CAYLEY GRAPHS AND CAYLEY SIGNED GRAPHS OVER FINITE COMMUTATIVE RINGS

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ให้ R เป็นริงสลับที่จำกัดซึ่งมีเอกลักษณ์ $1 \neq 0$ กราฟเคย์เลย์ยูนิแทรี เหนือ R เขียนแทนด้วย G_R เป็นกราฟที่มีเซตจุดยอดคือ R และเซตของเส้นเชื่อมคือ $\{\{a,b\} : a,b \in R$ และ $a-b \in R^{\times}\}$ เมื่อ $R^{\times}$ เป็นกรุปยูนิตของ R เราศึกษากราฟเครื่องหมายเคย์เลย์ยูนิแทรีเหนือริงสลับที่จำกัดและได้ริงสลับที่จำกัดทั้งหมดที่กราฟเครื่องหมายเคย์เลย์เหนือริงเหล่านั้นได้ดุล เราหาพลังงานของกราฟย่อยของกราฟเคย์เลย์ยูนิแทรีที่ถูกเหนี่ยวนำโดยการส่งกำลังสอง ยิ่งกว่านั้น เราได้ สเปกตรัมและ พลังงานของกราฟเคย์เลย์เหนือริงลูกโซ่จำกัด และ เราได้นำพลังงานนั้นมา ประยุกต์เพื่อหาผลลัพธ์เพิ่มเติมบนกราฟตัวหารร่วมมากบนริงผลหารของโดเมนที่มีการแยกตัวประกอบได้อย่างเดียว

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Let R be a finite commutative ring with identity $1 \neq 0$. The unitary Cayley graph of R, denoted by G_R, is the graph whose vertex set is R and the edge set $\{\{a,b\} : a,b \in R \text{ and } a-b \in R^\times\}$, where $R^\times$ is the group of units of R. We study a unitary Cayley signed graph and characterize all finite commutative ring in which their signed graphs are balanced. We find the energy of a subgraph of unitary Cayley graph induced by square mapping. Moreover, we determine the spectrum and obtain the energy of a Cayley graph over a finite chain ring and apply them to get further results on a gcd-graph over a quotient ring of a unique factorization domain.

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CONTENTS

CHAPTER I

PROLOGUE

1.1 Introduction

The study of ring-theoretic graphs has become an exciting research topic in the last two decades. This includes unitary Cayley graphs, integral circulant graphs, zero-divisor graphs and gcd-graphs. Mostly, the work is on determining its eigenvalues (which are real) and computing the sum of absolute values of the eigenvalues, called the energy of a graph. The energy is a graph parameter introduced by Gutman (see [17] and [11] for a good survey) arising from the Hückel molecular orbital approximation for the total π-electron energy. Nowadays, the energy of graph is studied for purely mathematical interest.

In Chapter II, we define the restricted unitary Cayley graph induced from the square mapping by $H_R = \mathrm{Cay}(R, T_R)$, where $T_R = K_R(R^\times)^{(2)}$, the product of K_R and $(R^\times)^{(2)}$ which are subgroups of $R^\times$, $(R^\times)^{(2)} = \{a^2 : a \in R^\times\}$ and $K_R = \{a \in R^\times : a^2 = 1\}$. The motivation of this graph comes from the quadratic unitary Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, \pm(\mathbb{Z}_n^\times)^{(2)})$ introduced by Beaudrap [6]. He bounded the diameter of such graphs and characterized the conditions on n for $\mathrm{Cay}(\mathbb{Z}_n, \pm(\mathbb{Z}_n^\times)^{(2)})$ to be perfect, i.e., the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph. When $n = p^s$, p

a prime and $s \in \mathbb{N}$, his graph and our graph coincide. The main purpose of the chapter is to obtain the eigenvalues and energy of the graphs. This is a joint work with Meemark and has been published in ScienceAsia [28].

In chapter III, we study the unitary Cayley signed graph defined to be an ordered pair $\mathcal{S}_R = (G_R, \sigma)$, where G_R is the unitary Cayley graph over R with signature $\sigma : E(G_R) \to \{1, -1\}$ given by

$$\sigma(\{a,b\}) = \begin{cases} 1, & \text{if } a \in R^{\times} \text{ or } b \in R^{\times}, \\ -1, & \text{otherwise.} \end{cases}$$

We give a criterion on R for S_R to be balanced (every cycle in $\mathcal{S}_R$ is positive) and a criterion for its line graph $L(\mathcal{S}_R)$ to be balanced. Moreover, we characterize all finite commutative rings with the property that the marked sigraph $(\mathcal{S}_R)_\mu$ is canonically consistent. We also give a characterization of all finite commutative rings where $\mathcal{S}_R$, $\eta(\mathcal{S}_R)$ and $L(\mathcal{S}_R)$ are hyperenergetic balanced. This result has been published in Journal of Algebra and Its Applications [29].

Finally, we consider the gcd-graph on a quotient ring of a unique factorization domain (UFD) introduced in [23] generalizes a gcd-graph or an integral circulant graph (i.e., its adjacency matrix is circulant and all eigenvalues are integers) defined over $\mathbb{Z}_n, n \geq 2$, (see [24, 40]). An integral circulant graph can also be considered as an extension of a unitary Cayley graph. This graph has been widely studied in many literatures, e.g., [5, 12, 17, 18, 21, 22, 34].

We determine the spectrum and obtain the energy of a Cayley graph over a finite chain ring in Chapter IV. This extends the idea of integral circulant graphs with prime power order presented in [34] in which they compute the energy via a sum of Ramanujan sums as studied in [24]. Our approach here is to examine all eigenvalues with multiplicities and then obtain the sum of their absolute values directly similar to [23]. We also show that the graph defined over a finite chain ring is indeed an integral circulant graph. The final section presents some applications of the energy found in Section 4.1. We gives further results on a gcd graph on a quotient ring of a unique factorization domain using a tensor product and a non-complete extended p-sum.

Some background knowledges and basic terminologies in algebraic graph theory are recalled in the following three sections. This includes results on the energy of a graph and properties of a unitary Cayley graph which we shall repeatedly use throughout this dissertation.

1.2 Terminologies in Graph Theory

A **graph** G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a finite set of objects called **vertices** and $E(G)$ is a set of 2-element subsets of $V(G)$, called **edges**. We will refer to $V(G)$ as the **vertex set** of G and to $E(G)$ as the **edge set** of G. The **order** of G is $|V(G)|$.

If $e = \{u, v\}$ is an edge of a graph G, we say that u and v are the **endvertices** of e and that u and v are **adjacent**. In this case, we also say that u and e are **incident**, as are v and e. Furthermore, if e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are **adjacent** edges. The **degree** of a vertex v in a graph G, denoted by $\deg v$, is the number of edges in G that are incident with v.

A graph G is **regular of degree** r if $\deg v = r$ for each vertex v of G. Such graphs are called r-**regular**. A graph is **complete** if every two of its vertices are adjacent. A complete graph of order n is therefore $(n-1)$-regular and has size $\binom{n}{2}$. We denote this graph by K_n.

An **isomorphism** from a graph G to a graph H is a bijection $f : V(G) \to V(H)$ such that $\{u, v\} \in E(G)$ if and only if $\{f(u), f(v)\} \in E(H)$. We say G is **isomorphic to** H, written $G \cong H$, if there is an isomorphism from G to H.

A **cycle** is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. The number of edges in a cycle is called its **length**.

A k-regular graph G with v vertices is said to be **strongly regular** with parameters (v, k, λ, μ) if there are integers λ and μ such that

(i) every two adjacent vertices have λ common neighbors, and

(ii) every two non-adjacent vertices have μ common neighbors.

Suppose $(H, +)$ is a group and $S \subseteq H$. The **Cayley graph** $\mathrm{Cay}(H, S)$ is a graph whose vertex set is H and the edge set $\{\{a, b\} : a, b \in H \text{ and } b - a \in S\}$.

1.3 Energy of Graphs

For a graph G with vertex set $\{v_1, \ldots, v_n\}$, the **adjacency matrix of** G, denoted by $A(G_R)$, is the $n \times n$ matrix $[a_{ij}]$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise,} \end{cases}$$

for all $i, j \in \{1, 2, \ldots, n\}$. The **eigenvalues** of G is defined to be the eigenvalues of the adjacency matrix of G. The set of all eigenvalues of G is called the **spectrum of** G. As is standard, if $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of a graph G of respective multiplicities $m_1, \ldots, m_k$, we use the notation $\operatorname{Spec} G = \begin{pmatrix} \lambda_1 & \ldots & \lambda_k \\ m_1 & \ldots & m_k \end{pmatrix}$ to describe the spectrum of G. We can explicitly determine the eigenvalues of a strongly regular graph (Section 10.2 of [15]) in the following lemma.

Lemma 1.3.1. *[15] A strongly regular graph with parameters* (v, k, λ, μ) *has exactly three eigenvalues:*

1. k *whose multiplicity is* 1,
2. $\frac{1}{2}\left[(\lambda - \mu) + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}\right]$ *whose multiplicity is* $\frac{1}{2}\left[(v - 1) - \frac{2k + (v-1)(\lambda - \mu)}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}\right]$, *and*
3. $\frac{1}{2}\left[(\lambda - \mu) - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}\right]$ *whose multiplicity is* $\frac{1}{2}\left[(v - 1) + \frac{2k + (v-1)(\lambda - \mu)}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}\right]$.

The sum of absolute values of all eigenvalues of a graph G is called the **energy of** G and denoted by $\mathcal{E}(G)$, i.e., $\mathcal{E}(G) = \sum m_i |\lambda_i|$. The energy is a graph parameter stemming from the Hückel molecular orbital approximation for the total π-electron energy (for survey on molecular graph energy see e.g., [17]). This concept was introduced by Gutman [16]. Later, the energy of graph was studied intensively in many literatures (see e.g., [31], [41], [43] and [23]).

For two graphs G and H, their **tensor product** $G \otimes H$ is the graph with vertex set $V(G) \times V(H)$, where (u, v) is adjacent to (u', v') if and only if u is adjacent to u' in G and v is adjacent to v' in H. The adjacency matrix of $G \otimes H$ is the Kronecker product of $A(G)$ and $A(H)$, i.e., $A(G \otimes H) = A(G) \otimes A(H)$.

Proposition 1.3.2. *[42] Let G and H be graphs. Suppose that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of G and $\mu_1, \ldots, \mu_m$ are the eigenvalues of H (repetition is possible). Then the eigenvalues of $G \otimes H$ are $\lambda_i \mu_j$, where $1 \leq i \leq n$ and $1 \leq j \leq m$. Moreover, $\mathcal{E}(G \otimes H) = \mathcal{E}(G)\mathcal{E}(H)$.*

The **complement of a graph** G, denoted by $\bar{G}$, is the graph with the same vertex set as G such that two vertices of $\bar{G}$ are adjacent if and only if they are not adjacent in G.

Proposition 1.3.3. *[15, 42] If a graph G with n vertices is k-regular, then G and $\bar{G}$ have the same eigenvectors. The eigenvalue associated with n-vector $\vec{1}_n$, whose entry are all 1, is k for G and $n - k - 1$ for $\bar{G}$. If $\vec{x} \neq \vec{1}_n$ is an eigenvector of G for eigenvalue λ of G, then its associated eigenvalue in $\bar{G}$ is $-1 - \lambda$.*

1.4 Unitary Cayley Graphs

Throughout this dissertation, every ring has the identity $1 \neq 0$. Let R be a finite commutative ring. The **unitary Cayley graph** of R, denoted by G_R, is the graph whose vertex set is R and the edge set $\{\{a, b\} : a, b \in R \text{ and } a - b \in R^\times\}$. We use $R^\times$ to denote the group of units of R. It is clear that G_R is regular of degree $|R^\times|$. We can also write G_R as $\mathrm{Cay}(R, R^\times)$.

A **local ring** is a commutative ring which has a unique maximal ideal. For a local ring R, we denote its unit group by $R^\times$ and it is straightforward that its unique maximal ideal $M = R \setminus R^\times$ consists of all non-unit elements. We also call the field R/M the **residue field of** R.

Example 1.4.1. If p is a prime, then $\mathbb{Z}_{p^s}$, $s \in \mathbb{N}$, is a local ring with the maximal ideal $p\mathbb{Z}_{p^s}$ and residue field $\mathbb{Z}_{p^s}/p\mathbb{Z}_{p^s}$ isomorphic to $\mathbb{Z}_p$. Moreover, every field is a local ring with maximal ideal $\{0\}$.

We also use the following well known fact of a finite commutative ring.

Theorem 1.4.2. *[10] A finite commutative ring R can be expressed as a direct product of local ring, i.e.,*

$$R \cong R_1 \times R_2 \times \cdots \times R_k,$$

where $k \in \mathbb{N}$, R_i is a local ring for $1 \leq i \leq k$. Consequently,

$$R^\times \cong R_1^\times \times R_2^\times \times \cdots \times R_k^\times.$$

The following theorem is Proposition 2.1 and Theorem 2.4 of [23]. It gives the eigenvalues and the energy of unitary Cayley graph of R.

Theorem 1.4.3. *[23] Let R be a finite commutative ring.*

1. *If R is a local ring with maximal ideal M, then*

$$\operatorname{Spec} G_R = \begin{pmatrix} |R^\times| & -|M| & 0 \\ 1 & \frac{|R|}{|M|} - 1 & \frac{|R|}{|M|}(|M|-1) \end{pmatrix}$$
$$= \begin{pmatrix} |R^\times| & -|M| & 0 \\ 1 & \frac{|R^\times|}{|M|} & \frac{|R|}{|M|}(|M|-1) \end{pmatrix}.$$

and $\mathcal{E}(G_R) = 2|R^\times|$. In particular, if F is the field with q elements, then F is a local ring with the maximal ideal $\{0\}$. So $|M| = |\{0\}| = 1$ and hence

$$\operatorname{Spec} G_F = \begin{pmatrix} q-1 & -1 \\ 1 & q-1 \end{pmatrix} = \begin{pmatrix} |F^\times| & -1 \\ 1 & |F^\times| \end{pmatrix}$$

and $\mathcal{E}(G_F) = 2(q-1)$.

