สมการเชิงฟังก์ชันของโคชีในโดเมนที่กำกัด

นายวัชรพล พิมพ์เสริฐ

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2546

ISBN 974-17-5382-9

ลิขสิทธิ์ของจุฬาลงกรณ์มหาวิทยาลัย

CAUCHY'S FUNCTIONAL EQUATION IN A RESTRICTED DOMAIN

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A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science in Mathematics
Department of Mathematics
Faculty of Science
Chulalongkorn University
Academic Year 2003
ISBN 974-17-5382-9

Thesis Title	Cauchy's functional equation in a restricted domain
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Field of Study	Mathematics
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วัชรพล พิมพ์เสริฐ : สมการเชิงฟังก์ชันของโคชีในโดเมนที่กำกัด (CAUCHY'S FUNCTIONAL EQUATION IN A RESTRICTED DOMAIN) อ.ที่ปรึกษา : อ. ดร.ศจี เพียรสกุล, อ.ที่ปรึกษาร่วม: รศ. ดร. วิเชียร เลาหโกศล, 35 หน้า. ISBN 974-17-5382-9.

ในปีค.ศ. 1964-1965 ปิโซและเชินเบิร์กทำการศึกษาที่สำคัญสองเรื่องเกี่ยวกับผลเฉลยของสมการเชิงฟังก์ชันของโคชีบนโดเมนที่กำกัดแน่นอน ซึ่งประกอบไปด้วยสมาชิกในรูปผลบวกเชิงเส้นของสมาชิกก่อกำเนิดในปริภูมิจำนวนจริง n มิติที่มีสัมประสิทธิ์เป็นจำนวนเต็ม สมาชิกก่อกำเนิดดังกล่าวสอดคล้องกับเงื่อนไขความเป็นอิสระเชิงเส้นสองข้อคือ เหนือจำนวนจริงและเหนือจำนวนตรรกยะ พบว่าผลเฉลยทั่วไปสามารถเขียนให้อยู่ในรูปผลบวกของฟังก์ชันเชิงเส้นและฟังก์ชันคาบได้

งานวิจัยนี้ได้ศึกษาผลเฉลยที่เป็นฟังก์ชันต่อเนื่องเอกรูปของสมการเชิงฟังก์ชันของโคชี ที่มีโดเมนเป็นเซตย่อยของฟีลด์จำนวนเชิงซ้อนที่ประกอบไปด้วยสมาชิกในรูปของผลบวกเชิงเส้นโดยมีสัมประสิทธิ์มาจากเซตย่อยของจำนวนเต็มเกาส์ ผลเฉลยที่ได้มีสมบัติใกล้เคียงกับผลเฉลยของปิโซและเชินเบิร์ก

วิธีการพิสูจน์ที่ใช้ ได้จากการวิเคราะห์วิธีการของปิโซและเชินเบิร์ก (1965) อย่างละเอียด โดยมีการปรับและเปลี่ยนเงื่อนไขเดิมของทฤษฎีบทที่ใช้ความอิสระเชิงเส้นของสมาชิกมาเป็น ความหนาแน่นของเซต

ภาควิชา **คณิตศาสตร์** ลายมือชื่อนิสิต……..………………................

สาขาวิชา **คณิตศาสตร์** ลายมือชื่ออาจารย์ที่ปรึกษา…………................

ปีการศึกษา 2546 ลายมือชื่ออาจารย์ที่ปรึกษาร่วม...……….................

4572483923 : MAJOR MATHEMATICS
KEY WORDS : FUNCTIONAL EQUATION / CAUCHY'S FUNCTIONAL EQUATION

WATCHARAPON PIMSERT : CAUCHY'S FUNCTIONAL EQUATION IN A RESTRICTED DOMAIN. THESIS ADVISOR : SAJEE PIANSKOOL, Ph.D., THESIS CO-ADVISOR : ASSOC. PROF. VICHIAN LAOHAKOSOL, Ph.D., 35 pp. ISBN 974-17-5382-9

In 1964-1965, Pisot and Schoenberg made two important studies on solutions of Cauchy's functional equation over certain restricted domains, consisting of linear integral combinations of generating elements in the real n-dimensional space. Such generating elements are subject to two independence conditions, one linear and the other rational. It was discovered that under suitable regularity on the functions, the most general solutions can be written as a linear function plus a periodic part.

In this thesis, we find uniformly continuous solutions of Cauchy's functional equation whose domain is a subset of the complex field comprising finite combinations over a subset of Gaussian integers. The solutions obtained are similar to those of Pisot and Schoenberg.

The proofs employed come from a detailed analysis of those used by Pisot and Schoenberg (1965) with a number of modification such as replacing some independence restriction by denseness.

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Department **Mathematics** Student's signature..............................
Field of study **Mathematics** Advisor's signature..............................
Academic year **2003** Co-advisor's signature..........................

ACKNOWLEDGEMENTS

I would like to express my profound gratitude and deep appreciation to Dr. Sajee Pianskool and Associate Professor Dr. Vichian Laohakosol, my thesis advisor and co-advisor, respectively, for their advice and encouragement. Sincere thanks and deep appreciation are also extended to Assistant Professor Dr. Patanee Udomkavanich, the chairman, and Assistant Professor Dr. Phichet Chaoha, a committee member, for their comments and suggestions. Also, I thank all teachers who have taught me all along.

In particular, I would like to express my deep gratitude to my parents, brother, sister, and friends for their encouragement throughout my graduate study.

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CHAPTER I

INTRODUCTION

An equation in which unknown are functions is called a *functional equation* and *Cauchy's functional equation* (CFE) is a functional equation, containing two variables x, y and one unknown function f of one variable, of the form $f(x+y) = f(x) + f(y)$. It is a generally known fact that the domain and the behavior of the solution function play important roles for obtaining a solution.

In the case that the domain of a solution f is the real n-dimensional space, $\mathbb{R}^n$, we know that the solution of CFE is a linear function under suitable regularity on the functions such as continuous at a point, bounded in an interval and monotone. There are certain other functional equations which can be transformed into a CFE. These are called *equations of Cauchy's type* which are of the following form (i) $f(x+y) = f(x)f(y)$, (ii) $f(xy) = f(x) + f(y)$ and (iii) $f(xy) = f(x)f(y)$.

In 1964, Pisot and Schoenberg investigated the monotone solution of Cauchy's functional equation $f(\sum_{i=1}^{k} u_i\alpha_i) = \sum_{i=1}^{k} f(u_i\alpha_i)$ (u_i are nonnegative integers) under various assumptions concerning the number k and the components $\alpha_i's$. In a successive paper, they considered the case where the domain of a solution of Cahchy's functional equation is a subset of $\mathbb{R}^n$, then the solution may not be a linear function.

Studying both of Pisot and Schoenberg's works, we find that the methods of proof are capable of extending to wider classes of functional equation. We treat here the CFE whose domain of solution is a subset of the complex field comprising finite combinations over a subset of Gaussian integers. We find that uniformly continuous solutions can be written as a linear function plus a periodic part.

In Chapter II, we introduce notation, definitions, auxiliary theorems used throughout this thesis.

In Chapter III, we solve for uniformly continuous solutions of CFE whose domain is a subset of the complex field. Uniformly continuous solutions of the three equations of Cauchy's type are also obtained as applications.

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CHAPTER II

PRELIMINARIES

The following symbols are standard:

$\mathbb{R}^n$ is the real n-dimensional space,

$\mathbb{C}$ is the complex field,

$\mathbb{R}$ is the set of all real numbers,

$\mathbb{Q}$ is the rational field,

$\mathbb{Z}$ is the set of all integers,

$\mathbb{N}$ is set of all natural numbers and

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2.1 Functional equation and Cauchy's functional equation (CFE)

In this section, we present definitions of functional equation and Cauchy's functional equation. Known general solutions of Cauchy's functional equation which satisfy some properties are given.

Definition 2.1.1. [**1**] An equation in which unknowns are functions is called a *functional equation.*

A function satisfying a functional equation on a given domain is called a *solution* of the equation on that domain.

Next, we give examples of a functional equation.

Example 2.1.2. *A solution* $f : \mathbb{Z} \to \mathbb{Z}$ *of the functional equation* $f(m + f(n)) = f(m) + n$ *is* $f(n) = n \quad (n \in \mathbb{Z})$.

Example 2.1.3. *A solution* $f : \mathbb{R} \to \mathbb{R}$ *of the functional equation* $f(x - f(y)) = 1 - x - y$ *is* $f(x) = \frac{1}{2} - x \quad (x \in \mathbb{R})$.

Definition 2.1.4. [1] A functional equation containing two variables x, y and one unknown function f of one variable is called a *Cauchy's functional equation* (abbreviated to CFE) if it is of the form $f(x + y) = f(x) + f(y)$.

The followings group of functional equations are called *equations of Cauchy's type* because they can be transformed into CFE by certain change of variables.

$$f(x + y) = f(x)f(y)$$

$$f(xy) = f(x) + f(y)$$

$$f(xy) = f(x)f(y).$$

For the first equation, if we let $g(x) = logf(x)$, then we obtain $g(x + y) = log(f(x + y)) = log(f(x)f(y)) = log(f(x)) + log(f(y)) = g(x) + g(y)$, so g satisfies Cauchy's functional equation.

