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LINEAR PRESERVERS ON HESSENBERG MATRICES

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ในงานวิจัยนี้ เราเน้นการศึกษาตัวคงสภาพแรงก์หนึ่งบนเมทริกซ์เฮสเซนเบิร์ก วัตถุประสงค์แรกคือเพื่อจำแนกการส่งเชิงเส้นซึ่งคงสภาพแรงก์หนึ่ง ผลที่ได้นี้นำไปสู่รูปแบบของการส่งเชิงเส้นซึ่งคงสภาพดีเทอร์มิแนนต์ซึ่งเป็นอีกหนึ่งวัตถุประสงค์ นอกจากนี้ยังให้รูปแบบของการส่งเชิงเส้นซึ่งคงสภาพค่าเฉพาะ ในท้ายที่สุด บนปริภูมิเดียวกัน เรายังให้การจำแนกลักษณะของการส่งการบวกอย่างทั่วถึงซึ่งคงสภาพแรงก์หนึ่ง

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....

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In this research, we focus on studying rank-1 preservers on Hessenberg matrices. The first purpose is to characterize linear maps preserving rank one. This result leads to attain the pattern of linear maps preserving determinants which is another purpose. Moreover, the form of linear maps preserving eigenvalues is given. Finally, on the same space, a characterization of surjective additive maps preserving rank one is provided.

Department : Mathematics and Computer Science	Student's Signature : ....
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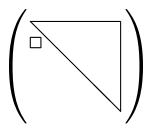
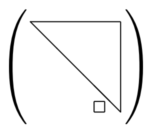
Last but not least, my sincere gratitude goes to my beloved parents and dearest sister for their unexceptionable encouragement throughout my study.

CONTENTS

NOTATION

m	a positive integer
n	a positive integer
$\mathbb{F}$	a field
$\mathbb{C}$	the field of complex numbers
A^t	the transpose of a matrix A
$A^{\sim}$	an $n \times n$ matrix (b_{ij}) such that $b_{ij} = a_{n+1-j,n+1-i}$ for any i, j where $A = (a_{ij})$
S	the function on $H_n(\mathbb{F})$ which maps A to $A^{\sim}$ for all $A \in H_n(\mathbb{F})$
$\rho(A)$	the rank of a matrix A
$\det(A)$	the determinant of a matrix A
E_{ij}	the elementary matrix having only one at the (i, j)-entry and zero for all other positions
$M_{mn}(\mathbb{F})$	the set of all $m \times n$ matrices over a field $\mathbb{F}$
$M_n(\mathbb{F})$	the set of all $n \times n$ matrices over a field $\mathbb{F}$
$T_n(\mathbb{F})$	the set of all $n \times n$ upper triangular matrices over a field $\mathbb{F}$
$H_n(\mathbb{F})$	the set of all $n \times n$ upper Hessenberg matrices over a field $\mathbb{F}$
$H_n^1(\mathbb{F})$	$\Big\{(a_{ij}) \in H_n(\mathbb{F}) \,\vert\, a_{21} \neq 0 \text{ and } a_{j+1,j} = 0 \text{ for all } j \in \{2, \dots, n-1\}\Big\}$, thus elements in this set are of the form 
$H_n^2(\mathbb{F})$	$\Big\{(a_{ij}) \in H_n(\mathbb{F}) \,\vert\, a_{n,n-1} \neq 0 \text{ and } a_{j+1,j} = 0 \text{ for all } j \in \{1, \dots, n-2\}\Big\}$, thus elements in this set are of the form 
Ω	$\{A \in H_n(\mathbb{F}) \,\vert\, \rho(A) = 1\}$
$x \otimes y$	xy^t where $x \in M_{m1}(\mathbb{F})$ and $y \in M_{n1}(\mathbb{F})$
$\operatorname{im} T$	the image of a function T

$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ & & \end{pmatrix}$	all empty entries of this matrix are all zeros
$\begin{pmatrix} \vert & & \vert \\ v_1 & \cdots & v_n \\ \vert & & \vert \end{pmatrix}_{m\times n}$	the j-column of this matrix is $v_j \in M_{m1}(\mathbb{F})$ for all $j \in \{1, \ldots, n\}$
$\begin{pmatrix} — & v_1 & — \\ & \vdots & \\ — & v_m & — \end{pmatrix}_{m\times n}$	the i-row of this matrix is $v_i \in M_{1n}(\mathbb{F})$ for all $i \in \{1, \ldots, m\}$

CHAPTER I
INTRODUCTION

1.1 Introduction

One of the most active research subjects in matrix theory in the last five decades is linear preserver problems (LPPs) which concern the categorization of linear maps on spaces of matrices or operators that leave certain properties, functions, subsets or relations invariant.

Research that is considered to be the beginning of LPPs is determinant preservers studied by Frobenius who gave the form of linear maps on square complex matrices preserving determinant in 1897, that is, if T is a linear map on $M_n(\mathbb{C})$ which is a determinant preserver (i.e., $\det(T(A)) = \det(A)$ for all matrices A), then there exist nonsingular $n \times n$ matrices P and Q such that $\det(PQ) = 1$ and either

$$T(A) = PAQ \quad \text{for all} \quad A \in M_n(\mathbb{C}) \qquad (*)$$

or

$$T(A) = PA^tQ \quad \text{for all} \quad A \in M_n(\mathbb{C}) \qquad (**)$$

where A^t denotes the transpose of a matrix A and vice versa.

Actually, there are many motivations for studying LPPs [15], such as

1) in order to characterize structures of linear maps T on a space V of matrices preserving one of the followings (called a general linear preserver problem):

• a function f defined on V (i.e., $f(T(A)) = f(A)$ for all $A \in V$; e.g., the determinant, the permanent);

• a subset Δ of V (i.e., $T(\Delta) \subseteq \Delta$; e.g., the set of idempotents of V, an algebraic set);

• a relation $\sim$ defined on V (i.e., $T(A) \sim T(B)$ whenever $A \sim B$; e.g., commutativity, similarity).

There are three major ways to do research on this type of LPPs. First, we can consider LPPs on matrix spaces over rings or semirings such as nonnegative integers and Boolean algebras. Second, we can consider additive maps or bilinear maps instead of linear maps. Finally, we can study other objects apart from functions, subsets or relations to be preserved.

2) in order to be a tool for solving some other mathematical problems such as problems in the area of differential equations and system theory.

3) in order to find conditions such that linear maps of the above forms $(*)$ or $(**)$ preserve a specific object.

However, one of the most studied subjects in LPPs is rank-1 preservers because rank-1 preservers play a pivotal role in investigating other questions about preservers such as commutativity preservers, rank-additivity preservers, spectrum preservers and determinant preservers ([6], [9], [14], [19], [20] and [23]). In 1959, Marcus and Moyls [18] gave the form of linear maps on $M_{mn}(\mathbb{F})$, the space of $m \times n$ matrices over an algebraically closed field $\mathbb{F}$ of characteristic 0, holding rank one by using multilinear algebra techniques, i.e., for a given linear map T on $M_{mn}(\mathbb{F})$ preserving rank one,

i) if $m \neq n$, then there exist nonsingular matrices P and Q such that $T(A) = PAQ$ for all $A \in M_{mn}(\mathbb{F})$; or

ii) if $m = n$, then there exist nonsingular matrices P and Q such that either $T(A) = PAQ$ for all $A \in M_{mn}(\mathbb{F})$ or $T(A) = PA^tQ$ for all $A \in M_{mn}(\mathbb{F})$.

Eight years later, Westwick [24] generalized these results to any algebraically closed fields. In 1977, Minc [19] reproved the theorem of Marcus and Moyls by using only elementary matrix theory. Nevertheless, two years before this Minc's research was published, Lim [17] provided a structure of all invertible linear maps preserving rank one over any fields. In 1985, this theorem of Lim was generalized by Waterhouse [22] to commutative rings with unit but the invertibility assumption was still remained. However, the invertibility assumption in the theorem of Lim could be omitted and the linear maps could be extended to linear maps between spaces of different dimensions. This result was proved by Li, Rodman and Šerml [16] in 2002. Furthermore, there are many authors studied on some subspaces of matrix spaces, for examples, in 1993, linear rank-1 preservers on Hermitian matrix spaces were revealed by Baruch and Loewy [2]. In 1998, the structure of linear rank-1 preservers on the space of upper triangular matrices over an arbitrary field was given by Chooi and Lim [8].

During the past twenty years, there are various research concerning additive preserver problems (APPs). APPs are problems similar to LPPs except these maps preserve the addition while preserving the scalar multiplication is not required. Commonly, additive rank-1 preservers are among the most studied subjects for examples Bell and Sourour [4] provided the structure of surjective additive rank-1 preservers on block triangular matrix algebras in 2000, Chooi and Lim [9] generalized some results of Bell and Sourour by studying additive rank-1 preservers on block triangular matrix spaces over any fields in 2006. Surjective additive rank-1 preservers on the full matrix algebra over any fields were characterized by Cao and Zhang [5] in 2004; besides, this result was applied to prove invertibility preservers, determinant preservers and characteristic polynomial preservers. Next year, the result of Li *et al.* [16] was extended by replacing linear maps with additive maps in the hypothetical condition which was shown by Zhang and Sze [25]. This work also generalized the result of Cao and Zhang [5]. In 2008, Gao and Zhang [13] found the structure of all additive maps preserving rank one between spaces of Hermitian matrices.

Found in [10], Karl Hessenberg, a German mathematician and engineer, called a square matrix (a_{ij}) *upper Hessenberg* if $a_{ij} = 0$ whenever $j + 1 < i$. One can see that upper triangular matrices, diagonal matrices and, for example, $\begin{pmatrix} 1&2&3&2 \\ 2&5&6&1 \\ 0&8&9&5 \\ 0&0&1&2 \end{pmatrix}$ are Hessenberg matrices. Obviously, upper Hessenberg matrices are quite the same as upper triangular matrices except the first one have zero entries below the first subdiagonal. In addition, some properties of triangular matrices are not true for Hessenberg matrices, for example, the product of Hessenberg matrices may not be a Hessenberg matrix as follows: $\begin{pmatrix} 1&2&3&2 \\ 2&5&6&1 \\ 0&8&9&5 \\ 0&0&1&2 \end{pmatrix} \begin{pmatrix} 1&1&1&1 \\ 1&1&1&1 \\ 0&0&1&1 \\ 0&0&1&1 \end{pmatrix} = \begin{pmatrix} 3&3&8&8 \\ 7&7&14&14 \\ 8&8&22&22 \\ 0&0&3&3 \end{pmatrix}$. Likewise upper triangular matrices, upper Hessenberg matrices over a field form a vector space.

One of the benefits of Hessenberg matrices is to alter QR algorithm (factorization matrices into triangular matrices) for the better (see [1] and [7]). This algorithm was devised by Francis in 1961 in order to replace LU factorization which is not stable absent pivoting. Nowadays, the QR algorithm is one of the most valuable algorithm to steadily compute the eigenvalues and corresponding eigenvectors or Schur vectors; furthermore, it is also the most popular approach for solving dense nonsymmetric eigenvalue problems. However, there is a limitation in size of a matrix in $M_n(\mathbb{F})$ because this method uses $\mathcal{O}(n^2)$ storage and runs in $\mathcal{O}(n^3)$ time. This shows that the QR algorithm is quite expensive. From whole reasons, Hessenberg matrices become a tool to make the algorithm practical since Hessenberg matrices have the forms closed to the forms of upper triangular matrices and are invariant under the QR algorithm. A new algorithm is called the Hessenberg QR algorithm by using Householder reflectors to first reduce every square matrix to an upper Hessenberg matrix. This new algorithm requires only $\mathcal{O}(n)$ storage and $\mathcal{O}(n^2)$ time. Besides, for integrable system in quantum mechanics, Hessenberg matrices are used as equipment in order to represent perturbed Hamiltonians for perturbation theory which is the technique used in the study of disturbed quantum systems see [26].

This dissertation focuses on rank-1 preservers on the space of all $n \times n$ upper Hessenberg matrices over a field which are studied in two main points; namely, linear maps and additive maps. Theorems regard linear rank-1 preservers on the space of matrices and the space of upper triangular matrices over any fields are given as follows. However, we introduce some basic notations first.

Let $M_{mn}(\mathbb{F})$ and $M_n(\mathbb{F})$ be the set of all $m \times n$ matrices over a field $\mathbb{F}$ and the set of all $n \times n$ matrices over a field $\mathbb{F}$, respectively. Let $T_n(\mathbb{F})$ and $H_n(\mathbb{F})$ be the set of all $n \times n$ upper triangular matrices over a field $\mathbb{F}$ and the set of all $n \times n$ upper Hessenberg matrices over a field $\mathbb{F}$, respectively. Then, it is clear that $T_n(\mathbb{F})$ and $H_n(\mathbb{F})$ are subspaces of $M_n(\mathbb{F})$ and $T_n(\mathbb{F})$ is also a subspace of $H_n(\mathbb{F})$. The transpose of a matrix A is denoted by A^t. Furthermore, $\rho(A)$ denotes the rank of a matrix A.

Definition 1.1. [3] A subspace V of any spaces of matrices is called a *rank*-1 *subspace* if each element in V is the zero matrix or has rank one. In addition, a map T on $H_n(\mathbb{F})$ is called a *rank*-1 *preserver* if $\rho(T(A)) = 1$ whenever $\rho(A) = 1$ for any $A \in H_n(\mathbb{F})$. Besides, a map T on $H_n(\mathbb{F})$ is called a *rank preserver* if T preserves all ranks.

Theorem 1.2. [17] *Let T be a linear rank-1 preserver on $M_{mn}(\mathbb{F})$. Then*
(i) $\operatorname{im} T$ *is a rank-1 subspace, or*
(ii) there exist nonsingular matrices P and Q such that

$$T(A) = PAQ \quad \text{for all} \quad A \in M_{mn}(\mathbb{F}) \tag{1.1}$$

or

$$T(A) = PA^tQ \quad \text{for all} \quad A \in M_{mn}(\mathbb{F}). \tag{1.2}$$

Note that the forms in (1.1) and (1.2) are alwalys called the "standard form".

In [8], for a matrix $A = (a_{ij})$ in $M_n(\mathbb{F})$, Chooi and Lim defined the $\sim$ of A, denoted by $A^\sim$, to be the matrix (b_{ij}) in $M_n(\mathbb{F})$ such that $b_{ij} = a_{n+1-j,n+1-i}$ for any i and j. For examples, $\begin{pmatrix} 1&2&3\\0&5&6\\0&8&9 \end{pmatrix}^\sim = \begin{pmatrix} 9&6&3\\8&5&2\\0&0&1 \end{pmatrix}$ and $\begin{pmatrix} 1&2&0&4\\3&5&6&1\\0&8&9&1\\0&0&0&8 \end{pmatrix}^\sim = \begin{pmatrix} 8&1&1&4\\0&9&6&0\\0&8&5&2\\0&0&3&1 \end{pmatrix}$. Observably, the diagonal line (not the main diagonal line) acts as the reflection-axis for remaining elements but the elements on this line are fixed. Furthermore, $(A+B)^\sim = A^\sim + B^\sim$, $(AB)^\sim = B^\sim A^\sim$, $(A^\sim)^\sim = A$ and $\rho(A) = \rho(A^\sim)$ for all $A, B \in M_n(\mathbb{F})$.

Theorem 1.3. [8] *Let T be a linear rank-1 preserver on $T_n(\mathbb{F})$. Then*
(i) $\operatorname{im} T$ *is an n-dimensional rank-1 subspace, or*
(ii) there exist nonsingular upper triangular matrices P and Q such that

$$T(A) = PAQ \quad \text{for all} \quad A \in T_n(\mathbb{F})$$

or

$$T(A) = PA^\sim Q \quad \text{for all} \quad A \in T_n(\mathbb{F}).$$

It is worth mention that the results in Theorems 1.2 and 1.3 are similar in the sense that matrices P and Q are nonsingular; however, there are many points different. First, P and Q in Theorem 1.2 are elements of $M_n(\mathbb{F})$ (maybe or maybe not in $T_n(\mathbb{F})$) but P and Q in Theorem 1.3 are elements of $T_n(\mathbb{F})$ because $\operatorname{im} T$ must be contained in $T_n(\mathbb{F})$. Second, in Theorem 1.3, the transpose of a matrix is replaced by the $\sim$ of a matrix since the transpose of upper triangular matrices are lower triangular matrices which are out of the considered spaces, $T_n(\mathbb{F})$. The $\sim$ of upper triangular matrices are upper triangular matrices so this symbol is thought in order to get that property.

For the part of additive rank-1 preservers, this work is motivated by the result of Cao and Zhang [5]. They provided the pattern of surjective additive maps preserving rank one on square matrices over an arbitrary field as follows.

Theorem 1.4. [5] *Let T be a surjective additive rank-1 preserver on $M_n(\mathbb{F})$. Then there exist a field automorphism θ on $\mathbb{F}$ and nonsingular matrices P and Q such that*

$$T(A) = PA^{\theta}Q \quad \textit{for all} \quad A \in M_n(\mathbb{F})$$

or

$$T(A) = P\big(A^{\theta}\big)^{t}Q \quad \textit{for all} \quad A \in M_n(\mathbb{F})$$

where $A^{\theta} = \big(\theta(a_{ij})\big)$ for all $A = (a_{ij}) \in M_n(\mathbb{F})$.

Notice that Theorems 1.2 and 1.4 are of the same types except that a field automorphism is required in Theorem 1.4.

We seperate this dissertation into four chapters. In Chapter I, we start with background of this research leading to why rank-1 preservers and Hessenberg matrices are chosen. Next, definitions and notation used frequently in this dissertation are given. The rest of Chapter I is dedicated to some basic properties of Hessenberg matrices.

Chapter II is devoted to characterize linear maps preserving rank-1 matrices as Theorem 2.29 which is one of the major goals in this dissertation. Besides, the forms of linear maps preserving all ranks of matrices, linear maps preserving determinants and linear maps preserving eigenvalues are exhibited by capability of Theorem 2.29 and the nonsingular condition as Corollaries 2.30 and 2.32 and the diagram in page 49, respectively.

The pattern of additive maps preserving rank-1 matrices are exposed as Theorem 3.7 under the surjectivity condition in Chapter III which is another of the major goals in this dissertation.

We are to summarize on our work in the final chapter.

Now, we quote some theorems about elementary properties of ranks of matrices.

Theorem 1.5. [12] *If P and Q are $m \times m$ and $n \times n$ nonsingular matrices and if A is an $m \times n$ matrix, then $\rho(PA) = \rho(A)$ and $\rho(AQ) = \rho(A)$.*

Theorem 1.6. [11] *The rank of a matrix A is the largest integer r such that A has an $r \times r$ submatrix B with* $\det B \neq 0$.

Theorem 1.7. [8] *For a positive integer $k \leq n$, if X is a k-dimensional rank-1 subspace of $M_n(\mathbb{F})$, then*

(i) $X = xM$ for some $0 \neq x \in M_{n1}(\mathbb{F})$ and k-dimensional subspace M of $M_{1n}(\mathbb{F})$, or

(ii) $X = Ny$ for some $0 \neq y \in M_{1n}(\mathbb{F})$ and k-dimensional subspace N of $M_{n1}(\mathbb{F})$.

1.2 Preliminaries

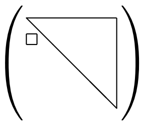
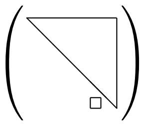
This section begins with some definitions and notation which are used through out in this dissertation. Then basic properties of upper Hessenberg matrices are given.

Recall that an $n \times n$ square matrix $A = (a_{ij})$ over a field $\mathbb{F}$ is called an *upper Hessenberg* if $a_{ij} = 0$ whenever $j + 1 < i$. Set

$$H_n^1(\mathbb{F}) = \Big\{(a_{ij}) \in H_n(\mathbb{F}) \,|\, a_{21} \neq 0 \text{ and } a_{j+1,j} = 0 \text{ for all } j \in \{2, \ldots, n-1\}\Big\} \text{ and}$$
$$H_n^2(\mathbb{F}) = \Big\{(a_{ij}) \in H_n(\mathbb{F}) \,|\, a_{n,n-1} \neq 0 \text{ and } a_{j+1,j} = 0 \text{ for all } j \in \{1, \ldots, n-2\}\Big\}.$$

Let us have a closed look at $H_n^1(\mathbb{F})$ and $H_n^2(\mathbb{F})$. Their elements are shown in the following pictures,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix} \in H_n^1(\mathbb{F}) \quad \text{and} \quad \begin{pmatrix} a_{11} & \cdots & a_{1,n-1} & a_{1n} \\ & \ddots & \vdots & \vdots \\ & & a_{n-1,n-1} & a_{n-1,n} \\ & & a_{n,n-1} & a_{nn} \end{pmatrix} \in H_n^2(\mathbb{F})$$

where $a_{ij} \in \mathbb{F}$ and the empty entries of each matrix are all zeros. Note that each type is quite similar to upper triangular matrices except elements in $H_n^1(\mathbb{F})$ and in $H_n^2(\mathbb{F})$ have excess positions at a_{21} and $a_{n,n-1}$, respectively. Then we rewrite the elements of these sets in other pictures as  and , respectively.

