ขอบเขตของการประมาณค่าแบบปกติสำหรับจำนวนการลดและจำนวนการผกผันของการเรียงสับเปลี่ยนแบบสุ่ม

นายวิชัยรัตน์ จันที

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2557

BOUNDS ON NORMAL APPROXIMATION FOR DESCENTS AND INVERSIONS OF RANDOM PERMUTATIONS

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A Dissertation Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2014

Thesis Title	BOUNDS ON NORMAL APPROXIMATION FOR DESCENTS AND INVERSIONS OF RANDOM PERMUTATIONS
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Field of Study	Mathematics
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วิชัยรัตน์ จันที : ขอบเขตของการประมาณค่าแบบปกติสำหรับจำนวนการลดและจำนวนการผกผันของการเรียงสับเปลี่ยนแบบสุ่ม. (BOUNDS ON NORMAL APPROXIMATION FOR DESCENTS AND INVERSIONS OF RANDOM PERMUTATIONS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : ศ.ดร.กฤษณะ เนียมมณี, 62 หน้า.

จุดประสงค์ของวิทยานิพนธ์ฉบับนี้ประกอบด้วยสองส่วน โดยส่วนที่หนึ่งเป็นการหาค่าคงตัวของขอบเขตของการประมาณค่าแบบปกติสำหรับจำนวนการลดและจำนวนการผกผันของการเรียงสับเปลี่ยนแบบสุ่มในงานของฟูลแมน ส่วนที่สองเป็นการหาขอบเขตการประมาณค่าแบบใหม่ซึ่งมีทั้งขอบเขตการประมาณค่าแบบสม่ำเสมอและไม่สม่ำเสมอ ซึ่งในส่วนของขอบเขตการประมาณค่าแบบสม่ำเสมอนั้นเราให้ค่าคงตัวที่ดีกว่าของฟูลแมน สำหรับในส่วนของการประมาณค่าแบบไม่สม่ำเสมอนั้นเราให้ขอบเขตแบบเชิงเส้นและขอบเขตแบบเลขชี้กำลัง

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต....................

สาขาวิชา คณิตศาสตร์ ลายมือชื่อ อ.ที่ปรึกษาหลัก....................

ปีการศึกษา 2557

5373887023 : MAJOR MATHEMATICS
KEYWORDS : STEIN'S METHOD / NORMAL APPROXIMATION / DESCENTS / INVERSIONS

WICHAIRAT CHUNTEE : BOUNDS ON NORMAL APPROXIMATION FOR DESCENTS AND INVERSIONS OF RANDOM PERMUTATIONS. ADVISOR : PROF. KRITSANA NEAMMANEE, Ph.D., 62 pp.

In this work, there are two objectives. First, we find the explicit constants for uniform bounds on normal approximation of the number of descents and the number of inversions given by Fulman (2004). Second, we give new bounds of such approximations. We give both uniform and non-uniform bounds. For uniform bounds we give constants which are better than Fulman's constants. In the part of non-uniform bounds we present both linear and exponential bounds.

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Field of Study : Mathematics Advisor's Signature..........................

Academic Year : 2014 Co-advisor's Signature..........................

ACKNOWLEDGEMENTS

I would like to thank Professor Dr.Kritsana Neammanee, my thesis advisor, for his suggestions and helpful advice in preparing and writing this thesis. I am also thankful to Associate Professor Dr.Imchit Termwuttipong, Associate Professor Dr.Songkiat Sumetkijakan, Assistant Professor Dr.Nattakarn Chaidee and Dr.Dawud Thongtha, my thesis committee, for their comments and assistance. Moreover, I would like to thank all teachers who have taught me all along. I would like to express my deep gratitude to my beloved family for their love.

Finally, I would like to thank the support of the Development and Promotion of Science and Technology Talents Project of Thailand.

CONTENTS

CHAPTER I

INTRODUCTION

We study two interesting statistics on the symmetric group S_n called descent and inversion. These statistics are used in considering some properties of a uniformly distributed random permutation π of the set $\{1, 2, \ldots, n\}$. The **number of descents** of π, denoted by $des(\pi)$, is defined as the number of pairs $(i, i+1)$ with $1 \leq i \leq n-1$ and $\pi(i) > \pi(i+1)$. It is closely related to Eulerian number $A(n, m)$, the number of permutations in the symmetric group S_n having exactly $m+1$ descents. The **number of inversions** of π, denoted by $inv(\pi)$, is defined as the number of pairs (i, j) with $1 \leq i < j \leq n$ and $\pi(i) > \pi(j)$. The number of permutations $\pi \in S_n$ in which exactly k elements are greater than the next element can be calculated as

$$\sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k+1-i)^n$$

and the number of permutations with k inversions is expressed by the coefficient of x^k in the expansion of the product

$$\prod_{m=1}^{n} \sum_{i=0}^{m-1} x^i$$

([4] pp. 8,43). The distribution of the number of descents can be approximated by the standard normal distribution function Φ which can be found in Pitman ([18]) and Tanny ([22]). Proofs of the asymptotic normality for the number of inversions was also shown in Bender([3]) and Diaconis ([10]).

In 2004, uniform bounds for normal approximation for the number of descents and the number of inversions were given by Fulman ([11]). He used an exchangeable pair and Rinott and Rotar's Theorem ([19]) to give the following results,

$$\Delta_{des(\pi)} \leq \frac{C_1}{\sqrt{n}} \qquad \text{and} \qquad \Delta_{inv(\pi)} \leq \frac{C_2}{\sqrt{n}} \tag{1.1}$$

where $\Delta_{des(\pi)} := \sup_z \Delta_{des(\pi),z}$, $\Delta_{inv(\pi)} := \sup_z \Delta_{inv(\pi),z}$,

$$\Delta_{des(\pi),z} := \left| P\left(\frac{des(\pi) - E(des(\pi))}{\sqrt{Var(des(\pi))}} \leq z \right) - \Phi(z) \right|$$

and

$$\Delta_{inv(\pi),z} := \left| P\left(\frac{inv(\pi) - E(inv(\pi))}{\sqrt{Var(inv(\pi))}} \leq z \right) - \Phi(z) \right|.$$

Following the arguments of Fulman ([11]), the constants C_1 and C_2 in (1.1) are 1096 and 5421, respectively. In our work, we use the technique from Neammanee and Rattanawong ([13]) to improve the constants C_1 and C_2. Our constants are 12.44 and 14.24, respectively. The following are our results.

Theorem 1.1.

$$\Delta_{des(\pi)} \leq \frac{12.44}{\sqrt{n}}.$$

Theorem 1.2.

$$\Delta_{inv(\pi)} \leq \frac{14.24}{\sqrt{n}}.$$

Moreover we use exchangeable pair given by Fulman ([11]) and Stein's method to find non-uniform bounds. In this thesis, we present two types of non-uniform bounds. The first type is of the polynomial form, which is called polynomial non-uniform bounds. The second type is of the exponential form.

Theorem 1.3 and Theorem 1.4 are our main results for polynomial non-uniform bounds.

Theorem 1.3. *For sufficiently large n and $z \in \mathbb{R}$,*

$$\Delta_{des(\pi),z} \leq \frac{1756}{(1+|z|)^3\sqrt{n}}.$$

Theorem 1.4. *For sufficiently large n and $z \in \mathbb{R}$,*

$$\Delta_{inv(\pi),z} \leq \frac{12160}{(1+|z|)^3 \sqrt{n}}.$$

Table 1.1 compares the constants of uniform and polynomial non-uniform bounds.

	uniform bounds	polynomial bounds				
		$z = 4.2$	$z = 8.5$	$z = 100$	$z = 1000$	$z = 10000$
Descent	$\frac{12.44}{\sqrt{n}}$	$\frac{12.44}{\sqrt{n}}$	$\frac{2.05}{\sqrt{n}}$	$\frac{1.71 \times 10^{-3}}{\sqrt{n}}$	$\frac{1.76 \times 10^{-6}}{\sqrt{n}}$	$\frac{1.76 \times 10^{-9}}{\sqrt{n}}$
Inversion	$\frac{14.24}{\sqrt{n}}$	$\frac{86.49}{\sqrt{n}}$	$\frac{14.24}{\sqrt{n}}$	$\frac{1.19 \times 10^{-2}}{\sqrt{n}}$	$\frac{1.22 \times 10^{-5}}{\sqrt{n}}$	$\frac{1.22 \times 10^{-8}}{\sqrt{n}}$

Table 1.1: Constants of uniform and polynomial non-uniform bounds.

In Table 1.1, we give an example to show our polynomial non-uniform bound is smaller than a uniform bound for large z.

The last part of our work, we give an exponential non-uniform bound. In fact, exponential bounds have been considered since 1975 by Petrov ([16]). He used Fourier transform and focused on independent and identically distributed random variable $X_1, \ldots, X_n$ with mean zero and variance one such that $Ee^{t|X_1|} < \infty$ for some $t > 0$ to show that

$$|P(W \leq z) - \Phi(z)| \leq \frac{C(1+z^2)e^{-\frac{z^2}{2}}}{\sqrt{n}},$$

where $W = \dfrac{X_1 + \cdots + X_n}{\sqrt{n}}$.

Throughout this work we reserve the notation C for a constant with assuming that the value may not be the same. However, in 2013 Chen, Fang and Shao ([8]) presented a result called Cramer-type moderate deviation result for dependent random variables. The result concerns with the Stein's method which is simpler than Fourier transform. In this thesis, we use the idea from Chen, Fang and

Shao ([8]) and exchangeable pairs given by Fulman ([11]) to give exponential non-uniform bounds. Theorem 1.5 and Theorem 1.6 are our main results.

Theorem 1.5. *For* $z \in \mathbb{R}$,

$$\Delta_{des(\pi),z} \leq \frac{1.061 + \left(1.265 + 3.78e^{\frac{2\sqrt{3}}{\sqrt{n}}}\right) e^{\frac{3}{4}\left(1+\frac{4.62}{\sqrt{n}}\right)} + 8.79e^{\frac{3}{2}\left(1+\frac{4.62}{\sqrt{n}}\right)}}{e^{\frac{|z|}{4}}\sqrt{n}}.$$

Moreover for sufficiently large n,

$$\Delta_{des(\pi),z} \leq \frac{51.25}{e^{\frac{|z|}{4}}\sqrt{n}}.$$

Theorem 1.6. *For* $z \in \mathbb{R}$,

$$\Delta_{inv(\pi),z} \leq \frac{0.008 + \left(1.388 + 0.46e^{\frac{6}{\sqrt{n}}}\right) e^{\frac{9}{4}\left(1+\frac{8}{\sqrt{n}}\right)} + 8.58e^{\frac{9}{2}\left(1+\frac{8}{\sqrt{n}}\right)}}{e^{\frac{|z|}{4}}\sqrt{n}}.$$

Moreover for sufficiently large n,

$$\Delta_{inv(\pi),z} \leq \frac{792.71}{e^{\frac{|z|}{4}}\sqrt{n}}.$$

Table 1.2 compares the constants of uniform and exponential non-uniform bounds.

	uniform bounds	exponential bounds			
		$z = 5.6$	$z = 16.1$	$z = 100$	$z = 1000$
Descent	$\frac{12.44}{\sqrt{n}}$	$\frac{12.44}{\sqrt{n}}$	$\frac{0.92}{\sqrt{n}}$	$\frac{7.12 \times 10^{-10}}{\sqrt{n}}$	$\frac{1.37 \times 10^{-107}}{\sqrt{n}}$
Inversion	$\frac{14.24}{\sqrt{n}}$	$\frac{196}{\sqrt{n}}$	$\frac{14.24}{\sqrt{n}}$	$\frac{1.11 \times 10^{-8}}{\sqrt{n}}$	$\frac{2.12 \times 10^{-106}}{\sqrt{n}}$

Table 1.2: Constants of uniform and exponential non-uniform bounds.

Table 1.3 and Table 1.4 compare the constants of polynomial and exponential non-uniform bounds for the number of descents and the number of inversions, respectively.

	$z=10$	$z=25$	$z=50$	$z=100$	$z=1000$
Polynomial bounds	$\frac{1.32}{\sqrt{n}}$	$\frac{0.10}{\sqrt{n}}$	$\frac{1.33\times10^{-2}}{\sqrt{n}}$	$\frac{1.71\times10^{-3}}{\sqrt{n}}$	$\frac{1.76\times10^{-6}}{\sqrt{n}}$
Exponential bounds	$\frac{4.21}{\sqrt{n}}$	$\frac{0.10}{\sqrt{n}}$	$\frac{1.91\times10^{-4}}{\sqrt{n}}$	$\frac{7.12\times10^{-10}}{\sqrt{n}}$	$\frac{1.37\times10^{-107}}{\sqrt{n}}$

Table 1.3: Constants of polynomial and exponential bounds for the number of descents.

	$z=10$	$z=44$	$z=70$	$z=100$	$z=1000$
Polynomial bounds	$\frac{9.14}{\sqrt{n}}$	$\frac{0.14}{\sqrt{n}}$	$\frac{3.4\times10^{-2}}{\sqrt{n}}$	$\frac{1.2\times10^{-2}}{\sqrt{n}}$	$\frac{1.22\times10^{-5}}{\sqrt{n}}$
Exponential bounds	$\frac{65.07}{\sqrt{n}}$	$\frac{0.14}{\sqrt{n}}$	$\frac{2\times10^{-5}}{\sqrt{n}}$	$\frac{1.11\times10^{-8}}{\sqrt{n}}$	$\frac{2.12\times10^{-106}}{\sqrt{n}}$

Table 1.4: Constants of polynomial and exponential bounds for the number of inversions.

Exponential bounds in Theorem 1.5 and Theorem 1.6 are much sharper than polynomial bounds in Theorem 1.3 and Theorem 1.4 in case of $z \geq 25$ and $z \geq 44$, respectively.

This thesis is organized into 8 chapters as follows. After this introduction chapter, we give preliminary results in Chapter 2. Exchangeable pairs in the Stein's method for normal approximation are obtained in Chapter 3. Moments of the number of Descents and Inversions are given in Chapter 4. Uniform bounds are presented in Chapter 5. Polynomial and exponential non-uniform bounds are presented in Chapter 6 and Chapter 7 respectively. The last chapter contains discussion on further research.

CHAPTER II

PRELIMINARIES

In this chapter, we give basic concepts in probability which will be used in our work. The proofs are omitted but can be found in [1], [2], [20] and [21].

2.1 Probability Space and Random Variables

Let Ω be a nonempty set and $\mathcal{F}$ be a σ-algebra of subsets of Ω .

Let $P : \mathcal{F} \to [0, 1]$ be a measure such that $P(\Omega) = 1$. Then $(\Omega, \mathcal{F}, P)$ is called a **probability space** and P, **a probability measure.** The set Ω is the **sample space** and the elements of $\mathcal{F}$ are called **events.** For any event A, the value $P(A)$ is called the **probability of** A.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is said to be a **random variable** if for every Borel set B in $\mathbb{R}$,

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}.$$

We shall usually use the notation $P(X \in B)$ in stead of $P(\{\omega \in \Omega | X(\omega) \in B\})$. In the case where $B = (-\infty, a]$ or $[a, b]$, $P(X \in B)$ is denoted by $P(X \leq a)$ or $P(a \leq X \leq b)$, respectively.

Let X be a random variable. A function $F : \mathbb{R} \to [0, 1]$ which is defined by

$$F(x) = P(X \leq x)$$

is called the **distribution function** of X.

Let X be a random variable with the distribution function F. X is said to be a **discrete random variable** if the image of X is countable and X is called a **continuous random variable** if F can be written in the form

$$F(x) = \int_{-\infty}^{x} f(t)dt \quad \text{for} \quad x \in \mathbb{R}$$

for some nonnegative integrable function f on $\mathbb{R}$. In this case, we say that f is the **probability density function** of X.

We will give some examples of random variable. We say that X is a **normal** random variable with parameter μ and σ^2, written as $X \sim \mathcal{N}(\mu, \sigma^2)$, if its probability density function is defined by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{1}{2\sigma^2}(x-\mu)^2\Big) \quad \text{for} \quad x \in \mathbb{R}.$$

Moreover, if $X \sim \mathcal{N}(0,1)$ then X is said to be the **standard normal** random variable.

