \leq & 0.05 \text{ lb/min} \end{array}\right\} \tag{4.38}$$

Inequalities (4.35) ensure the finiteness of $\mathbf{M}$ and $\mathbf{N}$, whereas (4.36) ensure that $\hat{e}_i$ and $\hat{u}_i$ are finite for all i. Inequalities (4.37) and (4.38) are sufficient conditions for ensuring that the design criteria (4.33) are satisfied.

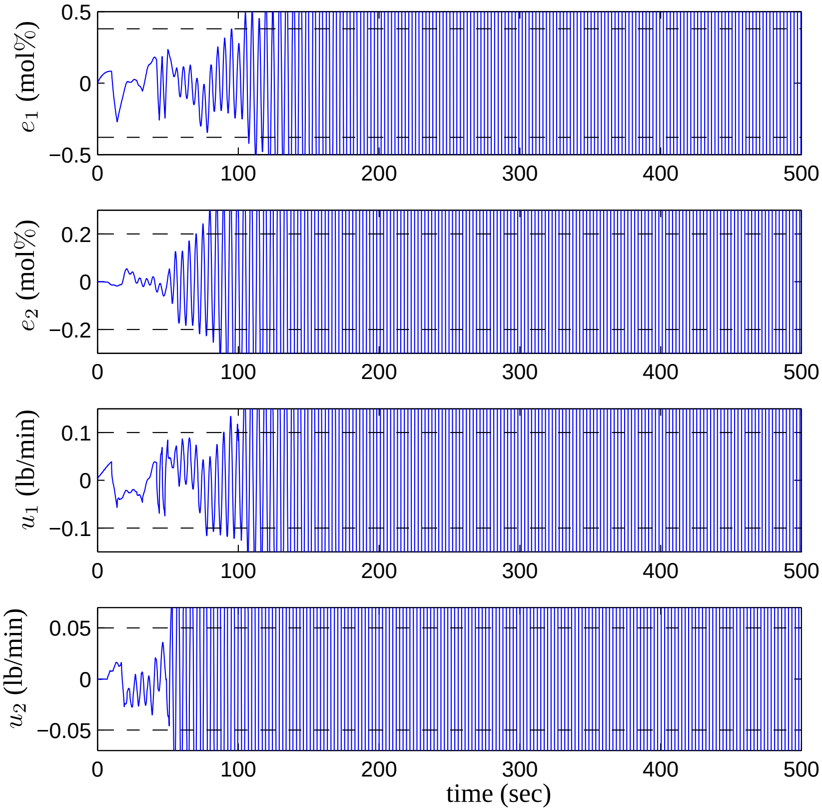

Figure 4.6: Responses of the original system using controller (4.34).

By using the MBP algorithm, a controller $\mathbf{K}(s, \mathbf{p})$ with

$$\begin{aligned} K_{11}(s, \mathbf{p}) &= \frac{0.03238s^2 + 0.0533s + 0.007084}{s(0.3652s + 1)} \\ K_{12}(s, \mathbf{p}) &= 0.03657 \\ K_{21}(s, \mathbf{p}) &= -0.00922 \\ K_{22}(s, \mathbf{p}) &= \frac{-0.2561s^2 - 0.1073s - 0.01}{s(20.73s + 1)} \end{aligned} \tag{4.39}$$

is found and the corresponding performance measures are

$$\mu_{11}(\mathbf{p}) = 0.0357, \quad \mu_{12}(\mathbf{p}) = 0.0497,$$

$$\mu_{21}(\mathbf{p}) = 0.1055, \quad \mu_{22}(\mathbf{p}) = 0.0767,$$

$$\nu_{11}(\mathbf{p}) = 0.0855, \quad \nu_{12}(\mathbf{p}) = 0.0671,$$

$$\nu_{21}(\mathbf{p}) = 0.0397, \quad \nu_{22}(\mathbf{p}) = 0.0387,$$

$$\frac{(1 - \mu_{22})\hat{e}_1^* + \mu_{12}\hat{e}_2^*}{\det(I - \mathbf{M})} = 0.3798 \text{ mol\%},$$

$$\frac{(1 - \mu_{11})\hat{e}_2^* + \mu_{21}\hat{e}_1^*}{\det(I - \mathbf{M})} = 0.1991 \text{ mol\%},$$

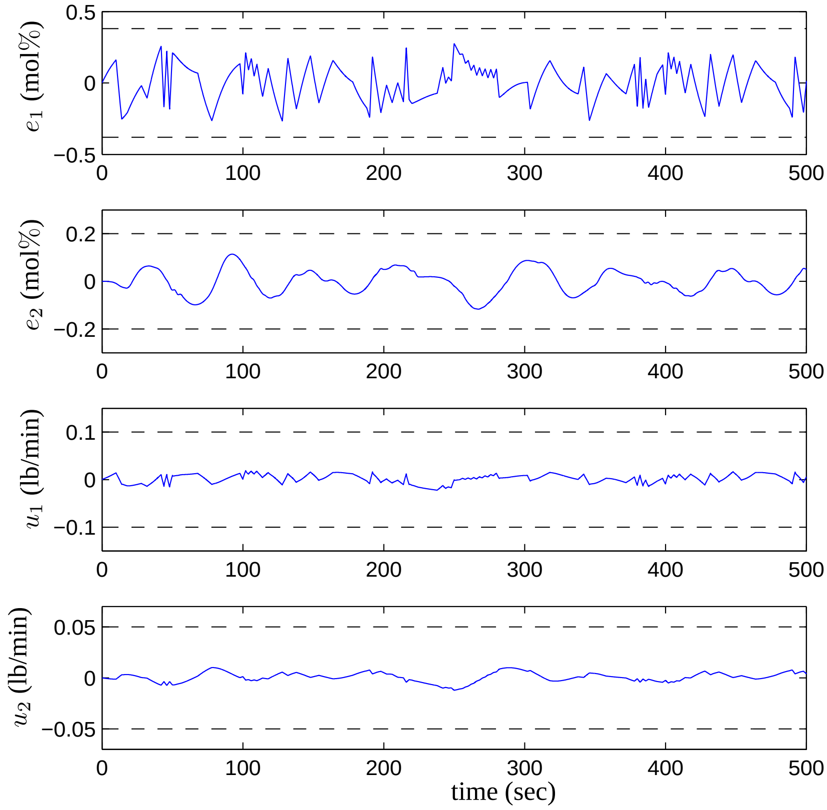

Figure 4.7: Responses of the original system using controller (4.39).

$$\frac{(1-\nu_{22})\hat{u}_1^* + \nu_{12}\hat{u}_2^*}{\det(I-\mathbf{N})} = 0.0376 \text{ lb/min},$$

$$\frac{(1-\nu_{11})\hat{u}_2^* + \nu_{21}\hat{u}_1^*}{\det(I-\mathbf{N})} = 0.0186 \text{ lb/min}.$$

The responses e_1, e_2, u_1 and u_2 of the original system are displayed in Figure 4.7. The peak values of e_1, e_2 ,u_1 and u_2 in response to $\hat{f}_1$ are 0.3791 mol%, 0.1547 mol%, 0.0343 lb/min and 0.0094 lb/min, respectively. Clearly, the design objectives are satisfied.

4.7 Conclusions and Discussion

This chapter derives a criterion of approximation for $n \times n$ feedback systems where the non-rational plant transfer matrix is replaced by a rational approximant during the design process and the system is subject to the possible inputs satisfying bounding conditions on their magnitude and slope. The design objective is to ensure that the error peaks $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n$ and the control peaks $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n$ always stay within the error bounds $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$ and the control bounds $\mathcal{U}_1, \mathcal{U}_2, \ldots,$ $\mathcal{U}_n$, respectively. For a chosen rational approximant matrix, the criterion provides useful sufficient conditions that are expressed as inequalities that can be solved in practice. When the plant and the control transfer matrices become scalar transfer functions, the results obtained in the paper turns out to be identical to Zakian's criterion of approximation for SISO systems. To illustrate the usefulness of the results, a controller design for a binary distillation column is carried out successfully by using the criterion in conjunction with the method of inequalities.

Although the system used in the numerical example to demonstrate the theory is the time-delay system, the theory can be used with other types of non-rational systems as well as long as the impulse response $[g_{ij}]$ is obtained. For various non-rational systems that can be founded in practice, see [12].

CHAPTER V

THEORY MAJORANTS FOR 2×2 VAGUE SYSTEMS

This chapter extends the theory of majorants for SISO vague systems to 2×2 vague systems.

5.1 Introduction

Consider the feedback system shown in Figure 5.1 and described by

$$\left.\begin{array}{rcl} u_i &=& k_{i1} \star e_1 + k_{i2} \star e_2 \\ e_i &=& f_i - g_{i1} \star u_1 - g_{i2} \star u_2 \end{array}\right\}, \quad i = 1, 2 \tag{5.1}$$

where the plant transfer matrix $\mathbf{G}(s) \triangleq [G_{ij}(s)]_{2\times 2}$ is known only to the extent that is belongs to a set $\mathcal{G} \subset \mathbb{R}^{2\times 2}(s)$ and $\mathbf{K}(s, \mathbf{p}) \triangleq [K(s, \mathbf{p})]_{2\times 2}$ is the transfer matrix of the controller with the design parameter $\mathbf{p} \in \mathbb{R}^N$. The vectors $\mathbf{e} = [e_1, e_2]^T$ and $\mathbf{u} = [u_1, u_2]^T$ are the error vector and the control vector of the system (5.1), respectively, and $k_{ij} : [0, \infty) \to \mathbb{R}$ and $g_{ij} : [0, \infty) \to \mathbb{R}$ denote the inverse Laplace transforms of $G_{ij}(s)$ and $K_{ij}(s, \mathbf{p})$, respectively.

Suppose that the input vector $\mathbf{f} = [f_1, f_2]^T$ is known only to the extent each element $f_i : [0, \infty) \to \mathbb{R}$ belongs to the the set $\mathcal{P}_i$ where $\mathcal{P}_i \subseteq \mathbb{L}_\infty$ for $i = 1, 2$.

Define, for the system (5.1), the performance measures $\hat{e}_1$, $\hat{e}_2$, $\hat{u}_1$ and $\hat{u}_2$ as follows.

$$\hat{e}_i \triangleq \sup_{f_1\in\mathcal{P}_1, f_2\in\mathcal{P}_2} \|e_i\|_\infty \ \text{ and } \ \hat{u}_i \triangleq \sup_{f_1\in\mathcal{P}_1, f_2\in\mathcal{P}_2} \|u_i\|_\infty \tag{5.2}$$

where $\hat{e}_i$ and $\hat{u}_i$ denote the peak values of e_i and u_i, respectively, for the spaces $\mathcal{P}_1$ and $\mathcal{P}_2$.