For each $i, j \in \{1, \dots, n\}$, let $E_{i,j}$ be the elementary matrix over $\mathbb{F}$ having only one at the (i,j)-entry and zero for all other positions. Then $\big\{E_{ij} \mid 1 \leq i, j \leq n \text{ and } i \leq j+1\big\}$ forms a basis of $H_n(\mathbb{F})$, thus we write $A = \sum_{\substack{1 \leq i,j \leq n \\ i \leq j+1}} a_{ij} E_{i,j}$ or $A = \sum a_{ij} E_{i,j}$ for short for all $A = (a_{ij}) \in H_n(\mathbb{F})$.

Let $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_n\}$ be the standard bases of $M_{n1}(\mathbb{F})$ and $M_{1n}(\mathbb{F})$, respectively. Then for each i and j, the elementary matrix E_{ij} is the product of e_i and f_j.

Recall that a subspace V of any spaces of matrices is called a *rank*-1 *subspace* if each element in V is the zero matrix or has rank one. In addition, a map T on $H_n(\mathbb{F})$ is called a *rank*-1 *preserver* if $\rho(T(A)) = 1$ whenever $\rho(A) = 1$ for any $A \in H_n(\mathbb{F})$. Furthermore, a map T is called a *rank preserver* if T preserves all ranks.

Henceforth, let $\Omega = \big\{A \in H_n(\mathbb{F}) \mid \rho(A) = 1\big\}$, i.e., Ω is the set of all rank-1 Hessenberg matrices.

Generally, every matrix of rank one in $M_{mn}(\mathbb{F})$ can be written as the product of a column vector and a row vector. In this dissertation, this product is denoted by

$$\begin{array}{ll} xy^t & \text{when } x \in M_{m1}(\mathbb{F}) \text{ and } y \in M_{n1}(\mathbb{F}), \\ xy & \text{when } x \in M_{m1}(\mathbb{F}) \text{ and } y \in M_{1n}(\mathbb{F}). \end{array}$$

For our convenient, xy^t is rewritten by $x \otimes y$. In addition, if $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^t$ in $M_{n1}(\mathbb{F})$ and $y = \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}^t$ in $M_{n1}(\mathbb{F})$, then

$$x \otimes y = \begin{pmatrix} x_1y_1 & \cdots & x_1y_n \\ \vdots & & \vdots \\ x_ny_1 & \cdots & x_ny_n \end{pmatrix} = \begin{pmatrix} | & & | \\ xy_1 & \cdots & xy_n \\ | & & | \end{pmatrix} = \begin{pmatrix} - & x_1y^t & - \\ & \vdots & \\ - & x_ny^t & - \end{pmatrix}$$

where $A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \in M_{mn}(\mathbb{F})$ stands for the matrix whose the j-column is $v_j \in M_{m1}(\mathbb{F})$ for all $j \in \{1, \ldots, n\}$ and $A = \begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & v_m & - \end{pmatrix} \in M_{mn}(\mathbb{F})$, the matrix whose the i-row is $v_i \in M_{1n}(\mathbb{F})$ for all $i \in \{1, \ldots, n\}$. Moreover, let

$$\begin{aligned} x \otimes M_{n1}(\mathbb{F}) &= \{x \otimes y \,|\, y \in M_{n1}(\mathbb{F})\} && \text{where} \quad x \in M_{m1}(\mathbb{F}), \\ M_{m1}(\mathbb{F}) \otimes y &= \{x \otimes y \,|\, x \in M_{m1}(\mathbb{F})\} && \text{where} \quad y \in M_{n1}(\mathbb{F}), \\ xM_{1n}(\mathbb{F}) &= \{xy \,|\, y \in M_{1n}(\mathbb{F})\} && \text{where} \quad x \in M_{m1}(\mathbb{F}) \quad \text{and} \\ M_{m1}(\mathbb{F})y &= \{xy \,|\, x \in M_{m1}(\mathbb{F})\} && \text{where} \quad y \in M_{1n}(\mathbb{F}). \end{aligned}$$

These sets are also rank-1 subspaces of $M_{mn}(\mathbb{F})$.

The following notation is adopted from [8]. For $s \in \{1, \ldots, n\}$, let

$$U_s = \left\{ \begin{pmatrix} x_1 & \cdots & x_s & 0 & \cdots & 0 \end{pmatrix}^t \,|\, x_i \in \mathbb{F} \text{ for all } i \in \{1, \ldots, s\} \right\}, \quad \text{and}$$

$$V_s = \left\{ \begin{pmatrix} 0 & \cdots & 0 & x_s & \cdots & x_n \end{pmatrix} \,|\, x_i \in \mathbb{F} \text{ for all } i \in \{s, \ldots, n\} \right\}; \quad \text{moreover,}$$

$$xV_s = \{xv \,|\, v \in V_s\} \quad \text{for each} \quad x \in M_{n1}(\mathbb{F}) \quad \text{and}$$

$$U_sy = \{uy \,|\, u \in U_s\} \quad \text{for each} \quad y \in M_{1n}(\mathbb{F}).$$

Clearly, for each $s \in \{1, \ldots, n\}$, U_s and V_s are subspaces of $M_{n1}(\mathbb{F})$ and $M_{1n}(\mathbb{F})$, respectively; furthermore, $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n$ and $V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_1$. Especially, $U_n = M_{n1}(\mathbb{F})$ and $V_1 = M_{1n}(\mathbb{F})$.

From now on, we omit "upper" in both Hessenberg matrices and triangular matrices. The rest of this section is devoted to investigate some properties of Hessenberg matrices as follows.

The following proposition and Lemma 1 in [5] are quite identical except (iii) and (iv) which are considered on different spaces. For Lemma 1, the full matrix spaces are attended but in this proposition, the Hessenberg matrix spaces are considered.

Proposition 1.8. *Let $x, y, u, v \in M_{n1}(\mathbb{F})$. The followings hold.*

(i) $x \otimes y = 0$ *if and only if* $x = 0$ *or* $y = 0$.

(ii) If $x \otimes y \neq 0$, *then* $x \otimes y = u \otimes v$ *if and only if there exists* $\alpha \in \mathbb{F} \setminus \{0\}$ *such that* $u = \alpha x$ *and* $y = \alpha v$.

(iii) If $x \otimes y + u \otimes v \in \Omega$, *then* $\{x, u\}$ *or* $\{y, v\}$ *is linearly dependent.*

(iv) For $n \geq 2$, *if* $u \neq 0$ *and* $v \neq 0$, *then there exists* $w \in \Omega$ *such that* $w \notin u \otimes M_{n1}(\mathbb{F}) \cup M_{n1}(\mathbb{F}) \otimes v$.

Proof. Let $x = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^t, y = \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix}^t, u = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix}^t$ and $v = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}^t$.

(i) It is clear that if $x = 0$ or $y = 0$, then $x \otimes y = 0$. Now, assume that $x \neq 0$. Then there exists $j \in \{1, \ldots, n\}$ such that $x_j \neq 0$. Then $x_j y_1 = x_j y_2 = \cdots = x_j y_n = 0$, thus $y_i = 0$ for all $i \in \{1, \ldots, n\}$. Hence $y = 0$.

(ii) Assume that $x \otimes y \neq 0$. It is trivial for the converse. Suppose that $x \otimes y = u \otimes v$. Then $x, y, u, v \neq 0$ and thus $v_j \neq 0$ and $x_l \neq 0$ for some $j, l \in \{1, \ldots, n\}$. It follows that $xy_i = uv_i$ for all $i \in \{1, \ldots, n\}$ since $\begin{pmatrix} | & & | \\ xy_1 & \cdots & xy_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ uv_1 & \cdots & uv_n \\ | & & | \end{pmatrix}$. Thus $y_j \neq 0$ and $u = \alpha_j x$ where $\alpha_j = \frac{y_j}{v_j} \neq 0$. Suppose that there exists $k \in \{1, \ldots, n\}$ such that $k \neq j$ and $v_k \neq 0$. Then, similarly, there exists $\alpha_k = \frac{y_k}{v_k} \neq 0$ and $\alpha_k x = \frac{y_k}{v_k} x = u = \alpha_j x$. As a result, $\alpha_k x_l = \alpha_j x_l$ with $x_l \neq 0$, so $u = \alpha x$ for some $\alpha \in \mathbb{F} \setminus \{0\}$.

It remains to show that $y = \alpha v$. Now, we obtain that $x \otimes y = u \otimes v = \alpha x \otimes v = x \otimes \alpha v$, that is, $\begin{pmatrix} - & x_1 y & - \\ & \vdots & \\ - & x_n y & - \end{pmatrix} = \begin{pmatrix} - & x_1(\alpha v) & - \\ & \vdots & \\ - & x_n(\alpha v) & - \end{pmatrix}$. Then $x_l y = x_l(\alpha v)$, and thus $y = \alpha v$.

(iii) Assume that $x \otimes y + u \otimes v \in \Omega$ and $\{x, u\}$ is linearly independent. Since $\rho(x \otimes y + u \otimes v) = 1$, it follows that any two rows of the matrix $x \otimes y + u \otimes v$ are linearly dependent, i.e.,

$$\{xy_i + uv_i, xy_j + uv_j\} \text{ is linearly dependent for all } i \neq j \text{ in } \{1, \ldots, n\}. \qquad (3)$$

For each $i \neq j$, let $\alpha_i, \beta_j \in \mathbb{F}$ not both zeros, without loss of generality, put $\alpha_i \neq 0$, such that

$$0 = \alpha_i(xy_i + uv_i) + \beta_j(xy_j + uv_j) = x(\alpha_i y_i + \beta_j y_j) + u(\alpha_i v_i + \beta_j v_j),$$

so $\alpha_i y_i + \beta_j y_j = 0$ and $\alpha_i v_i + \beta_j v_j = 0$ and then $y_i = \frac{-\beta_j}{\alpha_i} y_j$ and $v_i = \frac{-\beta_j}{\alpha_i} v_j$.

If there exists $l \in \{1, \ldots, n\}$ such that $v_l = 0$, then $\frac{-\beta_k}{\alpha_l} v_k = 0$ for some $k \in \{1, \ldots, n\} \setminus \{l\}$ because of $v_l = \frac{-\beta_k}{\alpha_l} v_k$. It follows that $\beta_k = 0$ or $v_k = 0$. If $\beta_k = 0$, then $y_l = 0$ because $\alpha_l \neq 0$. This shows that

$$\text{if } v_i = 0 \text{ for some } i, \text{then } y_i = 0 \text{ or } v_j = 0 \text{ for some } j \in \{1, \ldots, n\} \setminus \{i\}. \qquad (4)$$

We show that $\{y, v\}$ is linearly dependent. If $y = 0$ or $v = 0$, then $\{y, v\}$ is linearly dependent. Assume that $y \neq 0$ and $v \neq 0$.

Case 1: Assume that $y_i \neq 0$ for all $i \in \{1, \ldots, n\}$. Suppose that there exists $j \in \{1, \ldots, n\}$ such that $v_j = 0$. Then there exists $l \in \{1, \ldots, n\}$ with $l \neq j$ such that $v_l = 0$ by (4). Continue this process, we get $v = 0$ which is impossible. Hence $v_j \neq 0$ for all $j \in \{1, \ldots, n\}$. Thus $y = \begin{pmatrix} y_1 & \alpha_2 y_1 & \cdots & \alpha_n y_1 \end{pmatrix}^t$ and $v = \begin{pmatrix} v_1 & \alpha_2 v_1 & \cdots & \alpha_n v_1 \end{pmatrix}^t$. Then $y = \lambda v$ where $\lambda = \frac{y_1}{v_1} \neq 0$.

Case 2: There exists $i \in \{1, \ldots, n\}$ such that $y_i = 0$. Suppose that $v_j \neq 0$ for all $j \in \{1, \ldots, n\}$. Then there exists $l \in \{1, \ldots, n\}$ with $l \neq i$ such that $y_l = 0$ by (4). In the similar way, we obtain $y = 0$ which is impossible. Hence there exists $j \in \{1, \ldots, n\}$ such that $v_j = 0$. Since $y \neq 0$ and $v \neq 0$, there exist

$l, k \in \{1, \ldots, n\}$ such that $y_l \neq 0$ and $v_k \neq 0$. Without loss of generality, write

$$\begin{array}{rccccc}
 & i^{\text{th}} & j^{\text{th}} & k^{\text{th}} & l^{\text{th}} & \\
y = \Big(& 0 & y_j & y_k & y_{l \neq 0} & \Big)^t \\
v = \Big(& v_i & 0 & v_{k \neq 0} & v_l & \Big)^t
\end{array}$$

where other entries of y and v are arbitrary. In order to show that $\{v, y\}$ is linearly dependent, it suffices to prove that each position of v and y are related in the sense that for each $s \in \{1, \ldots, n\}$, the s-position of v is zero if and only if the s-position of y is zero.

First, by applying (3) on the j- and k-columns of y and v, there exist $\alpha_j, \beta_k \in \mathbb{F}$ not both zeros such that

$$0 = \alpha_j(xy_j + uv_j) + \beta_k(xy_k + uv_k) = x(\alpha_j y_j + \beta_k y_k) + u(\alpha_j v_j + \beta_k v_k).$$

Since $\{u, x\}$ is linearly independent, $v_j = 0$ and $v_k \neq 0$, it follows that $\beta_k = 0$ and hence $\alpha_j(xy_j) = 0$. Thus $y_j = 0$ because $\alpha_j \neq 0$ and x is not a zero vector. Besides, we know that $y_k \neq 0$ by considering (3) on the k- and l-columns of y and v, and the fact that $\{u, x\}$ is linearly independent, $y_l \neq 0$ and $v_k \neq 0$. Next, for $g \in \{1, \ldots, n\} \smallsetminus \{j, k\}$, we get that if $v_g = 0$, then $y_g = 0$ and if $v_g \neq 0$, then $y_g \neq 0$ by using the same technique on the g- and k-columns of y and v.

Hence for $s \in \{1, \ldots, n\}$, if $v_s = 0$, then $y_s = 0$; moreover, if $v_s \neq 0$, then $y_s \neq 0$. As a result, we obtain that for each $s \in \{1, \ldots, n\}$, the s-position of v is zero if and only if the s-position of y is zero. Finally, we consider only the positions of y which are not zero. Assume that there exists the p-position with $p \neq k$ such that $y_p \neq 0$. Then by making use of (3) on the p- and k-columns of y and v in the similar way, we obtain that $y_p = \frac{-\beta_k}{\alpha_p} y_k$ and $v_p = \frac{-\beta_k}{\alpha_p} v_k$ where $\alpha_p, \beta_k \in \mathbb{F} \smallsetminus \{0\}$. Similarly, each position of y and v which is not zero can be written as the product of a nonzero scalar and y_k and that of the same scalar and v_k, respectively. Hence $\{v, y\}$ is linearly dependent.

(iv) Assume that $n \geq 2$, $u \neq 0$ and $v \neq 0$. Then there exist $i, j \in \{1, \ldots, n\}$ such that $u_i \neq 0$ and $v_j \neq 0$.

Case 1: $i = 1$. Choose $w = E_{2l}$ where $l \neq j$. Then $w \in \Omega$. If $w \in u \otimes M_{n1}(\mathbb{F})$, then $w = u \otimes z$ where $z = \begin{pmatrix} z_1 & \cdots & z_n \end{pmatrix}^t$. From the first row of w, we get $z_k = 0$ for all $k \in \{1, \ldots, n\}$ which contradicts $u_2 z_l = 1$. Thus $w \notin u \otimes M_{n1}(\mathbb{F})$. Similarly, $w \notin M_{n1}(\mathbb{F}) \otimes v$. Hence $w \notin u \otimes M_{n1}(\mathbb{F}) \cup M_{n1}(\mathbb{F}) \otimes v$.

Case 2: $i \neq 1$. Choose $w = E_{1l}$ where $l \neq j$. Then $w \notin u \otimes M_{n1}(\mathbb{F}) \cup M_{n1}(\mathbb{F}) \otimes v$ by using the same manner. □

The following proposition demonstrates the form of every rank-1 subspace of $H_n(\mathbb{F})$.

Proposition 1.9. *Let k be a positive integer less than or equal to n and X a subset of $H_n(\mathbb{F})$. Then X is a k-dimensional rank-1 subspace if and only if there exist integers $s, t \in \{1, \ldots, n\}$ with $s \leq t + 1$ such that*

(i) there exist $0 \neq x \in U_s$ and a k-dimensional subspace M of V_t such that $X = xM$, or

(ii) there exist $0 \neq y \in V_t$ and a k-dimensional subspace N of U_s such that $X = Ny$.

Proof. The sufficient part is obvious. We now prove the necessary part. Assume that X is a k-dimensional rank-1 subspace of $H_n(\mathbb{F})$. Then it is a subspace of $M_n(\mathbb{F})$ and thus, by applying Theorem 1.7,

(1) $X = xM$ for some $0 \neq x \in M_{n1}(\mathbb{F})$ and k-dimensional subspace M of $M_{1n}(\mathbb{F})$, or

(2) $X = Ny$ for some $0 \neq y \in M_{1n}(\mathbb{F})$ and k-dimensional subspace N of $M_{n1}(\mathbb{F})$.

Consider (1). Let t be the largest positive integer such that $M \subseteq V_t$. Then there exists $v \in M$ such that its $(n - t + 1)$-component is nonzero. Since $xv \in xM = X \subseteq H_n(\mathbb{F})$ and from the definition of Hessenberg matrices, $x \in U_s$ for some $s \leq t + 1$.

Likewise, (2) can be done. □

We would like to point here out that the proof of Proposition 1.9 and that of Lemma 2.1 in [8] are similar although the spaces are different.

In general, for each $m \times n$ matrix A of rank $r \neq 0$, there exist nonsingular matrices P and Q in $M_m(\mathbb{F})$ and $M_n(\mathbb{F})$, respectively, such that $PAQ = I_r$ as Theorem 6.12 [21]. The analogous property in the sense of Hessenberg matrices is given.

Proposition 1.10. *If $A \in H_n(\mathbb{F})$ of rank $r \neq 0$, then there exist nonsingular matrices $P, Q \in T_n(\mathbb{F})$ such that $PAQ = \sum_{i=1}^{r} E_{s_i t_i}$ where $s_i, t_i \in \{1, \dots, n\}$ with $s_i \leq t_i + 1$ for all i and $s_i \neq s_j$, $t_i \neq t_j$ for all $i \neq j$.*

Proof. Let $A = (a_{ij})$ be a Hessenberg matrix of rank $r \neq 0$. Given $R_1, \dots, R_n$ and $C_1, \dots, C_n$ are the row vectors and column vectors of A, respectively. Let R_s be the first nonzero row vector from the last row of A and let a_{st} be the first nonzero entry from the left of R_s. Multiply R_s by a_{st}^{-1} and then for each $1 \leq i < s$, apply the row operation $R_i - a_{it}R_s \to R_i$, adding $-a_{it}$ times the s-row to the i-row.