2. *If $R = R_1 \times R_2 \times \cdots \times R_k$ and R_i is a local ring for all $i \in \{1, 2, \ldots, k\}$, then*

$$G_R \cong G_{R_1} \otimes G_{R_2} \otimes \cdots \otimes G_{R_k}$$

and hence

$$\begin{aligned} \mathcal{E}(G_R) &= \mathcal{E}(G_{R_1})\mathcal{E}(G_{R_2}) \ldots \mathcal{E}(G_{R_k}) \\ &= 2|R_1^\times| \cdot 2|R_2^\times| \cdot \ldots \cdot 2|R_k^\times| \\ &= 2^k |R_1^\times \times R_2^\times \times \cdots \times R_k^\times| \\ &= 2^k |R^\times|. \end{aligned}$$

CHAPTER II

EIGENVALUES AND ENERGY OF RESTRICTED UNITARY CAYLEY GRAPHS INDUCED FROM THE SQUARE MAPPING

2.1 Paley Graphs

Let R be a finite commutative ring. Consider the exact sequence of groups

$$1 \longrightarrow K_R \longrightarrow R^\times \xrightarrow{\theta} (R^\times)^{(2)} \longrightarrow 1,$$

where $\theta : a \mapsto a^2$ is the square mapping on $R^\times$ with kernel $K_R = \{a \in R^\times : a^2 = 1\}$ and $(R^\times)^{(2)} = \{a^2 : a \in R^\times\}$. Note that K_R consists of the identity and all elements of order two in $R^\times$. Let $T_R = K_R(R^\times)^{(2)}$, the product of K_R and $(R^\times)^{(2)}$ which are subgroups of $R^\times$. Define the subgraph H_R of the unitary Cayley graphs $G_R = \mathrm{Cay}(R, R^\times)$ by $H_R = \mathrm{Cay}(R, T_R)$, in which two vertices are adjacent if and only if their difference is in T_R. Since $-1 \in T_R$, the graph H_R is undirected. In addition, we observe that if $|R^\times|$ is odd, then $K_R = \{1\}$ and $R^\times = (R^\times)^{(2)}$, so $H_R = G_R$. All finite rings R with group of units $R^\times$ having an odd number of elements are completely determined by Dolžan [13].

The motivation of the graph defined above comes from the quadratic unitary Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, \pm(\mathbb{Z}_n^\times)^{(2)})$ introduced by Beaudrap [6]. He bounded the diameter of such graphs and characterized the conditions on n for $\mathrm{Cay}(\mathbb{Z}_n, \pm(\mathbb{Z}_n^\times)^{(2)})$ to be perfect. When $n = p^s$, p a prime and $s \in \mathbb{N}$, his graph and our graph H_R coincide. The main purpose here is to obtain the eigenvalues and energy of the graphs using an approach similar to [23].

Now, we give an example. Let q be a prime power such that $q \equiv 1 \mod 4$. The **Paley graph** is the graph whose vertex set is the finite field $\mathbb{F}_q$ with q elements and edge set is $\{\{a, b\} : a, b \in \mathbb{F}_q \text{ and } a - b \in (\mathbb{F}_q^\times)^{(2)}\}$. Since $\mathbb{F}_q$ is a field, $K_{\mathbb{F}_q} = \{\pm 1\}$. The congruence condition on q implies that -1 is a square in $\mathbb{F}_q$ and hence $T_{\mathbb{F}_q} = (\mathbb{F}_q^\times)^{(2)}$. Thus, the Paley graph is $H_{\mathbb{F}_q}$.

It is well known that the Paley graph is strongly regular as we are recalling in the next lemma ([15] p.221). Therefore, we obtain its eigenvalues from Lemma 1.3.1.

Lemma 2.1.1. *[15] Let q be a prime power such that $q \equiv 1 \mod 4$. The Paley graph over the finite field $\mathbb{F}_q$ is strongly regular with parameters $(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$ and*

$$\mathrm{Spec}\, H_{\mathbb{F}_q} = \begin{pmatrix} \frac{q-1}{2} & \frac{\sqrt{q}-1}{2} & \frac{-\sqrt{q}-1}{2} \\ 1 & \frac{q-1}{2} & \frac{q-1}{2} \end{pmatrix}.$$

The next lemma is obvious.

Lemma 2.1.2. *If q is a prime power congruent to 1 modulo 4, then $\mathcal{E}(H_{\mathbb{F}_q}) = \frac{q-1}{2}(1 + \sqrt{q})$.*

In what follows, we shall study the the energy of H_R when R is the ring of integers modulo n or R is the quotient ring of polynomials over a finite field. We prove the main theorem on the spectrum and energy of H_R in the next section.

The final section provides some computational examples using elementary number theory.

2.2 Energy of H_R

Let R be a finite commutative ring. Then R can be expressed as a direct sum of local rings, that is, $R \cong R_1 \times R_2 \times \cdots \times R_k$, where R_i is a local ring. Note that $R \cong R_1 \times R_2 \times \cdots \times R_k$ induces the isomorphisms

$$R^\times \cong R_1^\times \times R_2^\times \times \cdots \times R_k^\times \quad \text{and}$$

$$(R^\times)^{(2)} \cong (R_1^\times)^{(2)} \times (R_2^\times)^{(2)} \times \cdots \times (R_k^\times)^{(2)}.$$

In addition, $K_R \cong K_{R_1} \times K_{R_2} \times \cdots \times K_{R_k}$. This proves the following decomposition theorem.

Theorem 2.2.1. *Let R be a finite commutative ring. If $R = R_1 \times R_2 \times \cdots \times R_k$ and R_i is a local ring for each $i \in \{1, 2, \dots, k\}$, then*

$$H_R \cong H_{R_1} \otimes H_{R_2} \otimes \cdots \otimes H_{R_k}.$$

The above theorem tells us that we can concentrate only on H_R when R is a local ring.

Let R be a finite local ring with unique maximal ideal M. Then $R^\times = R \setminus M$ and $|R| = p^l$ for some $l \geq 1$ and p is a prime number. Note that the kernel of the homomorphism $\varphi : R^\times \to (R/M)^\times$ mapping a to $a + M$ is $1 + M$. Thus, we have the isomorphism $R^\times/(1 + M) \cong (R/M)^\times$. Recall that R/M is a field and $(R/M)^\times$ is cyclic. Since $|R^\times| = |R| - |M| = |M|(|R|/|M| - 1)$, $|M| = p^m$ for some

$m < l$ and $|M| = |1 + M|$, we have $1 + M$ is the Sylow p-subgroup of $R^{\times}$. Hence, $R^{\times} \cong (R/M)^{\times} \times (1 + M)$.

Assume that p is an odd prime. The above observation gives $K_R = \{\pm 1\}$ and leads us to distinguish two cases. If -1 is not a square in R, then $K_R(R^{\times})^{(2)} = R^{\times}$ and $H_R = G_R$.

Next, we suppose that -1 is a square in R. Since $|1 + M| = p^m$ and p is an odd prime, we have $(1+M)^{(2)} = 1+M$ so that $(R^{\times})^{(2)} \cong ((R/M)^{\times})^{(2)} \times (1+M)$. Write $R/M = \{r_1 + M, r_2 + M, \ldots, r_{p^{l-m}} + M\}$. Then for each $a \in R$, there is a unique i and $m_a \in M$ such that $a - r_i = m_a$. This yields the bijection $\tau : R \to R/M \times M$ by $\tau : a \mapsto (r_i + M, m_a)$ for all $a \in R$. Thus, for all $a, b \in R$, $(a - b) \in (R^{\times})^{(2)}$ if and only if $\tau(a - b) \in ((R/M)^{\times})^{(2)} \times M$. Hence, τ induces an isomorphism $H_R \cong H_{R/M} \otimes \mathcal{K}_{|M|}$, where $\mathcal{K}_{|M|}$ is the $|M|$-complete graph with a loop on each vertex. Note that $H_{R/M}$ is the Payley graph mentioned in Section 1. Furthermore, we know that the adjacency matrix of the $|M|$-complete graph with a loop on each vertex, $\mathcal{K}_{|M|}$, is the $|M| \times |M|$ matrix of all 1s, so

$$\operatorname{Spec} \mathcal{K}_{|M|} = \begin{pmatrix} |M| & 0 \\ 1 & |M| - 1 \end{pmatrix}.$$

and $\mathcal{E}(\mathcal{K}_{|M|}) = |M|$. Therefore, Lemma 2.1.2, Theorem 1.4.3 and Theorem 1.3.2 complete our main conclusions.

Theorem 2.2.2. *Let R be a finite local ring with unique maximal ideal M of characteristic odd prime p.*

1. *If -1 is not a square in R, then $H_R = G_R$ and $\mathcal{E}(H_R) = \mathcal{E}(G_R) = 2|R^{\times}|$.*

2. If -1 is a square in R, then $H_R \cong H_{R/M} \otimes \mathcal{K}_{|M|}$, where $\mathcal{K}_{|M|}$ is the $|M|$-complete graph with a loop on each vertex. Moreover, if $|R/M| = p^t$, then

$$\operatorname{Spec} H_R = \begin{pmatrix} \frac{|R^\times|}{2} & \frac{|M|(p^{t/2}-1)}{2} & \frac{|M|(-p^{t/2}-1)}{2} & 0 \\ 1 & \frac{p^t-1}{2} & \frac{p^t-1}{2} & |R|-p^t \end{pmatrix}$$

and $\mathcal{E}(H_R) = \dfrac{|R^\times|(p^{t/2}+1)}{2}$.

2.3 Examples

In this section, we provide two computational examples of Theorem 2.2.2 using elementary number theory. We present the results when $R = \mathbb{Z}_n$, the ring of integers modulo n and $R = A/fA$, where $A = \mathbb{F}_q[x]$, $q = p^s$ an odd prime power, $s \geq 1$, and f is a non-constant polynomial in A.

2.3.1 Quadratic residues of n

Let $n > 1$ be a positive integer. We write $G_n = G_{\mathbb{Z}_n}$, $K_n = K_{\mathbb{Z}_n}$, $T_n = T_{\mathbb{Z}_n}$ and $H_n = H_{\mathbb{Z}_n}$. We study the structure of the graph H_n and obtain its energy using the results discussed in the previous sections.

As usual, our work will start with the case when n is a prime power p^s. For $p = 2$, if $s = 1$, it is immediate that $H_2 = G_2$. If $s = 2$, we have $\mathbb{Z}_{2^2}^\times = \{\pm 1\} = K_{2^2}$ and $(\mathbb{Z}_{2^2}^\times)^{(2)} = \{1\}$, and thus $T_{2^2} = \{\pm 1\}$ and $H_4 = G_4$. If $s = 3$, we have $\mathbb{Z}_{2^3}^\times = \{\pm 1, \pm 3\} = K_{2^3}$ and $(\mathbb{Z}_{2^3}^\times)^{(2)} = \{1\}$, so $T_{2^3} = \mathbb{Z}_{2^3}^\times$ and $H_8 = G_8$. Finally, let $s \geq 4$. We recall the fact that $\mathbb{Z}_{2^s}^\times \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{s-2}}$. Since $(\mathbb{Z}_{2^s}^\times)^{(2)} \cong \mathbb{Z}_{2^{s-3}}$ and $K_{2^s} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we have $K_{2^s}(\mathbb{Z}_{2^s}^\times)^{(2)} \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{s-3}}$. Thus, $T_{2^s} = \pm(\mathbb{Z}_{2^s}^\times)^{(2)}$. It follows from Lemma 2 of [6] that $H_{2^s} \cong H_8 \otimes \mathcal{K}_{2^{s-3}}$. Hence, we have proved the next theorem.

Theorem 2.3.1. *The graphs $H_2 = G_2$, $H_4 = G_4$, $H_8 = G_8$ and for $s \geq 4$, we have*

$$H_{2^s} \cong H_8 \otimes \mathcal{K}_{2^{s-3}},$$

where $\mathcal{K}_{2^{s-3}}$ is the 2^{s-3}-complete graph with a loop on each vertex. Moreover, $\operatorname{Spec} H_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\operatorname{Spec} H_4 = \begin{pmatrix} 2 & -2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$, $\operatorname{Spec} H_8 = \begin{pmatrix} 2^2 & -2^2 & 0 \\ 1 & 1 & 6 \end{pmatrix}$, *and* $\operatorname{Spec} H_{2^s} = \begin{pmatrix} 2^{s-1} & -2^{s-1} & 0 \\ 1 & 1 & 2^s-2 \end{pmatrix}$ *for all $s \geq 4$. Furthermore, $\mathcal{E}(H_{2^s}) = 2^s$ for all $s \geq 1$.*

Next, let p be an odd prime and $s \geq 1$. Recall that -1 is a square in $\mathbb{Z}_{p^s}$ if and only if $p \equiv 1 \mod 4$. Theorem 2.2.2 directly gives the following results.

Theorem 2.3.2. *Let p be an odd prime and $s \geq 1$.*

1. *If $p \equiv 3 \mod 4$, then $H_{p^s} = G_{p^s}$ and $\mathcal{E}(H_{p^s}) = \mathcal{E}(G_{p^s}) = 2(p^s - p^{s-1})$.*

2. *If $p \equiv 1 \mod 4$, then $H_{p^s} = H_p \otimes \mathcal{K}_{p^{s-1}}$, where $\mathcal{K}_{p^{s-1}}$ is the p^{s-1}-complete graph with a loop on each vertex. Moreover,*

$$\operatorname{Spec} H_{p^s} = \begin{pmatrix} \frac{p^s-p^{s-1}}{2} & \frac{p^{s-1}(\sqrt{p}-1)}{2} & \frac{p^{s-1}(-\sqrt{p}-1)}{2} & 0 \\ 1 & \frac{p-1}{2} & \frac{p-1}{2} & p^s-p \end{pmatrix}$$

and $\mathcal{E}(H_{p^s}) = \frac{p^s-p^{s-1}}{2}(\sqrt{p}+1)$.

2.3.2 Quadratic residues of f

Let $\mathbb{F}_q$ be the finite field with $q = p^s$ elements of characteristic odd prime p. Let $A = \mathbb{F}_q[T]$, and let $f \in A$ be a non-constant polynomial. We write $G_f = G_{(A/fA)}$ and $H_f = H_{(A/fA)}$.

Let $P \in A$ be an irreducible polynomial and $e \geq 1$. Write $|P|$ for $q^{\deg P}$. We recall Theorem 1.10 of [33] for $d = 2$ in the next theorem.

Theorem 2.3.3. *[33] Let $a \in A$ and $e \geq 1$. Then the congruence $x^2 \equiv a \mod P^e$ has a solution if and only if $a^{\frac{|P|-1}{2}} \equiv 1 \mod P$ in A.*

This theorem gives the following criteria to determine if -1 is a square modulo P^e.