The design problem is to find $\mathbf{p}$ such that the following design criteria are satisfied.

$$\sup_{\mathbf{G}\in\mathcal{G}} \hat{e}_1 \le \mathcal{E}_1 \ \text{ and } \ \sup_{\mathbf{G}\in\mathcal{G}} \hat{e}_2 \le \mathcal{E}_2 \tag{5.3}$$

$$\sup_{\mathbf{G}\in\mathcal{G}} \hat{u}_1 \le \mathcal{U}_1 \ \text{ and } \ \sup_{\mathbf{G}\in\mathcal{G}} \hat{u}_2 \le \mathcal{U}_2 \tag{5.4}$$

where the bounds $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{U}_1$ and $\mathcal{U}_2$ are specified by designers.

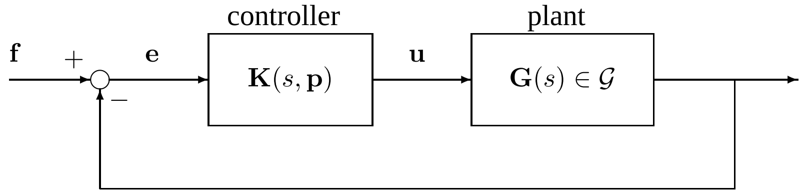

Figure 5.1: The two-input two-output feedback system given in (5.1).

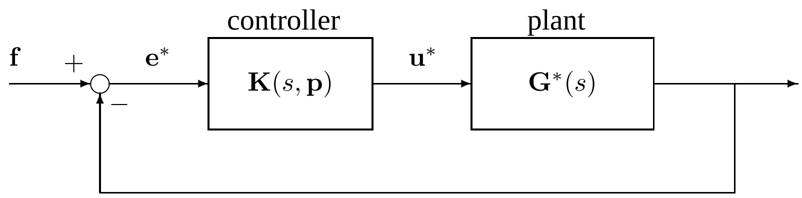

Figure 5.2: The nominal feedback system for (5.1).

5.2 Main Results

Let $\mathbf{G}^*(s) \triangleq [G_{ij}]_{2\times 2}$ be an approximant matrix of $\mathbf{G}(s)$, then replacing $\mathbf{G}(s)$ with $\mathbf{G}^*(s)$ yields the resulting system which is called a nominal system (see Figure 5.2) and described by

$$\left.\begin{array}{rcl} u_i^* &=& k_{i1} \star e_1^* + k_{i2} \star e_2^* \\ e_i^* &=& f_i - g_{i1} \star u_1^* + g_{i2} \star u_2^* \end{array}\right\}, \quad i = 1, 2 \tag{5.5}$$

where $\mathbf{e}^* = [e_1^*, e_2^*]^\top$ and $\mathbf{u}^* = [u_1^*, u_2^*]^\top$ are the error vector and the control vector of the nominal system (5.5), respectively, and $g_{ij}^* : [0, \infty) \to \mathbb{R}$ denotes the inverse Laplace transform of $G_{ij}^*(s)$.

Define, for the nominal system (5.5), the performance measures $\hat{e}_1^*$, $\hat{e}_2^*$, $\hat{u}_1^*$ and $\hat{u}_2^*$ as follows.

$$\hat{e}_i^* \triangleq \sup_{f_1 \in \mathcal{P}_1, f_2 \in \mathcal{P}_2} \|e_i^*\|_\infty \quad \text{and} \quad \hat{u}_i^* \triangleq \sup_{f_1 \in \mathcal{P}_1, f_2 \in \mathcal{P}_2} \|u_i^*\|_\infty \tag{5.6}$$

where $\hat{e}_i^*$ and $\hat{u}_i^*$ denote the peak values of e_i^* and u_i^*, respectively.

In the same fashion as in Chapter 4, define

$$\mathbf{X}(s) = [X_{ij}(s)]_{2\times 2} \triangleq [I + \mathbf{G}^*(s)\mathbf{K}(s)]^{-1} \tag{5.7}$$

$$\mathbf{Z}(s) = [Z_{ij}(s)]_{2\times 2} \triangleq \mathbf{G}(s) - \mathbf{G}^*(s) \tag{5.8}$$

$$\mathbf{W}(s) = [W_{ij}(s)]_{2\times 2} \triangleq \mathbf{X}(s)\mathbf{Z}(s)\mathbf{K}(s) \tag{5.9}$$

$$\mathbf{V}(s) = [V_{ij}(s)]_{2\times 2} \triangleq \mathbf{K}(s)\mathbf{X}(s)\mathbf{Z}(s) \tag{5.10}$$

$$\mathbf{M} = [\mu_{ij}]_{2\times 2}, \quad \mu_{ij} \triangleq \|w_{ij}\|_1 \tag{5.11}$$

$$\mathbf{N} = [\nu_{ij}]_{2\times 2}, \quad \nu_{ij} \triangleq \|v_{ij}\|_1 \tag{5.12}$$

where w_{ij} and v_{ij} are the inverse Laplace transforms of $W_{ij}(s)$ and $V_{ij}(s)$, respectively.

The following lemmas are the results have been published in [9]. The lemmas provide upper bounds of $\hat{e}_1$, $\hat{e}_2$, $\hat{u}_1$ and $\hat{u}_2$.

Lemma 5.1. *If* $\mu_{11} < 1, \mu_{22} < 1$ *and* $\det(I - \mathbf{M}) > 0$*, then it follows that*

$$\hat{e}_1 \leq \frac{(1 - \mu_{22})\hat{e}_1^* + \mu_{12}\hat{e}_2^*}{\det(I - \mathbf{M})}, \tag{5.13}$$

$$\hat{e}_2 \leq \frac{(1 - \mu_{11})\hat{e}_2^* + \mu_{21}\hat{e}_1^*}{\det(I - \mathbf{M})}. \tag{5.14}$$

Proof. The result readily follows from Lemma 4.1 in Chapter 4. ☐

Lemma 5.2. *If* $\nu_{11} < 1, \nu_{22} < 1$ *and* $\det(I - \mathbf{N}) > 0$*, then it follows that*

$$\hat{u}_1 \leq \frac{(1-\nu_{22})\hat{u}_1^* + \nu_{12}\hat{u}_2^*}{\det(I-\mathbf{N})}, \tag{5.15}$$

$$\hat{u}_2 \leq \frac{(1-\nu_{11})\hat{u}_2^* + \nu_{21}\hat{u}_1^*}{\det(I-\mathbf{N})}. \tag{5.16}$$

Proof. The result readily follows from Lemma 4.1 in Chapter 4. ☐

By noting that μ_{ij} and ν_{ij} depend on $\mathbf{G}(s) \in \mathcal{G}$, the following results is readily obtained from Lemmas 5.1 and 5.2.

Proposition 5.1. *Suppose that* $\hat{e}_1^* < \infty$ *and* $\hat{e}_2^* < \infty$*. Let* $\mu_{11} < 1$*,* $\mu_{22} < 1$ *and* $\det(I - \mathbf{M}) > 0$ *for any* $\mathbf{G}(s) \in \mathcal{G}$*. Then the design criteria* (5.3) *for the system* (5.1) *are satisfied if the following inequalities hold.*

$$\left.\begin{array}{rcl} \dfrac{\hat{e}_1^* + \sup\limits_{\mathbf{G}\in\mathcal{G}} \mu_{12}\hat{e}_2^*}{(1-\sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{11})(1-\sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{22}) - \sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{12}\sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{21}} & \leq & \mathcal{E}_1 \\ \dfrac{\hat{e}_2^* + \sup\limits_{\mathbf{G}\in\mathcal{G}} \mu_{21}\hat{e}_1^*}{(1-\sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{11})(1-\sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{22}) - \sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{12}\sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{21}} & \leq & \mathcal{E}_2 \end{array}\right\}. \tag{5.17}$$

Proof. Taking the supremum over the set $\mathcal{G}$ on the both sides of (5.13) yields

$$\begin{aligned} \sup_{\mathbf{G}\in\mathcal{G}} \hat{e}_1 &\leq \sup_{\mathbf{G}\in\mathcal{G}} \left[\frac{(1-\mu_{22})\hat{e}_1^* + \mu_{12}\hat{e}_2^*}{(1-\mu_{11})(1-\mu_{22}) - \mu_{12}\mu_{21}}\right] \\ &\leq \frac{\sup_{\mathbf{G}\in\mathcal{G}}[(1-\mu_{22})\hat{e}_1^* + \mu_{12}\hat{e}_2^*]}{\inf_{\mathbf{G}\in\mathcal{G}}[(1-\mu_{11})(1-\mu_{22}) - \mu_{12}\mu_{21}]} \\ &\leq \frac{(1-\inf\limits_{\mathbf{G}\in\mathcal{G}}\mu_{22})\hat{e}_1^* + \sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{12}\hat{e}_2^*}{(1-\sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{11})(1-\sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{22}) - \sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{12}\sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{21}} \\ &\leq \frac{\hat{e}_1^* + \sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{12}\hat{e}_2^*}{(1-\sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{11})(1-\sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{22}) - \sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{12}\sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{21}}. \end{aligned}$$

In the same way by taking the supremum over the set $\mathcal{G}$ on the both sides of (5.14), it can be shown that

$$\sup_{\mathbf{G}\in\mathcal{G}} \hat{e}_2 \leq \frac{\hat{e}_2^* + \sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{21}\hat{e}_1^*}{(1-\sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{11})(1-\sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{22}) - \sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{12}\sup\limits_{\mathbf{G}\in\mathcal{G}}\mu_{21}}.$$

This completes the proof of the proposition. ☐

Proposition 5.2. *Suppose that* $\hat{u}_1^* < \infty$ *and* $\hat{u}_2^* < \infty$*. Let* $\nu_{11} < 1$*,* $\nu_{22} < 1$ *and* $\det(I - \mathbf{N}) > 0$ *for any* $\mathbf{G}(s) \in \mathcal{G}$*. Then the original criteria* (5.4) *for the system* (5.1) *are satisfied if the following* *inequalities hold.*

$$\left.\begin{array}{rcl}\dfrac{\hat{u}_1^* + \sup\limits_{\mathbf{G}\in\mathcal{G}} \nu_{12}\hat{u}_2^*}{(1-\sup\limits_{\mathbf{G}\in\mathcal{G}}\nu_{11})(1-\sup\limits_{\mathbf{G}\in\mathcal{G}}\nu_{22}) - \sup\limits_{\mathbf{G}\in\mathcal{G}}\nu_{12}\sup\limits_{\mathbf{G}\in\mathcal{G}}\nu_{21}} & \leq & \mathcal{U}_1 \\ \dfrac{\hat{u}_2^* + \sup\limits_{\mathbf{G}\in\mathcal{G}} \nu_{21}\hat{u}_1^*}{(1-\sup\limits_{\mathbf{G}\in\mathcal{G}}\nu_{11})(1-\sup\limits_{\mathbf{G}\in\mathcal{G}}\nu_{22}) - \sup\limits_{\mathbf{G}\in\mathcal{G}}\nu_{12}\sup\limits_{\mathbf{G}\in\mathcal{G}}\nu_{21}} & \leq & \mathcal{U}_2\end{array}\right\}. \tag{5.18}$$