Next, for each $t < j \leq n$, we apply the column operation $C_j - \frac{a_{sj}}{a_{st}} C_t \to C_j$, adding $-\frac{a_{sj}}{a_{st}}$ times the t-column to the j-column. Let X and Y be the product of matrices obtained by these row operations and the product of matrices obtained by these column operations, respectively. Then X and Y are

$$
\begin{array}{cc}
 & \begin{array}{c} s^{\text{th}} \\ \downarrow \end{array} \\
s^{\text{th}}\rightarrow & \begin{pmatrix}
1 & & & -a_{1t}a_{st}^{-1} & & & \\
 & \ddots & & \vdots & & & \\
 & & 1 & -a_{s-1,t}a_{st}^{-1} & & & \\
 & & & a_{st}^{-1} & & & \\
 & & & & 1 & & \\
 & & & & & \ddots & \\
 & & & & & & 1
\end{pmatrix}
\end{array},
\qquad
\begin{array}{cc}
 & \begin{array}{c} t^{\text{th}} \\ \downarrow \end{array} \\
t^{\text{th}}\rightarrow & \begin{pmatrix}
1 & & & & & \\
 & \ddots & & & & \\
 & & 1 & -a_{s,t+1}a_{st}^{-1} & \cdots & -a_{sn}a_{st}^{-1} \\
 & & & 1 & & \\
 & & & & \ddots & \\
 & & & & & 1
\end{pmatrix}
\end{array},
$$

respectively, which are nonsingular triangular matrices such that $XAY = E_{st} + B$ with $B = \begin{pmatrix} U & V \\ & \end{pmatrix}$ where $U \in H_{s-1}(\mathbb{F})$ and $V \in M_{s-1,n-s+1}(\mathbb{F})$. By using the same argument with B, we get $X_2 B Y_2 = E_{s_2 t_2} + B_2$ where

$$X_2 = \begin{array}{r@{}c} & s_2^{\text{th}}\downarrow \\ s_2^{\text{th}}\rightarrow & \begin{pmatrix} 1 & & & -a_{1,t_2}a^{-1}_{s_2,t_2} & & & \\ & \ddots & & \vdots & & & \\ & & 1 & -a_{s_2-1,t_2}a^{-1}_{s_2,t_2} & & & \\ & & & a^{-1}_{s_2,t_2} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

$$\text{and } Y_2 = \begin{array}{r@{}c} & t_2^{\text{th}}\downarrow \\ t_2^{\text{th}}\rightarrow & \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & -a_{s_2,t_2+1}a^{-1}_{s_2,t_2} & \cdots & -a_{s_2,n}a^{-1}_{s_2,t_2} \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \end{array}, \ t_2 \neq t \text{ and } s_2 < s.$$

Furthermore, $X_2E_{st}Y_2 = \big(X_2e_s\big)\big(f_tY_2\big)$ which is the product of the s-column of X_2 and the t-row of Y_2, hence it is the product of e_s and f_t, which is E_{st}. This shows that

$$X_2E_{st}Y_2 = E_{st} \tag{1}$$

Continue the same process and thus the number of these methods is r times since $\rho(A) = r$. Let $P = X_r \cdots X_2X$ and $Q = YY_2 \cdots Y_r$. Then P and Q are nonsingular triangular matrices; moreover, in the same way with (1),

$$X_iE_{st}Y_i = E_{st} \quad \text{for all} \quad 3 \leq i \leq r,$$

and for $j \in \{2, \ldots, r\}$, we get

$$X_lE_{s_jt_j}Y_l = E_{s_jt_j} \quad \text{for all} \quad j+1 \leq l \leq r. \tag{2}$$

Then for $s_i, t_i \in \{1, \ldots, n\}$ with $s_i \leq t_i + 1$ for all i and $s_i \neq s_j$, $t_i \neq t_j$ when $i \neq j$,

$$PAQ = X_r \cdots X_2XAYY_2 \cdots Y_r$$

$$
\begin{aligned}
PAQ &= X_r \cdots X_2\big(E_{st} + B\big)Y_2 \cdots Y_r \\
&= X_r \cdots X_3\big(X_2E_{st}Y_2 + X_2BY_2\big)Y_3 \cdots Y_r \\
&= X_r \cdots X_3\big(E_{st} + X_2BY_2\big)Y_3 \cdots Y_r && \text{from (1)} \\
&= X_r \cdots X_3\big(E_{st} + E_{s_2t_2} + B_2\big)Y_3 \cdots Y_r \\
&= X_r \cdots X_4\big(X_3E_{st}Y_3 + X_3E_{s_2t_2}Y_3 + X_3B_2Y_3\big)Y_4 \cdots Y_r \\
&= X_r \cdots X_4\big(E_{st} + E_{s_2t_2} + X_3B_2Y_3\big)Y_4 \cdots Y_r && \text{from (2)} \\
&\vdots \\
&= \sum_{i=1}^{r} E_{s_it_i} && \text{where } E_{s_1t_1} = E_{st}.
\end{aligned}
$$

□

As a consequence, any Hessenberg matrix of rank one can be written as a product of $PE_{st}Q$ where P and Q are nonsingular triangular matrices and also are Hessenberg matrices. Thus, it is, in fact, the product of a column vector and a row vector, that is $A \in \Omega$ only if $A = xy$ for some nonzero vectors $x \in M_{n1}(\mathbb{F})$ and $y \in M_{1n}(\mathbb{F})$. Nevertheless, the if part holds if certain conditions are given.

Corollary 1.11. *Let $x \in M_{n1}(\mathbb{F})$ and $y \in M_{1n}(\mathbb{F})$. The matrix $xy \in \Omega$ if and only if there exist $l \in \{1, \ldots, n+1\}$ such that $x \in U_l$ and $y \in V_{l-1}$ where $V_0 = V_1$ and $U_{n+1} = U_n$.*

Proof. The converse is clear. Assume that $xy \in \Omega$. Then by Proposition 1.9, there exist integers $s, t \in \{1, \ldots, n\}$ with $s \leq t + 1$ such that $xy = ab$ for some $a \in U_s \smallsetminus \{0\}$ and $b \in V_t \smallsetminus \{0\}$. By Proposition 1.8 (ii), there exists $\alpha \in \mathbb{F} \smallsetminus \{0\}$ such that $x = \alpha a$ and $b = \alpha y$. Thus $x \in U_s$ and $y \in V_t$. Hence in case $s \in \{1, \ldots, n\}$, we obtain that $x \in U_s$ and $y \in V_{s-1}$ where $V_0 = V_1$ because of $s - 1 \leq t$ and $V_n \subseteq \cdots \subseteq V_1$. In case $t \in \{1, \ldots, n\}$, it follows that $y \in V_t$ and $x \in U_{t+1}$ where $U_{n+1} = U_n$ owing to $s \leq t + 1$ and $U_1 \subseteq \cdots \subseteq U_n$. Consequently, there exists $l \in \{1, \ldots, n+1\}$ such that $x \in U_l$ and $y \in V_{l-1}$. □

The following proposition guides us to observe forms of n-dimensional rank-1 subspaces of $H_n(\mathbb{F})$.

Proposition 1.12. *Every n-dimensional rank-1 subspace of $H_n(\mathbb{F})$ is one of the forms e_1V_1, U_nf_n, $(\alpha e_1+e_2)V_1$ or $U_n(f_{n-1}+\alpha f_n)$ for some $\alpha\in\mathbb{F}$.*

Proof. Let X be an n-dimensional rank-1 subspace of $H_n(\mathbb{F})$. Then by Proposition 1.9, there exist integers $s,t\in\{1,\ldots,n\}$ with $s\le t+1$ such that

(1) there exist $0\neq x\in U_s$ and an n-dimensional subspace M of V_t such that $X=xM$, or

(2) there exist $0\neq y\in V_t$ and an n-dimensional subspace N of U_s such that $X=Ny$.

For (1), it is well-known that M must be $M_{1n}(\mathbb{F})$ and then t is equal to one. Now, $1\le s\le 2$ so that $X=xV_1$ for some nonzero vector $x\in U_1$ or $X=xV_1$ for some nonzero vector $x\in U_2$.

Case 1: $X=xV_1$ for some nonzero vector $x\in U_1$. Then $x=\gamma e_1$ for some $\gamma\in\mathbb{F}\setminus\{0\}$, and thus $X=xV_1=\gamma e_1V_1=e_1V_1$.

Case 2: $X=xV_1$ for some nonzero vector $x\in U_2$. Then we can rewrite x as the form $\begin{pmatrix}x_1\\x_2\\0\\\vdots\\0\end{pmatrix}$. If $x_2=0$, then $x\in U_1$. Similar to Case 1, $X=e_1V_1$. Assume that $x_2\neq 0$. Then $x\in U_2\setminus U_1$. We get that $x=x_2(\frac{x_1}{x_2}e_1+e_2)=x_2(\alpha e_1+e_2)$ where $\alpha=\frac{x_1}{x_2}$. Thus $X=xV_1=x_2(\alpha e_1+e_2)V_1=(\alpha e_1+e_2)V_1$.

For (2), by using the same argument, we obtain that $X=U_ny$ for some nonzero vector $y\in V_n$ or $X=U_ny$ for some nonzero vector $y\in V_{n-1}$. And thus $X=U_nf_n$ or $X=U_n(f_{n-1}+\alpha f_n)$ for some $\alpha\in\mathbb{F}$, respectively. $\square$

From now on, for each $\alpha\in\mathbb{F}$, let

$$X=e_1V_1,\ Y=U_nf_n,\ X_\alpha=(\alpha e_1+e_2)V_1\ \text{ and }\ Y_\alpha=U_n(f_{n-1}+\alpha f_n).$$

Note that there are infinitely many spaces of each type X_α and Y_α.

Carefully, $(\alpha e_1+e_2)V_1$ and $(e_1+\alpha e_2)V_1$ are not similar. The space $(\alpha e_1+e_2)V_1$ has a connotation that the second row never can be the zero vector but the behavior of the first row depends on α. In a similar way, $(e_1+\alpha e_2)V_1$ informs that the first row never be zero but the second row is not the case.

Furthermore, the four patterns can be revealed as the following forms:

$$X = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ & & \\ & & \end{pmatrix} \middle| \, x_{1j} \in \mathbb{F} \text{ for all } j \in \{1, \dots, n\} \right\},$$

$$X_\alpha = \left\{ \begin{pmatrix} \alpha x_{21} & \cdots & \alpha x_{2n} \\ x_{21} & \cdots & x_{2n} \\ & & \\ & & \end{pmatrix} \middle| \, x_{2j} \in \mathbb{F} \text{ for all } j \in \{1, \dots, n\} \right\},$$

$$Y = \left\{ \begin{pmatrix} & & y_{1n} \\ & & \vdots \\ & & y_{nn} \end{pmatrix} \middle| \, y_{in} \in \mathbb{F} \text{ for all } i \in \{1, \dots, n\} \right\} \text{ and}$$

$$Y_\alpha = \left\{ \begin{pmatrix} & & y_{1,n-1} & \alpha y_{1,n-1} \\ & & \vdots & \vdots \\ & & y_{n,n-1} & \alpha y_{n,n-1} \end{pmatrix} \middle| \, y_{i,n-1} \in \mathbb{F} \text{ for all } i \in \{1, \dots, n\} \right\}.$$

However, we frequently write the above notation in these forms:

$$X = \left\{ \begin{pmatrix} - & x & - \\ & & \\ & & \end{pmatrix} \middle| \, x \in M_{1n}(\mathbb{F}) \right\}, \qquad X_\alpha = \left\{ \begin{pmatrix} - & \alpha x & - \\ - & x & - \\ & & \end{pmatrix} \middle| \, x \in M_{1n}(\mathbb{F}) \right\},$$

$$Y = \left\{ \begin{pmatrix} & & | \\ & & y \\ & & | \end{pmatrix} \middle| \, y \in M_{n1}(\mathbb{F}) \right\}, \text{ and} \quad Y_\alpha = \left\{ \begin{pmatrix} & & | & | \\ & & y & \alpha y \\ & & | & | \end{pmatrix} \middle| \, y \in M_{n1}(\mathbb{F}) \right\}.$$

In fact, $Y = X^\sim$ and $Y_\alpha = X_\alpha^\sim$ for any $\alpha \in \mathbb{F}$.

For each $\alpha \in \mathbb{F}$, we choose the bases of X, Y, X_α and Y_α, respectively, as follows: $\{E_{11}, E_{12}, \dots, E_{1n}\}$, $\{E_{1n}, E_{2n}, \dots, E_{nn}\}$, $\{\alpha E_{11}+E_{21}, \alpha E_{12}+E_{22}, \dots, \alpha E_{1n}+E_{2n}\}$ and $\{E_{1,n-1}+\alpha E_{1n}, E_{2,n-1}+\alpha E_{2n}, \dots, E_{n,n-1}+\alpha E_{nn}\}$.

Actually, in matrix theory, the product of upper triangular matrices is also upper triangular. Nevertheless, the product of upper Hessenberg matrices may no longer be upper Hessenberg. Then a condition forcing this property done is given.

Proposition 1.13. *For $n \geq 3$, let $A = (a_{ij})$ and $B = (b_{ij}) \in H_n(\mathbb{F})$. Then $AB \in H_n(\mathbb{F})$ if and only if $a_{i,i-1}b_{i-1,i-2} = 0$ for all $i \in \{3, 4, \ldots, n\}$.*

Equivalently, $AB \in H_n(\mathbb{F})$ if and only if $a_{i,i-1} = 0$ or $b_{i-1,i-2} = 0$ for all $i \in \{3, 4, \ldots, n\}$.

Proof. Assume that $AB = (c_{ij}) \in H_n(\mathbb{F})$. Let $j \in \{1, \ldots, n-2\}$. Then $c_{ij} = 0$ for all $i \in \{1, \ldots, n\}$ with $i > j+1$. However, for each $i \in \{1, \ldots, n\}$ with $i > j+1$,

$$c_{ij} = \sum_{s=1}^{n} a_{is}b_{sj} = \sum_{s=1}^{i-2} a_{is}b_{sj} + \sum_{s=i-1}^{n} a_{is}b_{sj} = \sum_{s=i-1}^{n} a_{is}b_{sj}$$

because $a_{is} = 0$ for all $s \in \{1, 2, \ldots, i-2\}$. Thus $0 = c_{ij} = \sum_{s=i-1}^{n} a_{is}b_{sj}$ for all $i > j+1$. In particular, put $i = j+2$. Then $i - 1 = j+1$ and then

$$\begin{aligned}
0 &= c_{j+2,j} \\
&= \sum_{s=j+1}^{n} a_{j+2,s}b_{sj} \\
&= a_{j+2,j+1}b_{j+1,j} \qquad \text{since } b_{sj} = 0 \text{ for all } s \in \{j+2, \ldots, n\} \\
&= a_{i,i-1}b_{i-1,i-2}.
\end{aligned}$$

Hence $a_{i,i-1}b_{i-1,i-2} = 0$ for all $i \in \{3, 4, \ldots, n\}$.

To prove the converse, assume that $a_{i,i-1}b_{i-1,i-2} = 0$ for all $i \in \{3, 4, \ldots, n\}$ and let $AB = (c_{ij})$. Fix $j \in \{1, \ldots, n-2\}$. Since $A, B \in H_n(\mathbb{F})$, we know that $a_{ij} = 0$ and $b_{ij} = 0$ for all $i > j+1$. Then $c_{ij} = 0$ for all $i \geq j+3$. It remains to show that $c_{ij} = 0$ when $i = j+2$. Let $i = j+2$. Then

$$\begin{aligned}
c_{ij} = c_{j+2,j} &= \sum_{s=1}^{n} a_{j+2,s}b_{sj} \\
&= \sum_{s=j+1}^{n} a_{j+2,s}b_{sj} \qquad \text{since } a_{j+2,s} = 0 \text{ for all } s \in \{1, 2, \ldots, j\} \\
&= a_{j+2,j+1}b_{j+1,j} \qquad \text{since } b_{sj} = 0 \text{ for all } s \in \{j+2, \ldots, n\} \\
&= 0 \qquad \text{since } a_{i,i-1}b_{i-1,i-2} = 0 \text{ for all } i \in \{3, 4, \ldots, n\}.
\end{aligned}$$

Thus $c_{ij} = 0$ for all $i > j+1$ and then $AB \in H_n(\mathbb{F})$. □

Proposition 1.14. *Let* $C = (c_{lk}) \in \Omega$. *If there exist* $0 \neq \alpha \in \mathbb{F}$ *and* $i, j \in \{1, \ldots, n\}$ *with* $i \leq j + 1$ *such that* $C + \alpha E_{ij} \in \Omega$, *then*

$$C = \begin{pmatrix} & c_{1j} & \\ & \vdots & \\ & c_{j+1,j} & \\ & 0 & \\ & \vdots & \\ & 0 & \end{pmatrix}_{n\times n} \quad \textit{or} \quad C = \begin{pmatrix} & & & & & \\ 0 & \cdots & 0 & c_{i,i-1} & \cdots & c_{i,n} \\ & & & & & \end{pmatrix}_{n\times n}.$$

Proof. Assume that there exist $0 \neq \alpha \in \mathbb{F}$ and $i, j \in \{1, \ldots, n\}$ with $i \leq j+1$ such that $C+\alpha E_{ij}$ has rank one. Since $\rho(C) = 1$, by Proposition 1.9, there exist integers $s, t \in \{1, \ldots, n\}$ with $s \leq t+1$ such that $C = \begin{pmatrix} x_1 \\ \vdots \\ x_s \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & y_t & \cdots & y_n \end{pmatrix} =: u_s v_t$.

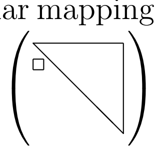
Now we can see C as $\begin{pmatrix} & x_1 y_t & \cdots & x_1 y_n \\ & \vdots & & \vdots \\ & x_s y_t & \cdots & x_s y_n \\ & & & \\ & & & \end{pmatrix}$.

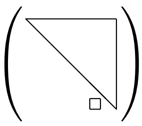
Moreover, $\alpha E_{ij} = \alpha e_i f_j$. Since $1 = \rho(C + \alpha E_{ij}) = \rho(u_s v_t + \alpha e_i f_j)$, we get that $\{u_s, \alpha e_i\}$ or $\{v_t, f_j\}$ is linearly dependent by (iii) of Proposition 1.8.

Case 1: Assume that $\{u_s, \alpha e_i\}$ is linearly dependent. Then there exist $\beta, \gamma \in \mathbb{F}$ not both zeros such that $\beta u_s + \gamma \alpha e_i = 0$. If $i > s$, then $\gamma\alpha = 0$ and then $\gamma = 0$, hence $\beta = 0$ which is impossible. Thus $i \leq s$. It follows that $\beta x_l = 0$ for all $l \neq i$. If $\beta = 0$, then $\gamma\alpha = 0$ which is impossible because $\alpha \neq 0$; moreover, β and γ cannot be zero simultaneously. Hence $x_l = 0$ for all $l \neq i$. Thus,

we rewrite C as
$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & y_t & \cdots & y_n \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & x_i y_t & \cdots & x_i y_n \end{pmatrix},$$
when $i \leq s \leq t+1$.

Case 2: Assume that $\{v_t, f_j\}$ is linearly dependent. By using the same manner, we get C as the form
$$\begin{pmatrix} c_{1j} \\ \vdots \\ c_{j+1,j} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$
$\square$

Finally, in this chapter, we take a closed look at a result of a particular mapping on $H_n(\mathbb{F})$. Recall that elements of $H_n^1(\mathbb{F})$ and $H_n^2(\mathbb{F})$ are of the forms 

and , respectively.

Proposition 1.15. *Let $A, B \in H_n(\mathbb{F})$ be nonsingular and $\varphi : H_n(\mathbb{F}) \to M_n(\mathbb{F})$ the map defined by $\varphi(X) = AXB$. Then $\operatorname{im}\varphi \subseteq H_n(\mathbb{F})$ if and only if $A \in H_n^1(\mathbb{F}) \cup T_n(\mathbb{F})$ and $B \in H_n^2(\mathbb{F}) \cup T_n(\mathbb{F})$.*

Proof. The sufficiency clearly holds. We prove the necessity. Assume that $\operatorname{im}\varphi \subseteq H_n(\mathbb{F})$. First, we show that $B \in H_n^2(\mathbb{F}) \cup T_n(\mathbb{F})$. Let $y \in V_t$ for some $1 \leq t \leq n$. If $t = 1$, then it is clear that $yB \in M_{1n}(\mathbb{F}) = V_1$. If $t = n$, choose $x = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^t$, then $x \in U_n$ and hence $xy \in H_n(\mathbb{F})$ by Corollary 1.11. From the fact that $Ax \in M_{n1}(\mathbb{F}) = U_n$ and $(Ax)(yB) = A(xy)B \in H_n(\mathbb{F})$ with $\rho\big((Ax)(yB)\big) = 1$, we get that $yB \in V_{n-1}$ by applying Corollary 1.11 again. This shows that if $y \in V_n$, then $yB \in V_{n-1}$.

Fix $t \in \{2, \ldots, n-1\}$. Then $xy \in H_n(\mathbb{F})$ for every $x \in U_{t+1}$ by Corollary 1.11. Since $(Ax)(yB) = A(xy)B \in H_n(\mathbb{F})$ and A is nonsingular, we conclude that

(i) the spaces $\{Ax \mid x \in U_{t+1}\}$ and U_{t+1} have the same dimensions which equals $t+1$ because $\{e_1, \ldots, e_{t+1}\}$ is a basis of U_{t+1} and $\{Ae_1, \ldots, Ae_{t+1}\}$ is a basis of $\{Ax \mid x \in U_{t+1}\}$, and

(ii) $yB \in V_t$ by the following reason.

Let $yB = \begin{pmatrix} u_1 & \cdots & u_t & \cdots & u_n \end{pmatrix}$. Suppose that $yB \notin V_t$. Then there exists $k < t$ such that $u_k \neq 0$. Write $y = \begin{pmatrix} 0 & \cdots & 0 & y_t & \cdots & y_n \end{pmatrix}$ and $B = (b_{ij})_{n \times n}$. Thus

$$\begin{aligned} u_k &= \sum_{l=t}^{n} y_l b_{lk} \\ &= \sum_{l=t}^{k+1} y_l b_{lk} && \text{since } b_{lk} = 0 \text{ when } l > k+1 \\ &= y_{k+1} b_{k+1,k} && \text{since } k+1 \leq t \text{ and } t \leq k+1. \end{aligned}$$

Now, k is the largest positive integer such that $yB \in V_k$ since $V_n \subseteq \cdots \subseteq V_2 \subseteq V_1$. Then $Ax \in U_{k+1}$ for all $1 \leq k \leq n-2$ owing to Corollary 1.11 and the fact that $t = k+1$. It follows that $Ax \in U_t$ for $2 \leq t \leq n-1$. This shows that if $x \in U_{t+1}$, then $Ax \in U_t$ for all $2 \leq t \leq n-1$. Hence the space $\{Ax \mid x \in U_{t+1}\}$ contains in U_t of which dimension equals t. Thus the dimension of this space is at most t contradicting (i). As a result, if $y \in V_t$, then $yB \in V_t$ for all $2 \leq t \leq n-1$.