We say that X is a **uniform** random variable on $\{x_1, x_2, \ldots, x_n\}$ if $P\left(X = x_i\right) = \dfrac{1}{n}$ for all $i = 1, 2, \ldots, n$.

2.2 Independence

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_\alpha$ be sub σ-algebras of $\mathcal{F}$ for all $\alpha \in \Lambda$. We say that $\{\mathcal{F}_\alpha | \alpha \in \Lambda\}$ is **independent** if and only if for any subset $J = \{j_1, j_2, \ldots, j_k\}$ of Λ and $A_m \in \mathcal{F}_{j_m}$ for $m = 1, \ldots, k$,

$$P\left(\bigcap_{m=1}^{k} A_m\right) = \prod_{m=1}^{k} P(A_m).$$

A set of random variables $\{X_\alpha |\ \alpha \in \Lambda\}$ is **independent** if $\{\sigma(X_\alpha) |\ \alpha \in \Lambda\}$ is independent, where $\sigma(X) = \sigma(\{X^{-1}(B) \mid B \text{ is a Borel subset of } \mathbb{R}\})$. We say that $X_1, X_2, \ldots, X_n$ are **independent** if $\{X_1, X_2, \ldots, X_n\}$ is **independent**.

Theorem 2.1. *Random variables* $X_1, X_2, \ldots, X_n$ *are* **independent** *if and only if for any Borel sets* $B_1, B_2, \ldots, B_n$ *we have*

$$P\left(\bigcap_{i=1}^{n}\{X_i \in B_i\}\right) = \prod_{i=1}^{n} P(X_i \in B_i).$$

Proposition 2.2. *If* $X_{ij}; i = 1, 2, \ldots, n; j = 1, 2, \ldots, m_i$ *are independent and* $f_i : \mathbb{R}^{m_i} \to \mathbb{R}$ *are measurable, then* $\{f_i\left(X_{i1}, X_{i2}, \ldots, X_{im_i}\right), i = 1, 2, \ldots, n\}$ *is independent.*

2.3 Expectation and Variance

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$.
If $\int_\Omega |X| dP < \infty$, then we define its **expected value** to be

$$E(X) = \int_\Omega X dP.$$

Proposition 2.3. *Let* X *be a random variable and* $E|X| < \infty$.

1. *If* X *is a discrete random variable, then* $E(X) = \sum_{x \in ImX} xP(X = x)$.
2. *If* X *is a continuous random variable with probability density function* f*, then*

$$E(X) = \int_{\mathbb{R}} x f(x) dx.$$

Let X be a random variable with $E(|X|^k) < \infty$. Then $E(|X|^k)$ is called the **k-th moment of X about the origin** and $E[X - E(X)]^k$ is called the **k-th moment of X about the mean**.

We call the second moment of X about the mean, the **variance** of X and denoted by $Var(X)$. Then

$$Var(X) = E[X - E(X)]^2.$$

Note that

1. $Var(X) \;=\; E(X^2) - E^2(X)$,

2. if $X \sim \mathcal{N}(\mu, \sigma^2)$, then $E(X) = \mu$ and $Var(X) = \sigma^2$.

Proposition 2.4. *If $X_1, X_2, \ldots, X_n$ are independent and $E|X_i| < \infty$ for $i = 1, 2, \ldots, n$, then*

1. $E(X_1X_2\cdots X_n) = E(X_1)E(X_2)\cdots E(X_n)$,

2. $Var(a_1X_1+a_2X_2+\cdots+a_nX_n) \;=\; a_1^2Var(X_1)+a_2^2Var(X_2)+\cdots+a_n^2Var(X_n)$ *for any real numbers $a_1, a_2, \ldots, a_n$.*

The following inequalities are useful in our work.

1. **Hölder's inequality :**

$$E(|XY|) \leq E^{\frac{1}{p}}(|X|^p)E^{\frac{1}{q}}(|Y|^q)$$

where $0 < p, q < \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $E(|X|^p) < \infty$, $E(|Y|^q) < \infty$.

2. **Chebyshev's inequality :**

$$P(\{|X - E(X)| \geq \varepsilon\}) \leq \frac{Var(X)}{\varepsilon^2} \quad \text{for all} \quad \varepsilon > 0$$

where $E(X^2) < \infty$.

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with $E\,|X| < \infty$ and $\mathcal{D}$ a sub σ-algebra of $\mathcal{F}$. Define a probability measure $P_{\mathcal{D}} : \mathcal{D} \to [0, 1]$ by

$$P_{\mathcal{D}}(A) = P(A)$$

and sign-measure $\mathcal{Q}_X : \mathcal{D} \to \mathbb{R}$ by

$$\mathcal{Q}_X(A) = \int_A X dP.$$

Then $\mathcal{Q}_X$ is absolutely continuous with respect to $P_{\mathcal{D}}$ and there exists a unique measurable function $E^{\mathcal{D}}(X)$ on $(\Omega, \mathcal{F}, \mathcal{D})$ such that

$$\int_A E^{\mathcal{D}}(X)dP_{\mathcal{D}} = \mathcal{Q}_X(A) = \int_A XdP \quad \text{for any } A \in \mathcal{D}.$$

We will say that $E^{\mathcal{D}}(X)$ is the **conditional expectation** of X with respect to D.

Moreover, for any random variables X and Y on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|) < \infty$, we will denote $E^{\sigma(Y)}(X)$ by $E^Y(X)$.

Theorem 2.5. *Let X be a random variable on probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|) < \infty$. Then the followings hold for any sub σ-algebra $\mathcal{D}$ of $\mathcal{F}$:*

1. *If X is random variable on (Ω, D, P_D), then $E^{\mathcal{D}}(X) = X$ a.s.$[P_D]$,*

2. *$E^{\mathcal{F}}(X) = X$ a.s.$[P]$,*

3. *If $\sigma(X)$ and $\mathcal{D}$ are independent, then $E^{\mathcal{D}}(X) = E(X)$ a.s.$[P_D]$.*

CHAPTER III

EXCHANGEABLE PAIR IN STEIN'S METHOD

One outstanding method in finding a Berry-Esseen bound is the Stein's method which is provided in 1972 by Stein ([20]). His method does not use Fourier transformation. It starts with the differential equation which is called the Stein's equation for normal approximation, i.e.,

$$g'(w) - wg(w) = \mathbb{I}_{(-\infty,z]}(w) - \Phi(z) \text{ for all } z \in \mathbb{R}, \tag{3.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous and piecewise differentiable function and

$$\mathbb{I}_{(-\infty,z]}(w) = \begin{cases} 1 & \text{if } w \leq z, \\ 0 & \text{if } w > z. \end{cases}$$

The solution g_z of (3.1) is given by

$$g_z(w) = \begin{cases} \sqrt{2\pi}e^{\frac{w^2}{2}}\Phi(w)[1-\Phi(z)] & \text{if } w \leq z, \\ \sqrt{2\pi}e^{\frac{w^2}{2}}\Phi(z)[1-\Phi(w)] & \text{if } w > z. \end{cases} \tag{3.2}$$

Hence,

$$g'_z(w) = \begin{cases} [1-\Phi(z)]\left[1+\sqrt{2\pi}we^{\frac{w^2}{2}}\Phi(w)\right] & \text{if } w < z, \\ \Phi(z)\left[-1+\sqrt{2\pi}we^{\frac{w^2}{2}}(1-\Phi(w))\right] & \text{if } w > z, \end{cases} \tag{3.3}$$

([6], pp. 252).

In our work, we need the following properties of g'_z:

$$|g'_z(w_1) - g'_z(w_2)| \leq 1 \quad \text{for all } w_1, w_2 \in \mathbb{R}, \tag{3.4}$$

$$|g'_z(w)| \leq 1 \quad \text{for all } w \in \mathbb{R}, \tag{3.5}$$

([1], pp. 10).

From (3.1), for any random variable W, we have

$$Eg'_z(W) - EWg_z(W) = P(W \leq z) - \Phi(z). \tag{3.6}$$

Then we can bound

$$|Eg'_z(W) - EWg_z(W)|$$

instead of

$$|P(W \leq z) - \Phi(z)|.$$

This technique is called the **Stein's method**.

The important technique in order to bound $|Eg'_z(W) - EWg_z(W)|$ is rewriting $EWg_z(W)$ in a suitable form. There are three approaches called exchangeable pair ([7],[19],[21]), size bias ([7],[12]) and zero bias ([7],[12]). In our work, we use exchangeable pair technique. The pair (W, W') is an **exchangeable pair** if

$$P(W \leq w_1, W' \leq w_2) = P(W \leq w_2, W' \leq w_1) \text{ for } w_1, w_2 \in \mathbb{R}.$$

Many authors (see for examples [7], pp.23) constructed an exchangeable pair W' of W which has the property that

$$E^W(W' - W) = -\lambda W \quad \text{with} \quad \lambda \in (0,1). \tag{3.7}$$

If (W, W') satisfies (3.7), then we have the following lemma.

Lemma 3.1. *([7], pp.22) Let (W, W') be an exchangeable pair satisfying* (3.7). *Then for a continuous and piecewise continuously differentiable function $g : \mathbb{R} \to \mathbb{R}$, we have*

$$EWg(W) = E\int_{-\infty}^{\infty} g'(W+t)\,K(t)\,dt$$

where

$$K(t) = \frac{1}{2\lambda}(W' - W)\{\mathbb{I}(0 \leq t \leq W' - W) - \mathbb{I}(W' - W \leq t \leq 0)\}$$

and $\mathbb{I}(A)$ *is given by*

$$\mathbb{I}(A)\,(w) = \begin{cases} 1 & \text{if } w \in A, \\ 0 & \text{if } w \notin A. \end{cases}$$

The exchangeable pair technique can be used to give uniform bounds and non-uniforms bound in normal approximation.

In 1997, Rinott and Rotar ([19]) used this technique and the Stein's method to give a theorem for normal approximation. Theorem 3.2 is one of their main results.

Theorem 3.2. *([19]) Let* W *be a random variable with* $EW = 0$ *and* $E\left(W^2\right) = 1$*. Let* W' *be an exchangeable pair of* W *satisfying* (3.7)*. Suppose in addition that* $|W' - W| \leq A$ *for some constant* A*. Then for* $z \in \mathbb{R}$*,*

$$|P\left(W \leq z\right) - \Phi\left(z\right)| \leq \frac{12}{\lambda}\sqrt{Var\left(E^W\left(W' - W\right)^2\right)} + 48\frac{A^3}{\lambda} + 8\frac{A^2}{\sqrt{\lambda}}.$$

At the end of this chapter, we present some examples of an exchangeable pair.

Example 3.3. ([7], pp.23) Let $X_1, \ldots, X_n$ be independent random variables and $X_1', \ldots, X_n'$ be an independent copy of $X_1, \ldots, X_n$. Let I be a uniform random variable on $\{1, \ldots, n\}$ which is independent to $X_1, \ldots, X_n, X_1', \ldots, X_n'$. Let $W = \sum_{i=1}^{n} X_i$. Define

$$W' = W - X_I + X_I'.$$

Then (W, W') is an exchangeable pair satisfying (3.7) with $\lambda = \frac{1}{n}$.

Example 3.4. ([21], pp. 38–39) Let $\{a_{ij}\}_{\{1 \leq i,j \leq n\}}$ be an array of real numbers. For a random permutation π on $\{1, 2, \ldots, n\}$, define

$$W = \sum_{i=1}^{n} a_{i\pi(i)}.$$

Let (J, K) be a uniformly distributed random variable of the set $\{(j,k)\,|1 \le j,k \le n, j \ne k\}$ so that J and K are independent of π.

Define π' by

$$\pi'(i) = \begin{cases} \pi(i) & \text{if } i \notin \{J, K\}, \\ \pi(K) & \text{if } i = J, \\ \pi(J) & \text{if } i = K. \end{cases}$$

Let

$$W' = \sum_{i=1}^{n} a_{i\pi'(i)} = W + a_{J,\pi(K)} + a_{K,\pi(J)} - a_{J,\pi(J)} - a_{K,\pi(K)}.$$

Then for all $a, b \in \mathbb{R}$, we have

$$\begin{aligned}
&P(W \le a, W' \le b) \\
&= \sum_{\substack{j,k \\ j \ne k}} \sum_{(l_1,l_2,\ldots,l_n)\in S_n} P\Bigg(\sum_i a_{il_i} \le a, a_{1l_1} + \cdots + a_{jl_k} + \cdots + a_{kl_j} + \cdots \\
&\qquad + a_{nl_n} \le b, (J,K) = (j,k), \pi = (l_1, l_2, \ldots, l_n) \Bigg) \\
&= \sum_{\substack{j,k \\ j \ne k}} \sum_{(l_1,l_2,\ldots,l_n)\in S_n} P\Bigg(\sum_i a_{il_i} \le a, a_{1l_1} + \cdots + a_{jl_k} + \cdots + a_{kl_j} + \cdots \\
&\qquad + a_{nl_n} \le b, (J,K) = (j,k), \\
&\qquad \pi = (l_1, \ldots, l_{j-1}, l_k, l_{j+1}, \ldots, l_{k-1}, l_j, l_{k+1} \ldots, l_n) \Bigg) \\
&= \sum_{\substack{j,k \\ j \ne k}} \sum_{(l_1,l_2,\ldots,l_n)\in S_n} P\Bigg(a_{1l_1} + \cdots + a_{jl_k} + \cdots + a_{kl_j} + \cdots + a_{nl_n} \le a, \\
&\qquad \sum_i a_{il_i} \le b, (J,K) = (j,k), \pi = (l_1, l_2, \ldots, l_n) \Bigg) \\
&= P(W' \le a, W \le b)
\end{aligned}$$

([15], pp.561-562). This implies that W' is an exchangeable pair of W and satisfies (3.7) with $\lambda = \dfrac{2}{n-1}$.

Example 3.5. ([11], pp.68-70) Let

$$X_1(\pi) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} M_{\pi(i)\pi(j)}$$

where π is a random permutation of the set $\{1, 2, \ldots, n\}$ and (M_{ij}) is a real $n \times n$ matrix defined by

$$M_{ij} = \begin{cases} -1 & \text{if } j = i+1, \\ 1 & \text{if } j = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

For a random permutation π on $\{1, 2, \ldots, n\}$, define

$$\pi'(i) = \begin{cases} \pi(i) & \text{if } i \notin \{I, I+1, \ldots, n\}, \\ \pi(i+1) & \text{if } i \in \{I, I+1, \ldots, n-1\}, \\ \pi(I) & \text{if } i = n, \end{cases} \tag{3.8}$$

where I is a uniformly distributed random variable on $\{1, 2, \ldots, n\}$. Then $X_1(\pi')$ is an exchangeable pair of $X_1(\pi)$ and satisfies (3.7) with $\lambda = \dfrac{2}{n}$.

Example 3.6. ([11], pp.68-70) Let

$$X_2(\pi) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} Q_{\pi(i)\pi(j)}$$

where π is a random permutation on $\{1, 2, \ldots, n\}$ and (Q_{ij}) is a real $n \times n$ matrix defined by

$$Q_{ij} = \begin{cases} -1 & \text{if } j > i, \\ 1 & \text{if } j < i, \\ 0 & \text{if } j = i. \end{cases}$$

For π' defined by (3.8), we have $X_2(\pi')$ is an exchangeable pair of $X_2(\pi)$ with $\lambda = \dfrac{2}{n}$.

CHAPTER IV

MOMENTS OF DESCENT AND INVERSION

In this chapter, we give auxiliary results of both the number of descents and the number of inversions. Recall that for a uniformly distributed random permutation π of the set $\{1, 2, \ldots, n\}$, the number of descents of π, denoted by $des\,(\pi)$, is defined as the number of pairs $(i, i+1)$ with $1 \le i \le n-1$ and $\pi\,(i) > \pi\,(i+1)$. The number of inversions of π, denoted by $inv\,(\pi)$, is defined as the number of pairs (i, j) with $1 \le i < j \le n$ and $\pi\,(i) > \pi\,(j)$.