Proof. Use Lemma 4.2 and the same technique used in Proposition 5.1. ☐

Define

$$\begin{aligned} A_{kl} &\triangleq \sup_{\mathbf{G}\in\mathcal{G}} \|z_{kl}\|_1,\ z_{kl} \triangleq g_{kl} - g_{kl}^*, \\ B_{kl} &\triangleq \sup_{\mathbf{G}\in\mathcal{G}} \{|z_{kl}(0)| + \|\dot{z}_{kl}\|_1\}. \end{aligned}$$

Upon noting the computational difficulty owing to supremal operations over the set $\mathcal{G}$, $\mathbf{M}$ is replaced by $\widetilde{\mathbf{M}} \triangleq [\tilde{\mu}_{ij}]_{2\times 2}$,

$$\tilde{\mu}_{ij} \triangleq \sum_{k=1}^{2}\sum_{l=1}^{2}\Big(A_{kl}|\bar{\sigma}_{lj}^{ik}| + B_{kl}||\sigma_{lj}^{ik} - \bar{\sigma}_{lj}^{ik}||_1\Big), \tag{5.19}$$

where σ_{lj}^{ik} is the unit-step response of $\big(K_{lj}X_{ik}\big)(s)$ and $\bar{\sigma}_{lj}^{ik}$ is the steady-state value of σ_{lj}^{ik}.

Now it is ready to state the main theorem on an upper bound for the error peak vector.

Theorem 5.1. *Suppose that* $\hat{e}_1^* < \infty$ *and* $\hat{e}_2^* < \infty$. *Let* $\tilde{\mu}_{11} < 1$, $\tilde{\mu}_{22} < 1$ *and* $\det(I - \widetilde{\mathbf{M}}) > 0$. *Then the original design criteria* (5.3) *for the actual system* (5.1) *are satisfied if the following inequalities hold.*

$$\frac{\hat{e}_1^* + \tilde{\mu}_{12}\hat{e}_2^*}{\det(I-\widetilde{\mathbf{M}})} \leq \mathcal{E}_1 \ \ \textit{and} \ \ \frac{\hat{e}_2^* + \tilde{\mu}_{21}\hat{e}_1^*}{\det(I-\widetilde{\mathbf{M}})} \leq \mathcal{E}_2. \tag{5.20}$$

Proof. It follows from (5.9) that

$$W_{ij}(s) = \sum_{k=1}^{2}\sum_{l=1}^{2} K_{lj}(s)X_{ik}(s)Z_{kl}(s).$$

Hence, $w_{ij}(t)$ is given by the input-output relation

$$\begin{aligned} w_{ij}(t) \;=\; & \sum_{k=1}^{2}\sum_{l=1}^{2}\Big\{z_{kl}(0)[\sigma_{lj}^{ik}(t) - \bar{\sigma}_{lj}^{ik}] + \\ & \int_0^t \dot{z}_{kl}(t-\tau)[\sigma_{lj}^{ik}(\tau) - \bar{\sigma}_{lj}^{ik}]d\tau + \bar{\sigma}_{lj}^{ik} z_{kl}(t)\Big\}. \end{aligned} \tag{5.21}$$

By a well-known result (see, for example, [13]) that

$$||x * y||_1 \leq ||x||_1||y||_1, \tag{5.22}$$

it follows from (5.7)–(5.9) and (5.11) and that

$$\begin{aligned} \sup_{\mathbf{G}\in\mathcal{G}} \mu_{ij} \;\leq\; & \sum_{k=1}^{2}\sum_{l=1}^{2}\Big\{\sup_{\mathbf{G}\in\mathcal{G}}\big[|z_{kl}(0)| + \|\dot{z}_{kl}\|_1\big]\|\sigma_{lj}^{ik} - \bar{\sigma}_{lj}^{ik}\|_1 \\ & + \sup_{\mathbf{G}\in\mathcal{G}}\|z_{kl}\|_1|\bar{\sigma}_{lj}^{ik}|\Big\}. \end{aligned} \tag{5.23}$$

Then it is easy to see from (5.19) that

$$\sup_{\mathbf{G}\in\mathcal{G}} \mu_{ij} \leq \tilde{\mu}_{ij}.$$

Therefore, by using the result in Proposition 5.1, if inequalities (5.20) hold, then the original design criteria (5.3) are satisfied. □

Similarly, the matix$\mathbf{N}$ is replaced by a majorant $\widetilde{\mathbf{N}} \triangleq [\tilde{\nu}_{ij}]_{2\times 2}$ where

$$\tilde{\nu}_{ij} = \sum_{k=1}^{2}\sum_{l=1}^{2}\left(A_{lj}|\bar{\sigma}_{ik}^{kl}| + B_{lj}||\sigma_{ik}^{kl} - \bar{\sigma}_{ik}^{kl}||_1\right). \tag{5.24}$$

The main theorem on the majorants for the control peak vector is stated as follows.

Theorem 5.2. *Suppose that* $\hat{u}_1^* < \infty$ *and* $\hat{u}_2^* < \infty$. *Let* $\tilde{\nu}_{11} < 1$, $\tilde{\nu}_{22} < 1$ *and* $\det(I - \widetilde{\mathbf{N}}) > 0$. *Then the original design criteria* (5.4) *for the actual system* (5.1) *are satisfied if the following inequalities hold.*

$$\frac{\hat{u}_1^* + \tilde{\nu}_{12}\hat{u}_2^*}{\det(I - \widetilde{\mathbf{N}})} \leq \mathcal{U}_1 \;\; \textit{and} \;\; \frac{\hat{u}_2^* + \tilde{\nu}_{21}\hat{u}_1^*}{\det(I - \widetilde{\mathbf{N}})} \leq \mathcal{U}_2. \tag{5.25}$$

Proof. Use the results in Proposition 5.2 and the technique used in Theorem 5.1. □

5.3 Finiteness of Approximation

Following the method of inequalities [30, 32, 33, 35], it is necessary that a search algorithm should start from a point $\mathbf{p} \in \mathbb{R}^N$ such that $\tilde{\mu}_{ij} < \infty$ and $\tilde{\nu}_{ij} < \infty$ for $i, j = 1, 2$. In this connection, the following lemmas provide useful sufficient conditions for ensuring that $\tilde{\mu}_{ij} < \infty$ and $\tilde{\nu}_{ij} < \infty$.

Let α_{ij}^{kl} denotes the abscissa of stability of the denominator of $K_{ij}(s)X_{kl}(s)$, which is defined by

$$\alpha_{ij}^{kl} \triangleq \max\{\text{Re } s : D_{ij}^{kl}(s) = 0\} \tag{5.26}$$

where $D_{ij}^{kl}(s)$ is the denominator of $K_{ij}(s)X_{kl}(s)$.

Lemma 5.3. *Assume that* $\mathbf{G}^*(s)$ *is a rational transfer matrix with time delay. Then* $\tilde{\mu}_{ij}$ *is finite if the following two conditions hold.*

(a) $||z_{kl}||_1 < \infty$ *for* $k, l = 1, 2$

(b) $\alpha_{lj}^{ik} < 0$ *for* $k, l = 1, 2$.

Proof. From the result in [1], it is known that retarded delay differential systems are BIBO stable if and only if $\alpha < 0$, where α denotes the abscissa of stability of the characteristic function $f(s)$, which is defined by

$$\alpha \triangleq \max\{\text{Re } s : f(s) = 0\}.$$

Condition (b) implies that the transfer function $X_{ij}(s)K_{kl}(s)$ is BIBO stable for all $k, l = 1, 2$. Then it follows that $|\bar{\sigma}_{lj}^{ik}| < \infty$ and $||\sigma_{lj}^{ik} - \bar{\sigma}_{lj}^{ik}||_1 < \infty$ for all $k, l = 1, 2$. Therefore, from the definition of $\tilde{\mu}_{ij}$ in (5.19), conditions (a) and (b) imply that $\tilde{\mu}_{ij} < \infty$. □

Lemma 5.4. *Assume that $\mathbf{G}^*(s)$ is a rational transfer matrix with time delay. Then $\tilde{\nu}_{ij}$ is finite if the following two conditions hold.*

(a) $||z_{lj}||_1 < \infty$ *for all* $l = 1, 2$

(b) $\alpha_{ik}^{kl} < 0$ *for all* $k, l = 1, 2.$

Proof. Use the definition of $\tilde{\nu}_{ij}$ in (5.24) and the same technique as in Lemma 5.3. □

Now it is ready to state the main theorem that provide useful sufficient conditions for ensuring that $\tilde{\mu}_{ij} < \infty$ and $\tilde{\nu}_{ij} < \infty$ for $i, j = 1, 2.$

Theorem 5.3. *Assume that $\mathbf{G}^*(s)$ is a rational transfer matrix with time delay. Then $\tilde{\mu}_{ij}$ and $\tilde{\nu}_{ij}$ are finite for all $i, j = 1, 2$ if the following two conditions hold.*

(a) $||z_{kl}||_1 < \infty$ *for all* $k, l = 1, 2$

(b) $\alpha_{kl}^{pq} < 0$ *for all* $k, l, p, q = 1, 2.$

Proof. It readily follows from Lemmas 5.3 and 5.4. □

From Theorems 5.1, 5.2 and 5.3, it readily follows that the solution of inequalities (5.3) and (5.4) involves three phases of computation as follows.

- Phase I : With an arbitrary starting point, find $\mathbf{p}_0$ satisfying

$$\alpha_{kl}^{pq}(\mathbf{p}_0) \le -\varepsilon \;\text{ for all } k, l, p, q = 1, 2 \tag{5.27}$$

  where $0 < \varepsilon \ll 1$ is given.