Known results about CFE are contained in the following theorems

Theorem 2.1.5. [2] *The general solution of a Cauchy's functional equation in the class of functions* $g : \mathbb{R}^n \to \mathbb{R}^m$*, continuous at a point, is given by* $g(x) = C \cdot x$ *for all* $x \in \mathbb{R}^n$ *where* C *is a constant* m *by* n *matrix and '$\cdot$'denotes the multiplication of a vector by a matrix.*

Corollary 2.1.6. [5] *If the Cauchy's functional equation* $f(x + y) = f(x) + f(y)$ *is satisfied for all reals* x, y *and if the function is continuous at a point, then* $f(x) = cx$ *for all real* x *where* c *is an arbitrary real constant.*

Theorem 2.1.7. [2] *The general continuous solution of Cauchy's functional equation for complex numbers is* $f(x) = ax + b\bar{x}$ *for all* $x \in \mathbb{C}$ *where* a, b *are arbitrary complex constants and* $\bar{x}$ *is the conjugate of* x.

2.2 CFE on a subset of $\mathbb{R}^n$

In 1964-1965, Pisot and Schoenberg proved that for given $\alpha_1, \alpha_, \ldots, \alpha_m \in \mathbb{R}^n$, a uniformly continuous solution of the CFE defined on the subset $S = \left\{ \sum_{i=1}^m u_i\alpha_i \mid u_i \in \mathbb{N}_0 \right\}$ of $\mathbb{R}^n$ need not be a linear function. Their main result reads as follows:

Theorem 2.2.1. [4] *Let* $\alpha_1, \alpha_2, \ldots, \alpha_m$ *be elements of* $\mathbb{R}^n$ $(n < m)$ *satisfying the following conditions :*

1. *every set of* n *among the* α_i *is linearly independent over* $\mathbb{R}$ *and*
2. *the elements* $\alpha_1, \alpha_2, \ldots, \alpha_m$, *are rationally independent, i.e., linearly independent over* $\mathbb{Q}$.

Let $S = \left\{ \sum_{i=1}^{m} u_i\alpha_i \mid u_i \in \mathbb{N}_0 \right\}$, B *a Banach space and* f *a map from* S *into* B.

If f *is a solution of the functional equation :* $f\left(\sum_{i=1}^{m} u_i\alpha_i\right) = \sum_{i=1}^{m} f(u_i\alpha_i)$ *(*u_i*'s are arbitrary nonnegative integers) and is uniformly continuous on* S, *then* $f(x)$ *admits a unique representation of the form* $f(x) = \lambda(x) + \sum_{i=1}^{m} \varphi_i(x)$ *for all* $x \in S$ *in which* λ *is a linear function from* $\mathbb{R}^n$ *into* B, *while, for each* $i = 1, 2, \ldots, m$, φ_i *is a function from*

$$S_i = \left\{ u_i\alpha_i + \sum_{\substack{j=1 \\ j\neq i}}^{m} k_j\alpha_j \mid u_i \in \mathbb{N}_0 \text{ and } k_j \in \mathbb{Z} \right\}$$

into B *and satisfies the following conditions :*

1. $\varphi_i(0) = 0$,

2. $\varphi_i(x + \alpha_j) = \varphi_i(x)$ $(j \neq i, x \in S_i)$ *and*

3. φ_i *is a uniformly continuous function on* S_i.

Dirichlet's and Kronecker's Theorems were applied in many places in the proof of Theorem 2.2.1. Besides, these are important tools for our research.

Theorem 2.2.2 (Dirichlet's Theorem). [3] *For given real numbers* $\xi_1, \xi_2, \ldots, \xi_k$ *and any positive number* ε*, we can find an integer* q *so that* $q\xi_i$ *differs from an integer, for every* i*, by less than* ε.

Theorem 2.2.3 (Kronecker 's Theorem). [3] *For each irrational number* α*, each real number* β*, each preassigned arbitrarily small number* $\varepsilon > 0$*, and arbitrarily large number* Ω*, there exist integers* p *and* n *with* $|\, n\, | \geq \Omega$ *and* $|\ \alpha n - \beta - p\ | < \varepsilon$.

Definition 2.2.4. A subset D of the complex field, is *dense* in $\mathbb{C}$ (or $\mathbb{R}$) if for each $z \in \mathbb{C}$ (or $\mathbb{R}$) and $\varepsilon > 0$, there exists $d \in D$ such that $|d - z| < \varepsilon$.

Example 2.2.5. *The set* $\mathbb{Z} + \mathbb{Z}\sqrt{3}$ *is dense in* $\mathbb{R}$.

Proof. To prove this, let $r \in \mathbb{R}$ and ε be a positive real number. By Kronecker's Theorem, we have $n, p \in \mathbb{Z}$ such that $|n\sqrt{3} - p - r| < \varepsilon$, so $\mathbb{Z} + \mathbb{Z}\sqrt{3}$ is dense in $\mathbb{R}$. □

CHAPTER III

CFE ON A SUBSET OF THE COMPLEX FIELD

This chapter contains 4 sections. In the first section, we solve for uniformly continuous solutions of a CFE on the set

$$S^+ = \Big\{ \sum_{m=1}^{n} (u_m + iv_m)\alpha_m \mid u_m, v_m \in \mathbb{N}_0 \Big\}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ ($n \in \mathbb{N}$ with $n \geq 2$), with S^+ dense in $\mathbb{C}$. We derive a corollary similar to Corollary 2.1.7.

In the second section, we give examples of elements in $\mathbb{R}^n$ which satisfy the conditions in Theorem 3.1.1.

In the third section, we find uniformly continuous solutions for equations of Cauchy's type.

In the last section, we compare Theorem 3.1.1 with Theorem 2.2.1.

3.1 Solutions of Cauchy's functional equation

This section is devoted to finding uniformly continuous solutions of a CFE on the set S^+, dense in $\mathbb{C}$. The following is the main result.

Theorem 3.1.1. *Let* $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ ($n \in \mathbb{N}$ *with* $n \geq 2$) *be such that the set*

$$S^+ = \Big\{ \sum_{m=1}^{n} (u_m + iv_m)\alpha_m \mid u_m, v_m \in \mathbb{N}_0 \Big\}$$

is dense in $\mathbb{C}$. If $f : S^+ \to \mathbb{C}$ is a uniformly continuous function satisfying the functional equation:

$$f\Big(\sum_{m=1}^{n}(u_m + iv_m)\alpha_m\Big) = \sum_{m=1}^{n}\{f(u_m\alpha_m) + f(iv_m\alpha_m)\}, \tag{3.1}$$

then f can be written as

$$f(x) = \lambda(x) + \sum_{m=1}^{n}\varphi_m(x), \tag{3.2}$$

where $\lambda : \mathbb{C} \to \mathbb{C}$ is a linear function over $\mathbb{C}$ and φ_m is a complex valued function ($m = 1, 2, \ldots, n$) defined on

$$S_m = \Bigg\{(u_{ma} + iu_{mb})\alpha_m + \sum_{\substack{j=1\\ j\neq m}}^{n}(k_{ja} + ik_{jb})\alpha_j \mid u_{ma}, u_{mb} \in \mathbb{N}_0; k_{ja}, k_{jb} \in \mathbb{Z}\Bigg\}$$

with the following properties:

1. *$\varphi_m(0) = 0$,*
2. *$\varphi_m(x + \alpha_j) = \varphi_m(x + i\alpha_j) = \varphi_m(x)$ ($j \neq m$ and $x \in S_m$) and*
3. *φ_m is uniformly continuous on S_m.*

Proof. Let f be a uniformly continuous solution of (3.1). There are 3 steps in the proof. For the first step, we show that

$$\lim_{\substack{|u+iv|\to\infty \\ u+iv \text{ in certain region of } \mathbb{C}}} \frac{f\big((u + iv)\alpha_m\big)}{u + iv} \text{ exists}$$

for all $m = 1, 2, \ldots, n$. Then, we define a linear function $\lambda : \mathbb{C} \to \mathbb{C}$ over $\mathbb{C}$ and show that λ is uniformly continuous on S^+. For the final step, we construct the functions φ_m, $m = 1, 2 \ldots, n$.

Step 1 Let ε be a positive real number. Because of the uniform continuity of f, there exists $\delta > 0$ such that

$$\forall\, z, w \in S^+, |z - w| < \delta \Rightarrow |f(z) - f(w)| < \varepsilon. \tag{3.3}$$

Since S^+ is dense in $\mathbb{C}$, there are $x, y \in S^+$ such that $|x - y| < \delta$. Let $x = \sum_{m=1}^{n}(c_{ma} + ic_{mb})\alpha_m$ and $y = \sum_{m=1}^{n}(d_{ma} + id_{mb})\alpha_m$ where $c_{ma}, c_{mb}, d_{ma}, d_{mb} \in \mathbb{N}_0$ for all m. Set $q_{m\tau} = c_{m\tau} - d_{m\tau}$ where $m = 1, 2, \ldots, n$ and $\tau = a, b$ and split $\{1, 2, \ldots, n\}$ into two disjoint sets I and J, $I \cup J = \{1, 2, \ldots, n\}$.

We claim that, for each $M \in \mathbb{N}$,

$$\begin{aligned}\Big|\sum_{t\in I}\big\{f((c_{ta} + ic_{tb})\alpha_t) &- f((d_{ta} + id_{tb})\alpha_t)\big\}\\ &+\frac{1}{M}\sum_{j\in J}\eta_{ja}\big\{f((w_{ja} + M|q_{ja}|)\alpha_j) - f(w_{ja}\alpha_j)\big\}\\ &+\frac{1}{M}\sum_{j\in J}\eta_{jb}\big\{f(i(w_{jb} + M|q_{jb}|)\alpha_j) - f(iw_{jb}\alpha_j)\big\}\Big| < \varepsilon \end{aligned} \tag{3.4}$$

where w_{ja}, w_{jb} are any nonnegative integers and $\eta_{j\tau} = 1$ if $q_{j\tau} \geq 0$, $\eta_{j\tau} = -1$ if $q_{j\tau} < 0$ for $j \in J$ and $\tau = a$ or b.