Corollary 2.3.4. *Let $\mathbb{F}_q$ be the finite field with $q = p^s$ elements of characteristic odd prime p. Let $P \in \mathbb{F}_q[T]$ be an irreducible polynomial and $e \geq 1$.*

1. *-1 is a quadratic non-residue of P^e if and only if ($p \equiv 3 \mod 4$ and $s \deg P$ is odd).*

2. *-1 is a quadratic residue of P^e if and only if ($p \equiv 1 \mod 4$) or ($p \equiv 3 \mod 4$ and $s \deg P$ is even).*

Theorem 2.2.2 gives eigenvalues and energy of the graph H_{P^e} for all $e \geq 1$ as follows.

Theorem 2.3.5. *Let $\mathbb{F}_q$ be the finite field with $q = p^s$ elements of characteristic odd prime p. Let $P \in \mathbb{F}_q[T]$ be an irreducible polynomial and $e \geq 1$.*

1. *If* ($p \equiv 3 \mod 4$ *and* $s \deg P$ *is odd*), *then* $H_{P^e} = G_{P^e}$ *and* $\mathcal{E}(H_{P^e}) = 2(|P|^e - |P|^{e-1})$.

2. *If* ($p \equiv 1 \mod 4$) *or* ($p \equiv 3 \mod 4$ *and* $s \deg P$ *is even*), *then* $H_{P^e} \cong H_P \otimes \mathcal{K}_{|P|^{e-1}}$, *where* $\mathcal{K}_{|P|^{e-1}}$ *is the* $|P|^{e-1}$*-complete graph with a loop on each vertex. Moreover,*

$$\operatorname{Spec} H_{P^e} = \begin{pmatrix} \frac{|P|^e - |P|^{e-1}}{2} & \frac{|P|^{e-1}(\sqrt{|P|}-1)}{2} & \frac{|P|^{e-1}(-\sqrt{|P|}-1)}{2} & 0 \\ 1 & \frac{|P|-1}{2} & \frac{|P|-1}{2} & |P|^e - |P| \end{pmatrix}$$

and $\mathcal{E}(H_{P^e}) = \frac{|P|^e - |P|^{e-1}}{2}(\sqrt{|P|}+1)$.

CHAPTER III

BALANCED UNITARY CAYLEY SIGRAPHS OVER FINITE COMMUTATIVE RINGS

3.1 Signed Graphs

A **signed graph** (or **sigraph** in short) is an ordered pair $\mathcal{S} = (S^u, \sigma)$, where S^u is a graph (V, E), called the **underlying graph** of $\mathcal{S}$ and σ is a function from the edge set E of S^u into the set $\{1, -1\}$, called a **signature** (or **sign** in short) of $\mathcal{S}$.

A sigraph is **all-positive** (respectively, **all-negative**) if all its edges are positive (negative). A sigraph is said to be **homogeneous** if it is either all-positive or all-negative and **heterogeneous** otherwise. Denote $d(v)$ the degree of vertex v and $d^-(v)$ the negative degree of a vertex v is the number of negative edges incident at v. A cycle in a sigraph $\mathcal{S}$ is said to be **positive** if it contains an even number of negative edges. A sigraph $\mathcal{S}$ is **balanced** if every cycle in $\mathcal{S}$ is positive.

We define the **unitary Cayley signed graph** (or **unitary Cayley sigraph** in short) to be an ordered pair $\mathcal{S}_R = (G_R, \sigma)$, where G_R is the unitary Cayley graph over R and $\sigma : E(G_R) \to \{1, -1\}$ given by

$$\sigma(\{a, b\}) = \begin{cases} 1, & \text{if } a \in R^\times \text{ or } b \in R^\times, \\ -1, & \text{otherwise.} \end{cases}$$

When $R = \mathbb{Z}_n$, $n > 1$, $\mathcal{S}_R = \mathcal{S}_n$ is the graph studied in [39] as we shall remark their results throughout the chapter. However, we obtain more general characterizations of R for the canonical consistent of the marked sigraph $(\mathcal{S}_R)_\mu$ in the next section (Theorem 3.3.4). Furthermore, hyperenergetic balanced sigraphs are studied in the final section.

3.2 Balanced Unitary Cayley Sigraphs

In this section, we work on conditions on R in which $\mathcal{S}_R$ to be balanced (Theorem 3.2.2) and the ones in which its line graph $L(\mathcal{S}_R)$ to be balanced (Theorem 3.2.8).

Recall that for a graph G with vertex set $\{v_1, \ldots, v_n\}$, the **adjacency matrix of** G, denoted by $A(G)$, is the $n \times n$ matrix $[a_{ij}]$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise,} \end{cases}$$

for all $i, j \in \{1, 2, \ldots, n\}$. The **adjacency matrix of a signed graph** is generalized the ordinary adjacency matrix by defining

$$a_{ij} = \begin{cases} \sigma(\{v_i, v_j\}), & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise,} \end{cases}$$

for all $i, j \in \{1, 2, \ldots, n\}$.

We begin with the following lemma.

Lemma 3.2.1. *Let R be a finite local commutative ring with $1 \neq 0$. Then the unitary Cayley sigraph $\mathcal{S}_R$ is an all-positive sigraph. Consequently, $A(G_R) = A(\mathcal{S}_R)$.*

Proof. Let R be a local ring with maximal ideal M. Suppose $\mathcal{S}_R$ has a negative edge $\{a, b\}$. Then a and b are in M, so $a - b \in M$. This contradicts the fact that a and b are adjacent. Hence, $\mathcal{S}_R$ is all-positive. □

For a finite commutative ring R with identity 1_R, we have $R \cong R_1 \times R_2 \times \cdots \times R_k$, where R_i is a local ring with maximal ideal M_i. Each R_i has the identity 1_i and we denote $n \cdot 1_i$ and $n \cdot 1_R$ by n. Recall Proposition 2.2 of [4] that if R is a local ring with maximal ideal M, then G_R is a complete multipartite graph whose partite sets are the cosets of M in R. A criterion for $\mathcal{S}_R$ to be balanced is as follows.

Theorem 3.2.2. *Let R be a finite commutative ring with $1 \neq 0$. The unitary Cayley sigraph $\mathcal{S}_R$ is balanced if and only if R is local or $|R_i| = 2|M_i|$ for some $i \in \{1, 2, \ldots, k\}$.*

Proof. If R is local, then by Lemma 3.2.1 we have $\mathcal{S}_R$ is all-positive, and hence balanced. Next, assume that $R = R_1 \times R_2 \times \cdots \times R_k$ such that $|R_i| = 2|M_i|$ for some i and we may assume $i = k$. Then G_R is bipartite with one partite set projects to M_k and another set projects to $R_k^\times$. Then every cycle in $\mathcal{S}_R$ has an even number of edge and of the form

$$C = (m_1, r_1, m_2, r_2, \ldots, m_l, r_l, m_1).$$

Any negative edge occurs when some $r_i \notin R^\times$ and this makes negative edge occur as an even number.

Next, suppose that $R = R_1 \times \cdots \times R_k$, where $k > 1$, R_i is a local ring and $|R_i/M_i| > 2$ for each $i \in \{1, 2, \ldots, k\}$. Then each G_{R_i} is a complete multipartite graph with $|R_i/M_i| > 2$ partite sets. Thus, each G_{R_i} has a cycle $(1_i, 0_i, a_i, 1_i)$, which mean $a_i - 0 \in R_i^{\times}$ and $a_i - 1 \in R_i^{\times}$ for all $i \in \{1, 2 \ldots, k\}$. Consider the element $\vec{x}_1 = (0_1, 1_2, 1_3, \ldots, 1_k)$, $\vec{x}_2 = (1_1, 0_2, 0_3, \ldots, 0_k)$ and $\vec{x}_3 = (a_1, a_2, \ldots, a_k)$ of $R = R_1 \times \cdots \times R_k$, we have a cycle $(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_1)$ in $\mathcal{S}_R$ with one negative edge $\{\vec{x}_1\vec{x}_2\}$. Thus, $\mathcal{S}_R$ in not balanced. $\square$

Recall that if $n > 1$, we have the decomposition

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{a_1}} \times \mathbb{Z}_{p_2^{a_2}} \times \cdots \times \mathbb{Z}_{p_k^{a_k}},$$

where $p_1 < p_2 < \cdots < p_k$ are primes and $a_1, a_2, \ldots, a_k \in \mathbb{N}$. If $p_1 = 2$, then $M_1 = 2\mathbb{Z}_{2^{a_1}}$ and $|\mathbb{Z}_{2^{a_1}}| = 2|M_1|$. If $p_1 > 2$, then $M_i = p_i\mathbb{Z}_{p_i^{a_i}}$ and $|\mathbb{Z}_{p_i^{a_i}}/M_i| = p_i > 2$ for all i. Hence, Theorem 3.2.2 yields Theorem 4 of [4]. We record it in the next corollary.

Corollary 3.2.3. *If $n > 1$ and $R = \mathbb{Z}_n$, then we write $\mathcal{S}_R = \mathcal{S}_n$ and $\mathcal{S}_n$ is balanced if and only if n is even or $n = p^r$ for some odd prime p and $r \in \mathbb{N}$.*

Remark. The condition "$|R_i| = 2|M_i|$ for some $i \in \{1, 2, \ldots, k\}$" in Theorem 3.2.2 is stronger than the one that "$|R|$ is even". For example, let $\mathbb{F}_4 = \{0, 1, a_1, a_2\}$ be the field with 4 elements and $R = \mathbb{F}_4 \times \mathbb{Z}_3$. Then $|R|$ is even but does not satisfy the criterion of Theorem 3.2.2. Consider the triangle $C = ((1, 0), (a_1, 1), (0, 2))$ in sigraph $\mathcal{S}_R$. The edge $\{(0, 2), (1, 0)\}$ is the only negative edge in C and thus $\mathcal{S}_R$ is not balanced.

The **negation** $\eta(\mathcal{S})$ of a sigraph $\mathcal{S}$ is the sigraph obtained from $\mathcal{S}$ by negating the sign of every edge of $\mathcal{S}$. Observe that if $\mathcal{S}$ is bipartite, then every cycle in

$\mathcal{S}$ has an even length. Therefore, the negation of a balanced bipartite graph is balanced.

Corollary 3.2.4. *Let $R = R_1 \times R_2 \times \cdots \times R_k$, where $R_1, \ldots, R_k$ are finite local commutative rings with $1 \neq 0$. The negation sigraph $\eta(\mathcal{S}_R)$ of the unitary Cayley sigraph $\mathcal{S}_R$ is balanced if and only if $|R_i| = 2|M_i|$ for some $i \in \{1, 2, \ldots, k\}$.*

Proof. Suppose $|R_i/M_i| > 2$ for all i. Let $\vec{0} = (0, 0, \ldots, 0)$, $\vec{1} = (1, 1, \ldots, 1)$ and x_3 be as in the proof of Theorem 3.2.2. Consider the triangle $C = (\vec{0}, \vec{1}, x_3)$ in $\mathcal{S}_R$. Then all it edges are positive, so we have a 3-cycle whose edges are negative in $\eta(\mathcal{S}_R)$. Hence, $\eta(\mathcal{S}_R)$ is not balanced.

Conversely, if $|R_i| = 2|M_i|$ for some i, then $\mathcal{S}_R$ is bipartite as shown in the proof of Theorem 3.2.2. Also, $\mathcal{S}_R$ is balanced. Hence, $\eta(\mathcal{S}_R)$ is balanced. □

Following Behzad and Chartrand [7], for a sigraph $\mathcal{S}$, we define its **line sigraph** $L(\mathcal{S})$ as the sigraph in which the edges of $\mathcal{S}$ are represented as vertices, two of these vertices are adjacent whenever the corresponding edges in S have a vertex in common, any such edge $\{a, b\}$ in $L(\mathcal{S})$ is defined to be negative whenever both a and b are negative edges in $\mathcal{S}$. A criterion for a balanced line sigraph is as follows.

Theorem 3.2.5. *[3] For a finite sigraph $\mathcal{S}$, its line sigraph $L(\mathcal{S})$ is balanced if and only if the following conditions hold:*

(i) for any cycle C in $\mathcal{S}$,

(a) if C is all-negative, then C has even length;

(b) if C is heterogeneous, then C has even number of negative sections with even length;

(ii) for $v \in S$, if $d(v) > 2$, then there is at most one negative edge incident at v in $\mathcal{S}$.

Recall that a unitary Cayley graph G_R over a finite commutative ring R is regular of degree $|R^\times|$. For a local ring R with maximal ideal M, $1 + M \subseteq R^\times$. Then $|M| \leq |R^\times|$. If $|R^\times| = 2$, then $|M| = 1$ or 2, and so $|R| = 3$ or 4. Thus, $|M| = 1$ when $R = \mathbb{Z}_3$ or $|M| = 2$ when ($R = \mathbb{Z}_4$ with $M = \{0, 2\}$) or ($R = \mathbb{Z}_2[x]/(x^2)$ with $M = \{0, x\}$) [35]. Moreover, $\mathcal{S}_{\mathbb{Z}_4}$ and $\mathcal{S}_{\mathbb{Z}_2[x]/(x^2)}$ are isomorphic sigraphs.

Furthermore, we observe that if R is a finite product of $\mathbb{Z}_2$'s, then G_R is regular of degree one and has no cycles. We shall need the following two lemmas in order to develop our final theorem (Theorem 3.2.8).