- Phase II : By starting from $\mathbf{p}_0$, find $\mathbf{p}_1$ satisfying (5.27) and

$$\left.\begin{array}{c} \alpha_{kl}^{pq}(\mathbf{p}_1) \le -\varepsilon \\ \tilde{\mu}_{11}(\mathbf{p}_1) < 1 \;\text{ and }\; \tilde{\mu}_{22}(\mathbf{p}_1) < 1 \\ \tilde{\nu}_{11}(\mathbf{p}_1) < 1 \;\text{ and }\; \tilde{\nu}_{22}(\mathbf{p}_1) < 1 \\ \det(I - \widetilde{\mathbf{M}}) > 0 \\ \det(I - \widetilde{\mathbf{N}}) > 0 \end{array}\right\} . \tag{5.28}$$

- Phase III : By starting from $\mathbf{p}_1$, find $\mathbf{p}$ satisfying (5.27), (5.28) and both the design criteria (5.20) and (5.25).

5.4 Numerical Example

In this section, the linearized model of a binary distillation column is obtained from linearization about an operating point [25]. The plant transfer matrix $\mathbf{G}(s)$ is given by

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{k_1 e^{-s}}{16.7s+1} & \dfrac{k_2 e^{-3s}}{as+1} \\ \dfrac{k_3 e^{-7s}}{10.9s+1} & \dfrac{k_4 e^{-2s}}{14.4s+1} \end{bmatrix}. \tag{5.29}$$

where $k_1 \in [11.6, 14.0]$, $k_2 \in [-20.8, -17.0]$, $k_3 \in [6.0, 7.2]$ and $k_4 \in [-21.4, -17.4]$.

Let the controller $\mathbf{K}(s, p)$ take the form

$$\mathbf{K}(s,p) = \begin{bmatrix} \dfrac{p_1(p_2 s+1)(p_3 s+1)}{s(p_4 s+1)} & p_9 \\ p_{10} & \dfrac{p_5(p_6 s+1)(p_7 s+1)}{s(p_8 s+1)} \end{bmatrix} \tag{5.30}$$

where $\mathbf{p} \triangleq [p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}]^T$ is a design parameter.

Assume the inputs f_1 belongs to the set $\mathcal{P}$ where

$$\mathcal{P} \triangleq \{f_1 : \|f_1\|_\infty \le 0.2 \ \text{ and } \ \|\dot{f}_1\|_\infty \le 0.2\}. \tag{5.31}$$

For simplicity, let $f_2 = 0$. It may be noted that in case that f_2 is not zero, one can use the principle of superposition to compute the peaks due to both inputs f_1 and f_2.

The main control objective is to ensure that, during the operation,

- the top product deviation e_1 stays within ± 0.50 mol%,
- the bottom product deviation e_2 stays within ± 0.30 mol%,
- the deviation of the reflux rate u_1 stays within ± 0.10 lb/min,
- the deviation of the reboiler rate u_2 stays within ± 0.20 lb/min.

Accordingly, the design criteria can be expressed as

$$\begin{aligned} \sup_{\mathbf{G}\in\mathcal{G}} \hat{e}_1 \le 0.50 \ \text{ and } \ \sup_{\mathbf{G}\in\mathcal{G}} \hat{e}_2 \le 0.30 \\ \sup_{\mathbf{G}\in\mathcal{G}} \hat{u}_1 \le 0.10 \ \text{ and } \ \sup_{\mathbf{G}\in\mathcal{G}} \hat{u}_2 \le 0.20. \end{aligned} \tag{5.32}$$

In our early study, we assume that all the parameters in (5.29) has uncertainties. It is found that the controller cannot be obtained by using inequalities (5.20) and (5.25) because we cannot find $\mathbf{p}$ such that the design criteria (5.28) are satisfied. Hence, for checking the effectiveness of the developed method, we have to know that the design problem has a solution or not. The design Section 5.4.1 are set up for checking the existence of a solution of the design problem.

5.4.1 Example 1: Design by varying the values of the parameter

For checking the existence of a solution, we will design the system with the plant (5.29) by varying the values of the parameters k_1, k_2, k_3 and k_4. The maximum value and the minimum value of each parameter are used to design for convenience. The values of the parameters in each case are shown in Tables 5.1 and 5.2.

Plant	$\mathbf{G}_1(s)$	$\mathbf{G}_2(s)$	$\mathbf{G}_3(s)$	$\mathbf{G}_4(s)$	$\mathbf{G}_5(s)$	$\mathbf{G}_6(s)$	$\mathbf{G}_7(s)$	$\mathbf{G}_8(s)$
k_1	11.6	11.6	11.6	11.6	11.6	11.6	11.6	11.6
k_2	−20.8	−20.8	−20.8	−20.8	−17.0	−17.0	−17.0	−17.0
k_3	6.0	6.0	7.2	7.2	6.0	6.0	7.2	7.2
k_4	−21.4	−17.4	−21.4	−17.4	−21.4	−17.4	−21.4	−17.4

Table 5.1: The values of uncertain parameters for $\mathbf{G}_1(s), \mathbf{G}_2(s), \ldots, \mathbf{G}_8(s)$.

Plant	$\mathbf{G}_9(s)$	$\mathbf{G}_{10}(s)$	$\mathbf{G}_{11}(s)$	$\mathbf{G}_{12}(s)$	$\mathbf{G}_{13}(s)$	$\mathbf{G}_{14}(s)$	$\mathbf{G}_{15}(s)$	$\mathbf{G}_{16}(s)$
k_1	14.0	14.0	14.0	14.0	14.0	14.0	14.0	14.0
k_2	−20.8	−20.8	−20.8	−20.8	−17.0	−17.0	−17.0	−17.0
k_3	6.0	6.0	7.2	7.2	6.0	6.0	7.2	7.2
k_4	−21.4	−17.4	−21.4	−17.4	−21.4	−17.4	−21.4	−17.4

Table 5.2: The values of uncertain parameters for $\mathbf{G}_9(s), \mathbf{G}_{10}(s), \ldots, \mathbf{G}_{16}(s)$.

Define $\hat{e}_1(\mathbf{G}_i, \mathbf{p}), \hat{e}_2(\mathbf{G}_i, \mathbf{p}), \hat{u}_1(\mathbf{G}_i, \mathbf{p})$ and $\hat{u}_2(\mathbf{G}_i, \mathbf{p})$ as peak values of e_1, e_2, u_1 and u_2, respectively, for $\mathbf{G}_i$, $i = 1, 2, \ldots, 16$. Now, according to the main control objective, the original design criteria (5.32) become

$$\left.\begin{array}{rclrcl} \hat{e}_1(\mathbf{G}_i,\mathbf{p}) & \le & 0.5, & \hat{e}_2(\mathbf{G}_i,\mathbf{p}) & \le & 0.3, \\ \hat{u}_1(\mathbf{G}_i,\mathbf{p}) & \le & 0.1, & \hat{u}_2(\mathbf{G}_i,\mathbf{p}) & \le & 0.2 \end{array}\right\}, \ i = 1, 2, \ldots, 16. \qquad (5.33)$$

In this example, a solution of the design inequalities (5.35) is obtained by using the MBP algorithm (see [34, 35] for the detail of the MBP algorithm). The peak values $\hat{e}_1(\mathbf{G}_i, \mathbf{p}), \hat{e}_2(\mathbf{G}_i, \mathbf{p})$, $\hat{u}_1(\mathbf{G}_i, \mathbf{p})$ and $\hat{u}_2(\mathbf{G}_i, \mathbf{p})$ associated with the possible set (5.31) are computed by the method developed in [24] where all the time responses use in the method are computed by employing the I_{MN} approximants [26, 28].

After a number of iterations, the MBP algorithm locates a design solution resulting in

$$\mathbf{K}(s,p) = \begin{bmatrix} \dfrac{0.1561s^2 + 0.1164s + 0.01969}{1.221s^2 + s} & 0.00257 \\ -0.00274 & -\left(\dfrac{0.7045s^2 + 0.1836s + 0.01144}{2.473s^2 + s}\right) \end{bmatrix}.$$

Plant	$\mathbf{G}_1(s)$	$\mathbf{G}_2(s)$	$\mathbf{G}_3(s)$	$\mathbf{G}_4(s)$	$\mathbf{G}_5(s)$	$\mathbf{G}_6(s)$	$\mathbf{G}_7(s)$	$\mathbf{G}_8(s)$
$\hat{e}_1(\mathbf{p})$	0.4228	0.4348	0.4145	0.4520	0.4458	0.4394	0.4283	0.4526
$\hat{e}_2(\mathbf{p})$	0.2140	0.1699	0.1989	0.1588	0.2336	0.1863	0.2168	0.1732
$\hat{u}_1(\mathbf{p})$	0.0614	0.0230	0.0531	0.0678	0.0735	0.0637	0.0664	0.0612
$\hat{u}_2(\mathbf{p})$	0.1138	0.1041	0.1134	0.1097	0.1241	0.1154	0.1244	0.1206

Table 5.3: The peak values $\hat{e}_1, \hat{e}_2, \hat{u}_1$ and $\hat{u}_2$ for the cases of $\mathbf{G}_1(s), \mathbf{G}_2(s), \ldots, \mathbf{G}_8(s)$.

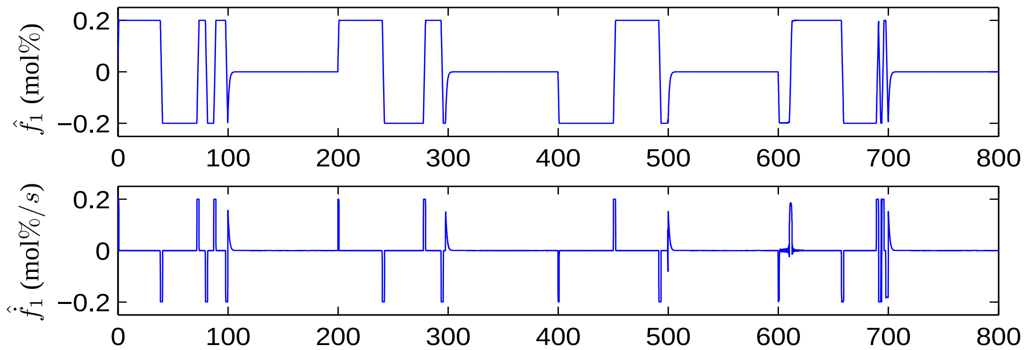

Figure 5.3: The waveforms of the test input $\hat{f}_1$ and its derivative.

The corresponding performance measures are shown in Table 5.3 and 5.4.