From the above proof, we conclude that

$$\text{if} \quad y \in V_t, \text{ then} \quad yB \in \begin{cases} V_1, & \text{when } \ t = 1; \\ V_t, & \text{when } \ 1 < t < n; \\ V_{n-1}, & \text{when } \ t = n. \end{cases} \tag{3}$$

In order to show that $B = (b_{ij}) \in H_n^2(\mathbb{F}) \cup T_n(\mathbb{F})$, from (3), we use only the fact that if $y \in V_t$, then $yB \in V_t$ for all $2 \leq t \leq n-1$. Let $l \in \{2, \ldots, n-1\}$ and $y = \begin{pmatrix} 0 & \cdots & 0 & y_l & \cdots & y_n \end{pmatrix} \in V_l$ such that $y_l \neq 0$. Then

$$yB = \begin{pmatrix} 0 & \cdots & 0 & \overset{(l-1)^{\text{th}}}{\overset{\downarrow}{y_l b_{l,l-1}}} & \cdots & \overset{n^{\text{th}}}{\overset{\downarrow}{\sum_{j=l}^{n} y_j b_{jn}}} \end{pmatrix} \in V_l.$$ Thereby, $y_l b_{l,l-1} = 0$ and then $b_{l,l-1} = 0$. It follows that $b_{l,l-1} = 0$ for all $2 \leq l \leq n-1$. For this reason, we

obtain $B \in H_n^2(\mathbb{F}) \cup T_n(\mathbb{F})$.

Next, we show that $A \in H_n^1(\mathbb{F}) \cup T_n(\mathbb{F})$. By using the same manner, we obtain that

$$\text{if} \quad x \in U_{t+1}, \text{ then } \quad Ax \in \begin{cases} U_{t+1}, & \text{when } 1 \leq t \leq n-2; \\ U_n, & \text{when } t = n-1. \end{cases}$$

However, proving that $A \in H_n^1(\mathbb{F}) \cup T_n(\mathbb{F})$ is enough to use the fact that if $x \in U_{t+1}$, then $Ax \in U_{t+1}$ for all $1 \leq t \leq n-2$. We rewrite this as if $x \in U_s$, then $Ax \in U_s$ for all $2 \leq s \leq n-1$.

Let $k \in \{2, \ldots, n-1\}$ and $y = \begin{pmatrix} x_1 & \cdots & x_k & 0 & \cdots & 0 \end{pmatrix}^t \in U_k$ such that $x_k \neq 0$. Write $A = (a_{ij})_{n \times n}$, then

$$Ax = \begin{array}{r} 1^{\text{st}}\rightarrow \\ \\ k+1^{\text{th}}\rightarrow \\ \\ \\ \\ \end{array} \begin{pmatrix} \sum_{j=1}^{k} a_{1j}x_j \\ \vdots \\ a_{k+1,k}x_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in U_k.$$ Thus $a_{k+1,k}x_k = 0$ and then $a_{k+1,k} = 0$.

It follows that $a_{k+1,k} = 0$ for all $2 \leq k \leq n-1$. For this reason, we obtain $A \in H_n^1(\mathbb{F}) \cup T_n(\mathbb{F})$. $\square$

CHAPTER II
LINEAR PRESERVERS ON HESSENBERG MATRICES

This chapter is devoted to characterize three types of linear maps; namely, rank-1 preservers, determinant preservers and eigenvalues preservers by seperating in two sections.

2.1 Rank-1 Preservers on Hessenberg Matrices

The aim of this section is to find a pattern of linear maps preserving rank-1 which is one of the major goals of this research as follows: Let T be a linear map on $H_n(\mathbb{F})$. Then T preserves rank-1 matrices if and only if

(i) $\operatorname{im} T$ is an n-dimensional rank-1 subspace, or

(ii) there exist nonsingular upper Hessenberg matrices P and Q such that $T(A) = PAQ$ for all $A \in H_n(\mathbb{F})$ or $T(A) = PA^{\sim}Q$ for all $A \in H_n(\mathbb{F})$.

Nevertheless, its proof is so tedious that it is divided in various results.

The following property is frequently used in the proof of several results in Chapter II.

Proposition 2.1. *Let $a, b \in M_{n1}(\mathbb{F})$. If $\{a, b\}$ is linearly independent, then there exists a nonsingular matrix $P \in M_n(\mathbb{F})$ such that $\begin{pmatrix} a & b \end{pmatrix} = P \begin{pmatrix} e_1 & e_2 \end{pmatrix}$.*

Proof. Assume that $\{a, b\}$ is linearly independent. Since $M_{n1}(\mathbb{F})$ is an n-dimensional vector space, we can extend $\{a, b\}$ to $\{a, b, v_3, \ldots, v_n\}$ which is a basis of $M_{n1}(\mathbb{F})$. Put $P = \begin{pmatrix} | & | & | & & | \\ a & b & v_3 & \ldots & v_n \\ | & | & | & & | \end{pmatrix}$. Then P is nonsingular and $\begin{pmatrix} a & b \end{pmatrix} = P \begin{pmatrix} e_1 & e_2 \end{pmatrix}$.

□

Recall that there are four main types of n-dimensional rank-1 subspaces of $H_n(\mathbb{F})$, namely, X, Y, X_α and Y_α for $\alpha \in \mathbb{F}$ as follows.

$$X = \left\{ \begin{pmatrix} - & x & - \\ & & \\ & & \end{pmatrix} \middle| x \in M_{1n}(\mathbb{F}) \right\}, \qquad X_\alpha = \left\{ \begin{pmatrix} - & \alpha x & - \\ - & x & - \\ & & \end{pmatrix} \middle| x \in M_{1n}(\mathbb{F}) \right\},$$

$$Y = \left\{ \begin{pmatrix} & & | \\ & & y \\ & & | \end{pmatrix} \middle| y \in M_{n1}(\mathbb{F}) \right\}, \text{ and } \quad Y_\alpha = \left\{ \begin{pmatrix} & | & | \\ & y & \alpha y \\ & | & | \end{pmatrix} \middle| y \in M_{n1}(\mathbb{F}) \right\}.$$

Sometimes, we use $\begin{pmatrix} \text{———} \\ \\ \end{pmatrix}$, $\begin{pmatrix} & \vert \\ & \vert \end{pmatrix}$, $\begin{pmatrix} \text{———} \\ \text{———} \\ \\ \end{pmatrix}$ and $\begin{pmatrix} & \vert & \vert \\ & \vert & \vert \end{pmatrix}$ to denote elements in X, Y, X_α and Y_α, respectively. However, $\begin{pmatrix} \text{———} \\ \text{———} \\ \\ \end{pmatrix}_\alpha$ and $\begin{pmatrix} & \vert & \vert \\ & \vert & \vert \end{pmatrix}_\alpha$ are used in order to emphasize the scalar α. In addition, from Chapter I, $A^\sim$ denotes the matrix (b_{ij}) in $M_n(\mathbb{F})$ such that $b_{ij} = a_{n+1-j,n+1-i}$ for all i and j where $A = (a_{ij}) \in M_n(\mathbb{F})$. In particular, $\begin{pmatrix} \diagdown & \vert \\ & \diagdown \end{pmatrix}^\sim$ is also an upper triangular; moreover, $\begin{pmatrix} \square\diagdown & \vert \\ & \diagdown \end{pmatrix}^\sim$ is an element in $H_n^2(\mathbb{F})$ and $\begin{pmatrix} \diagdown & \vert \\ & \diagdown\square \end{pmatrix}^\sim$ is contained in $H_n^1(\mathbb{F})$.

From now on, let S be the particular map defined on $H_n(\mathbb{F})$ by $S(A) = A^\sim$ for all $A \in H_n(\mathbb{F})$. Then S maps $\begin{pmatrix} \text{———} \\ \\ \end{pmatrix}$ to $\begin{pmatrix} & \vert \\ & \vert \end{pmatrix}$ and S maps $\begin{pmatrix} \text{———} \\ \text{———} \\ \\ \end{pmatrix}$ to $\begin{pmatrix} & \vert & \vert \\ & \vert & \vert \end{pmatrix}$ and vice versa. It is clear that S is a bijection linear rank-1 preserver on $H_n(\mathbb{F})$. Besides, $S \circ T$ and $T \circ S$ are linear rank-1 preservers on $H_n(\mathbb{F})$ for any linear rank-1 preserver T on $H_n(\mathbb{F})$.

In this section, T stands for a linear rank-1 preserver on $H_n(\mathbb{F})$. We first investigate connections among mappings matrices having only the first two rows of T; for example, knowing that T maps a matrix of the form $\begin{pmatrix} \text{———} \\ \text{———} \\ \\ \end{pmatrix}$ to a matrix of the form $\begin{pmatrix} \text{———} \\ \\ \end{pmatrix}$ compels T to map other matrices of the form $\begin{pmatrix} \text{———} \\ \text{———} \\ \\ \end{pmatrix}$ not into any matrices of the forms $\begin{pmatrix} & \vert \\ & \vert \end{pmatrix}$ and $\begin{pmatrix} & \vert & \vert \\ & \vert & \vert \end{pmatrix}$, see (i) in

Proposition 2.2. Furthermore, relationships of mappings matrices having only the last two columns of T are given in Proposition 2.3.

Proposition 2.2. *(i) If there exists $\alpha \in \mathbb{F}$ such that $T(X_\alpha) = X$, then $T(X_\beta) \notin \{Y, Y_\lambda\}_{\lambda\in\mathbb{F}}$ for all $\beta \in \mathbb{F}$.*

(ii) If there exists $\alpha \in \mathbb{F}$ such that $T(X_\alpha) = Y$, then $T(X_\beta) \notin \{X_\lambda\}_{\lambda\in\mathbb{F}}$ for all $\beta \in \mathbb{F}$.

(iii) If there exists $\alpha \in \mathbb{F}$ such that $T(X_\alpha) = X_\gamma$ for some $\gamma \in \mathbb{F}$, then $T(X_\beta) \notin \{Y_\lambda\}_{\lambda\in\mathbb{F}}$ for all $\beta \in \mathbb{F}$.

Proof. (i) Assume that $T(X_\alpha) = X$ for some $\alpha \in \mathbb{F}$. We obtain that for each $1 \leq j \leq n$,

$$T(\alpha E_{1j} + E_{2j}) = e_1 u_j = \begin{pmatrix} - & u_j & - \\ & & \\ & & \end{pmatrix}$$

for some $u_j \in V_1 \setminus \{0\}$. Suppose that there exists $\beta \in \mathbb{F}$ such that $T(X_\beta) = Y$. Thus for each $1 \leq t \leq n$,

$$T(\beta E_{1t} + E_{2t}) = w_t f_n = \begin{pmatrix} & & | \\ & & w_t \\ & & | \end{pmatrix}$$

for some $w_t \in U_n \setminus \{0\}$. Then $\{u_1, \dots, u_n\}$ and $\{w_1, \dots, w_n\}$ are linearly independent; otherwise, without loss of generality, there exist $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ not all zero such that $\lambda_1 u_1 + \cdots + \lambda_n u_n = 0$ and then

$$\begin{aligned}
0 &= \lambda_1 e_1 u_1 + \cdots + \lambda_n e_1 u_n \\
&= \lambda_1 T(\alpha E_{11} + E_{21}) + \cdots + \lambda_n T(\alpha E_{1n} + E_{2n}) \\
&= T\Big(\lambda_1(\alpha E_{11} + E_{21}) + \cdots + \lambda_n(\alpha E_{1n} + E_{2n})\Big) \\
&= T\begin{pmatrix} \lambda_1\alpha & \cdots & \lambda_n\alpha \\ \lambda_1 & \cdots & \lambda_n \\ & & \end{pmatrix},
\end{aligned}$$

which is a contradiction because T is a rank-1 preserver. In addition, for each $1 \leq t \leq n$, since $\rho\big((\alpha+\beta)E_{1t} + 2E_{2t}\big) = 1$ and T is a rank-1 preserver,

$$\begin{aligned} T\big((\alpha+\beta)E_{1t} + 2E_{2t}\big) &= T\left(\alpha E_{1t} + E_{2t}\right) + T\left(\beta E_{1t} + E_{2t}\right) \\ &= e_1 u_t + w_t f_n = \begin{pmatrix} - & u_t & - \\ & & \\ & & \end{pmatrix} + \begin{pmatrix} & & | \\ & & w_t \\ & & | \end{pmatrix} \end{aligned}$$

must have rank one. Thus $u_t \in V_n \setminus \{0\}$ or $w_t \in U_1 \setminus \{0\}$. Without loss of generality, let $u_1 \in V_n \setminus \{0\}$. Then $u_2, \ldots, u_n \notin V_n \setminus \{0\}$ so that $w_2, \ldots, w_n \in U_1 \setminus \{0\}$. This contradicts the linearly independence of $w_2, \ldots, w_n$ since $n \geq 3$. Hence $T(X_\beta) \neq Y$.

Next, suppose that there exist $\beta, \lambda \in \mathbb{F}$ such that $T(X_\beta) = Y_\lambda$. Then, for each $1 \leq t \leq n$,

$$T(\beta E_{1t} + E_{2t}) = z_t(f_{n-1} + \lambda f_n)$$

for some $z_t \in U_n \setminus \{0\}$. Thus for each $1 \leq t \leq n$, we get

$$T\big((\alpha+\beta)E_{1t} + 2E_{2t}\big) = e_1 u_t + z_t(f_{n-1} + \lambda f_n)$$

which has rank one so that $u_t \in V_{n-1} \setminus \{0\}$ or $z_t \in U_1 \setminus \{0\}$. Without loss of generality, let $z_1 \in U_1 \setminus \{0\}$. Then $u_i \in V_{n-1} \setminus \{0\}$ for any $i \geq 2$ and we obtain the form of u_i, namely, $u_i = \begin{pmatrix} 0 & \cdots & 0 & a_i & \lambda a_i \end{pmatrix}$ where $a_i \in \mathbb{F}$. Nevertheless, $\{u_1, \ldots, u_n\}$ is linearly independent which is a contradiction since $n \geq 3$. Hence $T(X_\beta) \neq Y_\lambda$ for any $\lambda \in \mathbb{F}$.

Similarly, (ii) and (iii) are proved. □

Proposition 2.3. *(i) If there exists $\alpha \in \mathbb{F}$ such that $T(Y_\alpha) = X$, then $T(Y_\beta) \notin \{Y, Y_\lambda\}_{\lambda \in \mathbb{F}}$ for all $\beta \in \mathbb{F}$.*

(ii) If there exists $\alpha \in \mathbb{F}$ such that $T(Y_\alpha) = Y$, then $T(Y_\beta) \notin \{X_\lambda\}_{\lambda \in \mathbb{F}}$ for all $\beta \in \mathbb{F}$.

(iii) If there exists $\alpha \in \mathbb{F}$ such that $T(Y_\alpha) = X_\gamma$ for some $\gamma \in \mathbb{F}$, then $T(Y_\beta) \notin \{Y_\lambda\}_{\lambda \in \mathbb{F}}$ for all $\beta \in \mathbb{F}$.

Proof. (i) Assume that $T(Y_\alpha) = X$ for some $\alpha \in \mathbb{F}$. Then $T \circ S(X_\alpha) = T(Y_\alpha) = X$. Since $T \circ S$ is a linear rank-1 preserver on $H_n(\mathbb{F})$, by applying Proposition 2.2 (i), we get that $T(Y_\beta) = T \circ S(X_\beta) \notin \{Y, Y_\lambda\}_{\lambda \in \mathbb{F}}$.

(ii) and (iii) can be proved in the same way. $\square$

Next, we draw our attention to find a relationship between the subspaces X and X_γ providing that for scalars α and β,

$$\begin{pmatrix} \text{------} \\ \text{------} \\ \end{pmatrix}_{\alpha} \xmapsto{T} \begin{pmatrix} \text{------} \\ \\ \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \text{------} \\ \text{------} \\ \end{pmatrix}_{\beta} \xmapsto{T} \begin{pmatrix} \text{------} \\ \text{------} \\ \end{pmatrix}_{\gamma}$$

as Proposition 2.4. Then we turn to a relationship between the subspaces X_γ and X_λ satisfying

$$\begin{pmatrix} \text{------} \\ \text{------} \\ \end{pmatrix}_{\alpha} \xmapsto{T} \begin{pmatrix} \text{------} \\ \text{------} \\ \end{pmatrix}_{\gamma} \quad \text{and} \quad \begin{pmatrix} \text{------} \\ \text{------} \\ \end{pmatrix}_{\beta} \xmapsto{T} \begin{pmatrix} \text{------} \\ \text{------} \\ \end{pmatrix}_{\lambda}$$

as Proposition 2.6. Besides, by ability of the map S, we also obtain the similar results on matrices having the last two columns see Propositions 2.5 and 2.7.

Proposition 2.4. *Let $\alpha, \beta, \gamma \in \mathbb{F}$ be such that $T(X_\alpha) = X$ and $T(X_\beta) = X_\gamma$. Moreover, for each $1 \leq j \leq n$, write*

$$T(\alpha E_{1j} + E_{2j}) = e_1 u_j \quad \text{and} \quad T(\beta E_{1j} + E_{2j}) = (\gamma e_1 + e_2) w_j$$

for some $u_j, w_j \in V_1 \setminus \{0\}$. Then, for each $1 \leq t \leq n$, there exists $\xi_t \in \mathbb{F} \setminus \{0\}$ such that $w_t = \xi_t u_t$.

Proof. Let $1 \leq t \leq n$. Then

$$\begin{aligned} T\big((\alpha + \beta) E_{1t} + 2E_{2t}\big) &= e_1 u_t + (\gamma e_1 + e_2) w_t = e_1 (u_t + \gamma w_t) + e_2 w_t \\ &= \begin{pmatrix} \text{—} & u_t + \gamma w_t & \text{—} \\ \text{—} & w_t & \text{—} \\ & & \end{pmatrix} \text{ has rank one.} \end{aligned}$$

If $\gamma = 0$, then there exists $a_t \in \mathbb{F} \setminus \{0\}$ such that $w_t = a_t u_t$. Assume that $\gamma \neq 0$. If $u_t + \gamma w_t = 0$, we get $w_t = -\frac{1}{\gamma} u_t$. If $u_t + \gamma w_t \neq 0$, then there exists $b_t \in \mathbb{F} \setminus \{0\}$ such that $b_t w_t = u_t + \gamma w_t$, so $u_t = (b_t - \gamma) w_t$ where $b_t - \gamma$ is not zero. $\square$

Proposition 2.5. *Let $\alpha, \beta, \gamma \in \mathbb{F}$ be such that $T(Y_\alpha) = Y$ and $T(Y_\beta) = Y_\gamma$. Moreover, for each $1 \le i \le n$, write*

$$T(E_{i,n-1} + \alpha E_{in}) = v_i f_n \quad \text{and} \quad T(E_{i,n-1} + \beta E_{in}) = z_i(f_{n-1} + \gamma f_n)$$

for some $v_i, z_i \in U_n \setminus \{0\}$. Then, for each $1 \le t \le n$, there exists $\xi_t \in \mathbb{F} \setminus \{0\}$ such that $z_t = \xi_t v_t$.

Proof. By assumption, $S \circ T \circ S(X_\alpha) = S \circ T(Y_\alpha) = S(Y) = X$ and $S \circ T \circ S(X_\beta) = S \circ T(Y_\beta) = S(Y_\gamma) = X_\gamma$. Since $S \circ T \circ S$ is still a linear rank-1 preserver, the proof is done by applying Proposition 2.4. □

Proposition 2.6. *Let $\alpha, \beta, \gamma, \lambda \in \mathbb{F}$ be such that $T(X_\alpha) = X_\gamma$ and $T(X_\beta) = X_\lambda$. Moreover, for each $1 \le j \le n$, write*

$$T(\alpha E_{1j} + E_{2j}) = (\gamma e_1 + e_2) u_j \quad \text{and} \quad T(\beta E_{1j} + E_{2j}) = (\lambda e_1 + e_2) w_j$$

for some $u_j, w_j \in V_1 \setminus \{0\}$. Then $\gamma = \lambda$ or, for each $1 \le t \le n$, there exists $\xi_t \in \mathbb{F} \setminus \{0\}$ such that $w_t = \xi_t u_t$.