For each permutation π,

let

$$U\,(\pi) := \frac{des\,(\pi) - E\,(des\,(\pi))}{\sqrt{Var\,(des\,(\pi))}}$$

and

$$V\,(\pi) := \frac{inv\,(\pi) - E\,(inv\,(\pi))}{\sqrt{Var\,(inv\,(\pi))}}.$$

Fulman ([11], p.66) showed that

$$U\,(\pi) = \frac{\sqrt{3}X_1\,(\pi^{-1})}{\sqrt{n+1}}, \tag{4.1}$$

where

$$X_1\,(\pi) = \sum_{i=1}^{n}\sum_{j=i+1}^{n} M_{\pi(i)\pi(j)}$$

and (M_{ij}) is a real $n \times n$ matrix defined by

$$M_{ij} = \begin{cases} -1 & \text{if } j = i+1, \\ 1 & \text{if } j = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, he also showed that

$$V(\pi) = \frac{\sqrt{18}X_2(\pi^{-1})}{\sqrt{n(n-1)(2n+5)}}, \tag{4.2}$$

where

$$X_2(\pi) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} Q_{\pi(i)\pi(j)}$$

and (Q_{ij}) is a real $n \times n$ matrix defined by

$$Q_{ij} = \begin{cases} -1 & \text{if } j > i, \\ 1 & \text{if } j < i, \\ 0 & \text{if } j = i. \end{cases}$$

Note that

$$E(des(\pi)) = \frac{n-1}{2}, Var(des(\pi)) = \frac{n+1}{12},$$
$$E(inv(\pi)) = \frac{\binom{n}{2}}{2} \text{ and } Var(inv(\pi)) = \frac{n(n-1)(2n+5)}{72}$$

([11], pp.71). For convenience, for $\pi \in S_n$, let

$$U := U(\pi), \tag{4.3}$$

and

$$V := V(\pi). \tag{4.4}$$

Then both (U', U) and (V', V), when $U'(\pi) := U(\pi')$ and $V'(\pi) := V(\pi')$ for π' defined in (3.8), are exchangeable pairs satisfying (3.7) with $\lambda = \frac{2}{n}$, i.e.

$$E^{\pi}(U' - U) = -\frac{2}{n}U, \tag{4.5}$$

$$E^{\pi}(V' - V) = -\frac{2}{n}V, \tag{4.6}$$

([11], pp.68-70). By the definition of U and V, we have the folowing facts.

$$E(U) = 0 \tag{4.7}$$

$$E(V) = 0 \tag{4.8}$$

$$Var(U) = EU^2 = 1 \tag{4.9}$$

$$Var(V) = EV^2 = 1. \tag{4.10}$$

From the construction of U' and V', we have the folowing properties;

$$|U' - U| \leq \frac{2\sqrt{3}}{\sqrt{n}} \tag{4.11}$$

$$|V' - V| \leq \frac{6}{\sqrt{n}} \tag{4.12}$$

([11], pp.72).

In Proposition 4.1 and Proposition 4.2, we give the bounds for the 4^{th} moment and 6^{th} moment of U and V, respectively.

Proposition 4.1. *Let U and V be as in* (4.3) *and* (4.4)*. Then*

(i) $EU^4 \leq 9 + \frac{31.18}{\sqrt{n}} + \frac{36}{n}$.

(ii) $EV^4 \leq 27 + \frac{162}{\sqrt{n}} + \frac{324}{n}$.

Proof. (i) Let $g(x) = x^3$. Then, by Lemma 3.1 and (4.5), we have

$$EU^4 = E\int_{-\infty}^{\infty} 3(U+t)^2 K(t)\,dt \tag{4.13}$$

where

$$K(t) = \frac{n}{4}(U' - U)\{\mathbb{I}(0 \leq t \leq U' - U) - \mathbb{I}(U' - U \leq t \leq 0)\}.$$

By (4.13) and the fact that

$$\int_{-\infty}^{\infty} t^k K(t)\,dt = \frac{n}{4(k+1)}(U' - U)^{k+2}, \tag{4.14}$$

we obtain

$$\begin{aligned} EU^4 &= 3E\int_{-\infty}^{\infty} U^2K(t)\,dt + 6EU\int_{-\infty}^{\infty} tK(t)\,dt + 3E\int_{-\infty}^{\infty} t^2K(t)\,dt \\ &= \frac{3n}{4}EU^2(U'-U)^2 + \frac{6n}{8}EU(U'-U)^3 + \frac{3n}{12}E(U'-U)^4. \end{aligned}$$

From (4.9) and (4.11) we have

$$EU^4 \le 9 + \frac{31.18}{\sqrt{n}} + \frac{36}{n}.$$

(ii) Using the same argument of (i), (4.6), (4.10) and (4.12), we complete the proof. □

Proposition 4.2. *Let U and V be as in (4.3) and (4.4). Then for sufficiently large n,*

(i) $EU^6 \le 621$.

(ii) $EV^6 \le 1319$.

Proof. (i) Similar to Proposition 4.1, we choose $g(x) = x^5$. Then

$$\begin{aligned} EU^6 &= 5E\int_{-\infty}^{\infty} (U+t)^4 K(t)\,dt \\ &= \frac{5n}{4}EU^4(U'-U)^2 + \frac{20n}{8}EU^3(U'-U)^3 + \frac{30n}{12}EU^2(U'-U)^4 \\ &\quad + \frac{20n}{16}EU(U'-U)^5 + \frac{5n}{20}E(U'-U)^6. \end{aligned}$$

By using the Hölder's inequality, (4.9), (4.11) and Proposition 4.1(i) we can show that for $n \ge 10$

$$\begin{aligned} EU^6 &\le 15EU^4 + \frac{104}{\sqrt{n}}\left(EU^4\right)^{\frac{1}{2}}\left(EU^2\right)^{\frac{1}{2}} + \frac{360}{n}EU^2 + \frac{312}{n\sqrt{n}} + \frac{432}{n^2} \\ &\le 621. \end{aligned}$$

(ii) Using (4.10), (4.12), Proposition 4.1(ii) and the same argument of (i), we can prove (ii). □

Proposition 4.3. *Let U and V be as in* (4.3) *and* (4.4). *Then*

(i) $E\left(U'-U\right)^4=\dfrac{48}{n\left(n+1\right)}$.

(ii) $E\left(V'-V\right)^4=\dfrac{1}{n^2}\left(\dfrac{1296}{25}-\dfrac{23904}{1225(n-1)}+\dfrac{15552}{175(2n+5)^2}-\dfrac{15696}{1225(2n+5)}\right)$.

Proof. (i) By the facts that, for each i

$$M_{ii_1}^{k_1}M_{ii_2}^{k_2}M_{ii_3}^{k_3}=0 \text{ for all distinct } i_1,i_2,i_3, \tag{4.15}$$

$$\sum_{\substack{i_1,i_2=1\\ i_1\neq i_2}}^{n} M_{i_1i_2}^4=2\left(n-1\right), \tag{4.16}$$

$$\sum_{\substack{i_1,i_1,i_3=1\\ i_1\neq i_2\neq i_3}}^{n} M_{i_1i_2}M_{i_1i_3}=-2\left(n-2\right), \tag{4.17}$$

$$\sum_{\substack{i_1,i_1,i_3=1\\ i_1\neq i_2\neq i_3}}^{n} M_{i_1i_2}^2M_{i_1i_3}^2=2\left(n-2\right), \tag{4.18}$$

we have

$$\begin{aligned}
&\sum_{l=1}^{n}E\left(\sum_{j=l+1}^{n}M_{\pi(l)\pi(j)}\right)^4\\
&=\sum_{l=1}^{n}E\left(\sum_{j=l+1}^{n}M_{\pi(l)\pi(j)}^4+4\sum_{\substack{j,k=l+1\\ j\neq k}}^{n}M_{\pi(l)\pi(j)}^3M_{\pi(l)\pi(k)}\right.\\
&\quad+3\sum_{\substack{j,k=l+1\\ j\neq k}}^{n}M_{\pi(l)\pi(j)}^2M_{\pi(l)\pi(k)}^2+6\sum_{\substack{j,k,m=l+1\\ j\neq k\neq m}}^{n}M_{\pi(l)\pi(j)}^2M_{\pi(l)\pi(k)}M_{\pi(l)\pi(m)}\\
&\quad\left.+\sum_{\substack{j,k,m,p=l+1\\ j\neq k\neq m\neq p}}^{n}M_{\pi(l)\pi(j)}M_{\pi(l)\pi(k)}M_{\pi(l)\pi(m)}M_{\pi(l)\pi(p)}\right)\\
&=\sum_{l=1}^{n}E\left(\sum_{j=l+1}^{n}M_{\pi(l)\pi(j)}^4+4\sum_{\substack{j,k=l+1\\ j\neq k}}^{n}M_{\pi(l)\pi(j)}^3M_{\pi(l)\pi(k)}\right.\\
&\quad\left.+3\sum_{\substack{j,k=l+1\\ j\neq k}}^{n}M_{\pi(l)\pi(j)}^2M_{\pi(l)\pi(k)}^2\right)
\end{aligned}$$

$$= \frac{1}{n(n-1)} \sum_{l=1}^{n} \left(\sum_{j=l+1}^{n} \sum_{\substack{i_1,i_2 \\ i_1 \neq i_2}}^{n} M_{i_1 i_2}^4 + \frac{4}{n-2} \sum_{\substack{j,k=l+1 \\ j \neq k}}^{n} \sum_{\substack{i_1,i_1,i_3 \\ i_1 \neq i_2 \neq i_3}}^{n} M_{i_1 i_2}^3 M_{i_1 i_3} \right.$$

$$\left. + \frac{3}{n-2} \sum_{\substack{j,k=l+1 \\ j \neq k}}^{n} \sum_{\substack{i_1,i_1,i_3 \\ i_1 \neq i_2 \neq i_3}}^{n} M_{i_1 i_2}^2 M_{i_1 i_3}^2 \right)$$

$$= \frac{n+1}{3}. \tag{4.19}$$

From this fact and the fact that for each fixed π, I,

$$X_1'(\pi) - X_1(\pi) = \sum_{j=I+1}^{n} -2M_{\pi(I)\pi(j)},$$

we have

$$\begin{aligned}
& E(U' - U)^4 \\
& = E\left(U'\left(\pi^{-1}\right) - U\left(\pi^{-1}\right)\right)^4 \\
& = \frac{9}{(n+1)^2} E\left(X_1'(\pi) - X_1(\pi)\right)^4 \\
& = \frac{9}{(n+1)^2} E E^{\pi}\left(X_1'(\pi) - X_1(\pi)\right)^4 \\
& = \frac{9}{n(n+1)^2} \sum_{l=1}^{n} E\left(\sum_{j=l+1}^{n} -2M_{\pi(l)\pi(j)}\right)^4 \\
& = \frac{144}{n(n+1)^2} \sum_{l=1}^{n} E\left(\sum_{j=l+1}^{n} M_{\pi(l)\pi(j)}\right)^4 \\
& = \frac{48}{n(n+1)}.
\end{aligned}$$

(ii) We can prove (ii) by using the same argument of (i) together with the following facts,

$$\sum_{\substack{i_1,i_2=1 \\ i_1 \neq i_2}}^{n} Q_{i_1 i_2}^4 = n(n-1),$$

$$\sum_{\substack{i_1,i_2,i_3=1 \\ i_1 \neq i_2 \neq i_3}}^{n} Q_{i_1 i_2} Q_{i_1 i_3} = \frac{n(n-1)(n-2)}{3},$$

$$\sum_{\substack{i_1,i_1,i_3=1\\ i_1\neq i_2\neq i_3}}^{n} Q_{i_1i_2}^2 Q_{i_1i_3}^2 = n\left(n-1\right)\left(n-2\right),$$

$$\sum_{\substack{i,i_1,i_2,i_3=1\\ i\neq,i_1\neq i_2\neq i_3}}^{n} Q_{ii_1}^2 Q_{ii_2} Q_{ii_3} = \frac{n\left(n-1\right)\left(n-2\right)\left(n-3\right)}{3},$$

$$\sum_{\substack{i,i_1,i_2,i_3,i_4=1\\ i\neq i_1\neq i_2\neq i_3\neq i_4}}^{n} Q_{ii_1} Q_{ii_2} Q_{ii_3} Q_{ii_4} = \frac{n\left(n-1\right)\left(n-2\right)\left(n-3\right)\left(n-4\right)}{5},$$

$$X_2'\left(\pi\right) - X_2\left(\pi\right) = \sum_{j=I+1}^{n} -2Q_{\pi(I)\pi(j)}$$ □

Proposition 4.4. *Let U and V be as in* (4.3) *and* (4.4)*. Then*

(i) $E\left(E^{\pi}\left(U'-U\right)^2\right)^2 = \frac{16}{n^2}\left(1+\frac{8}{5\left(n+1\right)}\right).$

(ii) $E\left(E^{\pi}\left(V'-V\right)^2\right)^2$
$= \frac{16}{n^2}\left(1-\frac{2064}{1225(n-1)}+\frac{2256}{625n}-\frac{864}{875(2n+5)^2}-\frac{29688}{30625(2n+5)}\right).$

Proof. (i) From the fact that

$$E^{\pi}\left(U'-U\right)^2 = \frac{12}{n\left(n+1\right)}\sum_{i=1}^{n}\left(\sum_{j=i+1}^{n} M_{\pi(i)\pi(j)}\right)^2 \tag{4.20}$$

([11], pp.72) and (4.19) we have

$$\begin{aligned}
&E\left(E^{\pi}\left(U'-U\right)^2\right)^2\\
&= \frac{144}{n^2\left(n+1\right)^2} E\left(\sum_{i=1}^{n}\left(\sum_{j=i+1}^{n} M_{\pi(i)\pi(j)}\right)^2\right)^2\\
&= \frac{144}{n^2\left(n+1\right)^2} E\left(\sum_{i=1}^{n}\left(\sum_{j=i+1}^{n} M_{\pi(i)\pi(j)}\right)^4\right.\\
&\quad \left.+\sum_{\substack{i,k=1\\ i\neq k}}^{n}\left(\sum_{j=i+1}^{n} M_{\pi(i)\pi(j)}\right)^2\left(\sum_{l=k+1}^{n} M_{\pi(k)\pi(l)}\right)^2\right)\\
&= \frac{144}{n^2\left(n+1\right)^2}\left(\frac{n+1}{3} + E\sum_{\substack{i,k=1\\ i\neq k}}^{n}\left(\sum_{j=i+1}^{n} M_{\pi(i)\pi(j)}\right)^2\left(\sum_{l=k+1}^{n} M_{\pi(k)\pi(l)}\right)^2\right).
\end{aligned} \tag{4.21}$$

In order to calculate $E \sum_{\substack{i,k=1 \\ i \neq k}}^{n} \left(\sum_{j=i+1}^{n} M_{\pi(i)\pi(j)} \right)^2 \left(\sum_{l=k+1}^{n} M_{\pi(k)\pi(l)} \right)^2$, we expand this sum as below.

$$
\begin{aligned}
& E \sum_{\substack{i,k=1 \\ i \neq k}}^{n} \left(\sum_{j=i+1}^{n} M_{\pi(i)\pi(j)} \right)^2 \left(\sum_{l=k+1}^{n} M_{\pi(k)\pi(l)} \right)^2 \\
&= E \sum_{\substack{i,k=1 \\ i \neq k}}^{n} \left(\sum_{j=i+1}^{n} M^2_{\pi(i)\pi(j)} + \sum_{\substack{j,m=i+1 \\ m \neq j}}^{n} M_{\pi(i)\pi(j)} M_{\pi(i)\pi(m)} \right) \\
&\quad \left(\sum_{l=k+1}^{n} M^2_{\pi(k)\pi(l)} + \sum_{\substack{l,s=k+1 \\ s \neq l}}^{n} M_{\pi(k)\pi(l)} M_{\pi(k)\pi(s)} \right) \\
&= E \sum_{\substack{i,k=1 \\ i \neq k}}^{n} \Bigg(\sum_{j=i+1}^{n} \sum_{l=k+1}^{n} M^2_{\pi(i)\pi(j)} M^2_{\pi(k)\pi(l)} \\
&\quad + \sum_{j=i+1}^{n} \sum_{\substack{l,s=k+1 \\ s \neq l}}^{n} M^2_{\pi(i)\pi(j)} M_{\pi(k)\pi(l)} M_{\pi(k)\pi(s)} \\
&\quad + \sum_{l=k+1}^{n} \sum_{\substack{j,m=i+1 \\ m \neq j}}^{n} M^2_{\pi(k)\pi(l)} M_{\pi(i)\pi(j)} M_{\pi(i)\pi(m)} \\
&\quad + \sum_{\substack{j,m=i+1 \\ m \neq j}}^{n} \sum_{\substack{l,s=k+1 \\ s \neq l}}^{n} M_{\pi(i)\pi(j)} M_{\pi(i)\pi(m)} M_{\pi(k)\pi(l)} M_{\pi(k)\pi(s)} \Bigg) \\
&= E \sum_{\substack{i,k=1 \\ i \neq k}}^{n} \Bigg(\sum_{j=i+1}^{n} \sum_{l=k+1}^{n} M^2_{\pi(i)\pi(j)} M^2_{\pi(k)\pi(l)} \\
&\quad + 2 \sum_{j=i+1}^{n} \sum_{\substack{l,s=k+1 \\ s \neq l}}^{n} M^2_{\pi(i)\pi(j)} M_{\pi(k)\pi(l)} M_{\pi(k)\pi(s)} \\
&\quad + \sum_{\substack{j,m=i+1 \\ j \neq m}}^{n} \sum_{\substack{l,s=k+1 \\ s \neq l}}^{n} M_{\pi(i)\pi(j)} M_{\pi(i)\pi(m)} M_{\pi(k)\pi(l)} M_{\pi(k)\pi(s)} \Bigg) \\
&=: B_{1M} + 2B_{2M} + B_{3M}. \qquad (4.22)
\end{aligned}
$$