To prove this inequality, let $j \in J$, $w_{ja}, w_{jb} \in \mathbb{N}_0$ and define $c_{j\tau}^{(k)}, d_{j\tau}^{(k)}$ for $\tau = a, b$ and $k \in \mathbb{N}$ as follows:

$$\begin{aligned}&\text{if } q_{j\tau} \geq 0, \text{ let} && c_{j\tau}^{(k)} = w_{j\tau} + kq_{j\tau}, && d_{j\tau}^{(k)} = w_{j\tau} + (k-1)q_{j\tau},\\ &\text{if } q_{j\tau} < 0, \text{ let} && c_{j\tau}^{(k)} = w_{j\tau} + (k-1)|q_{j\tau}|, && d_{j\tau}^{(k)} = w_{j\tau} + k|q_{j\tau}|.\end{aligned}$$

We see that $c_{m\tau}^{(k)}, d_{m\tau}^{(k)} \in \mathbb{N}_0$ and $q_{m\tau} = c_{m\tau}^{(k)} - d_{m\tau}^{(k)}$ for $k \in \mathbb{N}$, $\tau = a, b$ and $m = 1, 2, \ldots, n$.

For each $k \in \mathbb{N}$, we have

$$\sum_{t\in I}(c_{ta}+ic_{tb})\alpha_t+\sum_{j\in J}(c_{ja}^{(k)}+ic_{jb}^{(k)})\alpha_j \quad \text{and} \quad \sum_{t\in I}(d_{ta}+id_{tb})\alpha_t+\sum_{j\in J}(d_{ja}^{(k)}+id_{jb}^{(k)})\alpha_j$$

belong to S^+ and

$$\Big|\big\{\sum_{t\in I}(c_{ta}+ic_{tb})\alpha_t+\sum_{j\in J}(c_{ja}^{(k)}+ic_{jb}^{(k)})\alpha_j\big\}-\big\{\sum_{t\in I}(d_{ta}+id_{tb})\alpha_t+\sum_{j\in J}(d_{ja}^{(k)}+id_{jb}^{(k)})\alpha_j\big\}\Big|$$

$$=\Big|\sum_{t\in I}[(c_{ta}-d_{ta})+i(c_{tb}-d_{tb})]\alpha_t+\sum_{j\in J}[(c_{ja}^{(k)}-d_{ja}^{(k)})+i(c_{jb}^{(k)}-d_{jb}^{(k)})]\alpha_j\Big|$$

$$=\Big|\sum_{t\in I}(q_{ta}+iq_{tb})\alpha_t+\sum_{j\in J}(q_{ja}+iq_{jb})\alpha_j\Big|$$

$$=\Big|\sum_{m=1}^{n}(q_{ma}+iq_{mb})\alpha_m\Big|$$

$$=\Big|\sum_{m=1}^{n}[(c_{ma}-d_{ma})+i(c_{mb}-d_{mb})]\alpha_m\Big|$$

$$=\Big|\sum_{m=1}^{n}(c_{ma}+ic_{mb})\alpha_m-\sum_{m=1}^{n}(d_{ma}+id_{mb})\alpha_m\Big|$$

$$=|x-y|<\delta,$$

which, by (3.3), yields

$$\varepsilon>\Big|f\big(\sum_{t\in I}(c_{ta}+ic_{tb})\alpha_t+\sum_{j\in J}(c_{ja}^{(k)}+ic_{jb}^{(k)})\alpha_j\big)-f\big(\sum_{t\in I}(d_{ta}+id_{tb})\alpha_t+\sum_{j\in J}(d_{ja}^{(k)}+id_{jb}^{(k)})\alpha_j\big)\Big|$$

$$=\Big|\sum_{t\in I}f\big((c_{ta}+ic_{tb})\alpha_t\big)+\sum_{j\in J}f\big((c_{ja}^{(k)}+ic_{jb}^{(k)})\alpha_j\big)$$
$$-\sum_{t\in I}f\big((d_{ta}+id_{tb})\alpha_t\big)-\sum_{j\in J}f\big((d_{ja}^{(k)}+id_{jb}^{(k)})\alpha_j\big)\Big|$$

$$=\Big|\sum_{t\in I}\big\{f((c_{ta}+ic_{tb})\alpha_t)-f((d_{ta}+id_{tb})\alpha_t)\big\}+\sum_{j\in J}\big\{f((c_{ja}^{(k)}+ic_{jb}^{(k)})\alpha_j)-f((d_{ja}^{(k)}+id_{jb}^{(k)})\alpha_j)\big\}\Big|$$

Letting $k = 1, 2, \ldots, M$ and forming the arithmetic mean of the M quantities, we obtain the desired inequality (3.4).

Let $E = \sum_{t \in I} \big\{ f((c_{ta} + ic_{tb})\alpha_t) - f((d_{ta} + id_{tb})\alpha_t) \big\}$. There are 4 cases correspond to possible choices of q_{ja} and q_{jb} to be considered.

Case 1 $q_{ja},\, q_{jb} \geq 0$.

If $k = 1$, then

$$\Big| E + \sum_{j \in J} \Big\{ f\big((w_{ja} + q_{ja})\alpha_j\big) + f\big(i(w_{jb} + q_{jb})\alpha_j\big) - f\big((w_{ja} + 0 \cdot q_{ja})\alpha_j\big) - f\big(i(w_{jb} + 0 \cdot q_{jb})\alpha_j\big) \Big\} \Big| < \varepsilon$$

If $k = 2$, then

$$\Big| E + \sum_{j \in J} \Big\{ f\big((w_{ja} + 2q_{ja})\alpha_j\big) + f\big(i(w_{jb} + 2q_{jb})\alpha_j\big) - f\big((w_{ja} + q_{ja})\alpha_j\big) - f\big(i(w_{jb} + q_{jb})\alpha_j\big) \Big\} \Big| < \varepsilon$$

$$\vdots$$

In general, if $k = M$, then

$$\Big| E + \sum_{j \in J} \Big\{ f\big((w_{ja} + Mq_{ja})\alpha_j\big) + f\big(i(w_{jb} + Mq_{jb})\alpha_j\big) - f\big((w_{ja} + (M-1)q_{ja})\alpha_j\big) - f\big(i(w_{jb} + (M-1)q_{jb})\alpha_j\big) \Big\} \Big| < \varepsilon$$

Summing all the M inequalities, multiplying by $\frac{1}{M}$ and using the triangle inequality, we have

$$\varepsilon > \Big| E + \frac{1}{M} \sum_{j \in J} \Big\{ f\big((w_{ja} + Mq_{ja})\alpha_j\big) + f\big(i(w_{jb} + Mq_{jb})\alpha_j\big) - f(w_{ja}\alpha_j) - f(iw_{jb}\alpha_j) \Big\} \Big|$$

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$$= \big|E+\frac{1}{M}\sum_{j\in J}\big\{f\big((w_{ja}+Mq_{ja})\alpha_j\big)-f(w_{ja}\alpha_j)\big\}+\frac{1}{M}\sum_{j\in J}\big\{f\big(i(w_{jb}+Mq_{jb})\alpha_j\big)-f(iw_{jb}\alpha_j)\big\}\big|.$$

Applying the same technique to other cases, i.e., case $q_{ja}, q_{jb} < 0$, case $q_{ja} \geq 0, q_{jb} < 0$ and case $q_{ja} < 0, q_{jb} \geq 0$, the inequality (3.4) is similarly proved.

For fixed m, since $-(1+i) \in \mathbb{C}$ and S^+ is dense in $\mathbb{C}$, there exist $a_1, b_1, a_2, b_2, \ldots, a_n$, $b_n \in \mathbb{N}_0$ such that $\Big|\sum_{k=1}^{n}(a_k + ib_k)\alpha_k - (-(1+i)\alpha_m)\Big| < \delta$,
i.e.,

$$\Big|\sum_{\substack{k=1\\k\neq m}}^{n}(a_k + ib_k)\alpha_k + [(a_m+1) + i(b_m+1)]\alpha_m\Big| < \delta.$$

Taking $q_{ma} = a_m + 1 \geq 1 > 0$ and $q_{mb} = b_m + 1 \geq 1 > 0$ shows that for each m, there exist $q_{ka} = a_k \in \mathbb{N}_0,\ q_{kb} = b_k \in \mathbb{N}_0\ (k = 1, 2, \ldots, n, k \neq m)$ and that $q_{ma}, q_{mb} \in \mathbb{N}$ such that

$$|(q_{1a} + iq_{1b})\alpha_1 + \ldots + (q_{na} + iq_{nb})\alpha_n| < \delta.$$

From (3.4), (3.1) with $x = (q_{1a} + iq_{1b})\alpha_1 + \cdots + (q_{na} + iq_{nb})\alpha_n,\ y = 0,\ J = \{m\}$ and $I = \{1, 2, \ldots, n\}\smallsetminus\{m\}$, we have

$$\begin{aligned}\varepsilon > \Big|\sum_{t\in I}\big\{f(q_{ta}\alpha_t) + f(iq_{tb}\alpha_t) - f(0) - f(0)\big\}&\\ +\frac{1}{M}\big\{f((w_{ma} + Mq_{ma})\alpha_m) - f(w_{ma}\alpha_m)\big\}&\\ +\frac{1}{M}\big\{f(i(w_{mb} + Mq_{mb})\alpha_m) - f(iw_{mb}\alpha_m)\big\}\Big|&\end{aligned}$$

$$\begin{aligned}= \Big|\sum_{t\in I} f[(q_{ta} + iq_{tb})\alpha_t] + \frac{1}{M}f[((w_{ma} + iw_{mb}) + M(q_{ma} + iq_{mb}))\alpha_m]&\\ -\frac{1}{M}f[(w_{ma} + iw_{mb})\alpha_m]\Big|.&\qquad (3.5)\end{aligned}$$