Plant	$\mathbf{G}_9(s)$	$\mathbf{G}_{10}(s)$	$\mathbf{G}_{11}(s)$	$\mathbf{G}_{12}(s)$	$\mathbf{G}_{13}(s)$	$\mathbf{G}_{14}(s)$	$\mathbf{G}_{15}(s)$	$\mathbf{G}_{16}(s)$
$\hat{e}_1(\mathbf{p})$	0.4409	0.4632	0.4367	0.4799	0.4764	0.4694	0.4624	0.4853
$\hat{e}_2(\mathbf{p})$	0.2265	0.1772	0.2090	0.1669	0.2473	0.1953	0.2165	0.1828
$\hat{u}_1(\mathbf{p})$	0.0659	0.0628	0.0602	0.0605	0.0746	0.0678	0.0688	0.0647
$\hat{u}_2(\mathbf{p})$	0.1188	0.1142	0.1195	0.1122	0.1293	0.1194	0.1291	0.1220

Table 5.4: The peak values $\hat{e}_1, \hat{e}_2, \hat{u}_1$ and $\hat{u}_2$ for the cases of $\mathbf{G}_9(s), \mathbf{G}_{10}(s), \ldots, \mathbf{G}_{16}(s)$.

From the results in Tables 5.3 and 5.4, we can see that the design has a solution. The maximum peak values of e_1, e_2, u_1 and u_2 are 0.4853 mol$\%$, 0.2473 mol$\%$, 0.0746 lb/min and 0.1291 lb/min, respectively.

To verify the design, a simulation is carried out for the case in which the control system is subjected to a test input $f = [\hat{f}_1, 0]^T$ where the magnitude and the slope of $\hat{f}_1$ satisfy (5.31).

The waveforms the responses e_1, e_2, u_1 and u_2 are displayed in Figure 5.4 for $k_1 \in [11.6, 14.0]$, $k_2 \in [-20.8, -17.0]$, $k_3 \in [6.0, 7.2]$ and $k_4 \in [-21.4, -17.4]$. The maximal magnitudes of e_1, e_2, u_1 and u_2 in response to $\hat{f}_1$ are 0.4759 mol$\%$, 0.2345 mol$\%$, 0.0688 lb/min and 0.1005 lb/min, respectively. Clearly, the design objectives are satisfied. Hence, we can conclude that the design problem has a solution.

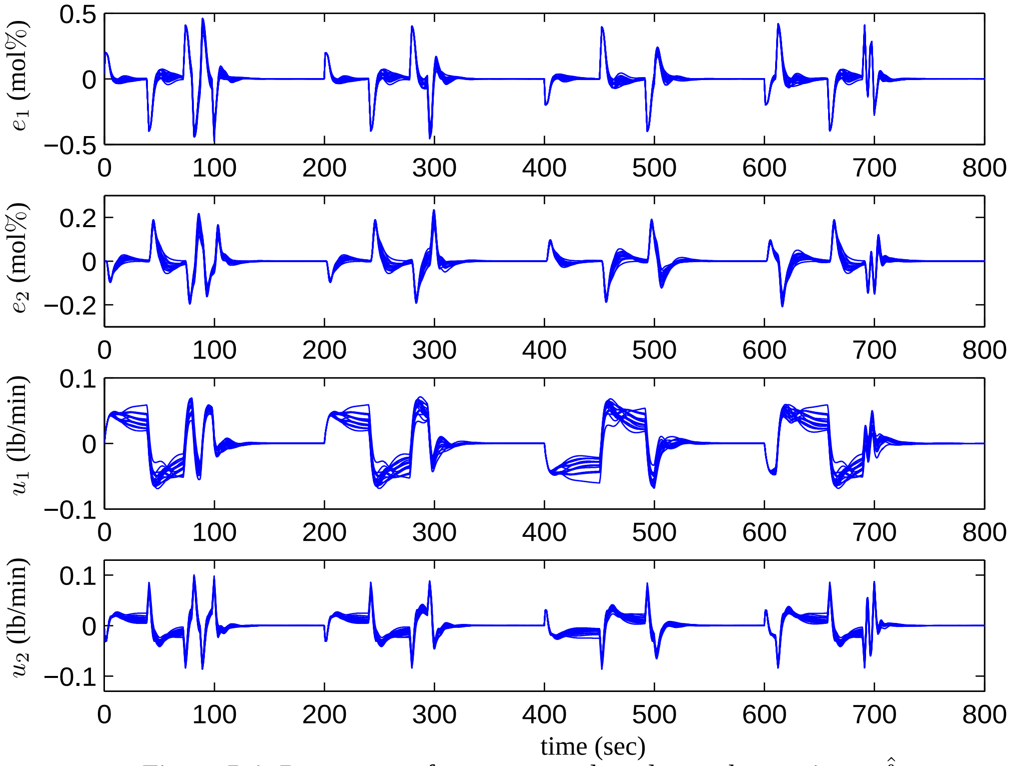

Figure 5.4: Responses of e_1, e_2, u_1 and u_2 due to the test input $\hat{f}_1$.

Now, because inequalities (5.20) and (5.25) in Theorems 5.1 and 5.2 cannot be used to solve the design problem, we will simplify the problem by reducing the plant uncertainties and relaxing the original design criteria (5.32) in Section 5.4.2.

5.4.2 Example 2: Design by using the theory of majorants

To reduce the plant uncertainties, the plant transfer matrix considered in this case is given by

$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{k_1 e^{-s}}{a_1 s + 1} & \dfrac{k_2 e^{-3s}}{a_2 s + 1} \\ \dfrac{k_3 e^{-7s}}{a_3 s + 1} & \dfrac{k_4 e^{-2s}}{a_4 s + 1} \end{bmatrix} \tag{5.34}$$

where the design is divided into four cases as follows.

- Case I: The transfer function $G_{11}(s)$ has parametric uncertainties. Parameter k_1 and a_1 vary 10% of their nominal values.

- Case II: The transfer function $G_{12}(s)$ has parametric uncertainties. Parameter k_2 and a_2 vary 10% of their nominal values.

- Case III: The transfer function $G_{21}(s)$ has parametric uncertainties. Parameter k_3 and a_3 vary 10% of their nominal values.

- Case IV: The transfer function $G_{22}(s)$ has parametric uncertainties. Parameter k_4 and a_4 vary 10% of their nominal values.

Parameter	Nominal Value	Case I	Case II	Case III	Case IV
k_1	12.8	$[11.6, 14]$	-	-	-
k_2	-18.9	-	$[-20.8, -17]$	-	-
k_3	6.6	-	-	$[6.0, 7.2]$	-
k_4	-19.4	-	-	-	$[-21.4, -17.4]$
a_1	16.7	$[15, 18.4]$	-	-	-
a_2	21	-	$[18.9, 23.1]$	-	-
a_3	10.9	-	-	$[9.8, 12]$	-
a_4	14.4	-	-	-	$[13.0, 15.8]$

Table 5.5: Nominal values and ranges of parameters for each case.

The nominal values and the ranges of the uncertain parameters in each case are shown in Table 5.5.

For this case, the control objectives is to ensure that, during the operation,

- the top product deviation e_1 stays within ± 1.0 mol$\%$,
- the bottom product deviation e_2 stays within ± 0.5 mol$\%$,
- the deviation of the reflux rate u_1 stays within ± 0.5 lb/min,
- the deviation of the reboiler rate u_2 stays within ± 0.2 lb/min.

Accordingly, the new design criteria can be expressed as

$$\begin{aligned} \sup_{\mathbf{G}\in\mathcal{G}} \hat{e}_1 \leq 1.00 \ \text{ and } \ \sup_{\mathbf{G}\in\mathcal{G}} \hat{e}_2 \leq 0.50 \\ \sup_{\mathbf{G}\in\mathcal{G}} \hat{u}_1 \leq 0.50 \ \text{ and } \ \sup_{\mathbf{G}\in\mathcal{G}} \hat{u}_2 \leq 0.20. \end{aligned} \tag{5.35}$$

Suppose that the plant transfer matrix $\mathbf{G}(s)$ is replaced by a fixed transfer matrix $\mathbf{G}^*(s)$ given by

$$\mathbf{G}^*(s) = \begin{bmatrix} \dfrac{12.8e^{-s}}{16.7s+1} & \dfrac{-18.9e^{-3s}}{21.0s+1} \\ \dfrac{6.6e^{-7s}}{10.9s+1} & \dfrac{-19.4e^{-2s}}{14.4s+1} \end{bmatrix}.$$

Following Theorems 5.1, 5.2 and 5.3, it is readily appreciated that the design problem is to find a value of $\mathbf{p}$ that satisfies

$$\alpha_{kl}^{pq}(\mathbf{p}) \leq -10^{-6}, \ \forall\, k, l, p, q = 1, 2 \tag{5.36}$$

$$\left.\begin{array}{ll} \tilde{\mu}_{11}(\mathbf{p}) \leq 0.5, & \tilde{\mu}_{22}(\mathbf{p}) \leq 0.5, \\ \tilde{\nu}_{11}(\mathbf{p}) \leq 0.5, & \tilde{\nu}_{22}(\mathbf{p}) \leq 0.5, \\ -\det(I - \widetilde{\mathbf{M}}) \leq -0.5, & -\det(I - \widetilde{\mathbf{N}}) \leq -0.5 \end{array}\right\} \tag{5.37}$$

$$\left.\begin{array}{rcl} \dfrac{(1-\tilde{\mu}_{22})\hat{e}_1^* + \tilde{\mu}_{12}\hat{e}_2^*}{\det(I-\widetilde{\mathbf{M}})} & \leq & 1.0 \text{ mol\%} \\ \dfrac{(1-\tilde{\mu}_{11})\hat{e}_2^* + \tilde{\mu}_{21}\hat{e}_1^*}{\det(I-\widetilde{\mathbf{M}})} & \leq & 0.5 \text{ mol\%} \end{array}\right\} \tag{5.38}$$

$$\left.\begin{array}{rcl} \dfrac{(1-\tilde{\nu}_{22})\hat{u}_1^* + \tilde{\nu}_{12}\hat{u}_2^*}{\det(I-\widetilde{\mathbf{N}})} & \leq & 0.5 \text{ lb/min} \\ \dfrac{(1-\tilde{\nu}_{11})\hat{u}_2^* + \tilde{\nu}_{21}\hat{u}_1^*}{\det(I-\widetilde{\mathbf{N}})} & \leq & 0.2 \text{ lb/min} \end{array}\right\}. \tag{5.39}$$

In each case, a solution of the design inequalities (5.38) and (5.39) is obtained by using the MBP algorithm. The peak values $\hat{e}_1^*(\mathbf{p}), \hat{e}_2^*(\mathbf{p}), \hat{u}_1^*(\mathbf{p})$ and $\hat{u}_2^*(\mathbf{p})$ associated with the possible set (5.31) are computed by the method developed in [24] where all the time responses in the design are computed by employing the I_{MN} approximants [26, 28].