Proof. Assume that $\gamma \ne \lambda$. Let $1 \le t \le n$. Since

$$\begin{aligned} T\big((\alpha+\beta)E_{1t} + 2E_{2t}\big) &= e_1 (\gamma u_t + \lambda w_t) + e_2 (u_t + w_t) \\ &= \begin{pmatrix} — & \gamma u_t + \lambda w_t & — \\ — & u_t + w_t & — \\ & & \end{pmatrix} \text{ has rank one,} \end{aligned}$$

there exists $a_t \in \mathbb{F} \setminus \{0\}$ such that $a_t (u_t + w_t) = \gamma u_t + \lambda w_t$ and then $(a_t - \gamma) u_t = (\lambda - a_t) w_t$. If $a_t - \gamma = 0$, then $\gamma = a_t$ and forces $\lambda - a_t = 0$ since $w_t \ne 0$. Thus $\lambda = a_t = \gamma$ which is impossible. Hence $a_t - \gamma$ and $\lambda - a_t$ are not zero and thus $w_t = \left(\frac{a_t - \gamma}{\lambda - a_t}\right) u_t$. □

Proposition 2.7. *Let $\alpha, \beta, \gamma, \lambda \in \mathbb{F}$ be such that $T(Y_\alpha) = Y_\gamma$ and $T(Y_\beta) = Y_\lambda$. Moreover, for each $1 \le i \le n$, write*

$$T(E_{i,n-1} + \alpha E_{in}) = v_i(f_{n-1} + \gamma f_n) \quad \text{and} \quad T(E_{i,n-1} + \beta E_{in}) = z_i(f_{n-1} + \lambda f_n)$$

for some $v_i, z_i \in U_n \setminus \{0\}$. Then $\gamma = \lambda$ or, for each $1 \le t \le n$, there exists $\xi_t \in \mathbb{F} \setminus \{0\}$ such that $z_t = \xi_t v_t$.

Proof. We can prove this in the same way as Proposition 2.5 by applying Proposition 2.6. $\square$

Now, we focus on results of $T(Y), T(X_\alpha)$ and $T(Y_\alpha)$ for any $\alpha \in \mathbb{F}$ provided the space $T(X)$ is given.

Proposition 2.8. *The following properties hold.*

(i) If $T(X) = X$, then $T(Y), T(Y_\beta) \notin \{X_a\}_{a\in\mathbb{F}}$ for all $\beta \in \mathbb{F}$ and $T(X_\alpha) \notin \{Y, Y_a\}_{a\in\mathbb{F}}$ for all $\alpha \in \mathbb{F}$.

(ii) If $T(X) = Y$, then $T(Y), T(Y_\beta) \notin \{Y_a\}_{a\in\mathbb{F}}$ for all $\beta \in \mathbb{F}$ and $T(X_\alpha) \notin \{X, X_a\}_{a\in\mathbb{F}}$ for all $\alpha \in \mathbb{F}$.

(iii) If $T(X) \in \{X_a\}_{a\in\mathbb{F}}$, then $T(Y) \neq X$, $T(Y_\beta) \neq X$ for all $\beta \in \mathbb{F}$ and $T(X_\alpha) \notin \{Y, Y_a\}_{a\in\mathbb{F}}$ for all $\alpha \in \mathbb{F}$.

(iv) If $T(X) \in \{Y_a\}_{a\in\mathbb{F}}$, then $T(Y) \neq Y$, $T(Y_\beta) \neq Y$ for all $\beta \in \mathbb{F}$ and $T(X_\alpha) \notin \{X, X_a\}_{a\in\mathbb{F}}$ for all $\alpha \in \mathbb{F}$.

Proof. (i) Assume that $T(X) = X$. Then for each $1 \leq j \leq n$, there exists $u_j \in V_1 \setminus \{0\}$ such that $T(E_{1j}) = e_1 u_j$. Suppose that $T(Y) = X_\gamma$ for some $\gamma \in \mathbb{F}$. Then for each $1 \leq i \leq n$, there exists $v_i \in V_1 \setminus \{0\}$ such that $T(E_{in}) = (\gamma e_1 + e_2)v_i$. Since $\begin{pmatrix} - & u_n & - \\ & & \\ & & \end{pmatrix} = e_1 u_n = T(E_{1n}) = (\gamma e_1 + e_2)v_1 = \begin{pmatrix} - & \gamma v_1 & - \\ - & v_1 & - \\ & & \end{pmatrix}$, we get $v_1 = 0$ which is absurd. Hence $T(Y) \notin \{X_a\}_{a\in\mathbb{F}}$.

Suppose that there exist $\beta, \gamma \in \mathbb{F}$ such that $T(Y_\beta) = X_\gamma$. Then for each $1 \leq i \leq n$, there exists $v_i \in V_1 \setminus \{0\}$ such that $T(E_{i,n-1} + \beta E_{in}) = (\gamma e_1 + e_2)v_i$. Thus $v_1 = 0$ which is a contradiction. Hence $T(Y_\beta) \notin \{X_a\}_{a\in\mathbb{F}}$ for all $\beta \in \mathbb{F}$.

Moreover, suppose that there exists $\alpha \in \mathbb{F}$ such that $T(X_\alpha) = Y$. Then, for each $1 \leq l \leq n$, there exists $w_l \in U_n \setminus \{0\}$ such that $T(\alpha E_{1l} + E_{2l}) = w_l f_n$. Since $T(E_{2l}) = w_l f_n - \alpha T(E_{1l}) = w_l f_n - \alpha e_1 u_l$, we get that $w_l \in U_1 \setminus \{0\}$ or $u_l \in V_n \setminus \{0\}$. Besides, $\{u_1, \ldots, u_n\}$ and $\{w_1, \ldots, w_n\}$ are linearly independent. This is impossible because $n \geq 3$. With the same reason, we can conclude that $T(X_\alpha) \notin \{Y_a\}_{a\in\mathbb{F}}$ for all $\alpha \in \mathbb{F}$.

(iii) Assume that $T(X) = X_\gamma$ for some $\gamma \in \mathbb{F}$. Then, for each $1 \le j \le n$, there exists $u_j \in V_1 \setminus \{0\}$ such that $T(E_{1j}) = (\gamma e_1 + e_2)u_j$. Moreover, $\{u_1, \dots, u_n\}$ is linearly independent. If $T(Y) = X$, then $u_n = 0$ which is impossible. If there exists $\beta \in \mathbb{F}$ such that $T(Y_\beta) = X$, then u_{n-1} and u_n are linearly dependent, again, leading to a contradiction. Hence $T(Y) \neq X$ and $T(Y_\beta) \neq X$ for all $\beta \in \mathbb{F}$.

Moreover, by the same argument, $T(X_\alpha) \notin \{Y, Y_a\}_{a\in\mathbb{F}}$ for all $\alpha \in \mathbb{F}$.

The proofs of (ii) and (iv) are obtained similarly to those of (i) and (iii), respectively. □

We obtain similar results of $T(X), T(X_\alpha)$ and $T(Y_\alpha)$ for any $\alpha \in \mathbb{F}$ on condition that the space $T(Y)$ is fixed.

Proposition 2.9. *The following properties hold.*

(i) If $T(Y) = X$, then $T(X), T(X_\alpha) \notin \{X_a\}_{a\in\mathbb{F}}$ for all $\alpha \in \mathbb{F}$ and $T(Y_\beta) \notin \{Y, Y_a\}_{a\in\mathbb{F}}$ for all $\beta \in \mathbb{F}$.

(ii) If $T(Y) = Y$, then $T(X), T(X_\alpha) \notin \{Y_a\}_{a\in\mathbb{F}}$ for all $\alpha \in \mathbb{F}$ and $T(Y_\beta) \notin \{X, X_a\}_{a\in\mathbb{F}}$ for all $\beta \in \mathbb{F}$.

(iii) If $T(Y) \in \{X_a\}_{a\in\mathbb{F}}$, then $T(X) \neq X$, $T(X_\alpha) \neq X$ for all $\alpha \in \mathbb{F}$ and $T(Y_\beta) \notin \{Y, Y_a\}_{a\in\mathbb{F}}$ for all $\beta \in \mathbb{F}$.

(iv) If $T(Y) \in \{Y_a\}_{a\in\mathbb{F}}$, then $T(X) \neq Y$, $T(X_\alpha) \neq Y$ for all $\alpha \in \mathbb{F}$ and $T(Y_\beta) \notin \{X, X_a\}_{a\in\mathbb{F}}$ for all $\beta \in \mathbb{F}$.

Proof. This is consequences of $T(Y) = (T \circ S)(X)$ and Proposition 2.8. □

Next, we consider the results of various combinations of given $T(X)$ and $T(Y)$.

Proposition 2.10. *Assume that $T(X) = X$.*

(i) If $T(Y) = Y$, then $T(X_\alpha) \neq X$ and $T(Y_\alpha) \neq Y$ for all $\alpha \in \mathbb{F}$.

(ii) If $T(Y) \in \{Y_a\}_{a\in\mathbb{F}}$, then $T(X_\alpha) \neq X$ for all $\alpha \in \mathbb{F}$.

Proof. Let $T(E_{1j}) = e_1 u_j$ for some $u_j \in V_1 \setminus \{0\}$ where $1 \leq j \leq n$.

(i) Assume that $T(Y) = Y$. For each $1 \leq i \leq n$, let $T(E_{in}) = v_i f_n$ for some $v_i \in U_n \setminus \{0\}$. Suppose that there exists $\alpha \in \mathbb{F}$ such that $T(X_\alpha) = X$. Then for each $1 \leq l \leq n$, $T(\alpha E_{1l} + E_{2l}) = e_1 w_l$ for some $w_l \in V_1 \setminus \{0\}$. For each $j \in \{1, \ldots, n\}$,

$$T(E_{2j}) = e_1 w_j - \alpha T(E_{1j}) = e_1 w_j - \alpha e_1 u_j = e_1(w_j - \alpha u_j).$$

Then $e_1 u_n = T(E_{1n}) = v_1 f_n$ and $e_1(w_n - \alpha u_n) = T(E_{2n}) = v_2 f_n$ which imply that v_1 and v_2 are elements in $U_1 \setminus \{0\}$ contradicting the linearly independence of $\{v_1, \ldots, v_n\}$. Hence $T(X_\alpha) \neq X$ for all $\alpha \in \mathbb{F}$. Similarly, we can show that $T(Y_\alpha) \neq Y$ for all $\alpha \in \mathbb{F}$.

(ii) This can be done by similar method of the proof of (i). $\square$

In addition, the following propositions can be proved in the same manner of the proof of Proposition 2.10.

Proposition 2.11. *Assume that* $T(X) = Y$*.*

(i) If $T(Y) = X$*, then* $T(X_\alpha) \neq Y$ *and* $T(Y_\alpha) \neq X$ *for all* $\alpha \in \mathbb{F}$*.*

(ii) If $T(Y) \in \{X_a\}_{a \in \mathbb{F}}$*, then* $T(X_\alpha) \neq Y$ *for all* $\alpha \in \mathbb{F}$*.*

Proposition 2.12. *(i) If* $T(X) = X_\alpha$ *for some* $\alpha \in \mathbb{F}$ *and* $T(Y) = Y$*, then* $T(Y_\beta) \neq Y$ *for all* $\beta \in \mathbb{F}$*.*

(ii) If $T(X) = Y_\alpha$ *for some* $\alpha \in \mathbb{F}$ *and* $T(Y) = X$*, then* $T(Y_\beta) \neq X$ *for all* $\beta \in \mathbb{F}$*.*

We find that $T(E_{2l})$ and $T(E_{k,n-1})$ for each $l, k \in \{1, 2, \ldots, n\}$ are necessary for the proof of the main result. The following propositions inform what they are under various conditions.

Proposition 2.13. *Let* $\alpha, \gamma, \lambda \in \mathbb{F}$ *be such that* $T(X) = X_\gamma$ *and* $T(X_\alpha) = X_\lambda$*. Moreover, write, for each* $1 \leq j \leq n$*,*

$$T(E_{1j}) = (\gamma e_1 + e_2) u_j \quad \text{and} \quad T(\alpha E_{1j} + E_{2j}) = (\lambda e_1 + e_2) w_j$$

for some $u_j, w_j \in V_1 \setminus \{0\}$. *Then there exists* $\sigma \in \mathbb{F}$ *such that, for each* $1 \leq l \leq n$,

$$T(E_{2l}) = \begin{cases} (\gamma e_1 + e_2)(w_l - \alpha u_l) \in X_\gamma, & \text{if } \gamma = \lambda; \\ (\sigma e_1 + e_2)a_l u_l \in X_\sigma, & \text{if } \gamma \neq \lambda \end{cases}$$

for some $a_l \in \mathbb{F} \setminus \{0\}$.

Proof. First, assume that $\gamma = \lambda$. Then for each $1 \leq l \leq n$,

$$T(E_{2l}) = (\lambda e_1 + e_2)w_l - \alpha T(E_{1l}) = (\gamma e_1 + e_2)(w_l - \alpha u_l) \in X_\gamma.$$

Next, assume that $\gamma \neq \lambda$.

Case 1: $\alpha = 0$. Let $1 \leq l \leq n$. Since

$$\begin{aligned} T(E_{1l} + E_{2l}) &= T(E_{1l}) + T(E_{2l}) = (\gamma e_1 + e_2)u_l + (\lambda e_1 + e_2)w_l \\ &= \begin{pmatrix} - & \gamma u_l & - \\ - & u_l & - \\ & & \end{pmatrix} + \begin{pmatrix} - & \lambda w_l & - \\ - & w_l & - \\ & & \end{pmatrix}, \end{aligned}$$

there exists $\eta_l \in \mathbb{F}$ such that $\eta_l(u_l + w_l) = \gamma u_l + \lambda w_l$. Thus $(\eta_l - \gamma)u_l = (\lambda - \eta_l)w_l$. Since $\gamma \neq \lambda$, we obtain that $\eta_l - \gamma \neq 0$ and $\lambda - \eta_l \neq 0$ so that $w_l = \left(\frac{\eta_l - \gamma}{\lambda - \eta_l}\right) u_l$. Hence $T(E_{2l}) = (\lambda e_1 + e_2)\left(\frac{\eta_l - \gamma}{\lambda - \eta_l}\right) u_l$ where $\frac{\eta_l - \gamma}{\lambda - \eta_l} \neq 0$.

Case 2: $\alpha \neq 0$. Note that

$$\begin{aligned} T(E_{21}) &= (\lambda e_1 + e_2)w_1 - \alpha T(E_{11}) = (\lambda e_1 + e_2)w_1 - \alpha(\gamma e_1 + e_2)u_1 \\ &= \begin{pmatrix} - & \lambda w_1 & - \\ - & w_1 & - \\ & & \end{pmatrix} - \begin{pmatrix} - & \alpha\gamma u_1 & - \\ - & \alpha u_1 & - \\ & & \end{pmatrix}. \end{aligned}$$

Thus there exists $\sigma \in \mathbb{F}$ such that $\sigma(w_1 - \alpha u_1) = \lambda w_1 - \alpha\gamma u_1$. Let $1 \leq l \leq n$. Then there exists $\eta_l \in \mathbb{F}$ (with $\eta_1 = \sigma$) such that

$$\eta_l(w_l - \alpha u_l) = \lambda w_l - \alpha\gamma u_l$$

so that $(\eta_l - \lambda)\, w_l = \alpha\,(\eta_l - \gamma)\, u_l$. Since $\gamma \neq \lambda$ and $\alpha \neq 0$, it follows that $\eta_l - \lambda \neq 0$ and $\eta_l - \gamma \neq 0$ and thus $w_l = \left(\frac{\alpha(\eta_l - \gamma)}{\eta_l - \lambda}\right) u_l$. Hence

$$T(E_{2l}) = (\lambda e_1 + e_2)w_l - \alpha(\gamma_1 + e_2)u_l$$

$$= (\lambda e_1 + e_2)\left(\frac{\alpha(\eta_l - \gamma)}{\eta_l - \lambda}\right) u_l - \alpha(\gamma e_1 + e_2) u_l$$
$$= (\eta_l e_1 + e_2)\left(\frac{\alpha(\lambda - \gamma)}{\eta_l - \lambda}\right) u_l \quad \text{where } \frac{\alpha(\lambda - \gamma)}{\eta_l - \lambda} \neq 0.$$

This shows that $T(E_{2l}) = (\eta_l e_1 + e_2)\left(\frac{\alpha(\lambda-\gamma)}{\eta_l-\lambda}\right) u_l$ for all $l \in \{1, \ldots, n\}$. It remains to show that $\eta_l = \sigma$ for all $l \in \{1, \ldots, n\}$. Let $l \in \{1, \ldots, n\}$. Since $T(E_{21} + E_{2l})$ has rank one, there exists $\vartheta_l \in \mathbb{F} \setminus \{0\}$ such that $\vartheta_l(\kappa_1 u_1 + \kappa_l u_l) = \eta_1 \kappa_1 u_1 + \eta_l \kappa_l u_l$ where $\kappa_l = \frac{\alpha(\lambda-\gamma)}{\eta_l-\lambda} \neq 0$. Then $(\vartheta_l - \eta_1)\kappa_1 u_1 + (\vartheta_l - \eta_l)\kappa_l u_l = 0$. Since $\{u_1, u_l\}$ is linearly independent, $\vartheta_l - \eta_1 = 0 = \vartheta_l - \eta_l$ and hence $\eta_1 = \vartheta_l = \eta_l$. This shows that $\eta_l = \eta_1 = \sigma$ for all $l \in \{1, \ldots, n\}$. □

Proposition 2.14. *Let $\alpha, \gamma, \lambda \in \mathbb{F}$ be such that $T(Y) = X_\gamma$ and $T(Y_\alpha) = X_\lambda$. Moreover, write, for each $1 \leq i \leq n$,*

$$T(E_{in}) = (\gamma e_1 + e_2) u_i \quad \textit{and} \quad T(E_{i,n-1} + \alpha E_{in}) = (\lambda e_1 + e_2) w_i$$

for some $u_i, w_i \in V_1 \setminus \{0\}$. Then there exists $\sigma \in \mathbb{F}$ such that, for each $1 \leq k \leq n$,

$$T(E_{k,n-1}) = \begin{cases} (\gamma e_1 + e_2)(w_k - \alpha u_k) \in X_\gamma, & \textit{if } \gamma = \lambda; \\ (\sigma e_1 + e_2) a_k u_k \in X_\sigma, & \textit{if } \gamma \neq \lambda \end{cases}$$

for some $a_k \in \mathbb{F} \setminus \{0\}$.

Proof. This is obtained similarly to the proof of Proposition 2.13 by replacing E_{1l} and E_{2l} by E_{kn} and $E_{k,n-1}$, respectively. □

Note that Proposition 2.14 can also be done by making use of Proposition 2.13 together with the fact that $T \circ S(X) = T(Y) = X_\gamma$ and $T \circ S(X_\alpha) = T(Y_\alpha) = X_\lambda$.

Proposition 2.15. *Let $\alpha, \gamma, \lambda \in \mathbb{F}$ be such that $T(Y) = Y_\gamma$ and $T(Y_\alpha) = Y_\lambda$. Moreover, write, for each $1 \leq i \leq n$,*

$$T(E_{in}) = v_i(f_{n-1} + \gamma f_n) \quad \textit{and} \quad T(E_{i,n-1} + \alpha E_{in}) = z_i(f_{n-1} + \lambda f_n)$$

for some $v_i, z_i \in U_n \setminus \{0\}$. Then there exists $\sigma \in \mathbb{F}$ such that, for each $1 \leq k \leq n$,

$$T(E_{k,n-1}) = \begin{cases} (z_k - \alpha v_k)(f_{n-1} + \gamma f_n) \in X_\gamma, & \textit{if } \gamma = \lambda; \\ a_k v_k (f_{n-1} + \sigma f_n) \in X_\sigma, & \textit{if } \gamma \neq \lambda \end{cases}$$

for some $a_k \in \mathbb{F} \setminus \{0\}$.

Proof. We can prove by applying $S \circ T \circ S$ with Proposition 2.13. $\square$

Propositions 2.16 and 2.17 can be shown as Proposition 2.13. Moreover, Propositions 2.18 and 2.19 can be proved by using $S \circ T \circ S$ with Propositions 2.16 and 2.17, respectively.

Proposition 2.16. *Let $\alpha, \gamma \in \mathbb{F}$ be such that $T(X) = X$ and $T(X_\alpha) = X_\gamma$. Moreover, write, for each $1 \leq j \leq n$,*

$$T(E_{1j}) = e_1 u_j \quad \text{and} \quad T(\alpha E_{1j} + E_{2j}) = (\gamma e_1 + e_2) w_j$$

for some $u_j, w_j \in V_1 \setminus \{0\}$. Then there exists $\sigma \in \mathbb{F}$ such that, for each $1 \leq l \leq n$, $T(E_{2l}) = (\sigma e_1 + e_2) a_l u_l \in X_\sigma$ for some $a_l \in \mathbb{F} \setminus \{0\}$.

Proposition 2.17. *Let $\alpha, \gamma \in \mathbb{F}$ be such that $T(X) = X_\gamma$ and $T(X_\alpha) = X$. Moreover, write, for each $1 \leq j \leq n$,*

$$T(E_{1j}) = (\gamma e_1 + e_2) u_j \quad \text{and} \quad T(\alpha E_{1j} + E_{2j}) = e_1 w_j$$

for some $u_j, w_j \in V_1 \setminus \{0\}$. Then there exists $\sigma \in \mathbb{F}$ such that, for each $1 \leq l \leq n$, $T(E_{2l}) = (\sigma e_1 + e_2) a_l u_l \in X_\sigma$ for some $a_l \in \mathbb{F} \setminus \{0\}$.