Consider the discrete strip

$$St(m) = \big\{(w_{ma}+iw_{mb})+M(q_{ma}+iq_{mb}) \mid M \in \mathbb{N}_0\ ,\ 0 \le w_{ma} < q_{ma} \text{ and } 0 \le w_{mb} < q_{mb}\big\}.$$

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Consider the sequence of values $\dfrac{f\big((u+iv)\alpha_m\big)}{u+iv}$, where $u+iv$ runs through $St(m)$. Let L be one of the limits of the sequence (e.g. *lim sup* or *lim inf*). For $u+iv \in St(m)$, letting $M \to \infty$ in (3.5), we have

$$\begin{aligned}\lim_{M\to\infty}\Big|\sum_{t\in I} & f[(q_{ta}+iq_{tb})\alpha_t] \\ & + \left\{\frac{(w_{ma}+iw_{mb})+M(q_{ma}+iq_{mb})}{M}\right\}\left\{\frac{f[((w_{ma}+iw_{mb})+M(q_{ma}+iq_{mb}))\alpha_m]}{(w_{ma}+iw_{mb})+M(q_{ma}+iq_{mb})}\right\} \\ & - \frac{1}{M}f[(w_{ma}+iw_{mb})\alpha_m]\Big| \le \varepsilon,\end{aligned}$$

i.e.,

$$\Big|\sum_{t\in I} f[(q_{ta}+iq_{tb})\alpha_t] + (q_{ma}+iq_{mb})L\Big| \le \varepsilon. \tag{3.6}$$

We now show that L is unique. Suppose that L_1, L_2 are any two limits of the sequence.

Let $E' = \sum_{t \in I} f[(q_{ta} + iq_{tb})\alpha_t]$, from(3.6) we obtain

$$\begin{aligned} 2\varepsilon = \varepsilon + \varepsilon &\geq \Big| E' + (q_{ma} + iq_{mb})L_1 \Big| + \Big| -1 \Big| \Big| E' + (q_{ma} + iq_{mb})L_2 \Big| \\ &\geq \Big| E' - E' + (q_{ma} + iq_{mb})L_1 - (q_{ma} + iq_{mb})L_2 \Big| \\ &= \Big| q_{ma} + iq_{mb} \Big| \Big| L_1 - L_2 \Big|. \end{aligned}$$

Since ε is arbitrary, $L_1 = L_2$, and Step 1 is complete.

Step 2 We define $\lambda : \mathbb{C} \to \mathbb{C}$ as follows: for each $x \in \mathbb{C}$,

we write

$$x = \sum_{m=1}^{n} (x_{ma} + ix_{mb})\alpha_m,$$

where $x_{ma}, x_{mb} \in \mathbb{R}$ and let

$$\lambda(x) = \sum_{m=1}^{n} (x_{ma} + ix_{mb})\lambda_m.$$

We show that

(a) λ is a function and

(b) λ is linear over $\mathbb{C}$.

To prove (a), it suffices to show that if

$$0 = \sum_{m=1}^{n} (x_{ma} + ix_{mb})\alpha_m$$

where $x_{ma}, x_{mb} \in \mathbb{R}$, then

$$\lambda(0) = \sum_{m=1}^{n} (x_{ma} + ix_{mb})\lambda_m = 0.$$

There are two cases to consider:

Case(I) : $x_{m\tau} = 0$ for all $m = 1, 2, \ldots, n$; for all $\tau = a, b$. This case is trivial.

Case(II) : There exists $x_{m'\tau'} \neq 0$ for some $m' \in \{1, 2, \ldots, n\}$ and $\tau' \in \{a, b\}$.

From Dirichlet's Theorem, for each $\nu \in \mathbb{N}$, there are $t^{(\nu)}, k_{m\tau}^{(\nu)} \in \mathbb{Z}$ with $t^{(\nu)} > 0$ such that

$$\left|t^{(\nu)} x_{m\tau} - k_{m\tau}^{(\nu)}\right| < \frac{1}{\nu} \qquad (m = 1, 2, \ldots, n\,; \tau = a, b). \tag{3.7}$$

We let $k_{m\tau}^{(\nu)} = 0$ if $x_{m\tau} = 0$. Note that we may choose $t^{(\nu)}$ such that $t^{(\nu)} \to \infty$ as $\nu \to \infty$, so that $k_{m\tau}^{(\nu)} \to \infty$ as $\nu \to \infty$ too. We see that

$$\begin{aligned}
\Big|\sum_{m=1}^{n}(k_{ma}^{(\nu)} + ik_{mb}^{(\nu)})\alpha_m\Big| &= \Big|\sum_{m=1}^{n}(k_{ma}^{(\nu)} + ik_{mb}^{(\nu)})\alpha_m - t^{(\nu)}(\sum_{m=1}^{n}(x_{ma}^{(\nu)} + ix_{mb}^{(\nu)})\alpha_m\Big| \\
&= \Big|\sum_{m=1}^{n}\{(k_{ma}^{(\nu)} - t^{(\nu)}x_{ma}) + i(k_{mb}^{(\nu)} - t^{(\nu)}x_{mb})\}\alpha_m\Big| \\
&\leq \sum_{m=1}^{n}\left|k_{ma}^{(\nu)} - t^{(\nu)}x_{ma}\right|\left|\alpha_m\right| + \sum_{m=1}^{n}\left|i\right|\left|k_{mb}^{(\nu)} - t^{(\nu)}x_{mb}\right|\left|\alpha_m\right| \\
&\leq \frac{1}{\nu}\sum_{m=1}^{n}\left|\alpha_m\right| + \frac{1}{\nu}\sum_{m=1}^{n}\left|\alpha_m\right| \\
&\leq \frac{2}{\nu}\sum_{m=1}^{n}\left|\alpha_m\right|.
\end{aligned}$$

Then

$$\lim_{\nu\to\infty}\left|\sum_{m=1}^{n}(k_{ma}^{(\nu)} + ik_{mb}^{(\nu)})\alpha_m\right| = 0. \tag{3.8}$$

Next, we show that

$$\lim_{\nu\to\infty}\frac{k_{m\tau}^{(\nu)}}{k_{m'\tau'}^{(\nu)}} = \frac{x_{m\tau}}{x_{m'\tau'}}.$$

From (3.7), for each $m = 1, 2, \ldots, n; \tau = a, b$ we have

$$-\frac{1}{\nu} < k_{m\tau}^{(\nu)} - t^{(\nu)} x_{m\tau} < \frac{1}{\nu},$$

i.e.,

$$t^{(\nu)} x_{m\tau} - \frac{1}{\nu} < k_{m\tau}^{(\nu)} < t^{(\nu)} x_{m\tau} + \frac{1}{\nu}.$$

Note that

- if $x_{m\tau} > 0$, then $t^{(\nu)} x_{m\tau} - \frac{1}{\nu} > 0$ for sufficiently large ν and
- if $x_{m\tau} < 0$, then $t^{(\nu)} x_{m\tau} + \frac{1}{\nu} < 0$ for sufficiently large ν.

Hence, for large ν,

$$\frac{t^{(\nu)} x_{m\tau} - \frac{1}{\nu}}{t^{(\nu)} x_{m'\tau'} + \frac{1}{\nu}} < \frac{k_{m\tau}^{(\nu)}}{k_{m'\tau'}^{(\nu)}} < \frac{t^{(\nu)} x_{m\tau} + \frac{1}{\nu}}{t^{(\nu)} x_{m'\tau'} - \frac{1}{\nu}}.$$

Thus

$$\lim_{\nu \to \infty} \frac{k_{m\tau}^{(\nu)}}{k_{m'\tau'}^{(\nu)}} = \frac{x_{m\tau}}{x_{m'\tau'}}. \tag{3.9}$$

If $x_{m\tau} > 0$, we choose ν such that $t^{(\nu)} x_{m\tau} > 1$, so $k_{m\tau}^{(\nu)} > 1 > 0$, and if $x_{m\tau} < 0$, we choose ν such that $t^{(\nu)} x_{m\tau} > -1$, so $k_{m\tau}^{(\nu)} < -1 < 0$. Then $\operatorname{sgn} k_{m\tau}^{(\nu)} = \operatorname{sgn} x_{m\tau}$ for sufficiently large ν when $m = 1, 2, \ldots, n$.