To verify the design in each case, a simulation is carried out for the case in which the control system is subjected to a test input $f = [\hat{f}_1, 0]^T$ where $\hat{f}_1$ is the worst case input in which their magnitude and the slope of $\hat{f}_1$ satisfy (5.31).

- **Case I:** $G_{11}(s)$ **has parametric uncertainties.**

  After a number of iterations, the MBP algorithm locates a design solution resulting in

$$\mathbf{K}(s,p) = \begin{bmatrix} \dfrac{1.265s^2 + 0.303s + 0.01766}{9.141s^2 + s} & 0.0032 \\ -0.0038 & -\left(\dfrac{0.4234s^2 + 0.3051s + 0.02628}{0.002817s^2 + s}\right) \end{bmatrix}.$$

  The corresponding performance measures are

$$\tilde{\mu}_{11}(\mathbf{p}) = 0.3348, \qquad \tilde{\mu}_{12}(\mathbf{p}) = 0.0113,$$

$$\tilde{\mu}_{21}(\mathbf{p}) = 0.1976, \qquad \tilde{\mu}_{22}(\mathbf{p}) = 0.0027,$$

$$\tilde{\nu}_{11}(\mathbf{p}) = 0.3375, \qquad \tilde{\nu}_{12}(\mathbf{p}) = 0,$$

$$\tilde{\nu}_{21}(\mathbf{p}) = 0.3156, \qquad \tilde{\nu}_{22}(\mathbf{p}) = 0,$$

$$\det(I-\mathbf{M}) = 0.6612, \quad \det(I-\mathbf{N}) = 0.6625,$$

$$\frac{(1-\tilde{\mu}_{22})\hat{e}_1^* + \tilde{\mu}_{12}\hat{e}_2^*}{\det(I-\widetilde{\mathbf{M}})} = 0.5998 \text{ mol\%}, \quad \frac{(1-\tilde{\mu}_1)\hat{e}_2^* + \tilde{\mu}_{21}\hat{e}_1^*}{\det(I-\widetilde{\mathbf{M}})} = 0.2276 \text{ mol\%},$$

$$\frac{(1-\tilde{\nu}_{22})\hat{u}_1^* + \tilde{\nu}_{12}\hat{u}_2^*}{\det(I-\widetilde{\mathbf{N}})} = 0.0888 \text{ lb/min}, \quad \frac{(1-\tilde{\nu}_{11})\hat{u}_2^* + \tilde{\nu}_{21}\hat{u}_1^*}{\det(I-\widetilde{\mathbf{N}})} = 0.0874 \text{ lb/min}.$$

  From the simulation results, the waveform of the test input $\hat{f}_1$ and those of the responses e_1, e_2, u_1 and u_2 due to $\hat{f}_1$ in Case I are displayed in Figure 5.5. The maximal magnitudes of e_1, e_2, u_1 and u_2 in response to $\hat{f}_1$ are 0.3947 mol%, 0.0751 mol%, 0.0428 lb/min and 0.0237 lb/min, respectively. Clearly, the design objectives are satisfied.

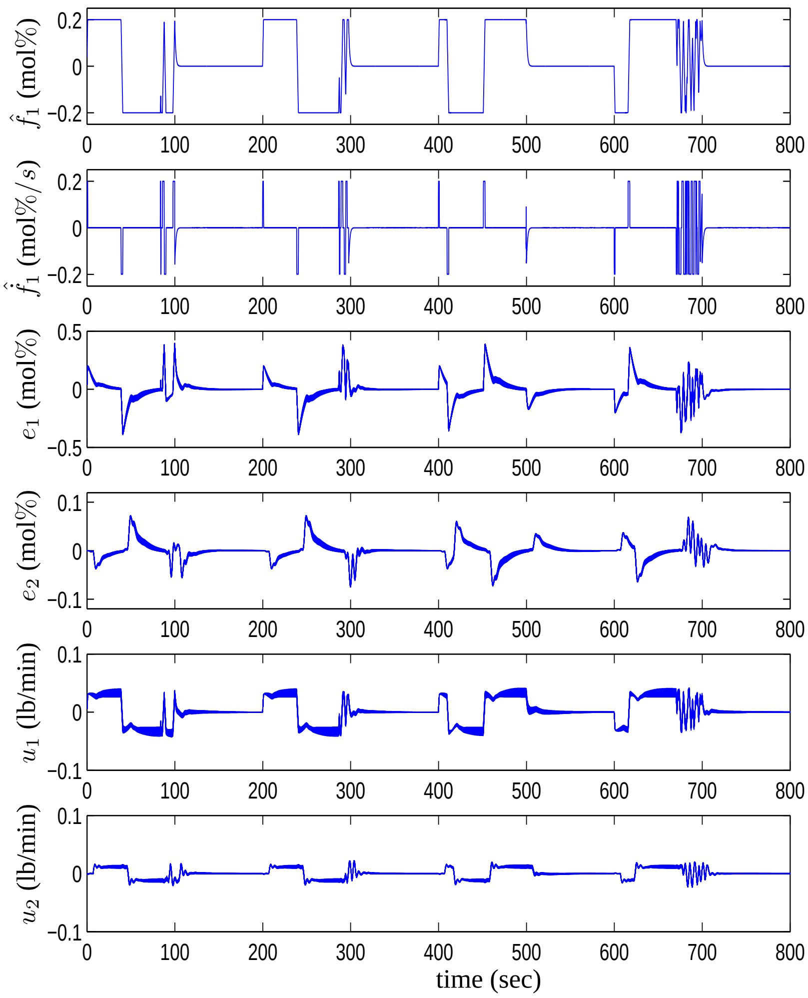

Figure 5.5: Responses of e_1, e_2, u_1 and u_2 due to the test input $\hat{f}_1$ for Case I.

- **Case II: $G_{12}(s)$ has parametric uncertainties.**

  After a number of iterations, the MBP algorithm locates a design solution resulting in

$$\mathbf{K}(s,p) = \begin{bmatrix} \dfrac{0.1561s^2 + 0.1164s + 0.01969}{1.221s^2 + s} & 0.00257 \\ -0.00274 & -\left(\dfrac{0.7045s^2 + 0.1836s + 0.01144}{2.473s^2 + s}\right) \end{bmatrix}.$$

  The corresponding performance measures are

$$\tilde{\mu}_{11}(\mathbf{p}) = 0.0099, \qquad \tilde{\mu}_{12}(\mathbf{p}) = 0.5576,$$

$$\tilde{\mu}_{21}(\mathbf{p}) = 0.0059, \qquad \tilde{\mu}_{22}(\mathbf{p}) = 0.3541,$$

$$\tilde{\nu}_{11}(\mathbf{p}) = 0, \qquad \tilde{\nu}_{12}(\mathbf{p}) = 0.5288,$$

$$\tilde{\nu}_{21}(\mathbf{p}) = 0, \qquad \tilde{\nu}_{22}(\mathbf{p}) = 0.3640,$$

$$\det(I - \mathbf{M}) = 0.6362, \quad \det(I - \mathbf{N}) = 0.6360,$$

$$\frac{(1-\tilde{\mu}_{22})\hat{e}_1^* + \tilde{\mu}_{12}\hat{e}_2^*}{\det(I-\widetilde{\mathbf{M}})} = 0.6864 \text{ mol\%}, \quad \frac{(1-\tilde{\mu}_{1})\hat{e}_2^* + \tilde{\mu}_{21}\hat{e}_1^*}{\det(I-\widetilde{\mathbf{M}})} = 0.1373 \text{ mol\%},$$
$$\frac{(1-\tilde{\nu}_{22})\hat{u}_1^* + \tilde{\nu}_{12}\hat{u}_2^*}{\det(I-\widetilde{\mathbf{N}})} = 0.0999 \text{ lb/min}, \quad \frac{(1-\tilde{\nu}_{11})\hat{u}_2^* + \tilde{\nu}_{21}\hat{u}_1^*}{\det(I-\widetilde{\mathbf{N}})} = 0.0195 \text{ lb/min}.$$

From the simulation results, the waveform of the test input $\hat{f}_1$ and those of the responses e_1, e_2, u_1 and u_2 due to $\hat{f}_1$ in Case II are displayed in Figure 5.6. The maximal magnitudes of e_1, e_2, u_1 and u_2 in response to $\hat{f}_1$ are 0.3931 mol%, 0.09007 mol%, 0.0350 lb/min and 0.0137 lb/min, respectively. Clearly, the design objectives are satisfied.

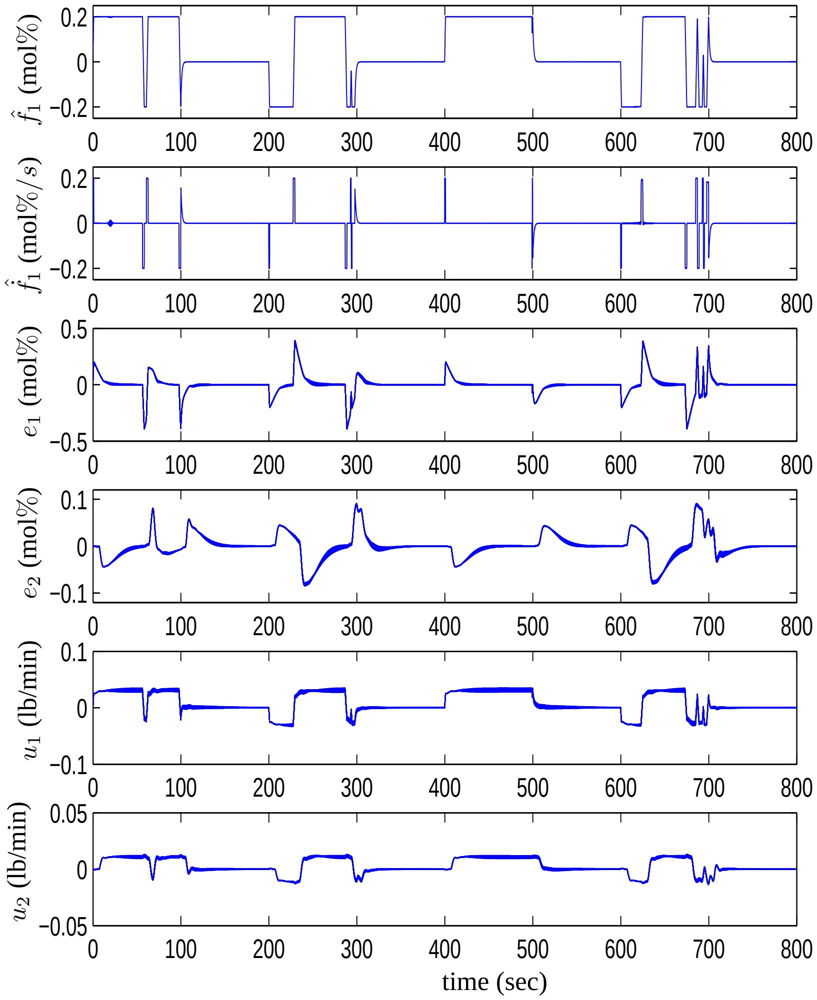

Figure 5.6: Responses of e_1, e_2, u_1 and u_2 due to the test input $\hat{f}_1$ for Case II.