From the facts that

$$\sum_{\substack{i_1,i_2,i_3=1\\ i_1\neq i_2\neq i_3}}^{n} M_{i_1i_2}^2 M_{i_3i_1}^2 = 2\left(n-2\right),$$

$$\sum_{\substack{i_1,i_2,i_3=1\\ i_1\neq i_2\neq i_3}}^{n} M_{i_1i_2}^2 M_{i_2i_3}^2 = 2\left(n-2\right),$$

$$\sum_{\substack{i_1,i_2,i_3=1\\ i_1\neq i_2\neq i_3}}^{n} M_{i_1i_2}^2 M_{i_3i_2}^2 = 2\left(n-2\right),$$

$$\sum_{\substack{i_1,i_2,i_3,i_4=1\\ i_1\neq i_2\neq i_3\neq i_4}}^{n} M_{i_1i_2}^2 M_{i_3i_4}^2 = 2\left(2n-4\right)\left(n-3\right),$$

we have

Cases	B_{1M}
$i=l, k\neq j, j\neq l$	$\frac{n-2}{3}$
$i\neq l, k=j, j\neq l$	$\frac{n-2}{3}$
$i\neq l, k\neq j, j=l$	$\frac{2}{3}\left(n-2\right)$
$i\neq l, k\neq j, j\neq l$	$\left(n-2\right)\left(n-3\right)$

This implies that

$$B_{1M} = \frac{10}{3} - \frac{11}{3}n + n^2. \tag{4.23}$$

Next we try to find B_{2M}. From the facts that

$$\sum_{\substack{i_1,i_2,i_3=1\\ i_1\neq i_2\neq i_3}}^{n} M_{i_1i_2}^2 M_{i_3i_1} M_{i_3i_2} = 0,$$

$$\sum_{\substack{i_1,i_2,i_3,i_4=1\\ i_1\neq i_2\neq i_3\neq i_4}}^{n} M_{i_1i_2}^2 M_{i_3i_1} M_{i_3i_4} = -2\left(n-3\right),$$

$$\sum_{\substack{i_1,i_2,i_3=1\\ i_1\neq i_2\neq i_3}}^{n} M_{i_1i_2}^2 M_{i_3i_2} M_{i_3i_1} = 0,$$

$$\sum_{\substack{i_1,i_2,i_3,i_4=1\\ i_1\neq i_2\neq i_3\neq i_4}}^{n} M_{i_1i_2}^2 M_{i_3i_4} M_{i_3i_1} = -2\left(n-3\right),$$

$$\sum_{\substack{i_1,i_2,i_3,i_4=1\\ i_1\neq i_2\neq i_3\neq i_4}}^{n} M_{i_1i_2}^2 M_{i_2i_3} M_{i_2i_4} = 0,$$

$$\sum_{\substack{i_1,i_2,i_3,i_4=1\\ i_1\neq i_2\neq i_3\neq i_4}}^{n} M_{i_1i_2}^2 M_{i_3i_2} M_{i_3i_4} = -2\left(n-3\right),$$

$$\sum_{\substack{i_1,i_2,i_3,i_4=1\\ i_1\neq i_2\neq i_3\neq i_4}}^{n} M_{i_1i_2}^2 M_{i_3i_4} M_{i_3i_2} = -2\left(n-3\right),$$

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5}}^{n} M_{i_1i_2}^2 M_{i_3i_4} M_{i_3i_5} = -4\left(n-3\right)\left(n-4\right),$$

we obtain

Cases	B_{2M}
$i=l, i\neq s, k\neq j, j\neq l, j=s$	0
$i=l, i\neq s, k\neq j, j\neq l, j\neq s$	$-\frac{n-3}{4}$
$i\neq l, i=s, k\neq j, j=l, j\neq s$	0
$i\neq l, i=s, k\neq j, j\neq l, j\neq s$	$-\frac{n-3}{4}$
$i\neq l, i\neq s, k=j, j\neq l, j\neq s$	0
$i\neq l, i\neq s, k\neq j, j=l, j\neq s$	$-\frac{10}{24}\left(n-3\right)$
$i\neq l, i\neq s, k\neq j, j\neq l, j=s$	$-\frac{10}{24}\left(n-3\right)$
$i\neq l, i\neq s, k\neq j, j\neq l, j\neq s$	$-8+\frac{14}{3}n-\frac{2}{3}n^2$

Hence

$$B_{2M} = -4 + \frac{10}{3}n - \frac{2}{3}n^2. \tag{4.24}$$

Finally we find B_{3M}. By the fact that

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5}}^{n} M_{i_1i_3} M_{i_1i_4} M_{i_2i_1} M_{i_2i_5} = 0,$$

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5}}^{n} M_{i_1i_3} M_{i_1i_4} M_{i_2i_5} M_{i_2i_1} = 0,$$

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5}}^{n} M_{i_1i_2}M_{i_1i_3}M_{i_2i_4}M_{i_2i_5} = 0,$$

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5}}^{n} M_{i_1i_3}M_{i_1i_2}M_{i_2i_4}M_{i_2i_5} = 0,$$

$$\sum_{\substack{i_1,i_2,i_3,i_4=1\\ i_1\neq i_2\neq i_3\neq i_4}}^{n} M_{i_1i_3}M_{i_1i_4}M_{i_2i_3}M_{i_2i_4} = 0,$$

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5}}^{n} M_{i_1i_2}M_{i_1i_3}M_{i_4i_2}M_{i_4i_5} = 2\left(n-4\right),$$

$$\sum_{\substack{i_1,i_2,i_3,i_4=1\\ i_1\neq i_2\neq i_3\neq i_4}}^{n} M_{i_1i_3}M_{i_1i_4}M_{i_2i_4}M_{i_2i_3} = 0,$$

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5}}^{n} M_{i_1i_2}M_{i_1i_3}M_{i_4i_5}M_{i_4i_2} = 2\left(n-4\right),$$

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5}}^{n} M_{i_1i_2}M_{i_1i_3}M_{i_4i_3}M_{i_4i_5} = 2\left(n-4\right),$$

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5}}^{n} M_{i_1i_2}M_{i_1i_3}M_{i_4i_5}M_{i_4i_3} = 2\left(n-4\right),$$

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5,i_6=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5\neq i_6}}^{n} M_{i_1i_2}M_{i_1i_3}M_{i_4i_5}M_{i_4i_6} = 4\left(20-9n+n^2\right),$$

we get the following results.

Cases	B_{3M}
$i=l, i\neq s$	0
$i\neq l, i=s$	0
$i\neq l, i\neq s, k=j, k\neq m, k\neq l, k\neq s$	0
$i\neq l, i\neq s, k\neq j, k=m, k\neq l, k\neq s$	0
$i\neq l, i\neq s, k\neq m, j\neq m, j=l, j\neq s, m\neq l, m=s$	0
$i\neq l, i\neq s, k\neq j, k\neq m, j\neq m, j=l, j\neq s, m\neq l, m\neq s$	$\frac{4}{15}\left(n-4\right)$

Cases	B_{3M}
$i \neq l, i \neq s, k \neq j, k \neq m, j \neq m, j \neq l, j = s, m = l, m \neq s$	0
$i \neq l, i \neq s, k \neq j, k \neq m, j \neq m, j \neq l, j = s, m \neq l, m \neq s$	$\frac{4}{15}(n-4)$
$i \neq l, i \neq s, k \neq j, k \neq m, j \neq m, j \neq l, j \neq s, m = l, m \neq s$	$\frac{4}{15}(n-4)$
$i \neq l, i \neq s, k \neq j, k \neq m, j \neq m, j \neq l, j \neq s, m \neq l, m = s$	$\frac{4}{15}(n-4)$
$i \neq l, i \neq s, k \neq j, k \neq m, j \neq m, j \neq l, j \neq s, m \neq l, m \neq s$	$\frac{80}{9} - 4n + \frac{4}{9}n^2$

This implies that

$$B_{3M} = \frac{208}{45} - \frac{44}{15}n + \frac{4}{9}n^2. \tag{4.25}$$

Therefore, by (4.23)–(4.25)

$$\begin{aligned}
&E \sum_{\substack{i,k=1 \\ i \neq k}}^{n} \left(\sum_{j=i+1}^{n} M_{\pi(i)\pi(j)} \right)^2 \left(\sum_{l=k+1}^{n} M_{\pi(k)\pi(l)} \right)^2 \\
&= \frac{10}{3} - \frac{11n}{3} + n^2 + 2\left(-4 + \frac{10}{3}n - \frac{2}{3}n^2 \right) + \frac{208}{45} - \frac{44}{15}n + \frac{4}{9}n^2 \\
&= -\frac{2}{45} + \frac{n}{15} + \frac{n^2}{9}.
\end{aligned}$$

Hence

$$E\left(E^{\pi} \left(U' - U \right)^2 \right)^2 = \frac{16}{n^2} \left(1 + \frac{8}{5(n+1)} \right).$$

(ii) To prove (ii), we follow the proof of (i) by replacing (4.20), (4.21) and (4.22) with

$$E^{\pi} \left(V' - V \right)^2 = \frac{72}{n^2 (n-1)(2n+5)} \sum_{i=1}^{n} \left(\sum_{j=i+1}^{n} Q_{\pi(i)\pi(j)} \right)^2,$$

$$\sum_{l=1}^{n} E \left(\sum_{j=l+1}^{n} Q_{\pi(l)\pi(j)} \right)^4 = n(n-1)\left(-764 + 961n - 249n^2 + 36n^3 \right)$$

and

$$B_{1Q}+2B_{2Q}+B_{3Q}:=E\sum_{\substack{i,k=1\\i\neq k}}^{n}\Bigg(\sum_{j=i+1}^{n}\sum_{l=k+1}^{n}Q^2_{\pi(i)\pi(j)}Q^2_{\pi(k)\pi(l)}$$
$$+2\sum_{j=i+1}^{n}\sum_{\substack{l,s=k+1\\s\neq l}}^{n}Q^2_{\pi(i)\pi(j)}Q_{\pi(k)\pi(l)}Q_{\pi(k)\pi(s)}$$
$$+\sum_{\substack{j,m=i+1\\j\neq m}}^{n}\sum_{\substack{l,s=k+1\\s\neq l}}^{n}Q_{\pi(i)\pi(j)}M_{\pi(i)\pi(m)}Q_{\pi(k)\pi(l)}Q_{\pi(k)\pi(s)}\Bigg).$$

From the facts that

$$\sum_{\substack{i_1,i_1,i_3=1\\i_1\neq i_2\neq i_3}}^{n}Q^2_{i_1i_2}Q^2_{i_3i_1}=n\left(n-1\right)\left(n-2\right),$$
$$\sum_{\substack{i_1,i_1,i_3=1\\i_1\neq i_2\neq i_3}}^{n}Q^2_{i_1i_2}Q^2_{i_2i_3}=n\left(n-1\right)\left(n-2\right),$$
$$\sum_{\substack{i_1,i_1,i_3=1\\i_1\neq i_2\neq i_3}}^{n}Q^2_{i_1i_2}Q^2_{i_3i_2}=n\left(n-1\right)\left(n-2\right),$$
$$\sum_{\substack{i_1,i_1,i_3,i_4=1\\i_1\neq i_2\neq i_3\neq i_4}}^{n}Q^2_{i_1i_2}Q^2_{i_3i_4}=n\left(n-1\right)\left(n-2\right)\left(n-3\right),$$

we have

Cases	B_1
$i=l,k\neq j,j\neq l$	$\frac{n(n-1)(n-2)}{6}$
$i\neq l,k=j,j\neq l$	$\frac{n(n-1)(n-2)}{6}$
$i\neq l,k\neq j,j=l$	$\frac{n(n-1)(n-2)}{3}$
$i\neq l,k\neq j,j\neq l$	$\frac{n(n-1)(n-2)(n-3)}{4}$

This implies that

$$B_{1Q}=\frac{n\left(n-1\right)\left(n-2\right)\left(3n-1\right)}{12}. \tag{4.26}$$

Since

$$\sum_{\substack{i_1,i_2,i_3=1\\ i_1\neq i_2\neq i_3}}^{n} Q_{i_1i_2}^2 Q_{i_3i_1} Q_{i_3i_2} = \frac{n\left(n-1\right)\left(n-2\right)}{3},$$

$$\sum_{\substack{i_1,i_2,i_3,i_4=1\\ i_1\neq i_2\neq i_3\neq i_4}}^{n} Q_{i_1i_2}^2 Q_{i_3i_1} Q_{i_3i_4} = \frac{n\left(n-1\right)\left(n-2\right)\left(n-3\right)}{3},$$

$$\sum_{\substack{i_1,i_2,i_3=1\\ i_1\neq i_2\neq i_3}}^{n} Q_{i_1i_2}^2 Q_{i_3i_2} Q_{i_3i_1} = \frac{n\left(n-1\right)\left(n-2\right)}{3},$$

$$\sum_{\substack{i_1,i_2,i_3,i_4=1\\ i_1\neq i_2\neq i_3\neq i_4}}^{n} Q_{i_1i_2}^2 Q_{i_3i_4} Q_{i_3i_1} = \frac{n\left(n-1\right)\left(n-2\right)\left(n-3\right)}{3},$$

$$\sum_{\substack{i_1,i_2,i_3,i_4=1\\ i_1\neq i_2\neq i_3\neq i_4}}^{n} Q_{i_1i_2}^2 Q_{i_2i_3} Q_{i_2i_4} = \frac{n\left(n-1\right)\left(n-2\right)\left(n-3\right)}{3},$$

$$\sum_{\substack{i_1,i_2,i_3,i_4=1\\ i_1\neq i_2\neq i_3\neq i_4}}^{n} Q_{i_1i_2}^2 Q_{i_3i_2} Q_{i_3i_4} = \frac{n\left(n-1\right)\left(n-2\right)\left(n-3\right)}{3},$$

$$\sum_{\substack{i_1,i_2,i_3,i_4=1\\ i_1\neq i_2\neq i_3\neq i_4}}^{n} Q_{i_1i_2}^2 Q_{i_3i_4} Q_{i_3i_2} = \frac{n\left(n-1\right)\left(n-2\right)\left(n-3\right)}{3},$$

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5}}^{n} Q_{i_1i_2}^2 Q_{i_3i_4} Q_{i_3i_5} = \frac{n\left(n-1\right)\left(n-2\right)\left(n-3\right)\left(n-4\right)}{3},$$

we obtain

Cases	B_2
$i=l, i\neq s, k\neq j, j\neq l, j=s$	$\frac{n(n-1)(n-2)}{18}$
$i=l, i\neq s, k\neq j, j\neq l, j\neq s$	$\frac{n(n-1)(n-2)(n-3)}{24}$
$i\neq l, i=s, k\neq j, j=l, j\neq s$	$\frac{n(n-1)(n-2)}{18}$
$i\neq l, i=s, k\neq j, j\neq l, j\neq s$	$\frac{n(n-1)(n-2)(n-3)}{24}$
$i\neq l, i\neq s, k=j, j\neq l, j\neq s$	$\frac{n(n-1)(n-2)(n-3)}{36}$
$i\neq l, i\neq s, k\neq j, j=l, j\neq s$	$\frac{5n(n-1)(n-2)(n-3)}{72}$
$i\neq l, i\neq s, k\neq j, j\neq l, j=s$	$\frac{5n(n-1)(n-2)(n-3)}{72}$
$i\neq l, i\neq s, k\neq j, j\neq l, j\neq s$	$\frac{n(n-1)(n-2)(n-3)(n-4)}{18}$