Let

$$U_\tau^+ = \{m \mid x_{m\tau} > 0\} = \{m \mid \operatorname{sgn} x_{m\tau} > 0\} \quad \text{and}$$
$$U_\tau^- = \{m \mid x_{m\tau} < 0\} = \{m \mid \operatorname{sgn} x_{m\tau} < 0\} \quad (\tau = a, b).$$

Rewriting (3.8) as

$$\lim_{\nu \to \infty} \Big| \big(\sum_{t \in U_a^+} k_{ta}^{(\nu)} \alpha_t + \sum_{t \in U_b^+} i k_{tb}^{(\nu)} \alpha_t \big) - \big(\sum_{s \in U_a^-} |k_{sa}^{(\nu)}| \alpha_s + \sum_{s \in U_b^-} i |k_{sb}^{(\nu)}| \alpha_s \big) \Big| = 0,$$

by uniform continuity and (3.1), we have

$$\lim_{\nu\to\infty}\Big|\Big\{\sum_{t\in U_a^+} f(k_{ta}^{(\nu)}\alpha_t)+\sum_{t\in U_b^+} f(ik_{tb}^{(\nu)}\alpha_t)\Big\}-\Big\{\sum_{s\in U_a^-} f(|k_{sa}^{(\nu)}|\alpha_s)+\sum_{s\in U_b^-} f(i|k_{sb}^{(\nu)}|\alpha_s)\Big\}\Big|=0$$

so that

$$\lim_{\nu\to\infty}\Big|\sum_{t\in U_a^+}\frac{f(k_{ta}^{(\nu)}\alpha_t)}{k_{m'\tau'}^{(\nu)}}\cdot\frac{k_{ta}^{(\nu)}}{k_{ta}^{(\nu)}}+\sum_{t\in U_b^+}\frac{f(ik_{tb}^{(\nu)}\alpha_t)}{k_{m'\tau'}^{(\nu)}}\cdot\frac{ik_{tb}^{(\nu)}}{ik_{tb}^{(\nu)}}$$
$$-\sum_{s\in U_a^-}\frac{f(|k_{sa}^{(\nu)}|\alpha_s)}{k_{m'\tau'}^{(\nu)}}\cdot\frac{|k_{sa}^{(\nu)}|}{|k_{sa}^{(\nu)}|}+\sum_{s\in U_b^-}\frac{f(i|k_{sb}^{(\nu)}|\alpha_s)}{k_{m'\tau'}^{(\nu)}}\cdot\frac{i|k_{sb}^{(\nu)}|}{i|k_{sb}^{(\nu)}|}\Big|=0.$$

which, by (3.9) and step 1, yield

$$\Big|\sum_{t\in U_a^+}\lambda_t\cdot\frac{x_{ta}}{x_{m'\tau'}}+\sum_{t\in U_b^+}\lambda_t\cdot\frac{ix_{tb}}{x_{m'\tau'}}-\sum_{s\in U_a^-}\lambda_s\cdot\frac{|x_{sa}|}{x_{m'\tau'}}-\sum_{s\in U_b^-}\lambda_s\cdot\frac{i|x_{sb}|}{x_{m'\tau'}}\Big|=0.$$

Hence

$$\sum_{m=1}^{n}\frac{(x_{ma}+ix_{mb})\lambda_m}{x_{m'\tau'}}=0,$$

i.e.,

$$\sum_{m=1}^{n}(x_{ma}+ix_{mb})\lambda_m=0.$$

Thus $\lambda(0)=0$, so λ is a function.

To prove (b), i.e., to show that λ is linear over $\mathbb{C}$, let $a,b,x,y\in\mathbb{C}$. Then there are $x_1,y_1,x_2,y_2\ldots,x_n,y_n\in\mathbb{C}$ such that $x=\sum_{m=1}^{n}x_m\alpha_m$ and $y=\sum_{m=1}^{n}y_m\alpha_m$.

By the definition of λ,

$$\begin{aligned}\lambda(ax+by) &= \lambda\big(a(\sum_{m=1}^{n} x_m\alpha_m) + b(\sum_{m=1}^{n} y_m\alpha_m)\big) &&= \lambda\big(\sum_{m=1}^{n}(ax_m+by_m)\alpha_m\big)\\ &= \sum_{m=1}^{n}(ax_m+by_m)\lambda_m &&= \sum_{m=1}^{n}(ax_m\lambda_m+by_m\lambda_m)\\ &= a\sum_{m=1}^{n} x_m\lambda_m + b\sum_{m=1}^{n} y_m\lambda_m &&= a\lambda(\sum_{m=1}^{n} x_m\alpha_m) + b\lambda(\sum_{m=1}^{n} y_m\alpha_m)\\ &= a\lambda(x)+b\lambda(y).\end{aligned}$$

Thus λ is a linear function over $\mathbb{C}$. Consequently,

$$\lambda(r_1+ir_2) = r_1\lambda(1)+r_2\lambda(i) \qquad \text{for all } r_1, r_2 \in \mathbb{R}.$$

Next, we show that λ is uniformly continuous on S^+. Let $\varepsilon > 0$ and $\Lambda = \max\{|\lambda(1)|, |\lambda(i)|\}$. Let $x = x_1+ix_2$ and $y = y_1+iy_2 \in S^+$ be such that $|x-y| < \frac{\varepsilon}{2\Lambda+1}$. Then

$$\begin{aligned}|\lambda(x)-\lambda(y)| &= |\lambda(x_1+ix_2)-\lambda(y_1+iy_2)|\\ &= |x_1\lambda(1)+x_2\lambda(i)-y_1\lambda(1)-y_2\lambda(i)|\\ &\le |x_1-y_1||\lambda(1)|+|x_2-y_2||\lambda(i)|\\ &\le |x-y|\Lambda+|x-y|\Lambda\\ &< \varepsilon.\end{aligned}$$

Hence λ is uniformly continuous on S^+.

Step 3 Define the function $\omega : S^+ \to \mathbb{C}$ by

$$\omega(x) = f(x)-\lambda(x) \qquad \text{for all } x \in S^+.$$

Then ω is uniformly continuous on S^+ since f and λ are. Next, we show that ω satisfies (3.1). Let $\sum_{m=1}^{n}(x_{ma}+ix_{mb})\alpha_m \in S^+$. Then

$$\begin{aligned}\omega\Big(\sum_{m=1}^{n}(x_{ma}+ix_{mb})\alpha_m\Big) &= f\Big(\sum_{m=1}^{n}(x_{ma}+ix_{mb})\alpha_m\Big) - \lambda\Big(\sum_{m=1}^{n}(x_{ma}+ix_{mb})\alpha_m\Big)\\ &= \sum_{m=1}^{n}\big\{f(x_{ma}\alpha_m)+f(ix_{mb}\alpha_m)\big\} - \sum_{m=1}^{n}\big\{(x_{ma}+ix_{mb})\lambda_m\big\}\\ &= \sum_{m=1}^{n}\Big\{\big(f(x_{ma}\alpha_m)-\lambda(x_{ma}\alpha_m)\big)+\big(f(ix_{mb}\alpha_m)-\lambda(ix_{mb}\alpha_m)\big)\Big\}\\ &= \sum_{m=1}^{n}\big\{\omega(x_{ma}\alpha_m)+\omega(ix_{mb}\alpha_m)\big\}.\end{aligned}$$

Hence ω satisfies (3.1) as desired. Moreover, for each $m = 1, 2, \ldots, n$, by Step 1 and the linearity of λ,

$$\lim_{\substack{|u+iv|\to\infty \\ u,v\in\ St(m)}} \frac{\omega((u+iv)\alpha_m)}{u+iv} = 0. \tag{3.10}$$

Now, we define a function $\varphi_m : S_m \to \mathbb{C}$ for all $m = 1, 2, \ldots, n$ by

1. $\varphi_m(0) = 0$,
2. $\varphi_m(x+\alpha_j) = \varphi_m(x+i\alpha_j) = \varphi_m(x)$ for all $j \neq m$ and $x \in S_m$ and
3. $\varphi_m\big((u_{ma}+iu_{mb})\alpha_m\big) = \omega\big((u_{ma}+iu_{mb})\alpha_m\big)$.

Note that for each m, φ_m has not been defined for the whole S_m. To extend the domain of φ_m to S_m, recall that

$$S_m = \Big\{(u_{ma}+iu_{mb})\alpha_m + \sum_{\substack{j=1\\ j\neq m}}^{n}(k_{ja}+ik_{jb})\alpha_j \Big| u_{ma}, u_{mb} \in \mathbb{N}_0; k_{ja}, k_{jb} \in \mathbb{Z}\Big\}.$$

and from the second property of φ_m, it follows that

$$\varphi_m\big(x + (k_{ja}+ik_{jb})\alpha_j\big) = \varphi_m(x) \tag{3.11}$$

where $x \in St(m)$, $k_{ja}, k_{jb} \in \mathbb{Z}$ and $j \in \{1, 2, \dots, n\} \setminus \{m\}$. This shows that $\varphi_m(x)$ exists for all $x \in S_m$.

It remains to confirm the shape of (3.2), for each $x = \sum_{m=1}^{n}(x_{ma} + ix_{mb})\alpha_m \in S^+$, we have

$$\begin{aligned} f(x) &= \lambda(x) + \omega(x) \\ &= \lambda(x) + \omega(\sum_{m=1}^{n}(x_{ma} + ix_{mb})\alpha_m) \\ &= \lambda(x) + \sum_{m=1}^{n}\omega((x_{ma} + ix_{mb})\alpha_m) \\ &= \lambda(x) + \sum_{m=1}^{n}\varphi_m((x_{ma} + ix_{mb})\alpha_m) \\ &= \lambda(x) + \sum_{m=1}^{n}\varphi_m\big((x_{ma} + ix_{mb})\alpha_m + \sum_{\substack{j=1 \\ j\neq m}}^{n}(x_{ja} + ix_{jb})\alpha_j\big) \\ &= \lambda(x) + \sum_{m=1}^{n}\varphi_m(x). \end{aligned}$$

To complete the proof, we are left only to show only that φ_m is uniformly continuous on S_m for $m = 1, 2, \dots, n$. Fix m, let $\varepsilon > 0$. Since ω is uniformly continuous on S^+, there exists $\delta^* > 0$ such that

$$\forall x, y \in S^+, |x - y| < \delta^* \Rightarrow |\omega(x) - \omega(y)| < \varepsilon.$$

Let

$$\zeta = (u_{ma} + iu_{mb})\alpha_m + \sum_{\substack{j=1 \\ j\neq m}}^{n}(k_{ja} + ik_{jb})\alpha_j$$

and

$$\eta = (v_{ma} + iv_{mb})\alpha_m + \sum_{\substack{j=1 \\ j\neq m}}^{n}(l_{ja} + il_{jb})\alpha_j$$

be elements of S_m such that $|\zeta-\eta|<\delta^*$.