- **Case III: $G_{21}(s)$ has parametric uncertainties.**

  After a number of iterations, the MBP algorithm locates a design solution resulting in

$$\mathbf{K}(s,p) = \begin{bmatrix} \dfrac{0.1561s^2 + 0.1164s + 0.01969}{1.221s^2 + s} & 0.00257 \\ -0.00274 & -\left(\dfrac{0.7045s^2 + 0.1836s + 0.01144}{2.473s^2 + s}\right) \end{bmatrix}.$$

The corresponding performance measures are

$$\tilde{\mu}_{11}(\mathbf{p}) = 0.4238, \qquad \tilde{\mu}_{12}(\mathbf{p}) = 0.0017,$$

$$\tilde{\mu}_{21}(\mathbf{p}) = 0.2365, \qquad \tilde{\mu}_{22}(\mathbf{p}) = 0.0016,$$

$$\tilde{\nu}_{11}(\mathbf{p}) = 0.4255, \qquad \tilde{\nu}_{12}(\mathbf{p}) = 0,$$

$$\tilde{\nu}_{21}(\mathbf{p}) = 0.6806, \qquad \tilde{\nu}_{22}(\mathbf{p}) = 0,$$

$$\det(I - \mathbf{M}) = 0.5748, \quad \det(I - \mathbf{N}) = 0.5745,$$

$$\frac{(1-\tilde{\mu}_{22})\hat{e}_1^* + \tilde{\mu}_{12}\hat{e}_2^*}{\det(I-\widetilde{\mathbf{M}})} = 0.6919 \text{ mol\%}, \quad \frac{(1-\tilde{\mu}_1)\hat{e}_2^* + \tilde{\mu}_{21}\hat{e}_1^*}{\det(I-\widetilde{\mathbf{M}})} = 0.2992 \text{ mol\%},$$

$$\frac{(1-\tilde{\nu}_{22})\hat{u}_1^* + \tilde{\nu}_{12}\hat{u}_2^*}{\det(I-\widetilde{\mathbf{N}})} = 0.1969 \text{ lb/min}, \quad \frac{(1-\tilde{\nu}_{11})\hat{u}_2^* + \tilde{\nu}_{21}\hat{u}_1^*}{\det(I-\widetilde{\mathbf{N}})} = 0.1826 \text{ lb/min}.$$

From the simulation results, the waveform of the test input $\hat{f}_1$ and those of the responses e_1, e_2, u_1 and u_2 due to $\hat{f}_1$ in Case III are displayed in Figure 5.7. The maximal magnitudes of e_1, e_2, u_1 and u_2 in response to $\hat{f}_1$ are 0.4060 mol%, 0.0903 mol%, 0.0477 lb/min and 0.0338 lb/min, respectively. Clearly, the design objectives are satisfied.

- **Case IV: $G_{22}(s)$ has parametric uncertainties.**

  After a number of iterations, the MBP algorithm locates a design solution resulting in

$$\mathbf{K}(s,p) = \begin{bmatrix} \dfrac{0.3362s^2 + 0.3391s + 0.07131}{0.5051s^2 + s} & 0.00111 \\ -0.00100 & -\left(\dfrac{0.1257s^2 + 0.0865s + 0.01464}{1.122s^2 + s}\right) \end{bmatrix}.$$

The corresponding performance measures are

$$\tilde{\mu}_{11}(\mathbf{p}) = 0.0016, \qquad \tilde{\mu}_{12}(\mathbf{p}) = 0.2343,$$

$$\tilde{\mu}_{21}(\mathbf{p}) = 0.0058, \qquad \tilde{\mu}_{22}(\mathbf{p}) = 0.4138,$$

$$\tilde{\nu}_{11}(\mathbf{p}) = 0, \qquad \tilde{\nu}_{12}(\mathbf{p}) = 1.1662,$$

$$\tilde{\nu}_{21}(\mathbf{p}) = 0, \qquad \tilde{\nu}_{22}(\mathbf{p}) = 0.4138,$$

$$\det(I - \mathbf{M}) = 0.5855, \quad \det(I - \mathbf{N}) = 0.5862,$$

$$\frac{(1-\tilde{\mu}_{22})\hat{e}_1^* + \tilde{\mu}_{12}\hat{e}_2^*}{\det(I-\widetilde{\mathbf{M}})} = 0.9792 \text{ mol\%}, \quad \frac{(1-\tilde{\mu}_1)\hat{e}_2^* + \tilde{\mu}_{21}\hat{e}_1^*}{\det(I-\widetilde{\mathbf{M}})} = 0.4915 \text{ mol\%},$$

$$\frac{(1-\tilde{\nu}_{22})\hat{u}_1^* + \tilde{\nu}_{12}\hat{u}_2^*}{\det(I-\widetilde{\mathbf{N}})} = 0.4955 \text{ lb/min}, \quad \frac{(1-\tilde{\nu}_{11})\hat{u}_2^* + \tilde{\nu}_{21}\hat{u}_1^*}{\det(I-\widetilde{\mathbf{N}})} = 0.0413 \text{ lb/min}.$$

From the simulation results, the waveform of the test input $\hat{f}_1$ and those of the responses e_1, e_2, u_1 and u_2 due to $\hat{f}_1$ in Case IV are displayed in Figure 5.8. The maximal magnitudes of e_1, e_2, u_1 and u_2 in response to $\hat{f}_1$ in Case IV are 0.5169 mol%, 0.2968 mol%, 0.1781 lb/min and 0.0254 lb/min, respectively. Clearly, the design objectives are satisfied.

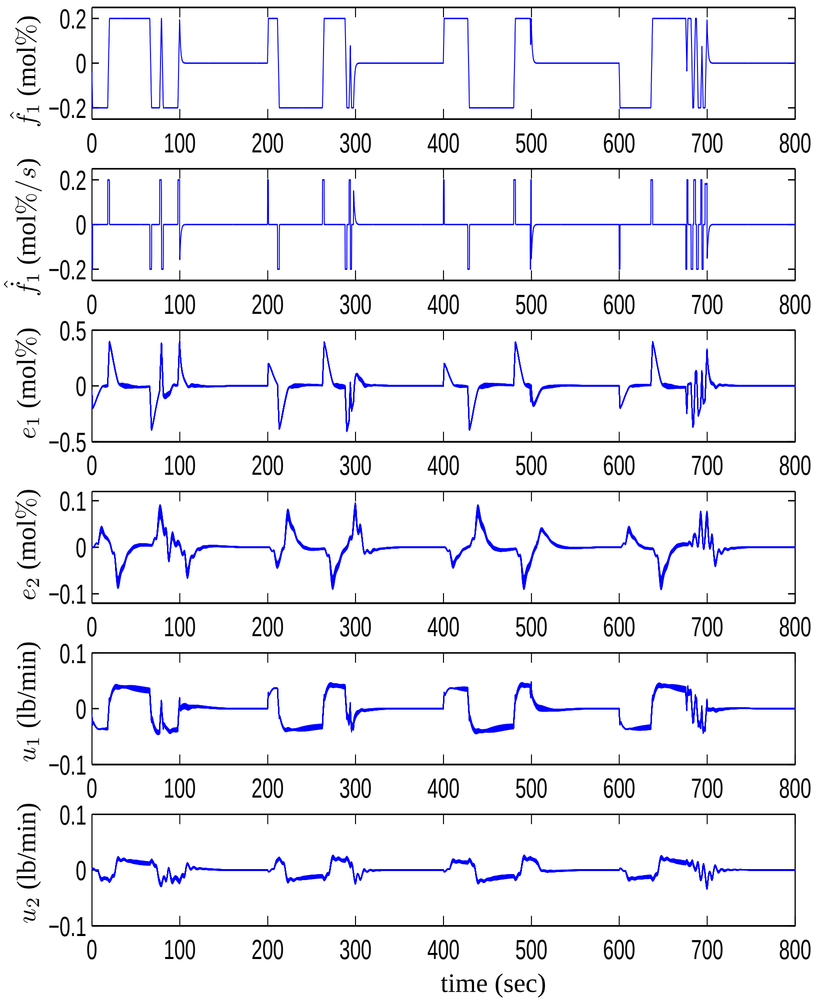

Figure 5.7: Responses of e_1, e_2, u_1 and u_2 due to the test input $\hat{f}_1$ for Case III.

From four cases of the design, it can be seen that the design by using inequalities (5.20) and (5.25) can give the satisfactory results for the original system. However, this method will be effective only in the case that the plant uncertainties are small enough. That is, the case that we can find $\mathbf{p}$ such that the design criteria (5.28) are satisfied.

Tables 5.6–5.9 show comparisons between the the upper bounds of $\hat{e}_1, \hat{e}_2, \hat{u}_1$ and $\hat{u}_2$ obtained from the design using inequalities (5.37), (5.38) and (5.39), and the maximal magnitudes of e_1, e_2, u_1 and u_2 obtained from the simulation results. We can see that there are reasonably differences between the upper bounds obtained from the design and the peak of the responses due to $\hat{f}_1$. This shows that the design inequalities have significant conservatism, which is caused by the derivation of the upper bounds $\hat{e}_i$ and $\hat{u}_i$.