Lemma 3.2.6. *If $R = \mathbb{Z}_2^{k-1} \times \mathbb{Z}_3$ and $k \geq 2$, then the sigraph $\mathcal{S}_R$ is a disjoint union of 2^{k-2} 6-cycles with only one cycle is isomorphic to $\mathcal{S}_{\mathbb{Z}_6}$ and other cycles (if any) are all-negative.*

Proof. Observe that G_R is regular of degree 2 and $G_{\mathbb{Z}_2 \times \mathbb{Z}_3}$ is isomorphic to $G_{\mathbb{Z}_6}$. For each $\vec{a} \in \mathbb{Z}_2^{k-1}$, we have an associated 6-cycle given by $C_{\vec{a}} = ((\vec{a}, 0), (\vec{1} - \vec{a}, 1), (\vec{a}, 2), (\vec{1} - \vec{a}, 0), (\vec{a}, 1), (\vec{1} - \vec{a}, 2)) = C_{\vec{1}-\vec{a}}$. If $\vec{a} \neq \vec{0}$ and $\vec{1}$, then $C_{\vec{0}} = C_{\vec{1}} = \mathcal{S}_{\mathbb{Z}_6}$. If $\vec{a} \neq \vec{0}$ and $\vec{1}$, then $(\vec{a}, 1), (\vec{a}, 2), (\vec{1} - \vec{a}, 1), (\vec{1} - \vec{a}, 2)$ are not units in R and $C_{\vec{a}} = C_{\vec{1}-\vec{a}}$ is all-negative. $\square$

Lemma 3.2.7. *If $R = \mathbb{Z}_2^{k-1} \times R_k$, $k \geq 2$ and $R_k = \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$, then the sigraph $\mathcal{S}_R$ is a disjoint union of 2^{k-1} 4-cycles with only one cycle is all-positive and other cycles (if any) are all-negative.*

Proof. Since $\mathcal{S}_{\mathbb{Z}_4}$ and $\mathcal{S}_{\mathbb{Z}_2[x]/(x^2)}$ are isomorphic sigraphs, it suffices to prove only for $R_k = \mathbb{Z}_4$. Observe that G_R is regular of degree 2 and $G_{\mathbb{Z}_4}$ is 4-cycle $(0, 1, 2, 3, 0)$. For each $\vec{a} \in \mathbb{Z}_2^{k-1}$, we have an associated 4-cycle given by $C_{\vec{a}} = ((\vec{a}, 0), (\vec{1} - \vec{a}, 1), (\vec{a}, 2), (\vec{1} - \vec{a}, 3), (\vec{a}, 0))$. Since $(\vec{1}, 1), (\vec{1}, 3) \in R^\times$, $C_{\vec{0}}$ is all-positive. If $\vec{a} \neq \vec{0}$, then $(\vec{1} - \vec{a}, 1), (\vec{1} - \vec{a}, 3)$ are not units in R and $C_{\vec{a}}$ is all-negative. □

Theorem 3.2.8. *Let R be a finite commutative ring with $1 \neq 0$ and let $\mathcal{S}_R$ be the unitary Cayley sigraph. Then the line sigraph $L(\mathcal{S}_R)$ is balanced if and only if R is local or $R = \mathbb{Z}_2^k$, $k \geq 2$ or $R = \mathbb{Z}_2^{k-1} \times R_k$, $k \geq 2$, and $R_k = \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.*

Proof. If R is a local ring, then all edges of $\mathcal{S}_R$ are positive, and hence all edges of $L(\mathcal{S}_R)$ are positive. If $R = \mathbb{Z}_2^k$ and $k \geq 2$, then $L(\mathcal{S}_R)$ has no edges and thus is balanced. Finally, if $R = \mathbb{Z}_2^{k-1} \times R_k$, $k \geq 2$ and $R_k = \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$, then by Lemma 3.2.7 and Theorem 3.2.5, $L(\mathcal{S}_R)$ is balanced.

Assume that $R = R_1 \times R_2 \times \cdots \times R_k$ is a product of local rings, where $k \geq 2$ and $R_k \neq \mathbb{Z}_2, \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. Then $|R^\times| \geq 2$. If $|R^\times| = 2$, then $R = \mathbb{Z}_2^{k-1} \times \mathbb{Z}_3$ and $L(\mathcal{S}_R)$ is not balanced by Lemma 3.2.6 and Theorem 3.2.5. Assume $|R^\times| > 2$. Then G_R is regular of degree greater than 2. In addition, $|R_k^\times| \geq 2$. Consider the vertex $(1, 1, \ldots, 1, 0)$ which is adjacent to $(0, 0, \ldots, 0, 1)$ and $(0, 0, \ldots, 0, a)$ for some $a \in R_k^\times$ and $a \neq 1$. Clearly, both edges are negative. Thus, $L(\mathcal{S}_R)$ is not balanced by Theorem 3.2.5. □

Example 3.2.9. If $n > 1$, then $\mathbb{Z}_n = \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_k^{r_k}}$, where $p_1 < p_2 < \cdots < p_k$ are primes and $r_1, r_2, \ldots, r_k \in \mathbb{N}$, so Theorem 3.2.8 implies that $L(\mathcal{S}_{\mathbb{Z}_n})$ is balanced if and only if $n = p^r$ for some prime number p and $r \in \mathbb{N}$. This result is Corollary 7 of [39].

3.3 $\mathcal{C}$-Consistent Unitary Cayley Sigraphs

A marked sigraph is an ordered pair $\mathcal{S}_\mu = (\mathcal{S}, \mu)$, where $\mathcal{S} = (S^u, \sigma)$ is a sigraph and $\mu : V(S^u) \to \{+, -\}$ is a function from vertex set $V(S^u)$ of S^u into the set $\{+, -\}$, called a **marking** of $\mathcal{S}$. A cycle C in $\mathcal{S}_\mu$ is said to be **consistent** if it contains an even number of negative vertices. A marked sigraph $\mathcal{S}$ is said to be **consistent** if every cycle is consistent. In particular, σ induces the natural marking μ_σ defined by

$$\mu_\sigma(v) = \prod_{e \in E_v} \sigma(e),$$

where E_v is the set of edges incident at v in $\mathcal{S}$. It is called the **canonical marking** of $\mathcal{S}$.

Now, if every vertex of a given sigraph $\mathcal{S}$ is canonically marked, then a cycle C in $\mathcal{S}$ is said to be **canonically consistent or $\mathcal{C}$-consistent** if it contains an even number of negative vertices and a sigraph $\mathcal{S}$ is said to be **$\mathcal{C}$-consistent** if every cycle is $\mathcal{C}$-consistent, that is, every cycle contains an even number of negative vertices.

Beineke and Harary [8, 9] were the first to pose the problem of characterizing consistent marked graphs, which was subsequently settled by Acharya [1, 2], Rao

[32] and Hoede [19]. Sinha and Garg discussed consistency of several sigraphs in [36, 37, 38, 39]. In this section, we obtain more general characterizations of R for the $\mathcal{C}$-consistent of the marked sigraph $(\mathcal{S}_R)_\mu$.

Lemma 3.3.1. *If $R = R_1 \times \cdots \times R_k$, $k \geq 1$ and $|M_i|$ is even for some $i \in \{1, \ldots, k\}$, then the marked sigraph $(\mathcal{S}_R)_\mu$ is $\mathcal{C}$-consistent.*

Proof. Without loss of generality, we may assume that $|M_1|$ is even and let $\vec{a} = (a_1, \ldots, a_k)$ be an element in R. If $a_i \in R_i^\times$ for all i or $a_i \in M_i$ for all i, then we are done. If not, we consider the following two cases.

Case 1. $a_1 \in M_1$. For $\vec{b} = (b_1, b_2, \ldots, b_k)$ in R, $\{\vec{a}, \vec{b}\}$ is negative if and only if $\vec{a} - \vec{b} \in R^\times$, $b_1 \in R_1^\times$ and $b_i \in M_i$ for some $i \in \{2, \ldots, k\}$. Hence, the number of negative edges of $\vec{a}$ is a multiple of $|R_1^\times|$, which is even.

Case 2. $a_1 \in R_1^\times$. Assume that $\vec{b} = (b_1, b_2, \ldots, b_k)$ and $\{\vec{a}, \vec{b}\}$ is negative. If $b_i \in M_i$ for some $i \in \{2, \ldots, k\}$, then the number of $\vec{b}$ is a multiple of $|R_1^\times|$ which is even. Assume $b_i \in R_i^\times$ for all $i \in \{2, \ldots, k\}$. Then the number of $\vec{b}$ is a multiple of $|M_1|$ since $a_1 \in R_1^\times$.

Thus, $d^-(v)$ is even for all $v \in R$, so every vertex is positively marked. Hence, $(\mathcal{S}_R)_\mu$ is $\mathcal{C}$-consistent. □

Lemma 3.3.2. *Let R be a finite commutative ring with $1 \neq 0$. If $R = R_1 \times \cdots \times R_k$, where $R_1, \cdots, R_k$ are local rings, $k \geq 1$ and $|R_i^\times|$ is even for each $i \in \{1, \ldots, k\}$, then the marked sigraph $(\mathcal{S}_R)_\mu$ is $\mathcal{C}$-consistent.*

Proof. Let $\vec{a} = (a_1, \ldots, a_k) \in R_1 \times \cdots \times R_k$. If $\vec{a} \in R^\times$, then $\vec{a}$ has no negative edges by definition. If $a_i \in M_i$ for all i, $\vec{a}$ also has no negative edges because any

$\vec{b} = (b_1, \ldots, b_k)$, which is adjacent to $\vec{a}$, we must have $b_i - a_i \in R_i^\times$, and hence $b_i \in R_i^\times$ for all i. Then a negative edge of $\vec{a}$ occurs when some $a_i \in R_i^\times$ and some $a_j \in M_j$. Observe that $d^+(\vec{a}) = \prod_i (|R_i^\times| - |M_i|) \prod_j (|R_j^\times|)$ is even. Then $d^-(\vec{a}) = |R^\times| - d^+(\vec{a})$ is even for all $\vec{a} \in R$. Hence, the graph $\mathcal{S}_R$ is $\mathcal{C}$-consistent. □

Lemma 3.3.3. *If R is a finite local commutative ring with $1 \neq 0$ and $|R^\times|$ is odd, then $R = \mathbb{F}_{2^r}$ for some $r \geq 1$.*

Proof. Recall that for a local ring R with maximal ideal M we have $|R| = p^r$, $|M| = p^m$ and $|R^\times| = |R| - |M| = p^r - p^m = p^m(p^{r-m} - 1)$, where p is prime and $m < r$. Since $p^m(p^{r-m} - 1)$ is odd, both p^m and $p^{r-m} - 1$ are odd, so $m = 0$ and $p = 2$. Thus, R is a field of 2^r elements. □

Theorem 3.3.4. *Let R be a finite commutative ring with $1 \neq 0$. Suppose that $R = R_1 \times R_2 \times \cdots \times R_k$, where each R_i is a local ring with maximal ideal M_i. Then the marked sigraph $(\mathcal{S}_R)_\mu$ is $\mathcal{C}$-consistent if and only if R satisfies one of the following conditions:*

(i) $k = 1$ *or* ($R = \mathbb{Z}_2^k$ *and* $k > 1$),

(ii) $k > 1$ *and* $|M_j|$ *is even for some* $j \in \{1, \ldots, k\}$,

(iii) $k > 1$ *and* $|R_i^\times|$ *is even for each* $i \in \{1, \ldots, k\}$,

(iv) $k > 1$ *and* $R = \mathbb{Z}_2^{k-1} \times \mathbb{Z}_3$.

Proof. $(\mathcal{S}_R)_\mu$ is clearly $\mathcal{C}$-consistent in the case (i). If R satisfies (iv), the result can be directly verified. Finally, if R satisfies (ii) or (iii), then $(\mathcal{S}_R)_\mu$ is $\mathcal{C}$-consistent by Lemma 3.3.1 and Lemma 3.3.2, respectively.

Next, we shall show that if R does not satisfy (i)–(iv), then $(\mathcal{S}_R)_\mu$ is not $\mathcal{C}$-consistent. Assume that R does not fulfill any conditions (i)–(iv) in the theorem. Then $k > 1$. Let $|R_i| = p_i^{r_i}$ for all $i \in \{1, 2, \ldots, k\}$, where $p_1 \leq p_2 \ldots \leq p_k$. If $p_1 > 2$, then $|R_i^\times|$ is even for all i, and so (iii) is true. Thus, p_1 must be 2. Since $|M_1|$ is odd, R_1 is a field of 2^{r_1} elements. Furthermore, Lemma 3.3.3 allows us to write

$$R = \mathbb{F}_{2^{r_1}} \times \cdots \times \mathbb{F}_{2^{r_s}} \times R_{s+1} \times \cdots \times R_k,$$

where $1 \leq r_1 \leq r_2 \leq \ldots \leq r_s$, $s \geq 1$, $|R_i| = p_i^{r_i}$ for all $i \in \{s+1, \ldots, k\}$ and $2 < p_{s+1} \leq p_{s+2} \leq \cdots \leq p_k$. The theorem will follow from the next series of lemmas in which we construct a cycle with odd negative vertices. □

Lemma 3.3.5. *Let R be a finite commutative ring with $1 \neq 0$. If $s = k$ and $r_1 > 1$, then $(\mathcal{S}_R)_\mu$ is not $\mathcal{C}$-consistent.*

Proof. Since for each $i \in \{1, 2, \ldots, k\}$ we have $|F_{2^{r_i}}| \geq 4$, there exist distinct elements $1 \neq a_i, b_i \in \mathbb{F}_{2^{r_i}}^\times$. Let $\vec{x}_1 = (0, \ldots, 0, 1)$, $\vec{x}_2 = (a_1, \ldots, a_{k-1}, a_k)$ and $\vec{x}_3 = (b_1, \ldots, b_{k-1}, b_k)$. Then $d^-(\vec{x}_1) = \prod_{i=1}^{k-1} |\mathbb{F}_{2^{r_i}}^\times|$ is odd and $d^-(\vec{x}_2) = d^-(\vec{x}_3) = 0$. Thus, we have a cycle $C = (\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_1)$ with odd negative vertices. □

Lemma 3.3.6. *Let R be a finite commutative ring with $1 \neq 0$. If $s = k$ and $r_1 = 1$, then $(\mathcal{S}_R)_\mu$ is not $\mathcal{C}$-consistent.*

Proof. Since $R \neq \mathbb{Z}_2^k$, there exists smallest $t < k$ such that $r_{t+1} > 1$. Similar to the proof of Lemma 3.3.5, we can choose elements

$\vec{x}_1 = (1_1, \dots, 1_t, 0_{t+1}, \dots, 0_k)$ with $d^-(\vec{x}_1) = |\mathbb{F}_{2^{r_1}}^\times| \times \cdots \times |\mathbb{F}_{2^{r_\ell}}^\times|$, which is odd,

$\vec{x}_2 = (0_1, \dots, 0_t, 1_{t+1}, \dots, 1_k)$ with $d^-(\vec{x}_2) = |R^\times| - d^+(\vec{x}_2)$, which is odd,

$\vec{x}_3 = (1_1, \dots, 1_t, a_{t+1}, \dots, a_k)$ with $d^-(\vec{x}_3) = 0$ and

$\vec{x}_4 = (0_1, \dots, 0_t, b_{t+1}, \dots, b_k)$ with $d^-(\vec{x}_4) = |R^\times| - d^+(\vec{x}_4)$, which is odd.