For each $j\neq m$, let $q_{ja}+iq_{jb}=(k_{ja}+ik_{jb})-(l_{ja}+il_{jb})$. Choose $u_{ja},u_{jb},v_{ja},v_{jb}\in\mathbb{N}_0$ such that

$$q_{ja}+iq_{jb}=(u_{ja}+iu_{jb})-(v_{ja}+iv_{jb}).$$

Let

$$x=(u_{ma}+iu_{mb})\alpha_m+\sum_{\substack{j=1\\ j\neq m}}^{n}(u_{ja}+iu_{jb})\alpha_j$$

and

$$y=(v_{ma}+iv_{mb})\alpha_m+\sum_{\substack{j=1\\ j\neq m}}^{n}(v_{ja}+iv_{jb})\alpha_j.$$

Then

$$\begin{aligned}
|\,x-y| &= \left|\Big\{(u_{ma}+iu_{mb})\alpha_m+\sum_{\substack{j=1\\ j\neq m}}^{n}(u_{ja}+iu_{jb})\alpha_j\Big\}-\Big\{(v_{ma}+iv_{mb})\alpha_m+\sum_{\substack{j=1\\ j\neq m}}^{n}(v_{ja}+iv_{jb})\alpha_j\Big\}\right|\\
&= \left|\Big\{(u_{ma}+iu_{mb})\alpha_m-(v_{ma}+iv_{mb})\alpha_m\Big\}+\sum_{\substack{j=1\\ j\neq m}}^{n}\Big\{(u_{ja}+iu_{jb})\alpha_j-(v_{ja}+iv_{jb})\alpha_j\Big\}\right|\\
&= \left|\Big\{(u_{ma}+iu_{mb})\alpha_m-(v_{ma}+iv_{mb})\alpha_m\Big\}+\sum_{\substack{j=1\\ j\neq m}}^{n}(q_{ja}+iq_{jb})\alpha_j\right|\\
&= \left|\Big\{(u_{ma}+iu_{mb})\alpha_m-(v_{ma}+iv_{mb})\alpha_m\Big\}+\sum_{\substack{j=1\\ j\neq m}}^{n}((k_{ja}+ik_{jb})-(l_{ja}+il_{jb}))\alpha_j\right|\\
&= \left|\Big\{(u_{ma}+iu_{mb})\alpha_m+\sum_{\substack{j=1\\ j\neq m}}^{n}(k_{ja}+ik_{jb})\alpha_j\Big\}-\Big\{(v_{ma}+iv_{mb})\alpha_m+\sum_{\substack{j=1\\ j\neq m}}^{n}(l_{ja}+il_{jb})\alpha_j\Big\}\right|\\
&= |\,\zeta-\eta|<\delta^*
\end{aligned}$$

Applying (3.4) with $f=\omega,\ I=\{m\},\ J=\{1,2,\dots,n\}\smallsetminus\{m\},\ w_{ja}=w_{jb}=0$ and $q_{j\tau}=u_{j\tau}-v_{j\tau}$ for $\tau=a,b$, we get

$$\Big|\omega\big((u_{ma}+iu_{mb})\alpha_m\big)-\omega\big((v_{ma}+iv_{mb})\alpha_m\big)$$
$$+\frac{1}{M}\sum_{j\in J}\eta_{ja}\big\{\omega(M|q_{ja}|\alpha_j)-0\big\}$$
$$+\frac{1}{M}\sum_{j\in J}\eta_{jb}\big\{\omega(iM|q_{jb}|\alpha_j)-0\big\}\Big|<\varepsilon. \tag{3.12}$$

Since the sequence of values $\dfrac{\omega(M|q_{ja}|\alpha_j)}{M}$ and $\dfrac{\omega(iM|q_{jb}|\alpha_j)}{M}$ are subsequences of the sequence of values $\dfrac{\omega((u+iv)\alpha_j)}{u+iv}$, where $u+iv$ runs through $St(m)$ and by (3.10),

$$\lim_{M\to\infty}\frac{\omega(M|q_{ja}|\alpha_j)}{M}=0=\lim_{M\to\infty}\frac{\omega(iM|q_{jb}|\alpha_j)}{M}\qquad\text{for each } j\in J. \tag{3.13}$$

Then by letting $M\to\infty$ in (3.12), we obtain from (3.13) that

$$\big|\omega\big((u_{ma}+iu_{mb})\alpha_m\big)-\omega\big((v_{ma}+iv_{mb})\alpha_m\big)\big|\le\varepsilon.$$

Note that

$$\varphi_m(\zeta)=\varphi_m\Big((u_{ma}+iu_{mb})\alpha_m+\sum_{\substack{j=1\\ j\neq m}}^{n}(k_{ja}+ik_{jb})\alpha_j\Big)$$
$$=\varphi_m\big((u_{ma}+iu_{mb})\alpha_m\big)=\omega\big((u_{ma}+iu_{mb})\alpha_m\big)$$

and

$$\varphi_m(\eta)=\varphi_m\Big((v_{ma}+iv_{mb})\alpha_m+\sum_{\substack{j=1\\ j\neq m}}^{n}(l_{ja}+il_{jb})\alpha_j\Big)$$
$$=\varphi_m\big((v_{ma}+iv_{mb})\alpha_m\big)=\omega\big((v_{ma}+iv_{mb})\alpha_m\big).$$

This leads to

$$\left|\varphi_m(\zeta) - \varphi_m(\eta)\right| \leq \varepsilon,$$

i.e., φ_m is uniformly continuous on S_m. $\square$

To obtain only linear solution, further restrictions must be put on the domain of solution. One such instance is the following corollary which may be regarded as analogues of Corollary 2.1.6 and Theorem 2.1.7.

Corollary 3.1.2. *Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ ($n \in \mathbb{N}$ with $n \geq 2$) satisfy the following properties:*

1. $S^+ = \left\{ \sum_{m=1}^{n} (u_m + iv_m)\alpha_m \mid u_m, v_m \in \mathbb{N}_0 \right\}$ *is dense in* $\mathbb{C}$ *and*
2. $T_m = \left\{ \sum_{\substack{j=1 \\ j \neq m}}^{n} (u_j + iv_j)\alpha_j \middle| u_j, v_j \in \mathbb{Z} \right\}$ *is dense in* $\mathbb{C}$ *for all* $m = 1, 2, \ldots, n$.

If $f : S^+ \to \mathbb{C}$ is a uniformly continuous solution of (3.1), then f is a linear function, i.e., there exists a linear function $\lambda : \mathbb{C} \to \mathbb{C}$ over $\mathbb{C}$ such that $f(x) = \lambda(x)$ for all $x \in S^+$.

Moreover, If $S^+ = \mathbb{C}$, then $f(z) = az + b\bar{z}$ where $z \in \mathbb{C}$, a, b are arbitrary complex constants and $\bar{z}$ is the conjugate of z.

Proof. From Theorem 3.1.1, f can be written in the form (3.2), i.e., $f(x) = \lambda(x) + \sum_{m=1}^{n} \varphi_m(x)$ where λ is a linear function over $\mathbb{C}$ and φ_m is a function satisfying the three properties as stated in Theorem 3.1.1. For each $m = 1, 2, \ldots, n$, let $u_m, v_m \in \mathbb{N}_0$ and $\varepsilon > 0$. Since T_m is dense in $\mathbb{C}$, there are $k_j \in \mathbb{Z}$ ($j = 1, \ldots, n$; $k \neq m$) such that

$$\left| \sum_{\substack{j=1 \\ j \neq m}}^{n} (u_j + iv_j)\alpha_j - (u_m + iv_m)\alpha_m \right| < \varepsilon.$$

By the uniform continuity and the properties of φ_m, we have $\Big|0 - \varphi_m(u_m + iv_m)\Big| = \Big|\varphi_m(\sum_{\substack{j=1 \\ j\neq m}}^{n}(u_j + iv_j)\alpha_j) - \varphi_m(u_m + iv_m)\Big| < \varepsilon$, so $\varphi_m(u_m + iv_m) = 0$. This implies that φ_m is the zero mapping, hence $f = \lambda$. Since for each $z = r_1 + ir_2 \in \mathbb{C}$, $\lambda(z) = \lambda(r_1 + ir_2) = r_1\lambda(1) + r_2\lambda(i)$, we have $\lambda(r_1 + ir_2) = r_1\lambda(1) + r_2\lambda(i) = \frac{z+\bar{z}}{2}\lambda(1) + \frac{z-\bar{z}}{2i}\lambda(i) = \frac{i\lambda(1)+\lambda(i)}{2i}z + \frac{i\lambda(1)-\lambda(i)}{2i}\bar{z} = az + b\bar{z}$, so $f(z) = az + b\bar{z}$ for all $z \in \mathbb{C}$. □

3.2 Examples

The following examples illustrate existence of $\alpha_i's$ in Theorem 3.1.1, i.e., there actually exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ ($n \in \mathbb{N}$ with $n \geq 2$) such that

$$S^+ = \Big\{\sum_{m=1}^{n}(u_m + iv_m)\alpha_m \mid u_m, v_m \in \mathbb{N}_0\Big\}$$

is dense in $\mathbb{C}$.

Example 3.2.1 (Case $n = 2$).