Proposition 2.18. *Let $\alpha, \gamma \in \mathbb{F}$ be such that $T(Y) = Y$ and $T(Y_\alpha) = Y_\gamma$. Moreover, write, for each $1 \leq i \leq n$,*

$$T(E_{in}) = v_i f_n \quad \text{and} \quad T(E_{i,n-1} + \alpha E_{in}) = z_i (f_{n-1} + \gamma f_n)$$

for some $v_i, z_i \in U_n \setminus \{0\}$. Then there exists $\sigma \in \mathbb{F}$ such that, for each $1 \leq k \leq n$, $T(E_{k,n-1}) = a_k v_k (f_{n-1} + \sigma f_n) \in Y_\sigma$ for some $a_k \in \mathbb{F} \setminus \{0\}$.

Proposition 2.19. *Let $\alpha, \gamma \in \mathbb{F}$ be such that $T(Y) = Y_\gamma$ and $T(Y_\alpha) = Y$. Moreover, write, for each $1 \leq i \leq n$,*

$$T(E_{in}) = v_i (f_{n-1} + \gamma f_n) \quad \text{and} \quad T(E_{i,n-1} + \alpha E_{in}) = z_i f_n$$

for some $v_i, z_i \in U_n \setminus \{0\}$. Then there exists $\sigma \in \mathbb{F}$ such that, for each $1 \leq k \leq n$, $T(E_{k,n-1}) = a_k v_k (f_{n-1} + \sigma f_n) \in Y_\sigma$ for some $a_k \in \mathbb{F} \setminus \{0\}$.

To obtain the proof of the main result, the series of Propositions 2.20–2.28 are needed. Now, allow us to state without proof Propositions 2.20–2.27 in order to see the overall results.

Proposition 2.20. *If $T(X) = X$, $T(Y) = Y$, $T(X_\alpha) \in \{X_a\}_{a\in\mathbb{F}}$ and $T(Y_\beta) \in \{Y_a\}_{a\in\mathbb{F}}$ for some $\alpha, \beta \in \mathbb{F}$, then there exist nonsingular upper triangular matrices P and Q such that $T(A) = PAQ$ for all $A \in H_n(\mathbb{F})$.*

Proposition 2.21. *If $T(X) = X$, $T(Y) \in \{Y_a\}_{a\in\mathbb{F}}$, $T(X_\alpha) \in \{X_a\}_{a\in\mathbb{F}}$ and $T(Y_\beta) \in \{Y, Y_a\}_{a\in\mathbb{F}}$ for some $\alpha, \beta \in \mathbb{F}$, then there exist a nonsingular upper triangular matrix P and a nonsingular matrix $Q \in H_n^2(\mathbb{F})$ such that $T(A) = PAQ$ for all $A \in H_n(\mathbb{F})$.*

Proposition 2.22. *If $T(X) = Y$, $T(Y) = X$, $T(X_\alpha) \in \{Y_a\}_{a\in\mathbb{F}}$ and $T(Y_\beta) \in \{X_a\}_{a\in\mathbb{F}}$ for some $\alpha, \beta \in \mathbb{F}$, then there exist nonsingular upper triangular matrices P and Q such that $T(A) = PA^{\sim}Q$ for all $A \in H_n(\mathbb{F})$.*

Proposition 2.23. *If $T(X) = Y$, $T(Y) \in \{X_a\}_{a\in\mathbb{F}}$, $T(X_\alpha) \in \{Y_a\}_{a\in\mathbb{F}}$ and $T(Y_\beta) \in \{X, X_a\}_{a\in\mathbb{F}}$ for some $\alpha, \beta \in \mathbb{F}$, then there exist a nonsingular matrix $P \in H_n^1(\mathbb{F})$ and a nonsingular upper triangular matrix Q such that $T(A) = PA^{\sim}Q$ for all $A \in H_n(\mathbb{F})$.*

Proposition 2.24. *If $T(X) \in \{X_a\}_{a\in\mathbb{F}}$, $T(Y) = Y$, $T(X_\alpha) \in \{X, X_a\}_{a\in\mathbb{F}}$ and $T(Y_\beta) \in \{Y_a\}_{a\in\mathbb{F}}$ for some $\alpha, \beta \in \mathbb{F}$, then there exist a nonsingular matrix $P \in H_n^1(\mathbb{F})$ and a nonsingular upper triangular matrix Q such that $T(A) = PAQ$ for all $A \in H_n(\mathbb{F})$.*

Proposition 2.25. *If $T(X) \in \{X_a\}_{a\in\mathbb{F}}$, $T(Y) \in \{Y_a\}_{a\in\mathbb{F}}$, $T(X_\alpha) \in \{X, X_a\}_{a\in\mathbb{F}}$ and $T(Y_\beta) \in \{Y, Y_a\}_{a\in\mathbb{F}}$ for some $\alpha, \beta \in \mathbb{F}$, then there exist nonsingular matrices $P \in H_n^1(\mathbb{F})$ and $Q \in H_n^2(\mathbb{F})$ such that $T(A) = PAQ$ for all $A \in H_n(\mathbb{F})$.*

Proposition 2.26. *If $T(X) \in \{Y_a\}_{a\in\mathbb{F}}$, $T(Y) = X$, $T(X_\alpha) \in \{Y, Y_a\}_{a\in\mathbb{F}}$ and $T(Y_\beta) \in \{X_a\}_{a\in\mathbb{F}}$ for some $\alpha, \beta \in \mathbb{F}$, then there exist a nonsingular upper triangular matrix P and a nonsingular matrix $Q \in H_n^2(\mathbb{F})$ such that $T(A) = PA^{\sim}Q$ for all $A \in H_n(\mathbb{F})$.*

Proposition 2.27. *If $T(X) \in \{Y_a\}_{a\in\mathbb{F}}$, $T(Y) \in \{X_a\}_{a\in\mathbb{F}}$, $T(X_\alpha) \in \{Y, Y_a\}_{a\in\mathbb{F}}$ and $T(Y_\beta) \in \{X, X_a\}_{a\in\mathbb{F}}$ for some $\alpha, \beta \in \mathbb{F}$, then there exist nonsingular matrices $P \in H_n^1(\mathbb{F})$ and $Q \in H_n^2(\mathbb{F})$ such that $T(A) = PA^{\sim}Q$ for all $A \in H_n(\mathbb{F})$.*

For the above propositions, we can divide these into three kinds from the pattern of their proofs. For each kind, the proof has the same step but part of details is quite different. The first kind is of Propositions 2.20, 2.21, 2.24 and 2.25 except the case that $T(X), T(X_\alpha) \in \{X_a\}_{a\in\mathbb{F}}$, $T(Y) \in \{Y_a\}_{a\in\mathbb{F}}$, and $T(Y_\beta) = Y$ for some $\alpha, \beta \in \mathbb{F}$. The second kind is of only the remaining case of Proposition 2.25 which is shown by using $S \circ T \circ S$. The last kind is of Propositions 2.22, 2.23, 2.26 and 2.27 which are done by using $S \circ T$. Thus only the proofs of Propositions 2.25 and 2.26 are given.

Proof. **(Proposition 2.25)** Assume that $T(X) \in \{X_a\}_{a\in\mathbb{F}}$, $T(Y) \in \{Y_a\}_{a\in\mathbb{F}}$, $T(X_\alpha) \in \{X, X_a\}_{a\in\mathbb{F}}$ and $T(Y_\beta) \in \{Y, Y_a\}_{a\in\mathbb{F}}$ for some $\alpha, \beta \in \mathbb{F}$. Then there are four cases to be considered:

(i) $T(X) \in \{X_a\}_{a\in\mathbb{F}}$, $T(Y) \in \{Y_a\}_{a\in\mathbb{F}}$, $T(X_\alpha) = X$ and $T(Y_\beta) = Y$ for some $\alpha, \beta \in \mathbb{F}$, or

(ii) $T(X) \in \{X_a\}_{a\in\mathbb{F}}$, $T(Y), T(Y_\beta) \in \{Y_a\}_{a\in\mathbb{F}}$ and $T(X_\alpha) = X$ for some $\alpha, \beta \in \mathbb{F}$, or

(iii) $T(X), T(X_\alpha) \in \{X_a\}_{a\in\mathbb{F}}$, $T(Y) \in \{Y_a\}_{a\in\mathbb{F}}$, and $T(Y_\beta) = Y$ for some $\alpha, \beta \in \mathbb{F}$, or

(iv) $T(X), T(X_\alpha) \in \{X_a\}_{a\in\mathbb{F}}$ and $T(Y), T(Y_\beta) \in \{Y_a\}_{a\in\mathbb{F}}$ for some $\alpha, \beta \in \mathbb{F}$.

As mentioned above, the proofs of Cases (i), (ii) and (iv) are similar and that of Case (iii) can be done by making use of $S \circ T \circ S$. Hence we prove only Case (iv) which is more delicate than other cases and then Case (iii).

Case (iv): Let $T(X) = X_\gamma$, $T(Y) = Y_\lambda$, $T(X_\alpha) = X_\delta$ and $T(Y_\beta) = Y_\mu$ for some $\gamma, \lambda, \delta, \mu \in \mathbb{F}$. For each $1 \le i, j, k, l \le n$, there exist $u_j, w_l \in V_1 \setminus \{0\}$ and $v_i, z_k \in U_n \setminus \{0\}$ such that

$$T(E_{1j}) = (\gamma e_1 + e_2)u_j, \qquad T(E_{in}) = v_i(f_{n-1} + \lambda f_n),$$

$$T(\alpha E_{1l} + E_{2l}) = (\delta e_1 + e_2)w_l \quad \text{and} \qquad T(E_{k,n-1} + \beta E_{kn}) = z_k(f_{n-1} + \mu f_n).$$

For each $2 < s \leq t+1 < n$, let $T(E_{st}) = \overline{u}_{st}\overline{v}_{st}$ where $\overline{u}_{st} \in M_{n1}(\mathbb{F}) \smallsetminus \{0\}$ and $\overline{v}_{st} \in M_{1n}(\mathbb{F}) \smallsetminus \{0\}$. Then

$$\begin{aligned} T(E_{1t} + E_{st}) &= (\gamma e_1 + e_2)u_t + \overline{u}_{st}\overline{v}_{st} \in \Omega, \\ \text{and} \;\; T(E_{st} + E_{sn}) &= \overline{u}_{st}\overline{v}_{st} + v_s(f_{n-1} + \lambda f_n) \in \Omega. \end{aligned}$$

Thus we obtain the following four cases from Proposition 1.8 (iii):

(i) $\{\gamma e_1 + e_2, \overline{u}_{st}\}$ and $\{\overline{u}_{st}, v_s\}$ are linearly dependent.

(ii) $\{\gamma e_1 + e_2, \overline{u}_{st}\}$ and $\{\overline{v}_{st}, f_{n-1} + \lambda f_n\}$ are linearly dependent.

(iii) $\{u_t, \overline{v}_{st}\}$ and $\{\overline{u}_{st}, v_s\}$ are linearly dependent.

(iv) $\{u_t, \overline{v}_{st}\}$ and $\{\overline{v}_{st}, f_{n-1} + \lambda f_n\}$ are linearly dependent.

However, Case (i) does not hold otherwise $\{v_1, v_s\}$ is linearly dependent contradicting the linearly independence of $\{v_1, \ldots, v_n\}$. With the same manner, Case (iv) cannot occur because of the linearly independence of $\{u_1, \ldots, u_n\}$. Suppose that Case (ii) holds. Then there exist $\varsigma \in \mathbb{F}$ and a nonsingular matrix $P \in M_n(\mathbb{F})$ from Proposition 2.1 such that

$$\begin{aligned} &T(E_{1t} + E_{1n} + E_{st} + E_{sn}) \\ &= (\gamma e_1 + e_2)u_t + (\gamma e_1 + e_2)u_n + \overline{u}_{st}\overline{v}_{st} + v_s(f_{n-1} + \lambda f_n) \\ &= (\gamma e_1 + e_2)u_t + (\gamma e_1 + e_2)u_n + \varsigma(\gamma e_1 + e_2)(f_{n-1} + \lambda f_n) + v_s(f_{n-1} + \lambda f_n) \\ &= (\gamma e_1 + e_2)\big(u_t + u_n + \varsigma\,(f_{n-1} + \lambda f_n)\big) + v_s(f_{n-1} + \lambda f_n) \\ &= \begin{pmatrix} \gamma e_1 + e_2 & v_s \end{pmatrix} \begin{pmatrix} u_t + u_n + \varsigma(f_{n-1} + \lambda f_n) \\ f_{n-1} + \lambda f_n \end{pmatrix} \\ &= P\begin{pmatrix} e_1 & e_2 \end{pmatrix} \begin{pmatrix} u_t + u_n + \varsigma(f_{n-1} + \lambda f_n) \\ f_{n-1} + \lambda f_n \end{pmatrix} = P\begin{pmatrix} u_t + u_n + \varsigma(f_{n-1} + \lambda f_n) \\ f_{n-1} + \lambda f_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times n}. \end{aligned}$$

Thus $\{u_t, f_{n-1} + \lambda f_n\}$ is linearly dependent which is a contradiction. Hence Case (ii) does not occur. Then Case (iii) must hold, i.e., for each $2 < s \leq t+1 < n$, $T(E_{st}) = \overline{u}_{st}\overline{v}_{st} = \epsilon_{st} v_s u_t$ where $\epsilon_{st} \in \mathbb{F} \setminus \{0\}$.

Since $(\gamma e_1 + e_2)u_n = T(E_{1n}) = v_1(f_{n-1} + \lambda f_n)$, we obtain that $u_n \in V_{n-1} \setminus \{0\}$ and $v_1 \in U_2 \setminus \{0\}$. Then there exists $\lambda_1 \in \mathbb{F} \setminus \{0\}$ such that

$$\lambda_1 u_n = f_{n-1} + \lambda f_n \quad \text{and} \quad \lambda_1 v_1 = \gamma e_1 + e_2. \tag{2.1}$$

In addition, $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ are linearly independent sets. By Propositions 2.13 and 2.15, there exist $\eta, \sigma \in \mathbb{F}$ such that for each $1 \leq l, k \leq n$

$$T(E_{2l}) = \begin{cases} (\gamma e_1 + e_2)(w_l - \alpha u_l), & \text{if } \gamma = \delta; \\ (\eta e_1 + e_2) a_l u_l, & \text{if } \gamma \neq \delta \end{cases}$$

$$T(E_{k,n-1}) = \begin{cases} (z_k - \beta v_k)(f_{n-1} + \lambda f_n), & \text{if } \lambda = \mu; \\ b_k v_k (f_{n-1} + \sigma f_n), & \text{if } \lambda \neq \mu \end{cases}$$

where $a_l, b_k \in \mathbb{F} \setminus \{0\}$. Then $\gamma \neq \delta$ and $\lambda \neq \mu$. Since $(\eta e_1 + e_2)a_{n-1}u_{n-1} = T(E_{2,n-1}) = b_2 v_2 (f_{n-1} + \sigma f_n)$, it follows that $u_{n-1} \in V_{n-1} \setminus \{0\}$ and $v_2 \in U_2 \setminus \{0\}$. Then there exist $\lambda_2, \lambda_3 \in \mathbb{F} \setminus \{0\}$ such that

$$\lambda_2 v_2 = \eta e_1 + e_2 \quad \text{and} \quad \lambda_3 u_{n-1} = f_{n-1} + \sigma f_n. \tag{2.2}$$

We obtain from (2.1) and (2.2) that

$$\begin{aligned} T(E_{1j}) &= \lambda_1 v_1 u_j, & T(E_{in}) &= \lambda_1 v_i u_n, \\ T(E_{2l}) &= \lambda_2 v_2 a_l u_l \quad \text{and} & T(E_{k,n-1}) &= \lambda_3 b_k v_k u_{n-1}. \end{aligned}$$

Next, for each $2 < i \leq j+1 < n$, there exists a nonsingular matrix $\overline{P} \in M_n(\mathbb{F})$ by applying Proposition 2.1 such that

$$\begin{aligned} &T\left(\sum_{j=i-1}^{n} (E_{1j} + E_{2j} + E_{ij})\right) \\ &= \sum_{j=i-1}^{n} \lambda_1 v_1 u_j + \sum_{j=i-1}^{n} \lambda_2 v_2 a_j u_j + \sum_{j=i-1}^{n-2} \epsilon_{ij} v_i u_j + \lambda_3 b_i v_i u_{n-1} + \lambda_1 v_i u_n \\ &= \lambda_1 v_1 \left(\sum_{j=i-1}^{n} u_j\right) + \lambda_2 v_2 \left(\sum_{j=i-1}^{n} a_j u_j\right) + v_i \left(\sum_{j=i-1}^{n-2} \epsilon_{ij} u_j + \lambda_3 b_i u_{n-1} + \lambda_1 u_n\right) \end{aligned}$$

$$= \begin{pmatrix} \lambda_1 v_1 & \lambda_2 v_2 & v_i \end{pmatrix} \begin{pmatrix} \sum_{j=i-1}^{n} u_j \\ \sum_{j=i-1}^{n} a_j u_j \\ \sum_{j=i-1}^{n-2} \epsilon_{ij} u_j + \lambda_3 b_i u_{n-1} + \lambda_1 u_n \end{pmatrix}$$

$$= \overline{P} \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} \begin{pmatrix} \sum_{j=i-1}^{n} u_j \\ \sum_{j=i-1}^{n} a_j u_j \\ \sum_{j=i-1}^{n-2} \epsilon_{ij} u_j + \lambda_3 b_i u_{n-1} + \lambda_1 u_n \end{pmatrix}$$

$$= \overline{P} \begin{pmatrix} \sum_{j=i-1}^{n} u_j \\ \sum_{j=i-1}^{n} a_j u_j \\ \sum_{j=i-1}^{n-2} \epsilon_{ij} u_j + \lambda_3 b_i u_{n-1} + \lambda_1 u_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times n} .$$

It follows that, for each i, there exist $\kappa_i, \zeta_i \in \mathbb{F} \setminus \{0\}$ such that

$$\kappa_i \left(\sum_{j=i-1}^{n} u_j \right) = \sum_{j=i-1}^{n} a_j u_j \text{ and } \zeta_i \left(\sum_{j=i-1}^{n} u_j \right) = \sum_{j=i-1}^{n-2} \epsilon_{ij} u_j + \lambda_3 b_i u_{n-1} + \lambda_1 u_n .$$

Thus $\lambda_2 a_j = \lambda_3 b_i = \epsilon_{ij} = \lambda_1$ so that $T(E_{ij}) = \lambda_1 v_i u_j$ for all $1 \leq i, j \leq n$ with $i \leq j+1$. Choose $P = \begin{pmatrix} | & | & & | \\ \lambda_1 v_1 & \lambda_1 v_2 & \cdots & \lambda_1 v_n \\ | & | & & | \end{pmatrix}$ and $Q = \begin{pmatrix} — & u_1 & — \\ & \vdots & \\ — & u_n & — \end{pmatrix}$. Then $Pe_i = \lambda_1 v_i$, $f_j Q = u_j$ and P, Q are nonsingular matrices. Besides, for each $1 \leq i, j \leq n$ with $i \leq j+1$, $T(E_{ij}) = \lambda_1 v_i u_j = Pe_i f_j Q = PE_{ij} Q$ which forces

$$\begin{aligned} T(A) = T\left(\sum_{\substack{1 \leq i,j \leq n \\ i \leq j+1}} a_{ij} E_{ij} \right) &= \sum_{\substack{1 \leq i,j \leq n \\ i \leq j+1}} a_{ij} T(E_{ij}) = \sum_{\substack{1 \leq i,j \leq n \\ i \leq j+1}} a_{ij} P E_{ij} Q \\ &= P \left(\sum_{\substack{1 \leq i,j \leq n \\ i \leq j+1}} a_{ij} E_{ij} \right) Q = PAQ \end{aligned}$$

for all $A \in H_n(\mathbb{F})$. By Proposition 1.15 and (2.1), $P \in H_n^1(\mathbb{F})$ and $Q \in H_n^2(\mathbb{F})$ as desired.