Thus, we have a cycle $(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, \vec{x}_1)$ with odd negative vertices. □

Lemma 3.3.7. *Let R be a finite commutative ring with $1 \neq 0$. If $s < k$ and $r_1 > 1$, then $(\mathcal{S}_R)_\mu$ is not $\mathcal{C}$-consistent.*

Proof. Similar to the proof of Lemma 3.3.5, we have an element $1 \neq b_i \in \mathbb{F}_{2^{r_i}}^\times$ for all $i \in \{1, 2, \dots, s\}$. Since $|R_j| \geq 3$ and $|R_j/M_j| \geq 3$, there exists $1 \neq a_j \in R_j^\times$ such that $a_j - 1 \in R_j^\times$ for all $j \in \{s+1, \dots, k\}$. Choose

$$\vec{x}_1 = (0_1, \dots, 0_s, 1_{s+1}, 1_{s+2}, \dots, 1_k),$$

$$\vec{x}_2 = (b_1, \dots, b_s, a_{s+1}, a_{s+2}, \dots, a_k) \text{ and}$$

$$\vec{x}_3 = (1_1, \dots, 1_s, 0_{s+1}, 0_{s+2}, \dots, 0_k).$$

Since

$$d^+(\vec{x}_1) = \prod_{i=s+1}^{k} (|R_i^\times| - |M_i|) \prod_{j=1}^{s} |\mathbb{F}_{2^{r_j}}^\times| \text{ and}$$
$$d^+(\vec{x}_3) = \prod_{i=s+1}^{k} |R_i^\times| (\prod_{j=1}^{s} |F_{2^{r_j}}{}^\times| - 1),$$

we have $d^+(\vec{x}_1)$ is odd and $d^+(\vec{x}_3)$ is even. It follows that $d^-(\vec{x}_1) = |R^\times| - d^+(\vec{x}_1)$ is odd, $d^-(\vec{x}_2) = 0$ and $d^-(\vec{x}_3) = |R^\times| - d^+(\vec{x}_3)$ is even. Thus, we have a cycle $C = (\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_1)$ with odd negative vertices. □

Lemma 3.3.8. *Let R be a finite commutative ring with $1 \neq 0$. If $s < k$, $r_1 = 1$ and $r_i \geq 2$ for some $i \in \{2, \ldots, s\}$, then $(\mathcal{S}_R)_\mu$ is not $\mathcal{C}$-consistent.*

Proof. Let t be the smallest number such that $r_{t+1} \geq 2$. Similar to the proofs of Lemmas 3.3.5 and 3.3.7, we can choose

$$\vec{x}_1 = (0_1, \ldots, 0_t, 0_{t+1}, \ldots, 0_s, 1_{s+1}, 1_{s+2}, \ldots, 1_k),$$

$$\vec{x}_2 = (1_1, \ldots, 1_t, b_{t+1}, \ldots, b_s, a_{s+1}, a_{s+2}, \ldots, a_k),$$

$$\vec{x}_3 = (0_1, \ldots, 0_t, d_{t+1}, \ldots, d_s, 0_{s+1}, 0_{s+2}, \ldots, 0_k)$$

$$\text{and } \vec{x}_4 = (1_1, \ldots, 1_t, 1_{t+1}, \ldots, 1_s, a_{s+1}, a_{s+2}, \ldots, a_k).$$

Then $d^-(\vec{x}_1) = |R^\times| - d^+(\vec{x}_1)$ is odd, $d^-(\vec{x}_2) = d^-(\vec{x}_4) = 0$ and $d^-(\vec{x}_3) = |R^\times| - d^+(\vec{x}_3)$ is even. Thus, we have a cycle $C = (\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, \vec{x}_1)$ with odd negative vertices. □

Lemma 3.3.9. *Let R be a finite commutative ring with $1 \neq 0$. If $s < k$, $r_i = 1$ for all $i \in \{1, 2, \ldots, s\}$ and $R \neq \mathbb{Z}_2^s \times \mathbb{Z}_3$, then $(\mathcal{S}_R)_\mu$ is not $\mathcal{C}$-consistent.*

Proof. We distinguish two cases.

Case 1. $k = s + 1$. Then $|R_k^\times| \geq 3$ and $|R_k/M_k| \geq 3$. Choose

$$\vec{x}_1 = (0_1, \ldots, 0_{k-1}, 1_k),\ \vec{x}_2 = (1_1, \ldots, 1_{k-1}, a_k)$$

$$\vec{x}_3 = (0_1, \ldots, 0_{k-1}, 0_k) \text{ and } \vec{x}_4 = (1_1, \ldots, 1_{k-1}, b_k)$$

for some distinct elements $1 \neq a_k, b_k \in R_k^\times$ such that $a_k - 1,\ b_k - 1 \in R_k^\times$. Observe that $d^+(\vec{x}_1) = |R_k^\times| - |M_k|$ is odd. Then $d^-(\vec{x}_1) = |R^\times| - d^+(\vec{x}_1)$ is odd and $d^-(\vec{x}_2) = d^-(\vec{x}_2) = d^-(\vec{x}_2) = 0$. Thus, we have a cycle $C = (\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, \vec{x}_1)$ with only one negative vertex.

Case 2. $k > s+1$. Similar to the proof of Lemma 3.3.7, we can choose

$$\vec{x}_1 = (0_1, \ldots, 0_s, 1_{s+1}, a_{s+2}, \ldots, a_k), \vec{x}_2 = (1_1, \ldots, 1_s, a_{s+1}, 0_{s+2}, \ldots, 0_k),$$

$$\vec{x}_3 = (0_1, \ldots, 0_s, 0_{s+1}, a_{s+2}, \ldots, a_k) \text{ and } \vec{x}_4 = (1_1, \ldots, 1_s, a_{s+1}, 1_{s+2}, \ldots, 1_k)$$

for some $1 \neq a_j \in R_j^\times$ such that $a_j - 1 \in R_j^\times$ for all $j \in \{s+1, \ldots, k\}$. Observe that $d^+(\vec{x}_1) = \prod_{i=s+1}^{k} (|R_i^\times| - |M_i|)$ is odd and $d^+(\vec{x}_3)$ is a multiple of $|R_{s+1}^\times|$, which is even. It follows that $d^-(\vec{x}_1) = |R^\times| - d^+(\vec{x}_1)$ is odd, $d^-(\vec{x}_2) = |R^\times|$, $d^-(\vec{x}_3) = |R^\times| - d^+(\vec{x}_3)$ and $d^-(\vec{x}_4) = 0$, which are all even. Thus, we have a cycle $C = (\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, \vec{x}_1)$ with odd negative vertices.

Hence, we have this last lemma and complete the proof of Theorem 3.3.4. □

Corollary 3.3.10. *Let $n > 1$. Then $(\mathcal{S}_{\mathbb{Z}_n})_\mu$ is $\mathcal{C}$-consistent if and only if n is odd, 2, 6 or a multiple of 4.*

Remark. Corollary 3.3.10 generalizes Theorem 20 of [39] without the restriction on the numbers of prime factors of n. Moreover, we have the following result on the negation of the sigraph $\mathcal{S}_R$. Its proof is a routine application of Theorem 3.3.4 and similar to the proof of Corollary 21 of [39].

Corollary 3.3.11. *Let $R = R_1 \times R_2 \times \cdots \times R_k$, where each R_i is a local ring with maximal ideal M_i. Then the negation sigraph $\eta(\mathcal{S}_R)$ is $\mathcal{C}$-consistent if and only if R satisfies one of the following conditions:*

(i) $k = 1$ *and R is not a finite field of characteristic* 2,

(ii) $k > 1$ *and $|M_j|$ is even for some $j \in \{1, \ldots, k\}$,*

(iii) $k > 1$ *and $|R_i^\times|$ is even for each $i \in \{1, \ldots, k\}$,*

(iv) $k > 1$ *and* $R = \mathbb{Z}_2^{k-1} \times \mathbb{Z}_3$ *or* $\mathbb{Z}_2^k$.

3.4 Hyperenergetic Balanced Sigraphs

Let $\mathcal{S} = (S^u, \sigma)$ be a sigraph. The **eigenvalues** of S^u, resp. $\mathcal{S}$, are the eigenvalues of its adjacency matrix defined in Section 3.2. The set of all eigenvalues is called the **spectrum** and sum of absolute values of all eigenvalues is called the **energy** of a graph or a sigraph, denoted by $\mathcal{E}(S^u)$ or $\mathcal{E}(\mathcal{S})$, respectively. A criterion for a sigraph to be balanced using spectrum was given by Gill and Acharya [14] as follows.

Theorem 3.4.1. *[14] For a finite sigraph* $\mathcal{S} = (S^u, \sigma)$, $\mathcal{S}$ *is balanced if and only if* S^u *and* $\mathcal{S}$ *have the same spectrum. In particular, a balanced signed graph has the same spectrum as its underlying graph.*

A graph G with n vertices is said to be **hyperenergetic** if its energy exceeds the energy of the complete graph K_n, or equivalently if $\mathcal{E}(G) > 2n - 2$. Hyperenergetic unitary Cayley graphs were completely characterized in [23]. We record it in the next theorem.

Theorem 3.4.2. *[23] Let* R *be a finite commutative ring, where* $R = R_1 \times R_2 \times \cdots \times R_k$ *and each* R_i *is a local ring with maximal ideal* M_i*. Assume that*

$$|R_1/M_1| \leq |R_2/M_2| \leq \cdots \leq |R_k/M_k|.$$

1. *For* $k = 1$, G_R *is not hyperenergetic.*

2. *For* $k = 2$, G_R *is hyperenergetic if and only if* $|R_1/M_1| \geq 3$ *and* $|R_2/M_2| \geq 4$.

3. *For* $k \geq 3$, G_R *is hyperenergetic if and only if* $(|R_{k-2}/M_{k-2}| \geq 3)$ *or* $(|R_{k-1}/M_{k-1}| \geq 3$ *and* $|R_k/M_k| \geq 4)$.

Applying Theorem 3.2.2 with the above results, we obtain the following characterization of a hyperenergetic balanced unitary Cayley sigraph.

Theorem 3.4.3. *Let* R *be a finite commutative ring, where* $R = R_1 \times R_2 \times \cdots \times R_k$ *and each* R_i *is a local ring with maximal ideal* M_i. *Assume that*

$$|R_1/M_1| \leq |R_2/M_2| \leq \cdots \leq |R_k/M_k|.$$

Then $\mathcal{S}_R$ *is a hyperenergetic balanced sigraph if and only if* R *satisfies one of the following conditions:*

(i) $k \geq 3$, $|R_1/M_1| = 2$ $|R_{k-1}/M_{k-1}| \geq 3$ *and* $|R_k/M_k| \geq 4$,

(ii) $k \geq 4$, $|R_1/M_1| = 2$ *and* $|R_{k-2}/M_{k-2}| \geq 3$.

Recall that a unitary Cayley sigraph $\mathcal{S}_R$ and its negation $\eta(\mathcal{S}_R)$ have the graph G_R as their underlying graph. Corollary 3.2.4 and Theorem 3.4.2 also yield the next corollary.

Corollary 3.4.4. *Let* R *be a finite commutative ring, where* $R = R_1 \times R_2 \times \cdots \times R_k$ *and each* R_i *is a local ring with maximal ideal* M_i. *Assume that*

$$|R_1/M_1| \leq |R_2/M_2| \leq \cdots \leq |R_k/M_k|.$$

Then the negation sigraph $\eta(\mathcal{S}_R)$ *is a hyperenergetic balanced sigraph if and only if* R *satisfies one of the following conditions:*

1. $k \geq 3$, $|R_1/M_1| = 2$ $|R_{k-1}/M_{k-1}| \geq 3$ *and* $|R_k/M_k| \geq 4$,

2. $k \geq 4$, $|R_1/M_1| = 2$ *and* $|R_{k-2}/M_{k-2}| \geq 3$.

Liu and Zhou ([26], Corollary 20) gave a condition for line graphs of unitary Cayley graphs $L(G_R)$ to be balanced as follows. We formulate it in our set-up as follows.

Theorem 3.4.5. *[26] Let R be a finite commutative ring, where $R = R_1 \times R_2 \times \cdots \times R_k$ and each R_i is a local ring with maximal ideal M_i. Assume that*

$$|R_1/M_1| \leq |R_2/M_2| \leq \cdots \leq |R_k/M_k|.$$

Then $L(G_R)$ is hyperenergetic if and only if one of the following conditions holds:

1. $|R^\times| \geq 4$,

2. $k = 1$ *and* $|R| = 2|M| \geq 8$,

3. $k \geq 2$, $2 = |R_1/M_1| = \cdots = |R_k/M_k|$ *and* $|R^\times| \geq 2$.

The above theorem and Theorem 3.2.8 directly give our final characterization of hyperenergetic balanced line sigraphs.

Theorem 3.4.6. *Let R be a finite commutative ring, where $R = R_1 \times R_2 \times \cdots \times R_k$ and each R_i is a local ring with maximal ideal M_i. Assume that*

$$|R_1/M_1| \leq |R_2/M_2| \leq \cdots \leq |R_k/M_k|.$$

Then the line sigraph $L(S_R)$ is hyperenergetic balanced if and only if one of the following conditions holds:

1. $k = 1$ *and* $|R| = 2|M| \geq 8$,

2. $k \geq 2$, $2 = |R_1/M_1| = \cdots = |R_k/M_k|$ *and* $|R^\times| \geq 2$.

CHAPTER IV

CAYLEY GRAPHS OVER A FINITE CHAIN RING AND GCD-GRAPHS

Let D be a unique factorization domain (UFD) and $c \in D$ a nonzero nonunit element. Assume that the commutative ring $D/(c)$ is finite. For a set $\mathcal{C}$ of proper divisors of c, we define the **gcd-graph**, $D_c(\mathcal{C})$, to be a graph whose vertex set is the quotient ring $D/(c)$ and edge set is

$$\{\{x + (c), y + (c)\} : x, y \in D \text{ and } \gcd(x - y, c) \in D^{\times}\mathcal{C}\}.$$

This gcd-graph on a quotient ring of a unique factorization domain (UFD) introduced in [23] generalizes a gcd-graph or an integral circulant graph (i.e., its adjacency matrix is circulant and all eigenvalues are integers) defined over $\mathbb{Z}_n, n \geq 2$, (see [24, 40]). An integral circulant graph can also be considered as an extension of a unitary Cayley graph. This graph has been widely studied in many literatures, e.g., [5, 12, 17, 18, 21, 22, 34].