We choose

$$\alpha_1 = \sqrt{2} \qquad \text{and} \qquad \alpha_2 = -1.$$

To show that $S^+ = \big\{(u_1 + iv_1)\alpha_1 + (u_2 + iv_2)\alpha_2 \mid u_1, v_1, u_2, v_2 \in \mathbb{N}_0\big\}$ is dense in $\mathbb{C}$, let $r_1 + ir_2 \in \mathbb{C}$ and $\varepsilon > 0$. By Kronecker's Theorem, there exist $u_1, p_1 \in \mathbb{N}, u_2, p_2 \in \mathbb{Z}$ such that $|u_1\sqrt{2} - u_2 - r_1| < \frac{\varepsilon}{2}$ and $|v_1\sqrt{2} - v_2 - r_2| < \frac{\varepsilon}{2}$ (u_1 and v_1 are large enough for $u_1\sqrt{2} - r_1 > 0$ and $v_1\sqrt{2} - r_2 > 0$, so that u_2 and v_2 are nonnegative integers). Thus $|(u_1 + iv_1)\alpha_1 + (u_2 + iv_2)\alpha_2 - (r_1 + ir_2)| < |u_1\sqrt{2} - u_2 - r_1| + |v_1\sqrt{2} - v_2 - r_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Therefore S^+ is dense in $\mathbb{C}$.

Example 3.2.2 (Case $n = 3$).

Let

$$\alpha_1 = 1 + i\sqrt{2},\ \alpha_2 = \sqrt{2} + i \quad \text{and} \quad \alpha_3 = -1 - i.$$

To show that $S^+ = \{\sum_{m=1}^{3}(u_m + iv_m)\alpha_m \mid u_m, v_m \in \mathbb{N}_0\}$ is dense in $\mathbb{C}$, let $r_1 + ir_2 \in \mathbb{C}$ and $\varepsilon > 0$. By Kronecker's Theorem, there exist $n_1, n_2 \in \mathbb{N},\ p_1, p_2 \in \mathbb{Z}$ such that $n_2 > |p_1|$, $|2n_1\sqrt{2} - 2p_1 - r_1| < \frac{\varepsilon}{2}$ and $|2n_2\sqrt{2} - 2p_2 - r_2| < \frac{\varepsilon}{2}$. Set $p = n_1 + |p_2|$, $q = p$, $a = n_2 - p_1, b = p - p_2 - n_1, c = p + n_1 - p_2$ and $d = n_2 + p_1$, hence p, q, a, b, c, d are nonnegative integers. Then

$$\begin{aligned}
&|(a+ib)(1+i\sqrt{2}) + (c+id)(\sqrt{2}+i) + (p+iq)(-1-i) - (r_1+ir_2)| \\
=&|(a - b\sqrt{2} + i(a\sqrt{2}+b) + c\sqrt{2} - d + i(c + d\sqrt{2}) - p + q + i(-p-q) - (r_1+ir_2)| \\
\le&|(c-b)\sqrt{2} + (a - d - p + q) - r_1| + |(a+d)\sqrt{2} + (b + c - p - q) - r_2| \\
=&|(p + n_1 - p_2 - p + p_2 + n_1)\sqrt{2} + (n_2 - p_1 - n_2 - p_1 - p + p) - r_1| \\
&+ |(n_2 - p_1 + n_2 + p_1)\sqrt{2} + (p - p_2 - n_1 + p + n_1 - p_2 - p - p) - r_2| \\
=&|2n_1\sqrt{2} - 2p_1 - r_1| + |2n_2\sqrt{2} - 2p_2 - r_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore S^+ is dense in $\mathbb{C}$.

Example 3.2.3 (Case $n = 4$).

Take

$$\alpha_1 = 1,\ \alpha_2 = \sqrt{2} + i, \alpha_3 = -i \quad \text{and} \quad \alpha_4 = 3 + i\sqrt{2}.$$

It is verified similarly to the previous examples that S^+ is dense in $\mathbb{C}$.

For the case $n > 4$, we can construct $\alpha_1, \alpha_2, \ldots, \alpha_n$ by letting $\alpha_1, \alpha_2, \alpha_3$ and α_4 be the same elements in Example 3.2.3 and the others $\alpha_i's$ arbitrary. Then, again, S^+ according to this choice is dense in $\mathbb{C}$. In the same vein, we can also construct $\alpha_1, \alpha_2, \ldots, \alpha_n$ using the choices of $\alpha_i's$ in Example 3.2.1 or Example 3.2.2.

Note that the $\alpha_i's$ defined in Examples 3.2.1 – 3.2.3 satisfy the hypothesises of Corollary 3.1.2. Consequently, any uniformly continuous solutions of Cauchy's functional equation must be linear.

The following example gives the $\alpha_i's$ such that S^+ is not dense in $\mathbb{C}$:

Example 3.2.4.

Let

$$\alpha_1 = 1 + i\sqrt{2},\ \alpha_2 = \sqrt{2} + i \quad \text{and} \quad \alpha_3 = \sqrt{3} + i\sqrt{2}.$$

Since elements in $S^+ = \{\sum_{m=1}^{3}(u_m + iv_m)\alpha_m \mid u_m, v_m \in \mathbb{N}_0\}$ is of the form

$$(u_1 - v_1\sqrt{2} - v_2\sqrt{2} + u_3\sqrt{3} - v_3\sqrt{2}) + i(u_1\sqrt{2} + v_1 + u_2\sqrt{2} + u_3\sqrt{2} + v_3\sqrt{3}),$$

the imaginary part is nonnegative real number, so S^+ can not be dense in $\mathbb{C}$.

3.3 Equations of Cauchy's type

We present next uniformly continuous solutions for equations of Cauchy's type introduced in Section 2.1. They are obtained from Theorem 3.1.1 by changing appropriate variables and domains.

Corollary 3.3.1. *Let* $S^+ = \left\{\prod_{m=1}^{n} \beta_m^{(u_m+iv_m)} \mid u_m, v_m \in \mathbb{N}_0\right\}$. *If* $g : S^+ \to \mathbb{C}$ *is a solution of the functional equation:*

$$g\left(\prod_{m=1}^{n} \beta_m^{(u_m+iv_m)}\right) = \sum_{m=1}^{n} \left\{g(\beta_m^{u_m}) + g(\beta_m^{iv_m})\right\} \tag{3.14}$$

under the conditions

1. $T^+ = \left\{\sum_{m=1}^{n}(u_m + iv_m)Log(\beta_m) \mid u_m, v_m \in \mathbb{N}_0\right\}$ *is dense in* $\mathbb{C}$ *and*

2. $g(e^x)$ *is uniformly continuous on* T^+,

then g *is of the form*

$$g(x) = \lambda(Log\ x) + \sum_{m=1}^{n} \varphi_m(Log\ x)$$

where $\lambda : \mathbb{C} \to \mathbb{C}$ *is a linear function over* $\mathbb{C}$ *and* φ_m *is a complex valued function* $(m = 1, 2, \ldots, n)$ *on*

$$S_m = \left\{ (u_{ma} + iu_{mb})Log(\beta_m) + \sum_{\substack{j=1 \\ j \neq m}}^{n} (k_{ja} + ik_{jb})Log(\beta_j) \mid u_{ma}, u_{mb} \in \mathbb{N}_0; k_{ja}, k_{jb} \in \mathbb{Z} \right\}$$

with the following properties:

1. $\varphi_m(0) = 0$,

2. $\varphi_m(x + Log\beta_j) = \varphi_m(x + iLog\beta_j) = \varphi_m(x)$ $(j \neq m$ *and* $x \in S_m)$ *and*

3. φ_m *is a uniformly continuous function on* S_m.

Proof. Let $\alpha_m = Log(\beta_m)$ and $f(x) = g(e^x)$. Then

$$\begin{aligned} g\left(\prod_{m=1}^{n} \beta_m^{(u_m+iv_m)}\right) &= g\left(e^{Log(\prod_{m=1}^{n} \beta_m^{(u_m+iv_m)})}\right) \\ &= f\left(Log(\prod_{m=1}^{n} \beta_m^{(u_m+iv_m)})\right) \\ &= f\left(\sum_{m=1}^{n} (u_m + iv_m)Log(\beta_m)\right) \\ &= f\left(\sum_{m=1}^{n} (u_m + iv_m)\alpha_m\right) \end{aligned}$$

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and

$$\sum_{m=1}^{n}\left\{g(\beta_m^{u_m})+g(\beta_m^{iv_m})\right\}=\sum_{m=1}^{n}\left\{g(e^{Log(\beta_m^{u_m})})+g(e^{Log(\beta_m^{iv_m})})\right\}$$
$$=\sum_{m=1}^{n}\{f(u_mLog(\beta_m))+f(iv_mLog(\beta_m))\}$$
$$=\sum_{m=1}^{n}\{f(u_m\alpha_m)+f(iv_m\alpha_m)\}$$

Then (3.14) is transformed into (3.1). Since T^+ is dense in $\mathbb{C}$ and $f(x)=g(e^x)$ is uniformly continuous on T^+, by Theorem 3.1.1, we have

$$f(x)=\lambda(x)+\sum_{m=1}^{n}\varphi_m(x),$$

where $\lambda:\mathbb{C}\to\mathbb{C}$ is a linear function over $\mathbb{C}$ and φ_m is a complex valued function $(m=1,2,\ldots,n)$ on

$$S_m=\left\{(u_{ma}+iu_{mb})\alpha_m+\sum_{\substack{j=1\\j\neq m}}^{n}(k_{ja}+ik_{jb})\alpha_j \mid u_{ma},u_{mb}\in\mathbb{N}_0;k_{ja},k_{jb}\in\mathbb{Z}\right\}$$

satisfying

1. $\varphi_m(0)=0$,
2. $\varphi_m(x+\alpha_j)=\varphi_m(x+i\alpha_j)=\varphi_m(x)$ $(j\neq m$ and $x\in S_m)$ and
3. φ_m is a uniformly continuous function on S_m.