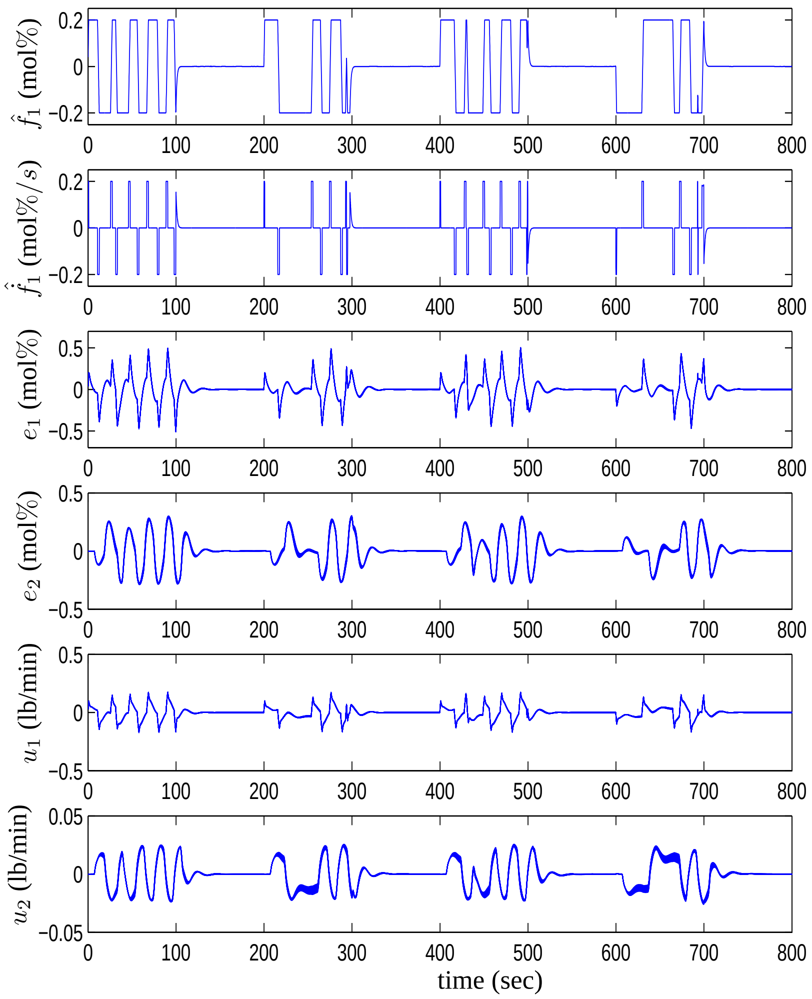

Figure 5.8: Responses of e_1, e_2, u_1 and u_2 due to the test input $\hat{f}_1$ for Case IV.

5.5 Conclusions and Discussion

This chapter presents the extension of Theory of majorants for SISO vague systems to the case of 2×2 vague systems by replacing an uncertain transfer matrix with a fixed transfer matrix. The principal design objective is to ensure that $e_1(t)$, $e_2(t)$, $u_1(t)$ and $u_2(t)$ stay within the ranges $\pm\mathcal{E}_1$, $\pm\mathcal{E}_2$, $\pm\mathcal{U}_1$ and $\pm\mathcal{U}_2$, respectively, for all time and for any possible input in the set $\mathcal{P}_1$ and $\mathcal{P}_2$ in spite of all uncertainties. The design problem is formulated using Zakian's theory of majorants [31] in conjunction with other theories in Zakian's framework [24,27,33,34] and can be expressed explicitly as a set of inequalities that can be solved in practice by numerical methods.

The numerical results in Section 5.4 may indicate that the design method using inequalities (5.20) and (5.25) is effective only when the number of plant uncertainties is small. However, Tables 5.6–5.9 show that the design inequalities have conservatism For this reason, it is interesting to develop, in the future, the design inequalities so that they can be used to design the system with the larger number of uncertainties or the conservatism can be reduced significantly in the design inequalities.

Response	**Peak value**	**Maximal magnitude**	**Difference ($\%$)**
e_1	0.5998	0.3947	$52.00\ \%$
e_2	0.2276	0.0751	$203.06\ \%$
u_1	0.0888	0.0428	$107.48\ \%$
u_2	0.0874	0.0237	$268.78\ \%$

Table 5.6: Comparison between upper the bounds of $\hat{e}_1, \hat{e}_2, \hat{u}_1$ and $\hat{u}_2$ and the maximal magnitudes of e_1, e_2, u_1 and u_2 due to $\hat{f}_1$ in Case I.

Response	**Peak value**	**Maximal magnitude**	**Difference ($\%$)**
e_1	0.6864	0.3931	$74.61\ \%$
e_2	0.1373	0.0901	$52.39\ \%$
u_1	0.0999	0.0350	$185.42\ \%$
u_2	0.0195	0.0137	$42.33\ \%$

Table 5.7: Comparison between upper the bounds of $\hat{e}_1, \hat{e}_2, \hat{u}_1$ and $\hat{u}_2$ and the maximal magnitudes of e_1, e_2, u_1 and u_2 due to $\hat{f}_1$ in Case II.

Response	**Peak value**	**Maximal magnitude**	**Difference ($\%$)**
e_1	0.6919	0.4060	$70.42\ \%$
e_2	0.2992	0.0903	$231.34\ \%$
u_1	0.1969	0.0477	$118.05\ \%$
u_2	0.1826	0.0338	$440.24\ \%$

Table 5.8: Comparison between the upper bounds of $\hat{e}_1, \hat{e}_2, \hat{u}_1$ and $\hat{u}_2$ and the maximal magnitudes of e_1, e_2, u_1 and u_2 due to $\hat{f}_1$ in Case III.

Response	**Peak value**	**Maximal magnitude**	**Difference ($\%$)**
e_1	0.9792	0.5169	$89.44\ \%$
e_2	0.4915	0.2968	$65.60\ \%$
u_1	0.4955	0.1781	$178.21\ \%$
u_2	0.0413	0.0254	$62.60\ \%$

Table 5.9: Comparison between the upper bounds of $\hat{e}_1, \hat{e}_2, \hat{u}_1$ and $\hat{u}_2$ and the maximal magnitudes of e_1, e_2, u_1 and u_2 due to $\hat{f}_1$ in Case IV.

CHAPTER VI

CONCLUSIONS

This thesis extends the theory of majorants, which consists of the criterion of approximation and majorants for vague systems, to the case of two-input two-output systems. The theory of majorants provides useful inequalities for designing feedback systems where the design objective is to ensure that the errors and the controller outputs of the systems always stay within their prescribed bounds whenever the inputs satisfy the magnitude and slope conditions.

The criterion of approximation for the SISO feedback systems is extended to the cases of two-input two-output feedback systems and then further extended to the case of MIMO feedback systems where non-rational transfer matrices are replaced by rational approximants during the design process so that reliable and efficient computational tools for rational systems can be fully utilized. Moreover, the criterion can be used with any types of non-rational systems whenever the impulse response matrix is obtained.

Based on the developed criterion, the theory of majorants for SISO vague systems is extended to the case of two-input two-output systems where the plant with uncertainties is replaced by a certain plant. The numerical examples show that the developed inequalities are effective when the number of parametric uncertainties is small enough. Furthermore, the search algorithm may fail to find a solution because the developed inequalities have conservatism.

To this end, from the numerical examples in Section 5.4, it is interesting to develop the inequalities for designing two-input two-output vague systems, so that they can be used effectively when the plant have more parametric uncertainties.

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APPENDIX

APPENDIX

The Moving-Boundaries-Process Algorithm [35] (see also [21]) Here we will describe the moving-boundaries process (MBP) algorithm, which has been used for solving the design inequalities in the numerical examples throughout the thesis.

Consider the design problem expressed in the form of inequalities

$$\phi_i(\mathbf{p}) \leq C_i \quad i = 1, 2, \ldots, m \tag{7.1}$$

where C_i are real numbers, $\mathbf{p}$ denotes a real vector $[p_1, p_2, \ldots, p_n]^T$ and ϕ_i are real functions of $\mathbf{p}$. Define the admissible set of the i^{th} inequality as

$$S_i \triangleq \{\mathbf{p} : \phi_i(\mathbf{p}) \leq C_i\}.$$

If there exists a point $\mathbf{p} \in \mathbb{R}^n$ that satisfies all the inequalities $\phi_i(\mathbf{p}) \leq C_i, i = 1, 2, \ldots, m$, then $\mathbf{p}$ is inside the set S defined by

$$S \triangleq \bigcap_{i=1}^{m} S_i.$$

The MBP algorithm is an iterative search, which proceed form an arbitrary initial point $\mathbf{p}_0$ to any point in set S. Let $\mathbf{p}^k$ denotes the value of $\mathbf{p}$ at the k^{th} iteration.

The MBP algorithm is stated as follows.

Algorithm 7.1. (Moving-boundary-process)

- *Initial step :* *Set $k = 0$ and choose an initial point $\mathbf{p}_0$. Then compute $\phi_i(\mathbf{p}), i = 1, 2, \ldots, m$.*
- *Step k :*
  - *(I) If $\phi_i(\mathbf{p}) \leq C_i, i = 1, 2, \ldots, m$, stop; otherwise generate a trial point $\tilde{\mathbf{p}}^k$.*
  - *(II) Compute $\phi_i(\tilde{\mathbf{p}}^k), i = 1, 2 \ldots, m$.*
  - *(III) If $\phi_i(\tilde{\mathbf{p}}^k) \leq \phi_i(\mathbf{p}^k), i = 1, 2, \ldots, m$, then:*
    - *(a) set $\mathbf{p}_{k+1} = \mathbf{p}^k$*
    - *(b) set $k = k + 1$*
    - *(c) go to step* (I)

    otherwise
    - *(a) generate another trial point $\tilde{\mathbf{p}}^k$*
    - *(b) go to step* (II).

In the thesis, Rosenbrock's method is used to generate the trial points $\tilde{\mathbf{p}}^k$. The detail of the method can be found in [22].

Biography

Tadchanon Chuman was born in Petchaburi, Thailand, in 1990. He received his Bachelor's degree in electrical engineering from Chulalongkorn University, in 2012. He has been granted a scholarship of the Honor Graduate Program for Electrical Engineering Students to pursue his Master's degree in electrical engineering at Chulalongkorn University, Thailand since 2012. He conducted his graduate study with the Control Systems Research Laboratory, Department of Electrical Engineering, Faculty of Engineering, Chulalongkorn University. His research interests include computer-aided control systems design by the method of inequalities and the principle of matching, and robust control.

List of Publications

1. T. Chuman and S. Arunsawatwong, "Application of Zakian's majorants to robust controller design for hydraulic force control systems," *Proc. 10th ECTI Conference*, Krabi, Thailand, (2013): 1–6.

2. T. Chuman, S. Arunsawatwong, "Criterion of approximation for designing 2×2 feedback systems with inputs satisfying bounding conditions," *Proc. 11th ECTI Conference*, Nakhon Ratchasima, Thailand, (2014). (accepted)