Therefore

$$B_{2Q} = \frac{n(n-1)(n-2)(2n^2-5n+1)}{36}. \tag{4.27}$$

Finally we find B_{3Q}. By the fact that

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5}}^{n} Q_{i_1i_2}Q_{i_1i_3}Q_{i_4i_1}Q_{i_4i_5} = \frac{n(n-1)(n-2)(n-3)(n-4)}{15},$$

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5}}^{n} Q_{i_1i_2}Q_{i_1i_3}Q_{i_4i_5}Q_{i_4i_1} = \frac{n(n-1)(n-2)(n-3)(n-4)}{15},$$

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5}}^{n} Q_{i_4i_1}Q_{i_4i_5}Q_{i_1i_2}Q_{i_1i_3} = \frac{n(n-1)(n-2)(n-3)(n-4)}{15},$$

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5}}^{n} Q_{i_4i_5}Q_{i_4i_1}Q_{i_1i_3}Q_{i_1i_2} = \frac{n(n-1)(n-2)(n-3)(n-4)}{15},$$

$$\sum_{\substack{i_1,i_2,i_3,i_4=1\\ i_1\neq i_2\neq i_3\neq i_4}}^{n} Q_{i_1i_2}Q_{i_1i_3}Q_{i_4i_2}Q_{i_4i_3} = \frac{n(n-1)(n-2)(n-3)}{3},$$

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5}}^{n} Q_{i_1i_2}Q_{i_1i_3}Q_{i_4i_2}Q_{i_4i_5} = \frac{2n(n-1)(n-2)(n-3)(n-4)}{15},$$

$$\sum_{\substack{i_1,i_2,i_3,i_4=1\\ i_1\neq i_2\neq i_3\neq i_4}}^{n} Q_{i_1i_2}Q_{i_1i_3}Q_{i_4i_3}Q_{i_4i_2} = \frac{n(n-1)(n-2)(n-3)}{3},$$

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5}}^{n} Q_{i_1i_2}Q_{i_1i_3}Q_{i_4i_5}Q_{i_4i_2} = \frac{2n(n-1)(n-2)(n-3)(n-4)}{15},$$

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5}}^{n} Q_{i_1i_3}Q_{i_1i_2}Q_{i_4i_2}Q_{i_4i_5} = \frac{2n(n-1)(n-2)(n-3)(n-4)}{15},$$

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5}}^{n} Q_{i_1i_3}Q_{i_1i_2}Q_{i_4i_5}Q_{i_4i_2} = \frac{2n(n-1)(n-2)(n-3)(n-4)}{15},$$

$$\sum_{\substack{i_1,i_2,i_3,i_4,i_5,i_6=1\\ i_1\neq i_2\neq i_3\neq i_4\neq i_5,i_6}}^{n} Q_{i_1i_3}Q_{i_1i_4}Q_{i_2i_5}Q_{i_2i_6} = \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{9}$$

we get the following results,

Cases	B_3
$i = l, i \neq s$	$\frac{n(n-1)(n-2)(n-3)(n-4)}{225}$
$i \neq l, i = s$	$\frac{n(n-1)(n-2)(n-3)(n-4)}{225}$
$i \neq l, i \neq s, k = j, k \neq m,$ $k \neq l, k \neq s$	$\frac{n(n-1)(n-2)(n-3)(n-4)}{225}$
$i \neq l, i \neq s, k \neq j, k = m$ $k \neq l, k \neq s$	$\frac{n(n-1)(n-2)(n-3)(n-4)}{225}$
$i \neq l, i \neq s, k \neq m, j \neq m,$ $j = l, j \neq s, m \neq l, m = s$	$\frac{n(n-1)(n-2)(n-3)}{18}$
$i \neq l, i \neq s, k \neq j, k \neq m, j \neq m,$ $j = l, j \neq s, m \neq l, m \neq s$	$\frac{4n(n-1)(n-2)(n-3)(n-4)}{225}$
$i \neq l, i \neq s, k \neq j, k \neq m, j \neq m,$ $j \neq l, j = s, m = l, m \neq s$	$\frac{n(n-1)(n-2)(n-3)}{18}$
$i \neq l, i \neq s, k \neq j, k \neq m, j \neq m,$ $j \neq l, j = s, m \neq l, m \neq s$	$\frac{4n(n-1)(n-2)(n-3)(n-4)}{225}$
$i \neq l, i \neq s, k \neq j, k \neq m, j \neq m,$ $j \neq l, j \neq s, m = l, m \neq s$	$\frac{4n(n-1)(n-2)(n-3)(n-4)}{225}$
$i \neq l, i \neq s, k \neq j, k \neq m, j \neq m,$ $j \neq l, j \neq s, m \neq l, m = s$	$\frac{4n(n-1)(n-2)(n-3)(n-4)}{225}$
$i \neq l, i \neq s, k \neq j, k \neq m, j \neq m,$ $j \neq l, j \neq s, m \neq l, m \neq s$	$\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{81}$

Therefore

$$B_{3Q} = \frac{n(n-1)(n-2)(n-3)(5n^2-9n+1)}{405}. \tag{4.28}$$

This completes the proof. □

CHAPTER V

UNIFORM BOUNDS

In this chapter, we first study about uniform bounds for the number of descents and the number of inversions. In 2004, uniform bounds for normal approximation for these random variables was given by Fulman ([11]). He used an exchangeable pair and Rinott and Rotar's Theorem ([19]). Fulman's results are shown below.

Theorem 5.1. ([*11*]) *There exists a positive constant* C_1 *such that*

$$\Delta_{des(\pi)} \leq \frac{C_1}{\sqrt{n}}.$$

Theorem 5.2. ([*11*]) *There exists a positive constant* C_2 *such that*

$$\Delta_{inv(\pi)} \leq \frac{C_2}{\sqrt{n}}.$$

5.1 Fulman's Constants

Our goal in this section is to calculate the constant C_1 in Theorem 5.1 and the constant C_2 in Theorem 5.2. Fulman ([11]) proved Theorem 5.1 and Theorem 5.2 by using Theorem 5.3 of Rinott and Rotar ([19]).

Theorem 5.3. *([19]) Let* W *be a random variable with* $EW = 0$ *and* $E\left(W^2\right) = 1$. *Let* W' *be an exchangeable pair of* W *satisfying* (3.7). *Suppose in addition that* $|W' - W| \leq A$ *for some constant* A. *Then for* $z \in \mathbb{R}$,

$$|P\left(W \leq z\right) - \Phi\left(z\right)| \leq \frac{12}{\lambda}\sqrt{Var\left(E^W\left(W' - W\right)^2\right)} + 48\frac{A^3}{\lambda} + 8\frac{A^2}{\sqrt{\lambda}}.$$

Fulman ([11]) showed that

$$E^U U' = \left(1 - \frac{2}{n}\right) U \ , \ |U' - U| \le \frac{2\sqrt{3}}{\sqrt{n+1}} \text{ and } Var\left(E^{\pi}\left(U' - U\right)^2\right) \le \frac{B_1}{n^3}$$

for some constant B_1.

Hence by Theorem 5.3,

$$\begin{aligned}
\Delta_{des(\pi),z} &\le 6n\sqrt{Var\left(E^{\pi}\left(U'-U\right)^2\right)} + 24n\left(\frac{2\sqrt{3}}{\sqrt{n}}\right)^3 + 4\sqrt{2}\sqrt{n}\left(\frac{2\sqrt{3}}{\sqrt{n}}\right)^2 \\
&\le \frac{6\sqrt{B_1}}{\sqrt{n}} + \frac{997.7}{\sqrt{n}} + \frac{67.9}{\sqrt{n}} \\
&\le \frac{6\sqrt{B_1}}{\sqrt{n}} + \frac{1065.6}{\sqrt{n}}. \qquad (5.1)
\end{aligned}$$

We see that the constant C_1 in Theorem 5.1 are provided when the constant B_1 is obtained. Now, we will calculate the constant B_1. By Proposition 4.4 and the fact that

$$E\left(U'-U\right)^2 = \frac{4}{n} \qquad (5.2)$$

([11], pp.68-70), we have

$$\begin{aligned}
&Var\left(E^{\pi}\left(U'-U\right)^2\right) \\
&= E\left(E^{\pi}\left(U'-U\right)^2\right)^2 - \left(EE^{\pi}\left(U'-U\right)^2\right)^2 \\
&= \frac{16}{n^2}\left(1 + \frac{8}{5(n+1)}\right) - \frac{16}{n^2} \\
&\le \frac{25.6}{n^3}. \qquad (5.3)
\end{aligned}$$

By (5.1) and (5.3), we obtain $C_1 = 1096$, i.e.

$$\Delta_{des(\pi),z} \le \frac{1096}{\sqrt{n}}.$$

Moreover, for the number of inversions, Fulman ([11]) showed that

$$E^V V' = \left(1 - \frac{2}{n}\right) V, |V' - V| \le \frac{6\sqrt{2}(n-1)}{\sqrt{n(n-1)(2n+5)}} \text{ and } Var\left(E^{\pi}\left(V'-V\right)^2\right) \le \frac{B_2}{n^3}$$

for some constant B_2.

To calculate the constant B_2, we describe the process below. By proposition 4.4 and the fact that

$$E\left(V'-V\right)^2 = \frac{4}{n} \tag{5.4}$$

([11], pp.68-70), we obtain

$$\begin{aligned}
&Var\left(E^{\pi}\left(V'-V\right)^2\right) \\
&= E\left(E^{\pi}\left(V'-V\right)^2\right)^2 - \left(EE^{\pi}\left(V'-V\right)^2\right)^2 \\
&= \frac{16}{n^2}\left(1-\frac{2064}{1225(n-1)}+\frac{2256}{625n}-\frac{864}{875(2n+5)^2}-\frac{29688}{30625(2n+5)}\right)-\frac{16}{n^2} \\
&= \frac{16}{n^2}\left(-\frac{2064}{1225(n-1)}+\frac{2256}{625n}-\frac{864}{875(2n+5)^2}-\frac{29688}{30625(2n+5)}\right) \\
&\leq \frac{30.8}{n^3}. \qquad (5.5)
\end{aligned}$$

By similar argument to the descent case, we obtain

$$\begin{aligned}
\Delta_{inv(\pi),z} &\leq 6n\sqrt{Var\left(E^{\pi}\left(V'-V\right)^2\right)} + 24n\left(\frac{6}{\sqrt{n}}\right)^3 + 4\sqrt{2}\sqrt{n}\left(\frac{6}{\sqrt{n}}\right)^2 \\
&\leq \frac{6\sqrt{30.8}}{\sqrt{n}} + \frac{5184}{\sqrt{n}} + \frac{204}{\sqrt{n}} \\
&\leq \frac{5421}{\sqrt{n}}.
\end{aligned}$$

It seems that the constant C_1 and C_2 are large. Thus, it is reasonable to find better constants. In Theorem 5.4 and Theorem 5.5, we improve the constants by reducing them to be 12.44 and 14.24, respectively.

5.2 Bound for the Number of Descents

In this section, we prove the following theorem.

Theorem 5.4.

$$\Delta_{des(\pi)} \leq \frac{12.44}{\sqrt{n}}.$$

To prove Theorem 5.4, we need some properties of U and U' proved in Chapter 4.

Proof of Theorem 5.4

Proof. From Lemma 3.1 and (4.5), we use the argument in ([13]) and then we obtain

$$\begin{aligned}\Delta_{des(\pi),z} &= \left|Eg'_z(U) - E(U)\,g_z(U)\right| \\ &= \left|Eg'_z(U) - E\int_{-\infty}^{\infty} g'_z(U+t)\,K(t)\,dt\right| \\ &\leq \left|Eg'_z(U) - Eg'_z(U)\int_{-\infty}^{\infty} K(t)\,dt\right| \\ &\quad + \left|E\int_{-\infty}^{\infty}\{g'_z(U) - g'_z(U+t)\}\,K(t)\,dt\right| \\ &=: |T_1| + |T_2|, \end{aligned} \tag{5.6}$$

where g_z is the solution of the Stein's equation defined by (3.2).

First we will bound $|T_1|$. From (5.2) and Proposition 4.4(i) we note that

$$\begin{aligned}& E\left(1 - \frac{n}{4}E^{\pi}(U'-U)^2\right)^2 \\ &= 1 - \frac{n}{2}E(U'-U)^2 + \frac{n^2}{16}E\left(E^{\pi}(U'-U)^2\right)^2 \\ &= -1 + \frac{n^2}{16}E\left(E^{\pi}(U'-U)^2\right)^2 \\ &= \frac{8}{5(n+1)}. \end{aligned} \tag{5.7}$$

From (3.5), (4.14) and (5.7) we have

$$\begin{aligned}|T_1| &= \left|Eg'_z(U) - Eg'_z(U)\int_{-\infty}^{\infty} K(t)\,dt\right| \\ &= \left|Eg'_z(U)\left(1 - \int_{-\infty}^{\infty} K(t)\,dt\right)\right| \\ &= \left|Eg'_z(U)\left(1 - \frac{n}{4}(U'-U)^2\right)\right| \\ &= \left|Eg'_z(U)\left(1 - \frac{n}{4}E^{\pi}(U'-U)^2\right)\right| \end{aligned}$$

$$\leq \sqrt{E\left(g_z'\left(U\right)\right)^2}\sqrt{E\left(1-\frac{n}{4}E^{\pi}\left(U'-U\right)^2\right)^2} \tag{5.8}$$

$$\leq \frac{1.27}{\sqrt{n}}. \tag{5.9}$$

Next, we will give a bound for $|T_2|$ by following the arguments of Neammanee and Rattanawong ([13], pp. 20-23). By the fact that

$$g_z'\left(w+s\right)-g_z'\left(w+t\right) \leq \begin{cases} 1 & \text{if } w+s<z, w+t>z, \\ \left(|w|+\frac{\sqrt{2\pi}}{4}\right)\left(|s|+|t|\right) & \text{if } s \geq t, \\ 0 & \text{otherwise,} \end{cases}$$

(See [6], pp. 247), yields

$$\begin{aligned} T_2 &= E\int_{-\infty}^{\infty}\left\{g_z'\left(U\right)-g_z'\left(U+t\right)\right\}K\left(t\right)dt \\ &\leq E\int_{\substack{U<z \\ U+t>z}} K\left(t\right)dt + E\int_{t\leq 0}\left(|U|+\frac{\sqrt{2\pi}}{4}\right)|t|\,K\left(t\right)dt \\ &=: M_1+M_2. \end{aligned}$$

For $\delta := |U'-U|$ and $z \in \mathbb{R}$, define a function $f_\delta : \mathbb{R} \to \mathbb{R}$ by

$$f_\delta\left(t\right) = \begin{cases} -\frac{3}{2}\delta & \text{if } t<z-2\delta, \\ -\frac{1}{2}\left(2z-\delta\right)+t & \text{if } z-2\delta \leq t \leq z+\delta, \\ \frac{3}{2}\delta & \text{if } t>z+\delta. \end{cases}$$

It is not difficult to see that

$$|f_\delta\left(t\right)| \leq \frac{3}{2}\delta \quad \text{for all } t \in \mathbb{R}, \tag{5.10}$$

and

$$f_\delta'\left(t\right) = \begin{cases} 1 & \text{if } z-2\delta<t<z+\delta, \\ 0 & \text{if } t<z-2\delta \text{ or } t>z+\delta. \end{cases} \tag{5.11}$$

Therefore,

$$\begin{aligned} E\int_{-\infty}^{\infty} f_\delta'(U+t)\,K(t)\,dt &\geq E\int_{\substack{z-\delta<U<z\\ |t|\leq\delta}} f_\delta'(U+t)\,K(t)\,dt \\ &= E\int_{\substack{z-\delta<U<z\\ |t|\leq\delta}} K(t)\,dt \\ &= E\int_{|t|\leq\delta} \mathbb{I}(z-\delta<U<z)\,K(t)\,dt. \end{aligned} \tag{5.12}$$

By Lemma 3.1, (4.5) and (5.12), we have that

$$\begin{aligned} E\int_{|t|\leq\delta} \mathbb{I}(z-\delta<U<z)\,K(t)\,dt &\leq E\int_{-\infty}^{\infty} f_\delta'(U+t)\,K(t)\,dt \\ &= EUf_\delta(U)\,. \end{aligned} \tag{5.13}$$

Note that $K(t)\geq 0$ and $K(t)=0$ for $|t|>|U'-U|$.