Case (iii): Let $T(X) = X_\gamma$, $T(Y) = Y_\lambda$, $T(X_\alpha) = X_\delta$ and $T(Y_\beta) = Y$ for some $\gamma, \lambda, \delta \in \mathbb{F}$. Note that $S \circ T \circ S$ is also a linear rank-1 preserver and

$$(S \circ T \circ S)(X) = S(T(Y)) = S(Y_\lambda) = X_\lambda,$$
$$(S \circ T \circ S)(Y) = S(T(X)) = S(X_\gamma) = Y_\gamma,$$
$$(S \circ T \circ S)(X_\beta) = S(T(Y_\beta)) = S(Y) = X,$$
$$(S \circ T \circ S)(Y_\alpha) = S(T(X_\alpha)) = S(X_\delta) = Y_\delta.$$

Applying Case (ii) yields that there exist nonsingular matrices $P \in H_n^1(\mathbb{F})$ and $Q \in H_n^2(\mathbb{F})$ such that $(S \circ T \circ S)(A) = PAQ$, that is $(T(A^\sim))^\sim = PAQ$ for all $A \in H_n(\mathbb{F})$. Put $B = A^\sim$. Thus $T(B) = \big(T(B)^\sim\big)^\sim = (PB^\sim Q)^\sim = Q^\sim BP^\sim$ where $Q^\sim \in H_n^1(\mathbb{F})$ and $P^\sim \in H_n^2(\mathbb{F})$. □

Proof. **(Proposition 2.26)** Assume that $T(X) = Y_\gamma$, $T(Y) = X$, $T(X_\alpha) = Y$ and $T(Y_\beta) = X_\lambda$ for some $\alpha, \beta, \gamma, \lambda \in \mathbb{F}$. Then

$$(S \circ T)(X) = S(Y_\gamma) = X_\gamma, \qquad (S \circ T)(Y) = S(X) = Y,$$
$$(S \circ T)(X_\alpha) = S(Y) = X \quad \text{and} \qquad (S \circ T)(Y_\beta) = S(X_\lambda) = Y_\lambda.$$

By Proposition 2.24, there exist a nonsingular matrix $P \in H_n^1(\mathbb{F})$ and a nonsingular upper triangular matrix Q such that $(S \circ T)(A) = PAQ$ for all $A \in H_n(\mathbb{F})$. As a result, $T(A) = (T(A)^\sim)^\sim = \big((S \circ T)(A)\big)^\sim = (PAQ)^\sim = Q^\sim A^\sim P^\sim$ for all $A \in H_n(\mathbb{F})$ where $Q^\sim$ is a nonsingular upper triangular matrix and $P^\sim \in H_n^2(\mathbb{F})$ is a nonsingular matrix.

The other result is obtained by applying $S \circ T$ and Proposition 2.24. □

There are two major results in the main theorem. One is the existence of nonsingular upper Hessenberg matrices satisfying some certain conditions. This can be obtained from Propositions 2.20–2.27. The other is the character of $\operatorname{im} T$ which can be done by making use of Proposition 2.28.

Proposition 2.28. *The following statements hold.*

(i) If there exist $\alpha, \beta \in \mathbb{F}$ such that $T(X) = T(Y) = T(X_\alpha) = T(Y_\beta) = X$, then $\operatorname{im} T = X$.

(ii) *If there exist* $\alpha, \beta \in \mathbb{F}$ *such that* $T(X) = T(Y) = T(X_\alpha) = T(Y_\beta) = Y$*, then* $\operatorname{im} T = Y$*.*

(iii) *If there exist* $\alpha, \beta \in \mathbb{F}$ *such that* $T(X), T(Y), T(X_\alpha), T(Y_\beta) \in \{X_a\}_{a \in \mathbb{F}}$*, then* $\operatorname{im} T = X_\gamma$ *for some* $\gamma \in \mathbb{F}$*.*

(iv) *If there exist* $\alpha, \beta \in \mathbb{F}$ *such that* $T(X), T(Y), T(X_\alpha), T(Y_\beta) \in \{Y_a\}_{a \in \mathbb{F}}$*, then* $\operatorname{im} T = Y_\gamma$ *for some* $\gamma \in \mathbb{F}$*.*

Proof. We prove (i) and (iii) only.

(i) Assume that there exist $\alpha, \beta \in \mathbb{F}$ such that $T(X) = T(Y) = T(X_\alpha) = T(Y_\beta) = X$. For each $1 \le i, j, k, l \le n$, there exist $u_j, v_i, w_l, z_k \in V_1 \smallsetminus \{0\}$ such that

$$T(E_{1j}) = e_1 u_j, \qquad T(E_{in}) = e_1 v_i,$$
$$T(\alpha E_{1l} + E_{2l}) = e_1 w_l \quad \text{and} \qquad T(E_{k,n-1} + \beta E_{kn}) = e_1 z_k.$$

Then for each $1 \le i, j \le n$,

$$T(E_{i,n-1}) = e_1 z_i - \beta T(E_{in}) = e_1 z_i - \beta e_1 v_i = e_1(z_i - \beta v_i) \in X$$
$$\text{and} \;\; T(E_{2j}) = e_1 w_j - \alpha T(E_{1j}) = e_1 w_j - \alpha e_1 u_j = e_1(w_j - \alpha u_j) \in X.$$

For each $2 < s \le t+1 < n$, let $T(E_{st}) = \overline{u}_{st}\overline{v}_{st}$ where $\overline{u}_{st} \in M_{n1}(\mathbb{F}) \smallsetminus \{0\}$ and $\overline{v}_{st} \in M_{1n}(\mathbb{F}) \smallsetminus \{0\}$. Since T is a rank-1 preserver, the ranks of the following matrices equal one:

$$T(E_{1t} + E_{st}) = e_1 u_t + \overline{u}_{st}\overline{v}_{st},$$
$$T(E_{st} + E_{sn}) = \overline{u}_{st}\overline{v}_{st} + e_1 v_s,$$
$$T(E_{2t} + E_{st}) = e_1(w_t - \alpha u_t) + \overline{u}_{st}\overline{v}_{st},$$
$$T(E_{st} + E_{s,n-1}) = \overline{u}_{st}\overline{v}_{st} + e_1(z_s - \beta v_s),$$
$$T(\alpha E_{1t} + E_{2t} + E_{st}) = e_1 w_t + \overline{u}_{st}\overline{v}_{st}.$$

We claim that $\{e_1, \overline{u}_{st}\}$ is linearly dependent. Suppose not. By Proposition 1.8 (iii), it follows that $\{u_t, \overline{v}_{st}\}$, $\{\overline{v}_{st}, v_s\}$, $\{w_t - \alpha u_t, \overline{v}_{st}\}$, $\{\overline{v}_{st}, z_s - \beta v_s\}$ and $\{w_t, \overline{v}_{st}\}$ are linearly dependent sets. Then there exists $\varsigma \in \mathbb{F}$ such that

$$T(\alpha E_{1t} + \alpha E_{1,n-1} + \alpha\beta E_{1n} + E_{2t} + E_{2,n-1} + \beta E_{2n} + E_{st} + E_{s,n-1} + \beta E_{sn})$$

$$= \alpha e_1 u_t + \alpha e_1 u_{n-1} + \alpha\beta e_1 u_n + e_1(w_t - \alpha u_t) + e_1(w_{n-1} - \alpha u_{n-1}) + \beta e_1(w_n - \alpha u_n)$$
$$+ \overline{u}_{st}\overline{v}_{st} + e_1(z_s - \beta v_s) + \beta e_1 v_s$$
$$= \varsigma e_1 \overline{v}_{st} + e_1 w_{n-1} + \beta e_1 w_n + \overline{u}_{st}\overline{v}_{st} = \begin{pmatrix} \varsigma e_1 & e_1 & \beta e_1 & \overline{u}_{st} \end{pmatrix} \begin{pmatrix} \overline{v}_{st} \\ w_{n-1} \\ w_n \\ \overline{v}_{st} \end{pmatrix}$$
$$= P\begin{pmatrix} \varsigma e_1 & e_1 & \beta e_1 & e_2 \end{pmatrix} \begin{pmatrix} \overline{v}_{st} \\ w_{n-1} \\ w_n \\ \overline{v}_{st} \end{pmatrix} = P \begin{pmatrix} \varsigma \overline{v}_{st} + w_{n-1} + \beta w_n \\ \overline{v}_{st} \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n\times n},$$

where $P \in M_n(\mathbb{F})$ is a nonsingular matrix obtained from the fact that $\{e_1, \overline{u}_{st}\}$ is linearly independent. Thus $\{w_{n-1} + \beta w_n, \overline{v}_{st}\}$ is linearly dependent and hence $\{w_{n-1} + \beta w_n, w_t\}$ is linearly dependent which is absurd. As a result, $\{e_1, \overline{u}_{st}\}$ is linearly dependent. This implies that for each $2 < s \leq t+1 < n$, $T(E_{st}) = \overline{u}_{st}\overline{v}_{st} = \epsilon_{st} e_1 \overline{v}_{st} \in X$ where $\epsilon_{st} \in \mathbb{F}\setminus\{0\}$.

Next, we are ready to show that $\operatorname{im} T = X$. If $A \in H_n(\mathbb{F})$, then $T(A) = T\big(\sum_{\substack{1\leq i,j\leq n \\ i\leq j+1}} \alpha_{ij} E_{ij}\big) = \sum_{\substack{1\leq i,j\leq n \\ i\leq j+1}} \alpha_{ij} T(E_{ij}) \in X$. Moreover, if $A \in X$, then $A \in \operatorname{im} T$ because $T(X) = X$. Hence we can conclude that $\operatorname{im} T = X$.

(iii) Assume that there exist $\alpha, \beta, \gamma, \lambda, \delta, \mu \in \mathbb{F}$ such that $T(X) = X_\gamma$, $T(Y) = X_\lambda$, $T(X_\alpha) = X_\delta$ and $T(Y_\beta) = X_\mu$. For each $1 \leq i, j, k, l \leq n$, there exist $u_j, v_i, w_l, z_k \in V_1 \setminus \{0\}$ such that

$$T(E_{1j}) = (\gamma e_1 + e_2)u_j, \qquad T(E_{in}) = (\lambda e_1 + e_2)v_i,$$
$$T(\alpha E_{1l} + E_{2l}) = (\delta e_1 + e_2)w_l \quad \text{and} \quad T(E_{k,n-1} + \beta E_{kn}) = (\mu e_1 + e_2)z_k.$$

Since $(\gamma e_1 + e_2)u_n = T(E_{1n}) = (\lambda e_1 + e_2)v_1$, we obtain that $u_n = v_1$ and $\gamma = \lambda$. It is clear that $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ are linearly independent sets. From Propositions 2.13 and 2.14, there exist $\eta, \sigma \in \mathbb{F}$ such that for each $1 \leq l, k \leq n$

$$T(E_{2l}) = \begin{cases} (\gamma e_1 + e_2)(w_l - \alpha u_l), & \text{if } \gamma = \delta; \\ (\eta e_1 + e_2)a_l u_l, & \text{if } \gamma \neq \delta \end{cases}$$

$$T(E_{k,n-1}) = \begin{cases} (\lambda e_1 + e_2)(z_k - \beta v_k), & \text{if } \lambda = \mu; \\ (\sigma e_1 + e_2) b_k v_k, & \text{if } \lambda \neq \mu \end{cases}$$

for some $a_l, b_k \in \mathbb{F} \setminus \{0\}$. If $\gamma \neq \delta$, then $(\eta e_1 + e_2) a_n u_n = T(E_{2n}) = (\lambda e_1 + e_2) v_2$ which implies that $a_n u_n = v_2$ and thus $a_n v_1 = v_2$ leading to a contradiction. If $\lambda \neq \mu$, then $b_1 u_n = u_{n-1}$ which contradicts the linearly independence of u_{n-1} and u_n. This shows that $\gamma = \delta$ and $\lambda = \mu$. Thus $\delta = \gamma = \lambda = \mu$ so that

$$\begin{aligned} &T(E_{1j}) = (\gamma e_1 + e_2) u_j, && T(E_{in}) = (\gamma e_1 + e_2) v_i, \\ &T(E_{2l}) = (\gamma e_1 + e_2)(w_l - \alpha u_l), && T(E_{k,n-1}) = (\gamma e_1 + e_2)(z_k - \beta v_k) \quad \text{and} \\ &T(\alpha E_{1l} + E_{2l}) = (\gamma e_1 + e_2) w_l. \end{aligned}$$

For each $2 < s \leq t + 1 < n$, let $T(E_{st}) = \overline{u}_{st} \overline{v}_{st}$ where $\overline{u}_{st} \in M_{n1}(\mathbb{F}) \setminus \{0\}$ and $\overline{v}_{st} \in M_{1n}(\mathbb{F}) \setminus \{0\}$. Then each of the followings

$$\begin{aligned} &T(E_{1t} + E_{st}) = (\gamma e_1 + e_2) u_t + \overline{u}_{st} \overline{v}_{st}, \\ &T(E_{st} + E_{sn}) = \overline{u}_{st} \overline{v}_{st} + (\gamma e_1 + e_2) v_s, \\ &T(E_{2t} + E_{st}) = (\gamma e_1 + e_2)(w_t - \alpha u_t) + \overline{u}_{st} \overline{v}_{st}, \\ &T(E_{st} + E_{s,n-1}) = \overline{u}_{st} \overline{v}_{st} + (\gamma e_1 + e_2)(z_s - \beta v_s), \\ &T(\alpha E_{1t} + E_{2t} + E_{st}) = (\gamma e_1 + e_2) w_t + \overline{u}_{st} \overline{v}_{st} \end{aligned}$$

has rank one. It can be shown similarly to the proof of (i) that $\{\gamma e_1 + e_2, u_{st}\}$ is linearly dependent and then $T(E_{st}) = u_{st} v_{st} = \epsilon_{st} (\gamma e_1 + e_2) v_{st} \in X_\gamma$ for some $\epsilon_{st} \in \mathbb{F} \setminus \{0\}$ for all $2 < s \leq t + 1 < n$. This leads to the conclusion that $\operatorname{im} T = X_\gamma$. $\square$

We now ready to prove the main theorem.

Theorem 2.29. *Let T be a linear map on $H_n(\mathbb{F})$. Then T preserves rank-1 matrices if and only if*

(i) $\operatorname{im} T$ *is an n-dimensional rank-1 subspace, or*

(ii) there exist nonsingular upper Hessenberg matrices P and Q such that $T(A) = PAQ$ for all $A \in H_n(\mathbb{F})$ or $T(A) = PA^{\sim}Q$ for all $A \in H_n(\mathbb{F})$.

Proof. The sufficiency is clear. We prove the necessity. Recall that every n-dimensional rank-1 subspace of $H_n(\mathbb{F})$ is one of the forms X, Y, X_α or Y_α for some $\alpha \in \mathbb{F}$. Since T is a rank-1 preserver, $T(X)$, $T(Y)$, $T(X_\alpha)$ and $T(Y_\alpha)$ must be n-dimensional rank-1 subspaces of $H_n(\mathbb{F})$ for any $\alpha \in \mathbb{F}$. There are four cases to be considered as the choices of $T(X)$.

Case 1: $T(X) = X$. Proposition 2.8 provides that there are three possibilities of $T(Y)$, i.e.,

$$T(Y) = X \quad \text{or} \quad T(Y) = Y \quad \text{or} \quad T(Y) \in \{Y_a\}_{a\in\mathbb{F}}.$$

Moreover,

$$T(X_\alpha) \notin \{Y, Y_a\}_{a\in\mathbb{F}} \quad \text{and} \quad T(Y_\beta) \notin \{X_a\}_{a\in\mathbb{F}} \quad \text{for any } \alpha, \beta \in \mathbb{F}. \tag{2.3}$$

Subcase 1.1: $T(Y) = X$.

Proposition 2.9 and (2.3) force that $T(X_\alpha) = X$ and $T(Y_\beta) = X$ for all $\alpha, \beta \in \mathbb{F}$. Consequently, $\operatorname{im} T = X$ by Proposition 2.28.

Subcase 1.2: $T(Y) = Y$.

Proposition 2.10 and (2.3) yield that $T(X_\alpha) \in \{X_a\}_{a\in\mathbb{F}}$ and $T(Y_\beta) \in \{Y_a\}_{a\in\mathbb{F}}$ for all $\alpha, \beta \in \mathbb{F}$. Thus there exist nonsingular upper triangular matrices P and Q such that $T(A) = PAQ$ for all $A \in H_n(\mathbb{F})$ by Proposition 2.20.

Subcase 1.3: $T(Y) \in \{Y_a\}_{a\in\mathbb{F}}$.

It follows from Proposition 2.9 and (2.3) that $T(X_\alpha) \in \{X_a\}_{a\in\mathbb{F}}$ and $T(Y_\beta) \in \{Y, Y_a\}_{a\in\mathbb{F}}$. Thus there exist a nonsingular upper triangular matrix P and a nonsingular matrix $Q \in H_n^2(\mathbb{F})$ such that $T(A) = PAQ$ for all $A \in H_n(\mathbb{F})$ by Proposition 2.21.

Case 2: $T(X) = Y$. Then there are three choices of $T(Y)$, namely,

$$T(Y) = X \quad \text{or} \quad T(Y) = Y \quad \text{or} \quad T(Y) \in \{X_a\}_{a\in\mathbb{F}}.$$

It can be shown parallel to Case1 that if $T(Y) = Y$, then $\operatorname{im} T = Y$. Otherwise, there exist nonsingular upper Hessenberg matrices P and Q such that $T(A) = PA^{\sim}Q$ for all $A \in H_n(\mathbb{F})$.

Case 3: $T(X) \in \{X_a\}_{a\in\mathbb{F}}$. If $T(Y) \in \{X_a\}_{a\in\mathbb{F}}$, then $\operatorname{im} T \in \{X_\gamma\}_{\gamma\in\mathbb{F}}$. For the others, there exist nonsingular upper Hessenberg matrices P and Q such that $T(A) = PAQ$ for all $A \in H_n(\mathbb{F})$.

Case 4: $T(X) \in \{Y_a\}_{a\in\mathbb{F}}$. Similarly, if $T(Y) \in \{Y_a\}_{a\in\mathbb{F}}$, then $\operatorname{im} T \in \{Y_\gamma\}_{\gamma\in\mathbb{F}}$; or else there exist nonsingular upper Hessenberg matrices P and Q such that $T(A) = PA^{\sim}Q$ for all $A \in H_n(\mathbb{F})$. □

From the above theorem, observingly, a matrix P must be only an element of $H_n^1(\mathbb{F})$ or $T_n(\mathbb{F})$ and a matrix Q must be only an element of $H_n^2(\mathbb{F})$ or $T_n(\mathbb{F})$.

In general, if there is a map preserving all ranks, then this map must preserve rank one. It is also true in the case of Hessenberg matrices.

Corollary 2.30. *Let T be a nonsingular linear map on $H_n(\mathbb{F})$. Then T is a rank preserver if and only if there exist nonsingular upper Hessenberg matrices P and Q such that $T(A) = PAQ$ for all $A \in H_n(\mathbb{F})$ or $T(A) = PA^{\sim}Q$ for all $A \in H_n(\mathbb{F})$.*

Proof. By using Theorem 2.29 and the fact that if T is a rank preserver, then T is a rank-1 preserver. For the converse, it is done on account of Theorem 1.5. □

2.2 Determinant Preservers and Eigenvalue Preservers

In this section, the pattern of linear maps preserving determinant and the pattern of linear maps preserving eigenvalues are found by the hand of Theorem 2.29. First of all, we show that a map preserving determinant preserves rank-1 under the condition that this map must be nonsingular.

Proposition 2.31. *If a nonsingular linear map on $H_n(\mathbb{F})$ preserves determinant, then it also preserves rank one.*

Proof. Assume that T is a nonsingular linear map on $H_n(\mathbb{F})$ preserving determinant. Let $A \in H_n(\mathbb{F})$ with $\rho(A) = 1$. Then by Proposition 1.10, there exist nonsingular triangular matrices P_1, P_2, Q_1 and Q_2 such that

$$P_1AQ_1 = E_{pq} \quad \text{where } p, q \in \{1, \ldots, n\} \text{ with } p \leq q+1 \text{ and}$$

$$P_2T(A)Q_2 = \sum_{i=1}^{r} E_{s_it_i} := Y$$

where $s_i, t_i \in \{1, \dots, n\}$ with $s_i \le t_i + 1$ for all i; moreover, $s_i \ne s_j$ and $t_i \ne t_j$ for $i \ne j$ providing that $\rho\big(T(A)\big) = r$. Define $\phi : H_n(\mathbb{F}) \to H_n(\mathbb{F})$ by $\phi(X) = P_2T(P_1^{-1}XQ_1^{-1})Q_2$. Then ϕ is a linear map and

$$\det \phi(X) = \big(\det X\big)\big(\det(P_2P_1^{-1}Q_1^{-1}Q_2)\big) = k \det X$$

where $k = \det(P_2P_1^{-1}Q_1^{-1}Q_2)$. Moreover, $\phi(E_{pq}) = Y$. Put

$$\text{Z:=}\begin{cases} \displaystyle\sum_{\substack{i=1\\ i\ne p}}^{n} E_{ii} & \text{, if } p = q; \\ \displaystyle\sum_{\substack{i=1\\ i\ne p,q}}^{n} E_{ii} + E_{qp} & \text{, if } p = q+1 \text{ or } q = p+1; \\ \displaystyle E_{1,p-1} + \sum_{\substack{i=1\\ i+1\ne p\\ i\ne q}}^{n-1} E_{i+1,i} + E_{q+1,n} & \text{, otherwise.} \end{cases}$$

Let $\alpha \in \mathbb{F}$. From the property of the determinant when using row operations, we get that $\det(\alpha E_{pq} + Z) = \pm \det \begin{pmatrix} \alpha & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = \pm\alpha$ and $\det \phi(\alpha E_{pq} + Z) = \det\big(\alpha Y + \phi(Z)\big)$, which is a polynomial $p(\alpha)$ in α of degree at most r. We get that $p(\alpha) = \det\phi(\alpha E_{pq} + Z) = k\det(\alpha E_{pq} + Z) = \pm k\alpha$ and then $1 \le r$, thus $\rho(A) \le \rho\big(T(A)\big)$. Since T is nonsingular and T preserves determinant, we gain that $\det B = \det\big(TT^{-1}(B)\big) = \det\big(T^{-1}(B)\big)$ for all $B \in H_n(\mathbb{F})$, that is T^{-1} preserves determinant. Similarly, $\rho(A) \le \rho\big(T^{-1}(A)\big)$ for all $A \in H_n(\mathbb{F})$, hence $\rho\big(T(A)\big) \le \rho(A)$. Thus $\rho\big(T(A)\big) = \rho(A) = 1$. □

Corollary 2.32. *Let T be a nonsingular linear map on $H_n(\mathbb{F})$. If T preserves determinant, then there exist nonsingular upper Hessenberg matrices P and Q such that $T(A) = PAQ$ for all $A \in H_n(\mathbb{F})$ or $T(A) = PA^{\sim}Q$ for all $A \in H_n(\mathbb{F})$.*

If there exist nonsingular upper Hessenberg matrices P and Q such that $\det(PQ) = 1$ and $T(A) = PAQ$ for all $A \in H_n(\mathbb{F})$ or $T(A) = PA^{\sim}Q$ for all

$A \in H_n(\mathbb{F})$, then T preserves determinant.