Since the number of sets of divisors $\mathcal{C}$ of $c = p_1^{s_1} \dots p_k^{s_k}$ can be very large, the energy of gcd-graph (over a $D/(c)$ or $\mathbb{Z}_n$) still not thoroughly studied. As usual, the common thing that ones would think of when they are working on a UFD is prime powers. We shall give the energy of gcd-graphs with the divisor set $\mathcal{C}$, where $\mathcal{C}$ consists of certain prime powers, by studying the energy of the

Cayley graph over the finite ring $D/(p_i^{s_i})$. When $D = \mathbb{Z}$, this graph is the integral circulant graph with prime power order studied by Sander and Sander in [34]. They derived a closed formula for its energy and worked on minimal and maximal energies for a fixed prime power p^s and varying divisor sets. While we have tried to extend their results to a Cayley graph over finite commutative rings, we come across a kind of rings which has a nice property on ideals, called a finite chain ring. The structure of this ring has been well studied, see [27, 30]. It is a finite local ring and generalizes the ring $D/(p^s)$ and the Galois ring $\mathbb{Z}_{p^s}[x]/(f(x))$, where $f(x)$ is a monic polynomial in $\mathbb{Z}_{p^s}[x]$ and the canonical reduction $\bar{f}(x)$ in $\mathbb{Z}_p[x]$ is irreducible. This explains why we first focus our work on a finite chain ring.

4.1 Cayley Graphs over a Finite Chain Ring

A *finite chain ring* is a finite commutative ring such that for any two ideals I_1 and I_2 of this ring, we have $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$. It is easy to see that a finite chain ring is a local ring. It also follows that:

Proposition 4.1.1. *[27] If R is a finite chain ring then R is a finite local principal ideal ring with maximal ideal M generated by $\theta \in M \smallsetminus M^2$.*

Example 4.1.2. If p is a prime, then $\mathbb{Z}_{p^s}$, $s \in \mathbb{N}$, is a finite chain ring. Every ideals of $\mathbb{Z}_{p^s}$ are of the form $p^a\mathbb{Z}_{p^s}$ where $a \in \mathbb{N}$. Moreover, $p^b\mathbb{Z}_{p^s} \subseteq p^a\mathbb{Z}_{p^s}$ when $b \geq a$.

Let R be a finite chain ring with unique maximal ideal M and residue field of q elements. Let s be the nilpotency of R, that is, the least positive integer such that $M^s = \{0\}$. It can be shown that we have the chain of ideals

$$R \supset M \supset M^2 \supset \cdots \supset M^s = \{0\}.$$

Write $R = M^0$. By Lemma 2.4 of [30], we also have $|M^i| = q^{s-i}$ for all $0 \leq i \leq s$, and so

$$|M^i/M^{i+1}| = q$$

for all $0 \leq i < s$. Thus, $|R| = q^s$. Moreover, M is principal generated by a $\theta \in M \setminus M^2$ and hence any element $x \in R$ can be written as

$$x = v_0 + v_1\theta + v_2\theta^2 + \cdots + v_{s-1}\theta^{s-1},$$

where $v_i \in \mathcal{V} = \{e_0, e_1, \ldots, e_{p^t-1}\}$, a fixed set of representatives of cosets in R/M. Let

$$\mathcal{C} = (M^{a_1} \setminus M^{a_1+1}) \cup (M^{a_2} \setminus M^{a_2+1}) \cup \cdots \cup (M^{a_r} \setminus M^{a_r+1}),$$

where $0 \leq a_1 < a_2 < \cdots < a_r \leq s-1$.

Consider the Cayley graph $\mathrm{Cay}(R, \mathcal{C})$ whose vertex set is R and $x, y \in R$ are adjacent if and only if $x-y \in \mathcal{C}$. This graph generalizes the gcd-graph defined over $\mathbb{Z}_{p^s}$ with the set $D = \{p^{a_1}, p^{a_2}, \ldots, p^{a_r}\}$ of proper divisors of p^s where two vertices $a, b \in \mathbb{Z}_{p^s}$ are adjacent if and only if $\gcd(b-a, p^s) = p^{a_i}$ for some $i \in \{1, 2, \ldots, r\}$ studied in [20] and [23]. The adjacency condition can be stated in terms of ideals as $b-a$ belongs to the ideal $p^{a_i}\mathbb{Z}$ but not in $p^{a_i+1}\mathbb{Z}$ for some $i \in \{1, 2, \ldots, r\}$.

For $x, y \in R$ of the forms

$$x = v_0 + v_1\theta + v_2\theta^2 + \cdots + v_{s-1}\theta^{s-1},$$

$$y = u_0 + u_1\theta + v_2\theta^2 + \cdots + u_{s-1}\theta^{s-1},$$

for some $v_i, u_j \in \mathcal{V}$, we have

$$x - y \in R \setminus M \Leftrightarrow v_0 \neq u_0.$$

Then the adjacency matrix for $\mathrm{Cay}(R, \mathcal{C})$ is

$$A_0 = \begin{array}{c} \begin{matrix} e_1 + M & e_2 + M & \cdots & e_q + M \end{matrix} \\ \begin{pmatrix} A_1 & B_1 & \cdots & B_1 \\ B_1 & A_1 & \cdots & B_1 \\ B_1 & B_1 & \cdots & B_1 \\ \vdots & \vdots & \ddots & \vdots \\ B_1 & B_1 & \cdots & A_1 \end{pmatrix} \end{array},$$

where

$$B_1 = \begin{cases} J_{q^{s-1} \times q^{s-1}} & \text{if } R \setminus M \subseteq \mathcal{C} \\ \mathbf{0}_{q^{s-1} \times q^{s-1}} & \text{if } R \setminus M \not\subseteq \mathcal{C}, \end{cases}$$

and A_1 is a $q^{s-1} \times q^{s-1}$ submatrix depending on M^i, $i \geq 1$. If $B_1 = \mathbf{0}_{q^{s-1} \times q^{s-1}}$, then

$$A_0 = I_q \otimes A_1 \qquad \text{(Process A)}$$

and if $B_1 = J_{q^{s-1} \times q^{s-1}}$, we have

$$A_0 = \overline{(I_q \otimes \overline{A}_1)}. \qquad \text{(Process B)}$$

Here, $J_{n \times n}$ is the matrix whose all entries are 1s and $\overline{X}$ of an adjacency matrix X of a graph G denotes the adjacency matrix $J - I - X$ of the complement graph of G. Next, we consider $x, y \in M$ such that

$$x = v_1\theta + v_2\theta^2 + \cdots + v_{s-1}\theta^{s-1},$$

$$y = u_1\theta + v_2\theta^2 + \cdots + u_{s-1}\theta^{s-1},$$

for some $v_i, u_j \in \mathcal{V}$. Then

$$x - y \in M \setminus M^2 \Leftrightarrow v_1 \neq u_1.$$

Similarly, we have submatrices

$$B_2 = \begin{cases} J_{q^{s-2}\times q^{s-2}} & \text{if } M \setminus M^2 \subseteq \mathcal{C} \\ \mathbf{0}_{q^{s-2}\times q^{s-2}} & \text{if } M \setminus M^2 \nsubseteq \mathcal{C}, \end{cases}$$

and A_2, which is a $q^{s-2} \times q^{s-2}$ submatrix depending on M^i for $i \geq 2$ such that

$$A_1 = \begin{cases} I_q \otimes A_2 & \text{if } B_2 = \mathbf{0}_{q^{s-2}\times q^{s-2}} \\ \overline{(I_q \otimes \overline{A}_2)} & \text{if } B_2 = J_{q^{s-2}\times q^{s-2}}. \end{cases}$$

Continuing these processes yields the sets of submatrices $\{A_1, \cdots, A_{s-1}\}$ and $\{B_1, \cdots, B_{s-1}\}$.

Lemma 4.1.3. *Let $i \in \{1, 2, \ldots, s-1\}$. Assume that*

$$\operatorname{Spec} A_i = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ m_1 & m_2 & \ldots & m_k \end{pmatrix}$$

with λ_1 is the largest eigenvalues. Then

$$\operatorname{Spec} \overline{(I_q \otimes \overline{A}_i)} = \begin{pmatrix} q^{s-i}(q-1) + \lambda_1 & \lambda_1 - q^{s-i} & \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ 1 & q-1 & q(m_1 - 1) & qm_2 & \ldots & qm_k \end{pmatrix}.$$

In particular, if $m_1 = 1$*, then*

$$\operatorname{Spec} \overline{(I_q \otimes \overline{A}_i)} = \begin{pmatrix} q^{s-i}(q-1)+\lambda_1 & \lambda_1 - q^{s-i} & \lambda_2 & \dots & \lambda_k \\ 1 & q-1 & qm_2 & \dots & qm_k \end{pmatrix}.$$

Proof. Observe that size of $A_i = |M^i| = q^{s-i}$ and the graph associated with A_i is regular. Then

$$\operatorname{Spec} \overline{A}_i = \begin{pmatrix} q^{s-i}-\lambda_1-1 & -1-\lambda_1 & -1-\lambda_2 & \dots & -1-\lambda_k \\ 1 & m_1-1 & m_2 & \dots & m_k \end{pmatrix}$$

which implies

$$\operatorname{Spec}(I_q \otimes \overline{A}_i) = \begin{pmatrix} q^{s-i}-\lambda_1-1 & -1-\lambda_1 & -1-\lambda_2 & \dots & -1-\lambda_k \\ q & q(m_1-1) & qm_2 & \dots & qm_k \end{pmatrix},$$

and so

$$\operatorname{Spec} \overline{(I_q \otimes \overline{A}_i)} = \left(\begin{matrix} q^{s-i+1}-(q^{s-i}-\lambda_1-1)-1 & -1-(q^{s-i}-\lambda_1-1) \\ 1 & q-1 \end{matrix} \right.$$

$$\left. \begin{matrix} -1-(-1-\lambda_1) & -1-(-1-\lambda_2) & \dots & -1-(-1-\lambda_k) \\ q(m_1-1) & qm_2 & \dots & qm_k \end{matrix} \right)$$

$$= \begin{pmatrix} q^{s-i+1}-q^{s-i}+\lambda_1 & \lambda_1-q^{s-i} & \lambda_1 & \lambda_2 & \dots & \lambda_k \\ 1 & q-1 & q(m_1-1) & qm_2 & \dots & qm_k \end{pmatrix}$$

by Propositions 1.3.2 and 1.3.3. □

Repeatedly applying Process A, Process B and Lemma 4.1.3 yield the following two lemmas.

Lemma 4.1.4. *Let* R *be a finite chain ring with unique maximal ideal* M*, residue field of* q *elements and of nilpotency* s*. Let*

$$\mathcal{C} = (M^{a_1} \smallsetminus M^{a_1+1}) \cup (M^{a_2} \smallsetminus M^{a_2+1}) \cup \cdots \cup (M^{a_r} \smallsetminus M^{a_r+1}),$$

with $0 \leq a_1 < a_2 < \cdots < a_r \leq s-1$*. If* $a_r = s-1$*, then the eigenvalues of* $\mathrm{Cay}(R, \mathcal{C})$ *are as follows:*

1. $(q-1)\sum_{i=1}^{r} q^{s-a_i-1}$ *with multiplicity* q^{a_1}*,*
2. $-q^{s-a_{k-1}-1} + (q-1)\sum_{i=k}^{r} q^{s-a_i-1}$ *with multiplicity* $q^{a_{k-1}}(q-1)$ *for* $k = 2, \ldots, r$*,*
3. $(q-1)\sum_{i=k}^{r} q^{s-a_i-1}$ *with multiplicity* $q^{a_k-a_{k-1}-1} - q^{a_{k-1}+1}$ *for* $k = 2, \ldots, r$*,*
4. -1 *with multiplicity* $q^{a_r}(q-1)$*.*

Proof. Since $a_r = s-1$, $A_{a_r} = A_{s-1}$ is an adjacency matrix of the complete graph on $|M^{a_r}| = |M^{s-1}| = q$ vertices and so

$$\mathrm{Spec}\, A_{a_r} = \mathrm{Spec}\, A_{s-1} = \begin{pmatrix} q-1 & -1 \\ 1 & q-1 \end{pmatrix}.$$

It follows from Proposition 1.3.2 and Lemma 4.1.3 that any eigenvalues of A_i except λ_1 which is the regular degree remain the same after process A and process B. So -1 is an eigenvalue of $\mathrm{Cay}(R, \mathcal{C})$ with multiplicity $q^{a_r}(q-1)$. Next, we consider the eigenvalue $q-1$ of A_{s-1}. We apply Process A until it reaches $a_{r-1}+1$ which makes its multiplicity to be $q^{a_r-a_{r-1}-1}$ and follow by Process B. By Lemma 4.1.3, the eigenvalues of $A_{a_{r-1}}$ that induces from $q-1$ are

1. $q^{s-a_{r-1}-1}(q-1)+(q-1) = q^{s-a_{r-1}-1}(q-1)+q^{s-a_r-1}(q-1)$ with multiplicity 1,
2. $q-1-q^{s-a_{r-1}-1}$ with multiplicity $q-1$,
3. $q-1$ with multiplicity $q(q^{a_r-a_{r-1}-1}-1)$.

By the same reason, $q-1-q^{s-a_{r-1}-1}$ and $q-1$ are eigenvalues of $\mathrm{Cay}(R,\mathcal{C})$ with multiplicity $q^{a_{r-1}}(q-1)$ and $q^{a_{r-1}+1}(q^{a_r-a_{r-1}-1}-1)=q^{a_r}-q^{a_{r-1}+1}$ respectively. Applying these processes to the eigenvalue $q^{s-a_{r-1}-1}(q-1)+(q-1)$ until it reaches a_{r-2} yields eigenvalues

1. $q^{s-a_{r-2}-1}(q-1)+q^{s-a_{r-1}-1}(q-1)+(q-1)$ with multiplicity 1,
2. $q^{s-a_{r-1}-1}(q-1)+(q-1)-q^{s-a_{r-2}-1}$ with multiplicity $q-1$,
3. $q^{s-a_{r-1}-1}(q-1)+(q-1)$ with multiplicity $q(q^{a_{r-1}-a_{r-2}-1}-1)$.