Thus we conclude from (5.2), (5.10), (5.12) and (5.13) that

$$\begin{aligned} M_1 &= E\int_{t>0} \mathbb{I}(z-t<U<z)\,K(t)\,dt \\ &\leq E\int_{-\infty}^{\infty} \mathbb{I}(z-t<U<z)\,K(t)\,dt \\ &= E\int_{|t|\leq|U'-U|} \mathbb{I}(z-t<U<z)\,K(t)\,dt \\ &\leq\ E\int_{|t|\leq|U'-U|} \mathbb{I}(z-|U'-U|<U<z)\,K(t)\,dt \\ &\leq EUf_{|U'-U|}(U) \\ &\leq E\,|U|\,\left|f_{|U'-U|}(U)\right| \\ &\leq \frac{3}{2}E\,|U|\,|U'-U| \\ &\leq \frac{3}{2}\sqrt{EU^2}\sqrt{E\,|U'-U|^2} \\ &\leq \frac{3}{\sqrt{n}}. \end{aligned} \tag{5.14}$$

Next we try to bound M_2.

$$\begin{aligned}M_2 &\leq E\int_{\mathbb{R}}\left(|U|+\frac{\sqrt{2\pi}}{4}\right)|t|\,K(t)\,dt\\ &= E\,|U|\int_{\mathbb{R}}|t|\,K(t)\,dt+\frac{\sqrt{2\pi}}{4}E\int_{\mathbb{R}}|t|\,K(t)\,dt\\ &=: M_{21}+M_{22}.\end{aligned}$$

From the fact that

$$\int_{-\infty}^{\infty}|t|\,K(t)\,dt=\frac{n}{8}\,|U'-U|^3\,, \tag{5.15}$$

Proposition 4.1(i) and Proposition 4.3(i), we have

$$\begin{aligned}M_{21} &= \frac{n}{8}E\,|U|\,|U'-U|^3\\ &\leq \frac{n}{8}\left(E\,|U|^4\right)^{\frac{1}{4}}\left(E\,|U'-U|^4\right)^{\frac{3}{4}}\\ &\leq \frac{n}{8}\,(76.18)^{\frac{1}{4}}\left(\frac{48}{n^2}\right)^{\frac{3}{4}}\\ &\leq \frac{6.74}{\sqrt{n}}\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}M_{22} &= \frac{\sqrt{2\pi}}{4}E\int_{\mathbb{R}}|t|\,K(t)\,dt\\ &= \frac{\sqrt{2\pi}}{4}\frac{n}{8}E\,|U'-U|^3\\ &\leq \frac{\sqrt{2\pi}}{4}\frac{n}{8}\left(E\,|U'-U|^4\right)^{\frac{3}{4}}\\ &\leq \frac{\sqrt{2\pi}n}{32}\left(\frac{48}{n^2}\right)^{\frac{3}{4}}\\ &\leq \frac{1.43}{\sqrt{n}}.\end{aligned} \tag{5.17}$$

Combining (5.14), (5.16) and (5.17), we get

$$T_2 \leq \frac{11.17}{\sqrt{n}}. \tag{5.18}$$

By the same argument as in (5.18) by using the fact that

$$g'_z(w+s) - g'_z(w+t) \geq \begin{cases} -1 & \text{if } w+s>z, w+t<z, \\ -\left(|w| + \frac{\sqrt{2\pi}}{4}\right)(|s|+|t|) & \text{if } s<t, \\ 0 & \text{otherwise,} \end{cases}$$

(See [6], pp. 247), we can show that

$$T_2 \geq \frac{-11.17}{\sqrt{n}}. \tag{5.19}$$

From (5.6), (5.9), (5.18) and (5.19), we get

$$\Delta_{des(\pi)} \leq \frac{12.44}{\sqrt{n}}.$$

□

5.3 Bound for the Number of Inversions

Theorem 5.5.

$$\Delta_{inv(\pi)} \leq \frac{14.24}{\sqrt{n}}.$$

Proof. For the the proof of Theorem 5.5, we can follow an argument of the proof of Theorem 5.4 by using Proposition 4.1(ii), Proposition 4.3(ii) and Proposition 4.4(ii).

Similar to (5.6) we note that

$$\begin{aligned} \Delta_{inv(\pi),z} &\leq \left| Eg'_z(V) - Eg'_z(V) \int_{-\infty}^{\infty} K(t)\,dt \right| \\ &\quad + \left| E\int_{-\infty}^{\infty} \{g'_z(V) - g'_z(V+t)\} K(t)\,dt \right| \\ &=: |T_1| + |T_2|. \end{aligned}$$

From (3.5), (4.12), (4.14), (5.5) and (5.8) we have

$$\begin{aligned} |T_1| &\leq \sqrt{E\left(g'_z(V)\right)^2} \sqrt{E\left(1 - \frac{n}{4}E^{\pi}\left(V'-V\right)^2\right)^2} \\ &\leq \frac{1.39}{\sqrt{n}}, \end{aligned}$$

and

$$
\begin{aligned}
T_2 &= E\int_{-\infty}^{\infty}\left\{g_z'(V)-g_z'(V+t)\right\}K(t)\,dt\\
&\le E\int_{\substack{V<z\\V+t>z}}K(t)\,dt + E\int_{t\le 0}\left(|V|+\frac{\sqrt{2\pi}}{4}\right)|t|\,K(t)\,dt\\
&\le E\int_{t>0}\mathbb{I}\left(z-t<V<z\right)K(t)\,dt + E\left|V\right|\int_{\mathbb{R}}|t|\,K(t)\,dt\\
&\quad+\frac{\sqrt{2\pi}}{4}E\int_{\mathbb{R}}|t|\,K(t)\,dt\\
&\le \frac{3}{2}\sqrt{EV^2}\sqrt{E\left|V'-V\right|^2}+\frac{n}{8}\left(E\left|V\right|^4\right)^{\frac{1}{4}}\left(E\left|V'-V\right|^4\right)^{\frac{3}{4}}\\
&\quad+\frac{\sqrt{2\pi}}{4}\frac{n}{8}\left(E\left|V'-V\right|^4\right)^{\frac{3}{4}}\\
&\le \frac{3}{\sqrt{n}}+\frac{8.33}{\sqrt{n}}+\frac{1.52}{\sqrt{n}}\\
&\le \frac{12.85}{\sqrt{n}}.
\end{aligned}
$$

Similarly we can show that

$$T_2 \ge -\frac{12.85}{\sqrt{n}}.$$

From these facts, we complete the proof. □

CHAPTER VI

POLYNOMIAL NON-UNIFORM BOUNDS

In this chapter, we give polynomial non-uniform bounds for the number of descents and inversions. These are our main results.

Theorem 6.1. *For sufficiently large* n *and* $z \in \mathbb{R}$*,*

$$\Delta_{des(\pi),z} \leq \frac{1756}{(1+|z|)^3\sqrt{n}}.$$

Theorem 6.2. *For sufficiently large* n *and* $z \in \mathbb{R}$*,*

$$\Delta_{inv(\pi),z} \leq \frac{12160}{(1+|z|)^3\sqrt{n}}.$$

Table 6.1 compares the constants of uniform and polynomial non-uniform bounds.

	uniform bounds	polynomial bounds				
		$z = 4.2$	$z = 8.5$	$z = 100$	$z = 1000$	$z = 10000$
Descent	$\frac{12.44}{\sqrt{n}}$	$\frac{12.44}{\sqrt{n}}$	$\frac{2.05}{\sqrt{n}}$	$\frac{1.71 \times 10^{-3}}{\sqrt{n}}$	$\frac{1.76 \times 10^{-6}}{\sqrt{n}}$	$\frac{1.76 \times 10^{-9}}{\sqrt{n}}$
Inversion	$\frac{14.24}{\sqrt{n}}$	$\frac{86.49}{\sqrt{n}}$	$\frac{14.24}{\sqrt{n}}$	$\frac{1.19 \times 10^{-2}}{\sqrt{n}}$	$\frac{1.22 \times 10^{-5}}{\sqrt{n}}$	$\frac{1.22 \times 10^{-8}}{\sqrt{n}}$

Table 6.1: Constants of uniform and polynomial non-uniform bounds.

In fact Theorem 6.1 and Theorem 6.2 are more efficient than Theorem 5.4 and Theorem 5.5 in the case of $z \geq 4.2$ and $z \geq 8.5$, respectively. This chapter is organized into 2 sections. The results for the number of descents and for the number of inversions are in Section 6.1 and Section 6.2, respectively.

6.1 Bound for the Number of Descents

To find a non-uniform bound, Chen and Shao ([6], pp. 248) show that for any random variable W such that $EW = 0$ and $EW^2 = 1$,

$$E\left|g_z'(W)\right| \leq \frac{C}{(1+z)^2}$$

when $z > 0$ and g_z' is defined by (3.3). In our work, we use their argument to find the explicit constant as in the following proposition.

Proposition 6.3. *Let W be either U or V. Then for $z \geq 4.2$, we have*

$$E\left|g_z'(W)\right|^2 \leq \frac{1}{(1+z)^6}\left(562 + 1.89E(1+W)^6\right).$$

Proof. Note that

$$g_z'(w) = \begin{cases} [1-\Phi(z)]\left[1+\sqrt{2\pi}we^{\frac{w^2}{2}}\Phi(w)\right] & \text{if } w < z, \\ \Phi(z)\left[-1+\sqrt{2\pi}we^{\frac{w^2}{2}}(1-\Phi(w))\right] & \text{if } w > z, \end{cases} \tag{6.1}$$

([6], pp. 252).

This implies that

$$\begin{aligned} E\left|g_z'(W)\right|^2 &= E\left|g_z'(W)\right|^2\mathbb{I}(W<0) + E\left|g_z'(W)\right|^2\mathbb{I}\left(0 \leq W \leq \frac{9z}{10}\right) \\ &\quad + E\left|g_z'(W)\right|^2\mathbb{I}\left(W > \frac{9z}{10}\right) \\ &=: R_1 + R_2 + R_3. \end{aligned}$$

From

$$0 \leq g_z'(w) \leq 1-\Phi(z) \text{ for } w < 0 \tag{6.2}$$

([6], pp. 252) and

$$1-\Phi(z) \leq \frac{e^{-\frac{z^2}{2}}}{z\sqrt{2\pi}} \text{ for } z > 0 \tag{6.3}$$

([1], pp. 11), we obtain

$$R_1 \leq E\left|g_z'(W)\right|^2 \leq \left(\frac{e^{-\frac{z^2}{2}}}{z\sqrt{2\pi}}\right)^2.$$

Since $r_1(z) := (1+z)^2 e^{-\frac{z^2}{2}}$ is decreasing on $[4.2, \infty)$,

$$\frac{e^{-\frac{z^2}{2}}}{z} \leq \frac{1.3e^{-\frac{z^2}{2}}}{1+z} \leq \frac{1.3r_1(4.2)}{(1+z)^3} = \frac{0.01}{(1+z)^3}.$$

Therefore

$$R_1 \leq \frac{0.01}{(1+z)^6}. \tag{6.4}$$

Next we will bound R_2. From (6.1) we have

$$0 \leq g_z'(w) \leq (1-\Phi(z))\left(1+\frac{9\sqrt{2\pi}}{10}ze^{\frac{81z^2}{200}}\right) \text{ for } 0 < w < \frac{9z}{10}.$$

Thus

$$\begin{aligned} R_2 &\leq \left[\left(\frac{e^{-\frac{z^2}{2}}}{z\sqrt{2\pi}}\right)\left(1+\frac{9\sqrt{2\pi}}{10}ze^{\frac{81z^2}{200}}\right)\right]^2 \\ &= \left(\frac{e^{-\frac{z^2}{2}}}{z\sqrt{2\pi}}+\frac{9}{10}e^{-\frac{19z^2}{200}}\right)^2. \end{aligned}$$

Since both $r_2(z) := (1+z)^3 e^{-\frac{19z^2}{200}}$ and $r_3(z) := (1+z)^2 e^{-\frac{119z^2}{200}}$ are decreasing on $[4.2, \infty)$,

$$e^{-\frac{19z^2}{200}} \leq \frac{r_2(4.2)}{(1+z)^3} = \frac{26.32}{(1+z)^3},$$

and

$$\frac{e^{-\frac{119z^2}{200}}}{z} \leq \frac{1.3e^{-\frac{119z^2}{200}}}{1+z} \leq \frac{1.3r_3(4.2)}{(1+z)^3} = \frac{0.01}{(1+z)^3}.$$

Therefore

$$R_2 \leq \frac{562}{(1+z)^6}. \tag{6.5}$$

By (3.5) and Chebyshev's inequality, we have

$$R_3 \leq \frac{1}{\left(1+\frac{9z}{10}\right)^6} E\left(1+W\right)^6$$
$$\leq \frac{1.89}{\left(1+z\right)^6} E\left(1+W\right)^6. \tag{6.6}$$

Combining (6.4)–(6.6), we complete the proof. □

Proposition 6.4. *Let $z \geq 4.2$ and $h : \mathbb{R} \to \mathbb{R}$ be defined by*

$$h\left(w\right) = \begin{cases} \left(\sqrt{2\pi}(1+w^2)e^{\frac{w^2}{2}}\left[1-\Phi\left(w\right)\right]-w\right)\Phi\left(z\right) & \text{if } w \geq z, \\ \left(\sqrt{2\pi}(1+w^2)e^{\frac{w^2}{2}}\Phi\left(w\right)+w\right)\left(1-\Phi\left(z\right)\right) & \text{if } w < z. \end{cases} \tag{6.7}$$

Then

(i) $h\left(w\right) \leq \dfrac{15.44}{\left(1+z\right)^3}$ *for* $w < \frac{3z}{4}$.

(ii) $h\left(w\right) \leq 1.001\left(1+z\right)$ *for* $\frac{3z}{4} < w < z$.

(iii) $h\left(w\right) \leq \dfrac{4}{\left(1+z\right)^3} \leq 1.001\left(1+z\right)$ *for* $w \geq z$.

Proof. 1. For $w < \frac{3z}{4}$, since h is increasing and non-negative, we obtain

$$h\left(w\right) \leq \left(\sqrt{2\pi}\left(1+\frac{9z}{16}^2\right)e^{\frac{9z^2}{32}}+z\right)\left(\frac{e^{-\frac{z^2}{2}}}{z\sqrt{2\pi}}\right)$$
$$\leq \left(\sqrt{2\pi}\left(1+z^2\right)e^{\frac{9z^2}{32}}+z\right)\left(\frac{e^{-\frac{z^2}{2}}}{z\sqrt{2\pi}}\right)$$
$$= \frac{\left(1+z^2\right)e^{-\frac{7z^2}{32}}}{z}+\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \tag{6.8}$$
$$\leq \left(1+z\right)e^{-\frac{7z^2}{32}}+\frac{0.01}{\left(1+z\right)^3}$$
$$\leq \frac{15.44}{\left(1+z\right)^3} \tag{6.9}$$

where the last two inequalities come from the fact that both $\left(1+z\right)^4 e^{-\frac{7z^2}{32}}$ and $\left(1+z\right)^3 e^{-\frac{z^2}{2}}$ are decreasing on $[4.2, \infty)$.

2. For $\frac{3z}{4} < w < z$, by (6.3) and (6.7) we have

$$\begin{aligned} h(w) &\le \left(\sqrt{2\pi}\left(1+z^2\right)e^{\frac{z^2}{2}} + z\right)\left(1-\Phi(z)\right) \\ &\le \frac{1+z^2}{z} + \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \\ &\le 1.001\,(1+z) \end{aligned} \tag{6.10}$$

where we use the fact that $z \ge 4.2$ and $\frac{e^{-\frac{z^2}{2}}}{1+z}$ is decreasing on $[4.2, \infty)$ in the last inequality.