Changing back the variables, the result follows. Using similar proof, we also obtain

$$g(x)=f(Logx)=\lambda(Log\ x)+\sum_{m=1}^{n}\varphi_m(Log\ x)$$

□

Corollary 3.3.2. *Let* $S^+ = \left\{ \sum_{m=1}^{n} (u_m + iv_m)\beta_m \mid u_m, v_m \in \mathbb{N}_0 \right\}$. *If* $g : S^+ \to \mathbb{C}$ *is a solution of functional equation:*

$$g\left(\sum_{m=1}^{n}(u_m + iv_m)\beta_m\right) = \prod_{m=1}^{n} g(u_m\beta_m)g(iv_m\beta_m)$$

under the conditions

1. S^+ *is dense in* $\mathbb{C}$ *and*

2. $Log\big(g(x)\big)$ *is uniformly continuous on* S^+,

then g *is of the form*

$$g(x) = exp\left(\lambda(x) + \sum_{m=1}^{n} \varphi_m(x)\right)$$

where $\lambda : \mathbb{C} \to \mathbb{C}$ *is a linear function over* $\mathbb{C}$ *and* φ_m *is a complex valued function* $(m = 1, 2, \ldots, n)$ *on*

$$S_m = \left\{(u_{ma} + iu_{mb})\beta_m + \sum_{\substack{j=1 \\ j\neq m}}^{n}(k_{ja} + ik_{jb})\beta_j \mid u_{ma}, u_{mb} \in \mathbb{N}_0; k_{ja}, k_{jb} \in \mathbb{Z}\right\}$$

with the following properties:

1. $\varphi_m(0) = 0$,

2. $\varphi_m(x + \beta_j) = \varphi_m(x + i\beta_j) = \varphi_m(x)$ $(j \neq m$ *and* $x \in S_m)$ *and*

3. φ_m *is a uniformly continuous function on* S_m.

Corollary 3.3.3. *Let* $S^+ = \left\{ \prod_{m=1}^{n} \beta_m^{(u_m+iv_m)} \mid u_m, v_m \in \mathbb{N}_0 \right\}$. *If* $g : S^+ \to \mathbb{C}$ *is a solution of functional equation:*

$$g\left(\prod_{m=1}^{n} \beta_m^{(u_m+iv_m)}\right) = \prod_{m=1}^{n} g(\beta_m^{u_m}) g(\beta_m^{iv_m})$$

under the conditions

1. $T^+ = \left\{ \sum_{m=1}^{n} (u_m + iv_m) Log(\beta_m) \mid u_m, v_m \in \mathbb{N}_0 \right\}$ *is dense in* $\mathbb{C}$ *and*
2. $Log\big(g(e^x)\big)$ *is uniformly continuous on* T^+,

then g *is of the form*

$$g(x) = exp\left(\lambda(Log\ x) + \sum_{m=1}^{n} \varphi_m(Log\ x)\right)$$

where $\lambda : \mathbb{C} \to \mathbb{C}$ *is a linear function over* $\mathbb{C}$ *and* φ_m *is a complex valued function* $(m = 1, 2, \ldots, n)$ *on*

$$S_m = \left\{ (u_{ma} + iu_{mb}) Log(\beta_m) + \sum_{\substack{j=1 \\ j \neq m}}^{n} (k_{ja} + ik_{jb}) Log(\beta_j) \mid u_{ma}, u_{mb} \in \mathbb{N}_0; k_{ja}, k_{jb} \in \mathbb{Z} \right\}$$

with the following properties:

1. $\varphi_m(0) = 0$,
2. $\varphi_m(x + Log\beta_j) = \varphi_m(x + iLog\beta_j) = \varphi_m(x)$ $(j \neq m$ *and* $x \in S_m)$ *and*
3. φ_m *is a uniformly continuous function on* S_m.

3.4 Remarks

The statements of Theorem 3.1.1 are similar to those of Theorem 2.2.1 when $n = 2$ but with different domains of solutions. We change domain from

$$S = \Big\{ \sum_{m=1}^{n} u_m \alpha_m \mid u_m \in \mathbb{N}_0 \Big\} \text{ in Theorem 2.2.1}$$

into

$$S^+ = \Big\{ \sum_{m=1}^{n} (u_m + iv_m)\alpha_m \mid u_m, v_m \in \mathbb{N}_0 \Big\} \text{ in Theorem 3.1.1}$$

and replace the condition from linear independence of $\alpha's$ into the denseness of set, i.e., S^+ is dense in $\mathbb{C}$. We found that the forms of uniformly continuous solutions, obtained from both theorems, are still the same but the properties of linear function λ and all φ_m for $m = 1, 2 \ldots, n$ are changed, for example, λ in Theorem 3.1.1 is a linear function over $\mathbb{C}$, but it is a linear function over $\mathbb{R}$ in Theorem 2.2.1.

A natural question is whether "$\mathbb{R}$ and $\mathbb{Q}$-linearly independence" are related to "denseness". The next two examples show that they are both rather unrelated.

Example 3.4.1.

In $\mathbb{R}^2$, let

$$\alpha_1 = (\sqrt{2}, 0), \alpha_2 = (\sqrt{2}, 1) \text{ and } \alpha_3 = (\sqrt{2}, \sqrt{3}).$$

To show that α_1, α_2 and α_3 are rationally independent, let $q_1, q_2, q_3 \in \mathbb{Q}$ be such that $q_1\alpha_1 + q_2\alpha_2 + q_3\alpha_3 = 0$. Then $q_1 + q_2 + q_3 = 0$ and $q_2 + q_3\sqrt{3} = 0$. Hence $q_3 = 0$ and so is q_2, implying $q_1 = 0$. For the other condition that any set of two among the $\alpha_i's$ are linearly independent over $\mathbb{R}$, let $r_{11}, r_{12}, r_{21}, r_{22}, r_{31}$ and $r_{32} \in \mathbb{R}$ be such that $r_{11}\alpha_1 + r_{12}\alpha_2 = 0$, $r_{21}\alpha_1 + r_{22}\alpha_3 = 0$ and $r_{31}\alpha_2 + r_{32}\alpha_3 = 0$. Hence $r_{11} + r_{12} = 0$, $r_{21} + r_{22} = 0$, $r_{31} + r_{32} = 0$, $r_{12} = 0$, $r_{22} = 0$ and $r_{31} + r_{32}\sqrt{3} = 0$, so $r_{11}, r_{12}, r_{21}, r_{22}, r_{31}$ and r_{32} are all zero.

Next, we will show that $\{k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 \mid k_1, k_2, k_3 \in \mathbb{Z}\}$ is not dense in $\mathbb{R}^2$. Since this set contains elements in the form $k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = (\sqrt{2}(k_1 + k_2 + k_3), k_2 + k_3\sqrt{3})$, the first coordinate is a multiple of $\sqrt{2}$ so this set can not be dense in $\mathbb{R}^2$.

Example 3.4.2.

In $\mathbb{R}$, let

$$\alpha_1 = 1, \alpha_2 = -1 \text{ and } \alpha_3 = \sqrt{2}.$$

We claim that $\{k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 \mid k_1, k_2, k_3 \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Let $r \in \mathbb{R}$ and $\varepsilon > 0$. By Kronecker's Theorem, there are integers n, p such that $|n\sqrt{2} - p - r| < \varepsilon$, hence choose integers k_1, k_2 and k_3 such that $k_1 - k_2 = -p$ and $k_3 = n$. Thus $\left|k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 - r\right| = \left|k_1 - k_2 + k_3\sqrt{2} - r\right| = \left| -p + n\sqrt{2} - r\right| < \varepsilon$, so we have the claim. Since $\alpha_1 + \alpha_2 + 0(\alpha_3) = 0$, α_1, α_2 and α_3 is linearly dependent over $\mathbb{Q}$.

We note that by the same proof in Theorem 2.2.1, if we replace the linear independence by the denseness of set, we also have the same result, i.e.,

Theorem 3.4.3. *Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be elements of $\mathbb{R}^n$ $(n < m)$, B be a Banach space and $S = \left\{ \sum_{i=1}^{m} u_i\alpha_i \mid u_i \in \mathbb{N}_0 \right\}$ be such that S is dense in $\mathbb{R}^n$.*

If f is a map from S into B and is a uniformly continuous solution on S of the functional equation : $f\big(\sum_{i=1}^{m} u_i\alpha_m\big) = \sum_{i=1}^{m} f(u_i\alpha_i)$ (u_i's are arbitrary nonnegative integers), then f admits a unique representation of the form $f(x) = \lambda(x) + \sum_{i=1}^{m} \varphi_i(x)$ for all $x \in S$ in which λ is a linear function from $\mathbb{R}^n$ into B, while φ_i is a function from

$$S_i = \left\{ u_i\alpha_i + \sum_{\substack{j=1 \\ j\neq i}}^{m} k_j\alpha_j \,\middle|\, u_i \in \mathbb{N}_0 \text{ and } k_j \in \mathbb{Z} \right\}$$

into B (for $i = 1, 2, \ldots, m$) and satisfies the conditions :

1. $\varphi_i(0) = 0$,

2. $\varphi_i(x + \alpha_j) = \varphi_i(x)$ $(j \neq i, x \in S_i)$ *and*

3. φ_i *is a uniformly continuous function on* S_i.

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