Continuing this argument, we obtain the eigenvalues of $\mathrm{Cay}(R,\mathcal{C})$ as follows:

1. $(q-1)\sum_{i=1}^{r} q^{s-a_i-1}$ with multiplicity q^{a_1},
2. $-q^{s-a_{k-1}-1}+(q-1)\sum_{i=k}^{r} q^{s-a_i-1}$ with multiplicity $q^{a_{k-1}}(q-1)$ for $k=2,\ldots,r$,
3. $(q-1)\sum_{i=k}^{r} q^{s-a_i-1}$ with multiplicity $q^{a_k}-q^{a_{k-1}+1}$ for $k=2,\ldots,r$,
4. -1 with multiplicity $q^{a_r}(q-1)$.

This completes the proof of the lemma. $\square$

Lemma 4.1.5. *Let R be a finite chain ring with unique maximal ideal M, residue field of q elements and of nilpotency s. Let*

$$\mathcal{C}=(M^{a_1}\setminus M^{a_1+1})\cup(M^{a_2}\setminus M^{a_2+1})\cup\cdots\cup(M^{a_r}\setminus M^{a_r+1}),$$

with $0\le a_1<a_2<\cdots<a_r\le s-1$. If $a_r\neq s-1$, then the eigenvalues of $\mathrm{Cay}(R,\mathcal{C})$ *are as follows:*

1. $(q-1)\sum_{i=1}^{r} q^{s-a_i-1}$ *with multiplicity* q^{a_1}*,*

2. $-q^{s-a_{k-1}-1} + (q-1)\sum_{i=k}^{r} q^{s-a_i-1}$ *with multiplicity* $q^{a_{k-1}}(q-1)$ *for* $k = 2, \ldots, r$*,*

3. $(q-1)\sum_{i=k}^{r} q^{s-a_i-1}$ *with multiplicity* $q^{a_k} - q^{a_{k-1}+1}$ *for* $k = 2, \ldots, r$*,*

4. $-q^{s-a_r-1}$ *with multiplicity* $q^{a_r}(q-1)$*,*

5. 0 *with multiplicity* $q^{a_r+1}(q^{s-a_r-1}-1)$*.*

Proof. Since $a_r \neq s-1$, $A_{a_r+1} = \mathbf{0}$, so $\overline{A}_{a_r+1}$ is an adjacency matrix the complete graph on $|M^{a_r+1}| = q^{s-a_r-1}$ vertices. Then

$$\operatorname{Spec} \overline{A}_{a_r+1} = \begin{pmatrix} q^{s-a_r-1}-1 & -1 \\ 1 & q^{s-a_r-1}-1 \end{pmatrix},$$

and hence

$$\operatorname{Spec} I_q \otimes \overline{A}_{a_r+1} = \begin{pmatrix} q^{s-a_r-1}-1 & -1 \\ q & q(q^{s-a_r-1}-1) \end{pmatrix}$$

and

$$\begin{aligned} \operatorname{Spec} A_{a_r} &= \operatorname{Spec} \overline{(I_q \otimes \overline{A}_{a_r+1})} \\ &= \begin{pmatrix} q^{s-a_r} - q^{s-a_r-1} & -q^{s-a_r-1} & 0 \\ 1 & q-1 & q(q^{s-a_r-1}-1) \end{pmatrix}. \end{aligned}$$

By Lemma 4.1.3, $-q^{s-a_r-1}$ and 0 are eigenvalues of $\operatorname{Cay}(R, \mathcal{C})$ with multiplicity $q^{a_r}(q-1)$ and $q^{a_r+1}(q^{s-a_r-1}-1)$. In addition, the eigenvalue $q^{s-a_r} - q^{s-a_r-1} = q^{s-a_r-1}(q-1)$ of A_{a_r} induces the eigenvalues of $A_{a_{r-1}}$ as follows:

1. $q^{s-a_{r-1}-1}(q-1) + q^{s-a_r-1}(q-1)$ with multiplicity 1,

2. $q^{s-a_r-1}(q-1) - q^{s-a_{r-1}-1}$ with multiplicity $q-1$,

3. $q^{s-a_r-1}(q-1)$ with multiplicity $q(q^{a_r-a_{r-1}-1}-1)$.

Similarly, $q^{s-a_r-1}(q-1)-q^{s-a_{r-1}-1}$ and $q^{s-a_r-1}(q-1)$ are eigenvalues of $\mathrm{Cay}(R,\mathcal{C})$ with multiplicity $q^{a_{r-1}}(q-1)$ and $q^{a_{r-1}+1}(q^{a_r-a_{r-1}-1}-1)$ respectively. Moreover, the eigenvalues $q^{s-a_{r-1}-1}(q-1)+q^{s-a_r-1}(q-1)$ of $A_{a_{r-1}}$ gives the following eigenvalues of $A_{a_{r-2}}$:

1. $q^{s-a_{r-2}-1}(q-1) + q^{s-a_{r-1}-1}(q-1) + q^{s-a_r-1}(q-1)$ with multiplicity 1,

2. $q^{s-a_{r-1}-1}(q-1) + q^{s-a_r-1}(q-1) - q^{s-a_{r-2}-1}$ with multiplicity $q-1$,

3. $q^{s-a_{r-1}-1}(q-1) + q^{s-a_r-1}(q-1)$ with multiplicity $q(q^{a_{r-1}-a_{r-2}-1}-1)$.

Repeating this process, we finally obtain the eigenvalues of $\mathrm{Cay}(R,\mathcal{C})$:

1. $(q-1)\sum_{i=1}^{r} q^{s-a_i-1}$ with multiplicity q^{a_1},

2. $-q^{s-a_{k-1}-1} + (q-1)\sum_{i=k}^{r} q^{s-a_i-1}$ with multiplicity $q^{a_{k-1}}(q-1)$ for $k = 2,\ldots,r$,

3. $(q-1)\sum_{i=k}^{r} q^{s-a_i-1}$ with multiplicity $q^{a_k} - q^{a_{k-1}+1}$ for $k = 2,\ldots,r$,

4. $-q^{s-a_r-1}$ with multiplicity $q^{a_r}(q-1)$,

5. 0 with multiplicity $q^{a_r+1}(q^{s-a_r-1}-1)$

as desired. □

Finally, we compute the energy of the graph $\mathrm{Cay}(R,\mathcal{C})$.

Theorem 4.1.6. *Let R be a finite chain ring with unique maximal ideal M, residue field of q elements and of nilpotency s. Let*

$$\mathcal{C} = (M^{a_1} \setminus M^{a_1+1}) \cup (M^{a_2} \setminus M^{a_2+1}) \cup \cdots \cup (M^{a_r} \setminus M^{a_r+1}),$$

with $0 \leq a_1 < a_2 < \cdots < a_r \leq s-1$. Then

$$\mathcal{E}(\mathrm{Cay}(R, \mathcal{C})) = 2(q-1)\left(q^{s-1}r - (q-1)\sum_{k=1}^{r-1}\sum_{i=k+1}^{r} q^{s-a_i+a_k-1}\right).$$

Proof. Observe that the eigenvalues and multiplicities of items (1)–(3) in Lemmas 4.1.4 and 4.1.5 are identical. Moreover, the product of the eigenvalue and its multiplicity in item (4) of Lemmas 4.1.4 and 4.1.5 are $-q^{s-1}(q-1)$. Then both cases have the same energy which can be obtained by a direct computation. □

Remark. When $R = \mathbb{Z}_{p^s}$, this result is Theorem 2.1 of [34].

4.2 Integral Circulant Graphs

In this section, we shall show that our Cayley graph over a finite chain ring is indeed an integral circulant graph. That is, it is isomorphic to a Cayley graph over $\mathbb{Z}_{p^\alpha}$ for some $\alpha \in \mathbb{N}$ which has a circulant adjacency matrix with integral eigenvalues.

Let R be a finite chain ring R with unique maximal ideal M and residue field of $q = p^t$ elements. Assume that R is of nilpotency s and M is generated by $\theta \in M \setminus M^2$. Then for each $x \in R$,

$$x = v_0 + v_1\theta + v_2\theta^2 + \cdots + v_{s-1}\theta^{s-1},$$

where $v_i \in \mathcal{V} = \{e_0, e_1, \ldots, e_{p^t-1}\}$, a fixed set of representatives of cosets in R/M, and

$$\mathcal{C}_1 = (M^{a_1} \smallsetminus M^{a_1+1}) \cup (M^{a_2} \smallsetminus M^{a_2+1}) \cup \cdots \cup (M^{a_r} \smallsetminus M^{a_r+1}),$$

with $0 \leq a_1 < a_2 < \cdots < a_r \leq s-1$. Note that the ring $\mathbb{Z}_{q^s} = \mathbb{Z}_{p^{ts}}$ is a finite chain ring with the chain

$$\mathbb{Z}_{p^{ts}} \supset p\mathbb{Z}_{p^{ts}} \supset p^2\mathbb{Z}_{p^{ts}} \supset \cdots \supset p^{ts-1}\mathbb{Z}_{p^{ts}} \supset p^{ts}\mathbb{Z}_{p^{ts}} = \{0\}$$

having

$$\mathbb{Z}_{p^{ts}} \supset p^t\mathbb{Z}_{p^{ts}} \supset p^{2t}\mathbb{Z}_{p^{ts}} \supset \cdots \supset p^{(s-1)t}\mathbb{Z}_{p^{ts}} \supset p^{st}\mathbb{Z}_{p^{ts}} = \{0\}$$

as a subchain. This observation implies that each $a \in \mathbb{Z}_{p^{st}}$ can be expressed as

$$a = c_0 + c_1p^t + c_2p^{2t} \cdots + c_{s-1}p^{(s-1)t},$$

where $c_i \in \{0, 1, \ldots, p^t - 1\}$. Let $g : e_i \mapsto i$ be a bijection from $\mathcal{V}$ onto $\{0, 1, \ldots, p^t - 1\}$. Let $\mathcal{C}_2 = \{p^{a_1t}, p^{a_1t+1}, \ldots, p^{a_1t+t-1}, \ldots, p^{a_rt}, p^{a_rt+1}, \ldots, p^{a_rt+t-1}\}$. We shall show that the graphs $\mathrm{Cay}(R, \mathcal{C}_1)$ and $\mathrm{Cay}(\mathbb{Z}_{p^{st}})$ are isomorphic. Define $f : \mathrm{Cay}(R, \mathcal{C}_1) \to \mathrm{Cay}(\mathbb{Z}_{p^{st}}, \mathcal{C}_2)$ by

$$f(v_0 + v_1\theta + \cdots + v_{s-1}\theta^{s-1}) = g(v_0) + g(v_1)p^t + g(v_2)p^{2t} + \cdots + g(v_{s-1})p^{(s-1)t}.$$

Then f is a well-defined bijection. To see that f is an isomorphism, we let

$$x = v_0 + v_1\theta + v_2\theta^2 + \cdots + v_{s-1}\theta^{s-1} \text{ and}$$

$$y = u_0 + u_1\theta + u_2\theta^2 + \cdots + u_{s-1}\theta^{s-1}.$$

Suppose that x and y are adjacent in $\mathrm{Cay}(R, \mathcal{C}_1)$. Then $x - y \in M^{a_i} \smallsetminus M^{a_i+1}$ for some a_i. This means $v_i = u_i$ for $i < a_i$ and $v_{a_i} \neq u_{a_i}$. Thus $g(v_i) = g(u_i)$

for $i < a_i$ and $g(v_{a_i}) \neq g(u_{a_i})$, so $f(x) - f(y) \in p^{a_i t}\mathbb{Z}_{p^{st}} \setminus p^{(a_i+1)t}\mathbb{Z}_{p^{st}}$. Then, as elements of $\mathbb{Z}$, $\gcd(f(x) - f(y), p^{st}) = p^j$ where $a_i t \leq j < (a_i + 1)t$ and thus $f(x)$ and $f(y)$ are adjacent in $\mathrm{Cay}(\mathbb{Z}_{p^{st}}, \mathcal{C}_2)$. Conversely, assume that $f(x)$ and $f(y)$ adjacent in $\mathrm{Cay}(\mathbb{Z}_{p^{st}}, \mathcal{C}_2)$. Then, as elements of $\mathbb{Z}$, $\gcd(f(x)-f(y), p^{st}) = p^j$ where $a_i t \leq j < (a_i + 1)t$ for some a_i. It follows that for

$$f(x) = g(v_0) + g(v_1)p^t + g(v_2)p^{2t} + \cdots + g(v_{s-1})p^{(s-1)t} \text{ and}$$
$$f(y) = g(u_0) + g(u_1)p^t + g(u_2)p^{2t} + \cdots + g(u_{s-1})p^{(s-1)t},$$

we have $g(v_i) = g(u_i)$ for $i < a_i$ and $g(v_{a_i}) \neq g(u_{a_i})$. Thus, $x - y \in M^{a_i} \setminus M^{a_i+1}$ and hence they are adjacent in $\mathrm{Cay}(R, \mathcal{C}_1)$. Hence, we have shown:

Proposition 4.2.1. *Let R be a finite chain ring with unique maximal ideal M, residue field of $q = p^t$ elements and of nilpotency s. Let*

$$\mathcal{C}_1 = (M^{a_1} \setminus M^{a_1+1}) \cup (M^{a_2} \setminus M^{a_2+1}) \cup \cdots \cup (M^{a_r} \setminus M^{a_r+1}),$$

with $0 \leq a_1 < a_2 < \cdots < a_r \leq s - 1$. Then

$$\mathrm{Cay}(R, \mathcal{C}_1) \cong \mathrm{Cay}(\mathbb{Z}_{p^{st}}, \mathcal{C}_2),$$

where $\mathcal{C}_2 = \{p^{a_1 t}, p^{a_1 t+1}, \ldots, p^{a_1 t+t-1}, \ldots, p^{a_r t}, p^{a_r t+1}, \ldots, p^{a_r t+t-1}\}$.