3. For $w \ge z$, Chen and Shao ([6], pp. 248-249) showed that

$$h(w) \le \frac{2}{1+w^3}.$$

We immediately obtain that

$$h(w) \le \frac{2}{1+z^3} \le \frac{4}{(1+z)^3} \le 1.001\,(1+z). \tag{6.11}$$

□

Proof of Theorem 6.1

Proof. Suppose that $z \ge 0$. In case of $z < 0$ we use the fact that $\Phi(z) = 1-\Phi(-z)$ and then apply the result to $-U$.

Case 1. $0 \le z \le 4.2$. By Theorem 5.4 and the fact that $1 \le \dfrac{140.61}{(1+z)^3}$ we have

$$\Delta_{des(\pi),z} \le \frac{12.44}{\sqrt{n}} \le \frac{1750}{(1+z)^3\sqrt{n}}. \tag{6.12}$$

Case 2. $z > 4.2$. Recall by (5.6) that

$$\begin{aligned} \Delta_{des(\pi),z} &\le \left| Eg_z'(U) - Eg_z'(U)\int_{-\infty}^{\infty} K(t)\,dt \right| \\ &\quad + \left| E\int_{-\infty}^{\infty} \left\{g_z'(U) - g_z'(U+t)\right\} K(t)\,dt \right| \\ &=: |T_1| + |T_2|. \end{aligned} \tag{6.13}$$

By (5.7), (5.8) and Proposition 6.3 we have

$$|T_1| \leq \frac{1.27}{(1+z)^3}\sqrt{562 + 1.89E\,(1+U)^6}. \tag{6.14}$$

Next we will bound $|T_2|$. Using the fact that

$$\begin{aligned}|g_z'(w+s) - g_z'(w+t) - \int_t^s h(w+u)\,du| \\ \leq \mathbb{I}\,(z - \max(s,t) < w < z - \min(s,t))\end{aligned}$$

where h is defined as in (6.7) ([6], pp. 250–251) we have

$$|T_2| \leq E\int_{-\infty}^{\infty} |\{g_z'(U) - g_z'(U+t)\}\,K(t)|\,dt \leq T_{21} + T_{22} \tag{6.15}$$

where

$$\begin{aligned} T_{21} &= E\int_{-\infty}^{\infty} \mathbb{I}\,(z - \max(0,t) < U < z - \min(0,t))\,K(t)\,dt \\ T_{22} &= E\int_{-\infty}^{\infty}\int_t^0 h(U+u)\,K(t)\,dudt. \end{aligned}$$

From now on, we let $\delta =: |U' - U|$.

We used the idea from Neammanee and Rattanawong ([14], pp. 38) to define $f_\delta : \mathbb{R} \to \mathbb{R}$ by

$$f_\delta(t) = \begin{cases} 0 & \text{if } t < z - 2\delta, \\ (1+t+\delta)^3\,(t - z + 2\delta) & \text{if } z - 2\delta \leq t \leq z + 2\delta, \\ 4\delta\,(1+t+\delta)^3 & \text{if } t > z + 2\delta. \end{cases}$$

Then

$$|f_\delta(t)| \leq 4\delta\,(1+|t|+\delta)^3 \text{ for all } t \in \mathbb{R} \tag{6.16}$$

and

$$f_\delta'(t) \geq \begin{cases} (1+z-\delta)^3 & \text{if } z - 2\delta < t < z + 2\delta, \\ 0 & \text{if } t < z - 2\delta \text{ or } t > z + 2\delta. \end{cases} \tag{6.17}$$

By Lemma 3.1, we have

$$E\int_{-\infty}^{\infty} f_\delta'(U+t)\,K(t)\,dt = EUf_\delta(U)\,. \tag{6.18}$$

From the facts that

$$K(t) = 0 \text{ for } |t| > \delta,$$

$$1.15\,(1+z-\delta)^3 \geq (1+z)^3 \text{ for sufficiently large } n,$$

(4.11), (6.16) – (6.18), we obtain

$$\begin{aligned}
|T_{21}| &\leq E\int_{|t|\leq\delta} \mathbb{I}\left(z-|t|<U<z+|t|\right)K(t)\,dt \\
&\leq E\int_{|t|\leq\delta} \mathbb{I}\left(z-\delta<U<z+\delta\right)K(t)\,dt \\
&\leq \frac{1.15}{(1+z)^3}E\int_{|t|\leq\delta} (1+z-\delta)^3\,\mathbb{I}\left(z-\delta<U<z+\delta\right)K(t)\,dt \\
&\leq \frac{1.15}{(1+z)^3}E\int_{|t|\leq\delta} f_\delta'(U+t)\,K(t)\,dt \\
&= \frac{1.15}{(1+z)^3}E\int_{-\infty}^{\infty} f_\delta'(U+t)\,K(t)\,dt \\
&= \frac{1.15}{(1+z)^3}EUf_\delta(U) \\
&\leq \frac{4.6}{(1+z)^3}E\,|U|\,|U'-U|\,(1+|U|+|U'-U|)^3 \\
&\leq \frac{15.94}{(1+z)^3\sqrt{n}}E\,|U|\left(1+|U|+\frac{2\sqrt{3}}{\sqrt{n}}\right)^3.
\end{aligned} \tag{6.19}$$

Next, we will estimate $|T_{22}|$. From (4.11), (5.15), Proposition 6.4(ii) and $\delta < \frac{z+1}{8}$ for sufficiently large n, we obtain

$$\begin{aligned}
&E\int_{-\infty}^{\infty}\int_t^0 h(U+u)\,K(t)\,\mathbb{I}\left(\frac{3z}{4}<U+u\leq z\right)dudt \\
&\leq 1.001\,(1+z)\,E\int_{-\infty}^{\infty}\int_t^0 K(t)\,\mathbb{I}\left(U+u>\frac{3z}{4}\right)dudt \\
&\leq 1.001\,(1+z)\,E\int_{-\infty}^{\infty}\int_t^0 K(t)\,\mathbb{I}\left(U+\delta>\frac{3z}{4}\right)dudt \\
&\leq 1.001\,(1+z)\,E\int_{-\infty}^{\infty}\int_t^0 K(t)\,\mathbb{I}\left(U>\frac{3z}{4}-\frac{z+1}{8}\right)dudt
\end{aligned}$$

$$\le 1.001\,(1+z)\,E\mathbb{I}\left(U>\frac{5z-1}{8}\right)\int_{-\infty}^{\infty}|t|\,K(t)\,dt$$
$$\le\ 1.001\,(1+z)\,\frac{n}{8}E\mathbb{I}\left(U>\frac{5z-1}{8}\right)|U'-U|^{3}$$
$$\le 5.21\,(1+z)\,\frac{1}{\sqrt{n}}\frac{E\,(1+U)^{4}}{\left(\frac{7}{8}+\frac{5z}{8}\right)^{4}}. \tag{6.20}$$

From the fact that

$$E\int_{-\infty}^{\infty}\int_{t}^{0}K(t)\,dudt\le E\int_{-\infty}^{\infty}|t|\,K(t)\,dt=\frac{n}{8}E\,|U'-U|^{3}, \tag{6.21}$$

(4.11), (4.14), Proposition 6.4(i), Proposition 6.4(iii) and (6.20) we obtain

$$|T_{22}|\le E\int_{-\infty}^{\infty}\int_{t}^{0}h(U+u)\,K(t)\,\mathbb{I}\left(U+u\le\frac{3z}{4}\text{ or }U+u>z\right)dudt$$
$$+E\int_{-\infty}^{\infty}\int_{t}^{0}h(U+u)\,K(t)\,\mathbb{I}\left(\frac{3z}{4}<U+u\le z\right)dudt$$
$$\le\frac{15.44}{(1+z)^{3}}E\int_{-\infty}^{\infty}\int_{t}^{0}K(t)\,dudt+5.21\,(1+z)\,\frac{1}{\sqrt{n}}\frac{E\,(1+U)^{4}}{\left(\frac{7}{8}+\frac{5z}{8}\right)^{4}}$$
$$\le\frac{1}{(1+z)^{3}\sqrt{n}}\left(80.23+34.15E\,(1+U)^{4}\right). \tag{6.22}$$

From (6.19) and (6.22), we have

$$|T_{2}|\le\frac{15.94}{(1+z)^{3}\sqrt{n}}E\,|U|\left(1+|U|+\frac{2\sqrt{3}}{\sqrt{n}}\right)^{3}$$
$$+\frac{1}{(1+z)^{3}\sqrt{n}}\left(80.23+34.15E\,(1+U)^{4}\right). \tag{6.23}$$

Hence, by Proposition 4.1(i), Proposition 4.2(i), (4.11), (6.14) and (6.23), for sufficiently large n, we obtain

$$\Delta_{des(\pi),z}\le\frac{1.27}{(1+z)^{3}}\sqrt{562+1.89E\,(1+U)^{6}}$$
$$+\frac{15.94}{(1+z)^{3}\sqrt{n}}E\,|U|\left(1+|U|+\frac{2\sqrt{3}}{\sqrt{n}}\right)^{3}$$
$$+\frac{1}{(1+z)^{3}\sqrt{n}}\left(80.23+34.15E\,(1+U)^{4}\right)$$
$$\le\frac{71}{(1+z)^{3}\sqrt{n}}+\frac{453}{(1+z)^{3}\sqrt{n}}+\frac{1232}{(1+z)^{3}\sqrt{n}}$$
$$\le\frac{1756}{(1+z)^{3}\sqrt{n}}.$$

□

6.2 Bound for the Number of Inversions

Using (4.12), Proposition 4.1(ii), Proposition 4.2(ii) and the argument of the proof of Theorem 6.1, we can prove Theorem 6.2 as below.

Case 1. $0 \leq z \leq 4.2$. By Theorem 5.5 and the fact that $1 \leq \dfrac{140.61}{(1+z)^3}$ we have

$$\Delta_{inv(\pi),z} \leq \frac{14.24}{\sqrt{n}} \leq \frac{2003}{(1+z)^3\sqrt{n}}.$$

Case 2. $z > 4.2$.

$$\begin{aligned}
\Delta_{inv(\pi),z} &\leq \frac{1.39}{(1+z)^3}\sqrt{562 + 1.89E\,(1+V)^6} \\
&\quad + \frac{27.6}{(1+z)^3\sqrt{n}} E\,|V| \left(1 + |V| + \frac{6}{\sqrt{n}}\right)^3 \\
&\quad + \frac{1}{(1+z)^3\sqrt{n}} \left(416.88 + 177.13E\,(1+V)^4\right) \\
&\leq \frac{110.07}{(1+z)^3\sqrt{n}} + \frac{1392.15}{(1+z)^3\sqrt{n}} + \frac{10657.21}{(1+z)^3\sqrt{n}} \\
&\leq \frac{12159.53}{(1+z)^3\sqrt{n}}.
\end{aligned}$$

From above two cases Theorem 6.2 has been proved.

CHAPTER VII

EXPONENTIAL NON-UNIFORM BOUNDS

Most of non-uniform bounds, the denominator may be of the polynomial form. In our work we make it to be exponential and give a better bound. In this chapter, we give exponential non-uniform bounds for the number of descents and inversions. These are main results.

Theorem 7.1. *For* $z \in \mathbb{R}$*,*

$$\Delta_{des(\pi),z} \le \frac{1.061 + \left(1.265 + 3.78e^{\frac{2\sqrt{3}}{\sqrt{n}}}\right) e^{\frac{3}{4}\left(1+\frac{4.62}{\sqrt{n}}\right)} + 8.79e^{\frac{3}{2}\left(1+\frac{4.62}{\sqrt{n}}\right)}}{e^{\frac{|z|}{4}}\sqrt{n}}.$$

Moreover for sufficiently large n*,*

$$\Delta_{des(\pi),z} \le \frac{51.25}{e^{\frac{|z|}{4}}\sqrt{n}}.$$

Theorem 7.2. *For* $z \in \mathbb{R}$*,*

$$\Delta_{inv(\pi),z} \le \frac{0.008 + \left(1.388 + 0.46e^{\frac{6}{\sqrt{n}}}\right) e^{\frac{9}{4}\left(1+\frac{8}{\sqrt{n}}\right)} + 8.58e^{\frac{9}{2}\left(1+\frac{8}{\sqrt{n}}\right)}}{e^{\frac{|z|}{4}}\sqrt{n}}.$$

Moreover for sufficiently large n*,*

$$\Delta_{inv(\pi),z} \le \frac{792.71}{e^{\frac{|z|}{4}}\sqrt{n}}.$$

The exponential bounds in Theorem 7.1 and Theorem 7.2 are much sharper than the polynomial bounds in Theorem 6.1 and Theorem 6.2 in the case of $z \ge 25$ and $z \ge 44$ respectively.

	uniform bounds	exponential bounds			
		$z = 5.6$	$z = 16.1$	$z = 100$	$z = 1000$
Descent	$\frac{12.44}{\sqrt{n}}$	$\frac{12.44}{\sqrt{n}}$	$\frac{0.92}{\sqrt{n}}$	$\frac{7.12 \times 10^{-10}}{\sqrt{n}}$	$\frac{1.37 \times 10^{-107}}{\sqrt{n}}$
Inversion	$\frac{14.24}{\sqrt{n}}$	$\frac{196}{\sqrt{n}}$	$\frac{14.24}{\sqrt{n}}$	$\frac{1.11 \times 10^{-8}}{\sqrt{n}}$	$\frac{2.12 \times 10^{-106}}{\sqrt{n}}$

Table 7.1: Constants of uniform and exponential non-uniform bounds.

Table 7.2 and Table 7.3 compare the constants of polynomial and exponential non-uniform bounds for the number of descents and the number of inversions, respectively.

	$z = 10$	$z = 25$	$z = 50$	$z = 100$	$z = 1000$
Polynomial bounds	$\frac{1.32}{\sqrt{n}}$	$\frac{0.10}{\sqrt{n}}$	$\frac{1.33 \times 10^{-2}}{\sqrt{n}}$	$\frac{1.71 \times 10^{-3}}{\sqrt{n}}$	$\frac{1.76 \times 10^{-6}}{\sqrt{n}}$
Exponential bounds	$\frac{4.21}{\sqrt{n}}$	$\frac{0.10}{\sqrt{n}}$	$\frac{1.91 \times 10^{-4}}{\sqrt{n}}$	$\frac{7.12 \times 10^{-10}}{\sqrt{n}}$	$\frac{1.37 \times 10^{-107}}{\sqrt{n}}$

Table 7.2: Constants of polynomial and exponential bounds for the number of descents.

	$z = 10$	$z = 44$	$z = 70$	$z = 100$	$z = 1000$
Polynomial bounds	$\frac{9.14}{\sqrt{n}}$	$\frac{0.14}{\sqrt{n}}$	$\frac{3.4 \times 10^{-2}}{\sqrt{n}}$	$\frac{1.2 \times 10^{-2}}{\sqrt{n}}$	$\frac{1.22 \times 10^{-5}}{\sqrt{n}}$
Exponential bounds	$\frac{65.07}{\sqrt{n}}$	$\frac{0.14}{\sqrt{n}}$	$\frac{2 \times 10^{-5}}{\sqrt{n}}$	$\frac{1.11 \times 10^{-8}}{\sqrt{n}}$	$\frac{2.12 \times 10^{-106}}{\sqrt{n}}$

Table 7.3: Constants of polynomial and exponential bounds for the number of inversions.

The proofs of Theorem 7.1 and Theorem 7.2 are separated into 2 sections as described below.

7.1 Bound for the Number of Descents

Before proving a non-uniform bound, we need Lemma 7.3 which gives an exponential bound of Ee^U. The proof of Lemma 7.3 uses the idea from Lemma 5.1 of [8].

Lemma 7.3. *For $n \geq 12$, $Ee^U \leq e^{\frac{3}{2}\left(1+\frac{4.62}{\sqrt{n}}\right)}$.*

Proof. For $s \in (0, \infty)$, let $f_s : \mathbb{R} \to \mathbb{R}$ be defined by $f_s(u) = e^{su}$ and $h : (0, \infty) \to \mathbb{R}$ be defined by $h(t) = Ee^{tU}$. From (4.11) and (4.14), we have

$$\int_{-\infty}^{\infty} K(t)\, dt = \frac{n}{4}\left(U' - U\right)^2 \leq 3.$$

From Lemma 3.1, (4.5) and $K(t) = 0$ for $|t| > \delta := |U' - U|$, we obtain

$$\begin{aligned} h'(s) &= EUe^{sU} \\ &= EUf_s(U) \\ &= E\int_{-\infty}^{\infty} f_s'(U+t)\, K(t)\, dt \\ &= sE\int_{-\infty}^{\infty} e^{s(U+t)} K(t)\, dt \\ &= sE\int_{|t|\leq\delta} e^{s(U+t)} K(t)\, dt \\ &\leq sE\int_{|t|\leq\delta} e^{s(U+\delta)} K(t)\, dt \\ &\leq 3se^{s\delta} Ee^{sU} \\ &\leq 3sEe^{sU} + 3sEe^{sU}\left|e^{s\delta} - 1\right| \\ &\leq 3sEe^{sU} + 6s^2\delta Ee^{sU} \end{aligned}$$

where we have used the inequality $|e^x - 1| \leq 2|x|$ for $|x| \leq 1$ to get the last inequality. Therefore

$$\frac{h'(s)}{h(s)} \leq 3s + 6\delta s^2$$
$$\ln h(s) \leq \frac{3s^2}{2} + 2\delta s^3$$
$$h(s) \leq e^{\frac{3s^2}{2}+2\delta s^3}.$$

Hence $Ee^U = h(1) \leq e^{\frac{3}{2}\left(1+\frac{4.62}{\sqrt{n}}\right)}$. □

Proof of Theorem 7.1

Proof. By the same reason of Theorem 6.1, it suffices to prove the Theorem 7.1 in case of $z \geq 0$. Note that $1 \leq \frac{3.67}{e^{\frac{z}{4}}}$ for all $0 \leq z \leq 5.2$. Hence, by Theorem 5.4 we have

$$\Delta_{des(\pi),z} \leq \frac{12.44}{\sqrt{n}} \leq \frac{45.65}{e^{\frac{z}{4}}\sqrt{n}}. \tag{7.1}$$

Suppose that $z > 5.2$. By (5.6) we note that

$$\begin{aligned}\Delta_{des(\pi),z} &\leq \left| Eg_z'(U) - Eg_z'(U) \int_{-\infty}^{\infty} K(t)\, dt \right| \\ &\quad + \left| E \int_{-\infty}^{\infty} \{g_z'(U) - g_z'(U+t)\} K(t)\, dt \right| \\ &=: |T_1| + |T_2|. \end{aligned} \tag{7.2}$$

By (3.3), (3.5) and (6.3), we have

$$\begin{aligned} E|g_z'(U)|^2 &\leq E|g_z'(U)| \\ &= E|g_z'(U)|\, \mathbb{I}\left(U \leq \frac{z}{2}\right) + E|g_z'(U)|\, \mathbb{I}\left(U > \frac{z}{2}\right) \\ &\leq \left(1 + \sqrt{\frac{\pi}{2}} z e^{\frac{z^2}{8}}\right)(1 - \Phi(z)) + P\left(U > \frac{z}{2}\right) \\ &\leq \left(1 + \sqrt{\frac{\pi}{2}} z e^{\frac{z^2}{8}}\right)\left(\frac{e^{-\frac{z^2}{2}}}{z\sqrt{2\pi}}\right) + e^{-\frac{z}{2}} Ee^U \\ &= \frac{e^{-\frac{z^2}{2}}}{z\sqrt{2\pi}} + \frac{e^{-\frac{3z^2}{8}}}{2} + e^{-\frac{z}{2}} Ee^U \\ &\leq \left(2.68 \times 10^{-4} + Ee^U\right) e^{-\frac{z}{2}}, \end{aligned} \tag{7.3}$$

where we have applied the fact that $e^{\frac{z}{2}}e^{-\frac{3z^2}{8}} \leq 5.32\times 10^{-4}$ and $e^{\frac{z}{2}}e^{-\frac{z^2}{2}} \leq 1.81 \times 10^{-5}$ on $[5.2, \infty)$ to obtain the last inequality.

From (5.7), (5.8) and (7.3) we have

$$|T_1| \leq \sqrt{E\left(g_z'(U)\right)^2}\sqrt{\frac{8}{5n}} \leq \frac{0.021 + 1.265\left(Ee^U\right)^{\frac{1}{2}}}{e^{\frac{z}{4}}\sqrt{n}}. \tag{7.4}$$

Next, we will give a bound of $|T_2|$. From (6.15) we have

$$|T_2| \leq \ E\int_{-\infty}^{\infty} \left|\left\{g_z'(U) - g_z'(U+t)\right\} K(t)\right| dt \leq T_{21} + T_{22} \tag{7.5}$$

where

$$T_{21} = E\int_{-\infty}^{\infty} \mathbb{I}\left(z - \max(0,t) < U < z - \min(0,t)\right) K(t)\, dt$$
$$T_{22} = E\int_{-\infty}^{\infty}\int_{t}^{0} h(U+u) K(t)\, dudt.$$

Let $f_\delta : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_\delta(t) = \begin{cases} 0 & \text{if } t < z - 2\delta \\ e^{\frac{t}{2}}(t - z + 2\delta) & \text{if } z - 2\delta \leq t \leq z + 2\delta \\ 4\delta e^{\frac{t}{2}} & \text{if } t > z + 2\delta. \end{cases}$$

Then

$$f_\delta'(t) \geq \begin{cases} e^{\frac{t}{2}} & \text{if } z - 2\delta < t < z + 2\delta \\ 0 & \text{if } t < z - 2\delta \text{ and } t > z + 2\delta \end{cases} \tag{7.6}$$

and

$$|f_\delta(t)| \leq 4\delta e^{\frac{t}{2}} \quad \text{for all } t \in \mathbb{R}. \tag{7.7}$$

Then by Lemma 3.1, (4.11), (7.6), (7.7) and the facts that

$$K(t) = 0 \text{ for } \ |t| > \delta,$$

$$e^{\frac{z-2\delta}{2}} \geq e^{\frac{z}{2}-\frac{2\sqrt{3}}{\sqrt{n}}},$$

we obtain

$$\begin{aligned}
|T_{21}| &\leq E\int_{|t|\leq\delta} \mathbb{I}\left(z-|t|<U<z+|t|\right)K\left(t\right)dt \\
&\leq E\int_{|t|\leq\delta} \mathbb{I}\left(z-\delta<U<z+\delta\ \right)K\left(t\right)dt \\
&\leq \frac{e^{\frac{2\sqrt{3}}{\sqrt{n}}}}{e^{\frac{z}{2}}}E\int_{|t|\leq\delta} e^{\frac{U+t}{2}}\mathbb{I}\left(z-\delta<U<z+\delta\ \right)K\left(t\right)dt \\
&\leq \frac{e^{\frac{2\sqrt{3}}{\sqrt{n}}}}{e^{\frac{z}{2}}}E\int_{|t|\leq\delta} f_\delta'\left(U+t\right)K\left(t\right)dt \\
&= \frac{e^{\frac{2\sqrt{3}}{\sqrt{n}}}}{e^{\frac{z}{2}}}E\int_{-\infty}^{\infty} f_\delta'\left(U+t\right)K\left(t\right)dt \\
&= \frac{e^{\frac{2\sqrt{3}}{\sqrt{n}}}}{e^{\frac{z}{2}}}EUf_\delta\left(U\right) \\
&\leq \frac{4e^{\frac{2\sqrt{3}}{\sqrt{n}}}}{e^{\frac{z}{2}}}E\left|U\right|e^{\frac{U}{2}}\left|U'-U\right| \\
&\leq \frac{13.86e^{\frac{2\sqrt{3}}{\sqrt{n}}}}{e^{\frac{z}{2}}\sqrt{n}}\left(E\left|U\right|^2\right)^{\frac{1}{2}}\left(Ee^U\right)^{\frac{1}{2}} \\
&\leq \frac{3.78e^{\frac{2\sqrt{3}}{\sqrt{n}}}\left(Ee^U\right)^{\frac{1}{2}}}{e^{\frac{z}{4}}\sqrt{n}}. \qquad (7.8)
\end{aligned}$$

To bound $|T_{22}|$, we note that for $z > 5.2$ and $w < \frac{3z}{4}$, by using (6.7) and (6.8) we have

$$h\left(w\right) \leq \frac{\left(1+z^2\right)e^{-\frac{7z^2}{32}}}{z} + \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \leq \frac{0.20}{e^{\frac{z}{2}}} \qquad (7.9)$$

where we have used the fact that $e^{\frac{z}{2}}e^{-\frac{7z^2}{32}} \leq 0.037$, $ze^{\frac{z}{2}}e^{-\frac{7z^2}{32}} \leq 0.189$ and $e^{\frac{z}{2}}e^{-\frac{z^2}{2}} \leq 1.81\times10^{-5}$ on $[5.2,\infty)$.

By (4.11), (4.14), (5.15), Proposition 6.4(ii), Proposition 6.4(iii) and $\delta < \frac{z}{4}$ for sufficiently large n, we have

$$\begin{aligned}
&E\int_{-\infty}^{\infty}\int_t^0 h\left(U+u\right)K\left(t\right)\mathbb{I}\left(U+u>\frac{3z}{4}\right)dudt \\
&\leq 1.001\left(1+z\right)E\int_{-\infty}^{\infty}\int_t^0 K\left(t\right)\mathbb{I}\left(U+u>\frac{3z}{4}\right)dudt
\end{aligned}$$

$$\leq 1.001\,(1+z)\,E\int_{-\infty}^{\infty}\int_{t}^{0}K(t)\,\mathbb{I}\left(U+\delta>\frac{3z}{4}\right)dudt$$
$$\leq 1.001\,(1+z)\,E\int_{-\infty}^{\infty}\int_{t}^{0}K(t)\,\mathbb{I}\left(U>\frac{z}{2}\right)dudt$$
$$\leq\ 1.001\,(1+z)\,\frac{n}{8}E\mathbb{I}\left(U>\frac{z}{2}\right)|U'-U|^{3}$$
$$\leq\frac{5.202}{\sqrt{n}}\,(1+z)\,\frac{Ee^{U}}{e^{\frac{z}{2}}}.$$

From the fact that $\frac{1+z}{e^{\frac{z}{4}}}$ is decreasing on $[5.2,\infty)$ we obtain

$$E\int_{-\infty}^{\infty}\int_{t}^{0}h\,(U+u)\,K(t)\,\mathbb{I}\left(U+u>\frac{3z}{4}\right)dudt\leq\frac{8.79Ee^{U}}{e^{\frac{z}{4}}\sqrt{n}}.\tag{7.10}$$

Thus, from (6.21), (7.9) and (7.10) we obtain

$$\begin{aligned}|T_{22}|&\leq E\int_{-\infty}^{\infty}\int_{t}^{0}h\,(U+u)\,K(t)\,\mathbb{I}\left(U+u\leq\frac{3z}{4}\right)dudt\\&\quad+E\int_{-\infty}^{\infty}\int_{t}^{0}h\,(U+u)\,K(t)\,\mathbb{I}\left(\frac{3z}{4}<U+u\right)dudt\\&\leq\frac{0.20}{e^{\frac{z}{2}}}E\int_{-\infty}^{\infty}\int_{t}^{0}K(t)\,dudt+\frac{8.79Ee^{U}}{e^{\frac{z}{4}}\sqrt{n}}\\&\leq\frac{1.04}{e^{\frac{z}{4}}\sqrt{n}}+\frac{8.79Ee^{U}}{e^{\frac{z}{4}}\sqrt{n}}.\end{aligned}\tag{7.11}$$

From (7.5), (7.8) and (7.11) we have

$$|T_{2}|\leq\frac{1.04+3.78e^{\frac{2\sqrt{3}}{\sqrt{n}}}\left(Ee^{U}\right)^{\frac{1}{2}}+8.79Ee^{U}}{e^{\frac{z}{4}}\sqrt{n}}.\tag{7.12}$$

Hence, by Lemma 7.3, (7.1), (7.2), (7.4) and (7.12), the theorem is proved. □

7.2 Bound for the Number of Inversions

Proof of Theorem 7.2. Using (4.12) we can follow an idea of Lemma 7.3 to show that $Ee^{V}\leq e^{\frac{9}{2}\left(1+\frac{8}{\sqrt{n}}\right)}$. Then Theorem 7.2 can be proved by this fact and the argument of Theorem 7.1 as described below. When $z\leq 5.2$ we can show that

$$\Delta_{des(\pi),z}\leq\frac{14.24}{\sqrt{n}}\leq\frac{53}{e^{\frac{z}{4}}\sqrt{n}}.$$

Thus we suppose that $z > 5.2$. Similar to (5.6) we note that

$$\begin{aligned}
\Delta_{inv(\pi),z} \leq & \left| Eg_z'(V) - Eg_z'(V) \int_{-\infty}^{\infty} K(t)\,dt \right| \\
& + \left| E \int_{-\infty}^{\infty} \left\{ g_z'(V) - g_z'(V+t) \right\} K(t)\,dt \right| \\
\leq & \left| Eg_z'(V) - Eg_z'(V) \int_{-\infty}^{\infty} K(t)\,dt \right| \\
& + E \int_{-\infty}^{\infty} \mathbb{I}\left(z - \max(0,t) < V < z - \min(0,t)\right) K(t)\,dt \\
& + E \int_{-\infty}^{\infty} \int_{t}^{0} h(V+u)\,K(t)\,dudt. \\
\leq & \sqrt{E\left(g_z'(V)\right)^2} \sqrt{\frac{51.84}{n}} + \frac{0.46 e^{\frac{2\sqrt{3}}{\sqrt{n}}} \left(Ee^V\right)^{\frac{1}{2}}}{e^{\frac{z}{4}}\sqrt{n}} + \frac{0.002}{e^{\frac{z}{4}}\sqrt{n}} + \frac{8.58 Ee^V}{e^{\frac{z}{4}}\sqrt{n}}. \\
\leq & \frac{0.006 + 1.388\left(Ee^V\right)^{\frac{1}{2}}}{e^{\frac{z}{4}}\sqrt{n}} + \frac{0.46 e^{\frac{2\sqrt{3}}{\sqrt{n}}} \left(Ee^V\right)^{\frac{1}{2}}}{e^{\frac{z}{4}}\sqrt{n}} + \frac{0.002}{e^{\frac{z}{4}}\sqrt{n}} + \frac{8.58 Ee^V}{e^{\frac{z}{4}}\sqrt{n}}.
\end{aligned}$$

The proof of Theorem 7.2 is now complete. □

CHAPTER VIII

FURTHER RESEARCH

In this thesis, the number of descents and the number of inversions must be defined on a random permutation π of the set $\{1, 2, \ldots, n\}$. It involves one to one property of function. In 2007, these two statistics have been generalized to reduce one-to-one property. It is based on multiset in which Conger and Viswanath ([9]) used to analyze the arrangement of molecule in Human genome. The range of π in their work is a multiset generalized the Fulman's work to random permutation $\pi : \{1, 2, \ldots, n\} \to \{1^{n_1}, 2^{n_2}, \ldots, k^{n_k}\}$ when $n_1 + n_2 + \cdots + n_k = n$, here

$$\{1^{n_1}, 2^{n_2}, \ldots, k^{n_k}\} =: \{1_1, 1_2, \ldots, 1_{n_1}, 2_1, 2_2, \ldots, 2_{n_2}, \ldots, k_1, k_2, \ldots, k_{n_k}\}.$$

They gave a bound for normal approximation to be rate of order $\frac{1}{\sqrt{n}}$. An open problem is "Can we improve it to be a non-uniform bound?"

Moreover these statistics can be generalized as described below. The number of ***d*-descents** of π, denoted by $des_d(\pi)$, is defined as the number of pairs (i, j) with $i < j \leq i + d$ and $\pi(i) > \pi(j)$. In particular, 1-descents correspond to descents in the traditional sense, and $(n-1)$-descents correspond to inversions. In 2008, Miklós Bóna ([5]) used Janson's dependency criterion to prove that the distribution of d-descents of permutations of length n converge to a normal distribution as n goes to infinity. In 2011, John Pike ([17]) provided an explicit formula for the mean and variance of these statistics and obtain bounds on the rate of convergence using Stein's method. His result shows that the distribution of d-descents in a random permutation converges to the normal distribution on the order of $\frac{1}{\sqrt{n}}$ when d is fixed. It is an open problem to find a non-uniform bound.

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