4.3 GCD-graphs over a UFD

Let D be a unique factorization domain (UFD) and $c \in D$ a nonzero nonunit element. Assume that the commutative ring $D/(c)$ is finite. Write $c = p_1^{s_1} \ldots p_k^{s_k}$ as a product of irreducible elements. We now study the gcd-graph $D_c(\mathcal{C})$. Suppose that for each $i \in \{1, 2, \ldots, k\}$, there exists a set $\mathcal{C}_i = \{p_i^{a_{i1}}, p_i^{a_{i2}}, \ldots, p_i^{a_{ir_i}}\}$, with $0 \le a_{i1} < a_{i2} < \cdots < a_{ir_i} \le s_i - 1$ so that

$$\mathcal{C} = \{p_1^{a_{1t_1}} \cdots p_k^{a_{kt_k}} \ : \ t_i \in \{1, 2, \ldots, r_i\} \text{ for all } i \in \{1, 2, \ldots, k\}\}.$$

Then for $x, y \in D/(c)$,

$$x \text{ is adjacent to } y \Leftrightarrow \gcd(x-y, c) \in D^\times \mathcal{C} \Leftrightarrow \gcd(x-y, p_i^{s_i}) \in D^\times \mathcal{C}_i \text{ for all } i.$$

This implies that

$$D_c(\mathcal{C}) = \mathrm{Cay}(D/(p_1^{s_1}), \mathcal{C}_1) \otimes \cdots \otimes \mathrm{Cay}(D/(p_k^{s_k}), \mathcal{C}_k),$$

where each factor on the right is the Cayley graph over the finite chain ring $D/(p_i^{s_i})$ which we have already computed the energy in Section 4.1. Recall from Proposition 1.3.2 that $\mathcal{E}(G \otimes H) = \mathcal{E}(G)\mathcal{E}(H)$ for two graphs G and H. Therefore, we have the following theorem.

Theorem 4.3.1. *Let D be a UFD and a nonzero nonunit $c = p_1^{s_1} \ldots p_k^{s_k} \in D$ factored as a product of irreducible elements. Assume that $D/(c)$ is finite and for each $i \in \{1, 2, \ldots, k\}$, there exists a set $\mathcal{C}_i = \{p_i^{a_{i1}}, p_i^{a_{i2}}, \ldots, p_i^{a_{ir_i}}\}$, with $0 \le a_{i1} < a_{i2} < \cdots < a_{ir_i} \le s_i - 1$ such that*

$$\mathcal{C} = \{p_1^{a_{1t_1}} \cdots p_k^{a_{kt_k}} \ : \ t_i \in \{1, 2, \ldots, r_i\} \textit{ for all } i \in \{1, 2, \ldots, k\}\}.$$

Then

$$\mathcal{E}(D_c(\mathcal{C})) = \mathcal{E}(D_{p_1^{s_1}}(\mathcal{C}_1)) \ldots \mathcal{E}(D_{p_k^{s_k}}(\mathcal{C}_k)).$$

Remark. Recall that if a matrix A has eigenvalues $\lambda_1, \ldots, \lambda_n$, then the eigenvalues of $A + I$ are $\lambda_1 + 1, \ldots, \lambda_n + 1$. Hence, one can obtain the energy of the gcd-graph in Theorem 4.3.1 when $\mathcal{C}_i$ contains $p_i^{s_i}$ using this fact and the eigenvalues computed in Lemma 4.1.4 or 4.1.5.

Now, we study the case where some $\mathcal{C}_j = \{p_j^{s_j}\}$. To compute the energy in this case, we shall use the graph operation which more general than the tensor product called a non-complete extended p-sum [25] defined as follows.

Given a set $B \subseteq \{0,1\}^k$ and graphs $G_1, \ldots, G_k$, the **NEPS (non-complete extended p-sum)** of these graphs with respect to basis B, $G = \text{NEPS}(G_1, \ldots, G_k ; B)$, has its vertex set as the Cartesian product of the vertex sets of the individual graphs, i.e., $V(G) = V(G_1) \times \cdots \times V(G_k)$. Two distinct vertices $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ are adjacent in G if and only if there exists some k-tuple $(\beta_1, \ldots, \beta_k) \in B$ such that $x_i = y_i$ whenever $\beta_i = 0$ and $x_i,\ y_i$ are distinct and adjacent in G_i whenever $\beta_i = 1$. In particular, when $B = \{(1, 1, \ldots, 1)\}$,

$$\text{NEPS}(G_1, \ldots, G_k; B) = G_1 \otimes G_2 \otimes \cdots \otimes G_k.$$

The eigenvalues of the graph $\text{NEPS}(G_1, \ldots, G_k; B)$ is presented in the next theorem.

Theorem 4.3.2. *[12] Let $G_1, \ldots, G_k$ be graphs with $n_1, \ldots, n_k$ vertices respectively. Furthermore, for $i \in \{1, \ldots, k\}$ let $\lambda_{i1}, \ldots, \lambda_{in_i}$ be the eigenvalues of G_i. Then, the spectrum of the graph $G = \text{NEPS}(G_1, \ldots, G_k; B)$ with respect to basis B consists of all possible values*

$$\mu_{i_1,\ldots,i_k} = \sum_{(\beta_1,\ldots,\beta_k)\in B} \lambda_{1i_1}^{\beta_1} \cdots \lambda_{ki_k}^{\beta_k}$$

with $1 \le i_l \le n_l$ for $1 \le l \le k$.

Next, we consider $c = p_1^{s_1} \dots p_k^{s_k}$ written as a product of irreducible elements. Suppose that $l \leq k$ and for each $i \in \{1, 2, \dots, l\}$, there exists a set $\mathcal{C}_i = \{p_i^{a_{i1}}, p_i^{a_{i2}}, \dots, p_i^{a_{ir_i}}\}$, with $0 \leq a_{i1} < a_{i2} < \cdots < a_{ir_i} \leq s_i - 1$ so that

$$\mathcal{C}' = \{p_1^{a_{1t_1}} \cdots p_l^{a_{lt_l}} p_{l+1}^{s_{l+1}} \cdots p_k^{s_k} : t_i \in \{1, 2, \dots, r_i\} \text{ for all } i \in \{1, 2, \dots, l\}\}.$$

Then

$$D_c(\mathcal{C}') = \mathrm{NEPS}(D_{p_1^{s_1}}(\mathcal{C}_1), D_{p_2^{s_2}}(\mathcal{C}_2), \dots, D_{p_k^{s_k}}(\mathcal{C}_k); \{(\underbrace{1, \dots, 1}_{l}, \underbrace{0, \dots, 0}_{k-l})\}),$$

where $C_j = \{p_j^{s_j}\}$ for $l < j \leq k$. By Theorem 4.3.2, all eigenvalues of $D_c(\mathcal{C}')$ are the eigenvalues of

$$\mathrm{Cay}(D/(p_1^{s_1}), \mathcal{C}_1) \otimes \cdots \otimes \mathrm{Cay}(D/(p_l^{s_l}), \mathcal{C}_l)$$

each repeated $\prod_{j=l+1}^{k} |D/(p_j^{s_j})|$ times. Hence, it follows from Theorem 4.3.1 that:

Theorem 4.3.3. *Let D be a UFD and a nonzero nonunit $c = p_1^{s_1} \dots p_k^{s_k} \in D$ factored as a product of irreducible elements. Assume that $D/(c)$ is finite, $l \leq k$ and for each $i \in \{1, 2, \dots, l\}$, there exists a set $\mathcal{C}_i = \{p_i^{a_{i1}}, p_i^{a_{i2}}, \dots, p_i^{a_{ir_i}}\}$, with $0 \leq a_{i1} < a_{i2} < \cdots < a_{ir_i} \leq s_i - 1$ such that*

$$\mathcal{C}' = \{p_1^{a_{1t_1}} \cdots p_l^{a_{lt_l}} p_{l+1}^{s_{l+1}} \cdots p_k^{s_k} : t_i \in \{1, 2, \dots, r_i\} \textit{ for all } i \in \{1, 2, \dots, l\}\}.$$

Then

$$\mathcal{E}(D_c(\mathcal{C}')) = \mathcal{E}(D_{p_1^{s_1}}(\mathcal{C}_1)) \cdots \mathcal{E}(D_{p_l^{s_l}}(\mathcal{C}_l)) \prod_{j=l+1}^{k} |D/(p_j^{s_j})|.$$

REFERENCES

[1] Achraya B. D.: A characterization of consistent marked graphs, *Nat. Acad. Sci. Lett.*, 6(12), 431–440 (1983).

[2] Achraya B. D.: Some further properties of consistent marked graphs, *Indian J. of Pure Appl. Math.*, 15(8), 837–842 (1984).

[3] Achraya M., Sinha D.: A characterization of sigraphs whose line sigraphs and jump sigraphs are switching equivalent, *Graph Theory Notes N.Y.*, XLIV:6, 30–34 (2003).

[4] Akhtar R., Boggess M., Jackson-Henderson T., Jiménez I., Karpman R., Kinzel A., Pritikin D.: On the unitary Cayley graph of a finite ring, *The Electronic J. Comb.*, 16(1), #R117 (2009).

[5] Bašić M., Petković M. D.: Some classes of integral circulant graphs either allowing or not allowing perfect state transfer, *Appl. Math. Lett.*, 22, 1609–1615 (2010).

[6] Beaudrap N.: On restricted unitary Cayley graphs and symplectic transformations modulo n, *The Electronic J. Comb.*, 17, #R69 (2010).

[7] Behzad M., Chartrand G. T.: Line coloring of signed graphs, *Elem. Math.*, 24(3), 49–52 (1969).

[8] Beineke L. W., Harary F.: Consistency in marked graphs, *J. Math. Psychol.*, 18(3), 260–269 (1978).

[9] Beineke L. W., Harary F.: Consistent graphs with signed points, *Riv. Math. per. Sci. Econom. Sociol.*, 1, 81–88 (1978).

[10] Bini G., Flamini F.: *Finite Commutative Rings and Their Applications*, Springer, New York, 2002.

[11] Brualdi R. A.: *Energy of a Graph*, `http://www.public.iastate.edu/lhogben/energyB.pdf`.

[12] Cvetković D., Doob M., Sachs H.: *Spectra of Graphs, 87: Theory and Application (Pure & Applied Mathematics)*, 3rd edn, Johann Ambrosius Barth Verlag, 1995.

[13] Dolžan D.: Group of units in a finite ring, *J. Pure Appl. Algebra*, 170, 175–183 (2001).

[14] Gill M.K., Acharya B.D.: A recurrence formula for computing the charateristic polynomial of a sigraph, *J. Comb., Inf. & Syst. Sci.*, (5), 68-72 (1980).

[15] Godsil C., Royle G.: *Algebraic Graph Theory*, Springer, New York, 2001.

[16] Gutman I.: The energy of a graph, *Ber. Math. Stat. Sekt. Forschungszent. Graz*, 103, 1–22 (1978).

[17] Gutman I.: *The Energy of a Graph: Old and New Results, Algebraic Combinatorics and Applications*, Springer, New York, 2001.

[18] Gutman I., Kiani D., Mirzakhah M., Zhou B.: On incidence energy of a graph, *Linear Algebra Appl.*, 431, 1223–1233 (2009).

[19] Hoede C.: A characterization of consistent marked sigraphs, *J. Graph Theory*, 16(1), 17-23 (1992).

[20] Ilić A.: The energy of unitary Cayley graphs, *Linear Algebra Appl.*, 431, 1881–1889 (2009).

[21] Ilić A., Bašić M., Gutman I.: Triply Equienergetic Graphs, *MATCH Commun. Math. Comput. Chem.*, 64, 189–200 (2010).

[22] Jooyandeh M. R., Kiani D., Mirzakhah M.: Incidence energy of a graph, *MATCH Commun. Math. Comput. Chem.*, 62, 561–572 (2009).

[23] Kiani D., Aghaei M. M. H., Meemark Y., Suntornpoch B.: Energy of unitary Cayley graphs and gcd-graphs, *Linear Algebra Appl.*, 435(6), 1336–1343 (2011).

[24] Klotz W., Sander T.: Some properties of unitary Cayley graphs, *The Electronic J. Comb.*, 14, #R45 (2007).

[25] Klotz W., Sander T.: GCD-graphs and NEPS of complete graphs, *Ars Math. Contemp.*, (6), 289–299 (2013).

[26] Liu X., Zhou S.: Spectral properties of unitary Cayley graphs of finite commutative rings, *The Electronic J. Comb.*, 19(4), #P13 (2012).

[27] McDonald B. R.: *Finite Rings with Identity*, Marcel Dekker, New York, 1974.

[28] Meemark Y., Suntornpoch B.: Eigenvalues and energy of restricted unitary Cayley graphs induced from the square mapping, *ScienceAsia*, 39, 649–652 (2013).

[29] Meemark Y., Suntornpoch B.: Balanced unitary Cayley sigraphs over finite commutative rings, *J Algebra Appl.*, 13(5), 1350152 (12 pages) (2014).

[30] Norton G. H., Sălăgean A.: On the structure of linear and cyclic codes over finite chain rings, *Appl. Algebra Engng. Comm. Comput.*, 10(6), 489–506 (2000).

[31] Ramane H. S., Walikar H. B., Rao S. B., Acharya B. D., Hampiholi P. R., Jog S. R., Gutman I.: Spectra and energies of iterated line graphs of regular graphs, *Appl. Math. Lett.*, 18, 679–682 (2005).

[32] Rao S. B.: Characterizations of harmonious marked graphs and consistent nets, *J. Comb., Inf. & Syst. Sci.*, 9(2), 97-112 (1984).

[33] Rosen M.: *Number Theory in Function Fields*, Springer, New York, 2002.

[34] Sander J. W., Sander T.: The energy of integral circulant graphs with prime power order, *Appl. Anal. Discrete Math.*, 5, 22–36 (2011).

[35] Singmaster D., Bloom D. M.: Rings of order four, *The American Mathematical Monthly*, 71(8), 918–920 (1964).

[36] Sinha D., Garg P.: Canonical consistency of signed line structures, *Graph Theory Notes N.Y.*, 59, 22–27 (2010).

[37] Sinha D., Garg P.: Balance and consistency of total signed graphs, *Indian J. Mathematics*, 53(1), 71–81 (2011).

[38] Sinha D., Garg P.: Some results on semi-total signed graphs, *Discuss. Math. Graph Theory*, 31(4), 625-638 (2011).

[39] Sinha D., Garg P.: On the unitary Cayley signed graphs, *The Electronic J. Comb.*, 18, #P229 (2011).

[40] So W.: Integral circulant graphs, *Discrete Math.*, 306, 153–158 (2006).

[41] Stevanović D., Indulal G.: The distance spectrum and energy of the compositions of regular graphs, *Appl. Math. Lett.*, 22, 1136–1140 (2009).

[42] West D. B.: *Introduction to Graph Theory*, 2nd edn, Prentice-Hall, 2000.

[43] Ye L.: The energy of a type of lattice, *Appl. Math. Lett.*, 24, 145–148 (2011).

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