Proof. By using Proposition 2.31 and Theorem 2.29. □

Next, the relations between maps preserving determinants and maps preserving eigenvalues are manifest.

Proposition 2.33. *If a linear map on $H_n(\mathbb{F})$ preserves determinants and maps the identity matrix into itself, then it also preserves eigenvalues.*

Proof. Let T be a linear map preserving determinants such that $T(I_n) = I_n$ and $A \in H_n(\mathbb{F})$. Then

$$\begin{aligned}
\lambda \text{ is an eigenvalue of } A \quad &\Leftrightarrow \quad \det(A - \lambda I_n) = 0 \\
&\Rightarrow \quad \det T(A - \lambda I_n) = 0 \\
&\Leftrightarrow \quad \det\big(T(A) - \lambda T(I_n)\big) = 0 \\
&\Leftrightarrow \quad \det\big(T(A) - \lambda I_n\big) = 0 \\
&\Leftrightarrow \quad \lambda \text{ is an eigenvalue of } T(A).
\end{aligned}$$

Hence T preserves eigenvalues. □

Proposition 2.34. *If $\mathbb{F}$ is an algebraically closed field, then a linear map on $H_n(\mathbb{F})$ preserving eigenvalues also preserves determinants.*

Proof. Let T be a linear map preserving eigenvalues and $A \in H_n(\mathbb{F})$. By applying Jordan canonical form, we obtain that the product of all eigenvalues of A is equal to the determinant of A. Accordingly, $\det A = \det\big(T(A)\big)$. Hence T preserves determinants. □

In addition, the relation between preserving eigenvalues and the form:

> $T(A) = PAQ$ for all $A \in H_n(\mathbb{F})$ or $T(A) = PA^{\sim}Q$ for all $A \in H_n(\mathbb{F})$ where P and Q are nonsingular upper Hessenberg matrices (*)

is given. The necessity is done by capability of Proposition 2.34 and Corollary 2.32. For the sufficiency, Corollary 2.32 and Proposition 2.33 are used as tools, see the following picture.

$$\begin{tikzcd}
\boxed{\begin{array}{l}\text{preserving}\\ \text{eigenvalues}\end{array}} \arrow[r, shift left, "\mathbb{F}\ \text{is algebraically closed}"] & \boxed{\begin{array}{l}\text{preserving}\\ \text{determinants}\end{array}} \arrow[l, shift left, "T(I_n)=I_n"] \arrow[r, shift left, "T\ \text{is nonsingular}"] & \boxed{\text{the form } (*)} \arrow[l, shift left, "\det(PQ)=1"]
\end{tikzcd}$$

Finally, a relation between determinants and traces is given.

Proposition 2.35. *If a linear map on $H_n(\mathbb{F})$ preserves determinants and maps the identity matrix into itself, then it also preserves traces.*

Proof. Let $A \in H_n(\mathbb{F})$. Assume that T preserves determinants. Then for each $x \in \mathbb{F}$,

$$\det(A - xI_n) = \det\big(T(A) - xT(I_n)\big) = \det\big(T(A) - xI_n\big).$$

However, the coefficients of x^{n-1} of $\det(A - xI_n)$ and $\det\big(T(A) - xI_n\big)$ are $\mathrm{tr}A$ and $\mathrm{tr}\big(T(A)\big)$, respectively. Hence T preserves traces. $\square$

Similarly, we can write the above relationship as follows.

$$\begin{tikzcd}
\boxed{\text{the form } (*)} \arrow[r, "\det(PQ)=1"] & \boxed{\begin{array}{l}\text{preserving}\\ \text{determinants}\end{array}} \arrow[r, "T(I_n)=I_n"] & \boxed{\begin{array}{l}\text{preserving}\\ \text{traces}\end{array}}
\end{tikzcd}$$

CHAPTER III

ADDITIVE PRESERVERS ON HESSENBERG MATRICES

In this chapter, rank-1 preservers are still investigated however linear maps are replaced by surjective additive maps. Recall that a map φ on a space V is *additive* if $\varphi(a+b) = \varphi(a) + \varphi(b)$ for any elements a and b in V. Furthermore,

$$x \otimes M_{n1}(\mathbb{F}) = \{x \otimes y \mid y \in M_{n1}(\mathbb{F})\} \qquad \text{where } x \in M_{m1}(\mathbb{F}),$$

$$M_{m1}(\mathbb{F}) \otimes y = \{x \otimes y \mid x \in M_{m1}(\mathbb{F})\} \qquad \text{where } y \in M_{n1}(\mathbb{F}) \quad \text{and}$$

$$\Omega = \{A \in H_n(\mathbb{F}) \mid \rho(A) = 1\}.$$

For certain mappings on $H_n(\mathbb{F})$, relationships between the first row and the last column of each matrix in $H_n(\mathbb{F})$ are shown as follows. Note that for a space V of matrices, set $V^t = \{A^t \mid A \in V\}$.

Lemma 3.1. *Let φ be a surjective additive rank-1 preserver on $H_n(\mathbb{F})$. Then*

(i) there exist $s, q \in \{1, \dots, n\}$ with $s \leq 2$ and $q \geq n-1$, nonzero elements $x_1 \in U_s$ and $v_n \in V_q^t$ and injective additive maps g_1, c_n on $M_{n1}(\mathbb{F})$ such that

$$\varphi(e_1 \otimes z) = x_1 \otimes g_1(z) \quad \textit{for all} \quad z \in M_{n1}(\mathbb{F})$$

$$\textit{and} \quad \varphi(z \otimes e_n) = c_n(z) \otimes v_n \quad \textit{for all} \quad z \in M_{n1}(\mathbb{F}), \ \textit{or}$$

(ii) there exist $s, q \in \{1, \dots, n\}$ with $s \leq 2$ and $q \geq n-1$, nonzero elements $u_n \in U_s$ and $y_1 \in V_q^t$ and injective additive maps d_1, h_n on $M_{n1}(\mathbb{F})$ such that

$$\varphi(e_1 \otimes z) = d_1(z) \otimes y_1 \quad \textit{for all} \quad z \in M_{n1}(\mathbb{F})$$

$$\textit{and} \quad \varphi(z \otimes e_n) = u_n \otimes h_n(z) \quad \textit{for all} \quad z \in M_{n1}(\mathbb{F}).$$

Proof. Since $e_1 \otimes M_{n1}(\mathbb{F})$ is a rank-1 subspace and φ preserves all rank-1 matrices, it follows from Proposition 1.9 that there exist $s, q \in \{1, \ldots, n\}$ with $s \leq q+1$ such that

$$\varphi\Big(e_1 \otimes M_{n1}(\mathbb{F})\Big) = x_1 \otimes M \text{ for some nonzero } x_1 \in U_s \text{ and subspace } M \text{ of } V_q^t,$$

$$\text{or} \quad \varphi\Big(e_1 \otimes M_{n1}(\mathbb{F})\Big) = N \otimes y_1 \text{ for some nonzero } y_1 \in V_q^t \text{ and subspace } N \text{ of } U_s.$$

This implies that there exists a 1-1 additive map $g_1 : M_{n1}(\mathbb{F}) \to V_q^t$ such that

$$\varphi(e_1 \otimes z) = x_1 \otimes g_1(z) \quad \text{for all} \quad z \in M_{n1}(\mathbb{F}), \tag{*}$$

or there exists a 1-1 additive map $d_1 : M_{n1}(\mathbb{F}) \to U_s$ such that

$$\varphi(e_1 \otimes z) = d_1(z) \otimes y_1 \quad \text{for all} \quad z \in M_{n1}(\mathbb{F}). \tag{**}$$

For (*), the map g_1 is 1-1 because for each $u, v \in M_{n1}(\mathbb{F})$ such that $g_1(u) = g_1(v)$, then $\varphi(e_1 \otimes u) = x_1 \otimes g_1(u) = x_1 \otimes g_1(v) = \varphi(e_1 \otimes v)$ and thus $\varphi\big(e_1 \otimes (u-v)\big) = 0$; hence, $u = v$ since φ is a rank-1 preserver. Furthermore, the map g_1 is additive because φ is additive. By virtue of the injectivity of g_1 and the conditions of s and q, we get $q = 1$ and $s \leq 2$ for (*); otherwise, $s = n$ and $n-1 \leq q \leq n$ for (**).

Hence we can say that there exist $s \in \{1, \ldots, n\}$ with $s \leq 2$, a nonzero element $x_1 \in U_s$ and a 1-1 additive map $g_1 : M_{n1}(\mathbb{F}) \to M_{n1}(\mathbb{F})$ such that

$$\varphi(e_1 \otimes z) = x_1 \otimes g_1(z) \quad \text{for all} \quad z \in M_{n1}(\mathbb{F}), \tag{1}$$

or there exist $q \in \{1, \ldots, n\}$ with $q \geq n-1$, a nonzero element $y_1 \in V_q^t$ and a 1-1 additive map $d_1 : M_{n1}(\mathbb{F}) \to M_{n1}(\mathbb{F})$ such that

$$\varphi(e_1 \otimes z) = d_1(z) \otimes y_1 \quad \text{for all} \quad z \in M_{n1}(\mathbb{F}). \tag{2}$$

Similarly, since $M_{n1}(\mathbb{F}) \otimes e_n$ is a rank-1 subspace, there exist $l \in \{1, \ldots, n\}$ with $l \leq 2$, a nonzero element $u_n \in U_l$ and a 1-1 additive map $h_n : M_{n1}(\mathbb{F}) \to M_{n1}(\mathbb{F})$ such that

$$\varphi(z \otimes e_n) = u_n \otimes h_n(z) \quad \text{for all} \quad z \in M_{n1}(\mathbb{F}), \tag{3}$$

or there exist $k \in \{1, \ldots, n\}$ with $k \geq n-1$, a nonzero element $v_n \in V_k^t$ and a 1-1 additive map $c_n : M_{n1}(\mathbb{F}) \to M_{n1}(\mathbb{F})$ such that

$$\varphi(z \otimes e_n) = c_n(z) \otimes v_n \quad \text{for all} \quad z \in M_{n1}(\mathbb{F}). \tag{4}$$

There are four possible cases. It sufficies to show that

(i) (1) and (3) cannot hold simultaneously, and

(ii) (2) and (4) cannot hold simultaneously.

However, their proofs are similar, so we prove only (i).

Suppose that (1) and (3) hold simultaneously. Since $x_1 \otimes g_1(e_n) = \varphi(e_1 \otimes e_n) = u_n \otimes h_n(e_1)$, by (ii) of Proposition 1.8, there exists a nonzero $\alpha \in \mathbb{F}$ such that $x_1 = \alpha u_n$. Thus $\varphi(e_1 \otimes z) = \alpha u_n \otimes g_1(z) = u_n \otimes \alpha g_1(z) \in u_n \otimes M_{n1}(\mathbb{F})$ for all $z \in M_{n1}(\mathbb{F})$.

Case 1: Suppose that $\varphi(\Omega) \subseteq u_n \otimes M_{n1}(\mathbb{F})$. In general, each Hessenberg matrix is the sum of finitely many rank-1 matrices. Then $\varphi\big(H_n(\mathbb{F})\big) \subseteq u_n \otimes M_{n1}(\mathbb{F})$ which contradicts the surjectivity of φ.

Case 2: $\varphi(\Omega) \nsubseteq u_n \otimes M_{n1}(\mathbb{F})$. Then there exist nonzero $x, y, u, v \in M_{n1}(\mathbb{F})$ with $x \otimes y \in \Omega$ such that $\varphi(x \otimes y) = u \otimes v$ and $\{u, u_n\}$ is linearly independent. We know that

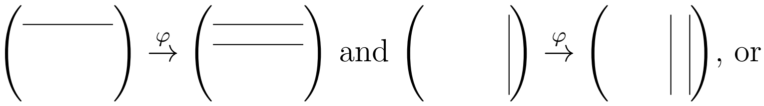
$$
\begin{aligned}
&\varphi(x \otimes y) = u \otimes v \in \Omega,\\
&\varphi\big((x + e_1) \otimes y\big) = u \otimes v + \alpha u_n \otimes g_1(y) \in \Omega,\\
&\varphi\big(x \otimes (y + e_n)\big) = u \otimes v + u_n \otimes h_n(y) \in \Omega \qquad \text{and}\\
&\varphi\big((x + e_1) \otimes (y + e_n)\big) = u \otimes v + \alpha u_n \otimes g_1(y) + u_n \otimes h_n(y) + u_n \otimes h_n(e_1) \in \Omega,
\end{aligned}
$$

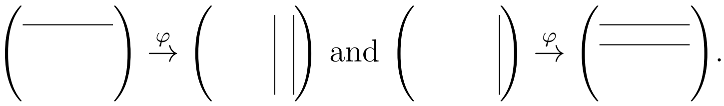
by using (iii) of Proposition 1.8 repeatedly we obtain that $\big\{v, g_1(y)\big\}$, $\big\{v, h_n(y)\big\}$ and $\big\{v, h_n(e_1)\big\}$ are linearly dependent so that there exists a nonzero $\beta \in \mathbb{F}$ such that $v = \beta h_n(e_1)$, and hence $\varphi(x \otimes y) = u \otimes \beta h_n(e_1) = \beta u \otimes h_n(e_1) \in M_{n1}(\mathbb{F}) \otimes h_n(e_1)$.

As a conclusion, $\varphi(\Omega) \subseteq u_n \otimes M_{n1}(\mathbb{F}) \cup M_{n1}(\mathbb{F}) \otimes h_n(e_1)$, it follows that $\varphi\big(H_n(\mathbb{F})\big) \subseteq u_n \otimes M_{n1}(\mathbb{F}) \cup M_{n1}(\mathbb{F}) \otimes h_n(e_1)$ which contradicts the surjectivity of φ by (iv) of Proposition 1.8. $\square$

The previous lemma indicates that, for a surjective additive rank-1 preserver φ on $H_n(\mathbb{F})$, the mapping of the last column (i.e., of the form $z \otimes e_n$) of every Hessenberg matrix via φ depends on the mapping of its first row (i.e., of the form

$e_1 \otimes z$). However, there are only two types of the mapping of the first row via φ see the followings.

(i) 

(ii) 

Next, the first two following lemmas explain the form of the mapping of each column when given the character of the mapping of the first row. Another next two following lemmas inform in case that the mapping of the last column is given. However, the proofs of Lemmas 3.2–3.5 use the same method, thereby we prove only Lemma 3.5.

Lemma 3.2. *Let φ be a surjective additive rank-1 preserver on $H_n(\mathbb{F})$ satisfying the condition (1) in the proof of Lemma 3.1. Then, for $1 \le i \le n-1$, there exist $p_i, r_i \in \{1, \dots, n\}$ with $i+1 \le p_i \le r_i + 1 \le n+1$, a nonzero element $v_i \in V_{r_i}^t$ and an injective additive map $c_i : U_{i+1} \to U_{p_i}$ such that*

$$\varphi(z \otimes e_i) = c_i(z) \otimes v_i \quad \textit{for all} \quad z \in U_{i+1}.$$

Lemma 3.3. *Let φ be a surjective additive rank-1 preserver on $H_n(\mathbb{F})$ satisfying the condition (2) in the proof of Lemma 3.1. Then, for $1 \le i \le n-1$, there exist $p_i, r_i \in \{1, \dots, n\}$ with $1 \le p_i \le r_i + 1 \le n - i + 1$, a nonzero element $u_i \in U_{p_i}$ and an injective additive map $h_i : U_{i+1} \to V_{r_i}^t$ such that*

$$\varphi(z \otimes e_i) = u_i \otimes h_i(z) \quad \textit{for all} \quad z \in U_{i+1}.$$

Lemma 3.4. *Let φ be a surjective additive rank-1 preserver on $H_n(\mathbb{F})$ satisfying the condition (3) in the proof of Lemma 3.1. Then, for $2 \le i \le n$, there exist $l_i, k_i \in \{1, \dots, n\}$ with $n - i + 2 \le l_i \le k_i + 1 \le n+1$, a nonzero element $y_i \in V_{k_i}^t$ and an injective additive map $d_i : V_{i-1}^t \to U_{l_i}$ such that*

$$\varphi(e_i \otimes z) = d_i(z) \otimes y_i \quad \textit{for all} \quad z \in V_{i-1}^t.$$

Lemma 3.5. *Let φ be a surjective additive rank-*1 *preserver on $H_n(\mathbb{F})$ satisfying the condition (4) in the proof of Lemma 3.1. Then, for $2 \leq i \leq n$, there exist $l_i, k_i \in \{1, \ldots, n\}$ with $1 \leq l_i \leq k_i + 1 \leq i$, a nonzero element $x_i \in U_{l_i}$ and an injective additive map $g_i : V_{i-1}^t \to V_{k_i}^t$ such that*

$$\varphi(e_i \otimes z) = x_i \otimes g_i(z) \quad \textit{for all} \quad z \in V_{i-1}^t.$$

Proof. Let $2 \leq i \leq n$. Since $\varphi(e_i \otimes V_{i-1}^t)$ is a rank-1 subspace, by the same way of the proof in Lemma 3.1, we obtain that there exist $l_i, k_i \in \{1, \ldots, n\}$ with $l_i \leq k_i + 1$, such that either there exist a nonzero element $x_i \in U_{l_i}$ and a 1-1 additive map $g_i : V_{i-1}^t \to V_{k_i}^t$ such that

$$\varphi(e_i \otimes z) = x_i \otimes g_i(z) \quad \text{for all} \quad z \in V_{i-1}^t, \tag{5}$$

or there exist a nonzero element $y_i \in V_{k_i}^t$ and a 1-1 additive map $d_i : V_{i-1}^t \to U_{l_i}$ such that

$$\varphi(e_i \otimes z) = d_i(z) \otimes y_i \quad \text{for all} \quad z \in V_{i-1}^t. \tag{6}$$

We show that (6) does not occur. Suppose that (6) holds. Then we obtain by applying (4) and (6) that $d_i(e_n) \otimes y_i = \varphi(e_i \otimes e_n) = c_n(e_i) \otimes v_n$ for each i. By (ii) of Proposition 1.8, there exists $\alpha \neq 0$ in $\mathbb{F}$ such that $y_i = \alpha v_n$ and hence $\varphi(e_i \otimes e_n) = d_i(e_n) \otimes \alpha v_n \in U_{l_i} \otimes v_n$. Next, let $x, y, u, v \in M_{n1}(\mathbb{F})$ be nonzero such that $\varphi(x \otimes y) = u \otimes v$ and $\{v, v_n\}$ is linearly independent. Since

$$\begin{aligned}
&\varphi\big(x \otimes y\big) = u \otimes v \in \Omega, \\
&\varphi\big((x + e_i) \otimes y\big) = u \otimes v + d_i(y) \otimes \alpha v_n \in \Omega, \\
&\varphi\big(x \otimes (y + e_n)\big) = u \otimes v + c_n(x) \otimes v_n \in \Omega, \quad \text{and} \\
&\varphi\big((x + e_i) \otimes (y + e_n)\big) = u \otimes v + d_i(y) \otimes \alpha v_n + c_n(x) \otimes v_n + c_n(e_i) \otimes v_n \in \Omega,
\end{aligned}$$

it follows that there exists $\beta \neq 0$ in $\mathbb{F}$ such that $u = \beta c_n(e_i)$ and thus $\varphi(x \otimes y) = \beta c_n(e_i) \otimes v \in c_n(e_i) \otimes M_{n1}(\mathbb{F})$. Accordingly, $\varphi(\Omega) \subseteq U_{l_i} \otimes v_n \cup c_n(e_i) \otimes M_{n1}(\mathbb{F})$ contradicting (iv) of Proposition 1.8.

In addition, from (5), since g_i is 1-1, we get that $\dim V_{i-1} \leq \dim V_{k_i}$ which equals $n - k_i + 1$. Thus $V_{i-1} \subseteq V_{k_i}$. Recall that $V_n \subseteq \cdots \subseteq V_1$, hereupon, $k_i \leq i - 1$ and thus $l_i \leq k_i + 1 \leq i$. $\square$

The following lemma results from the combination of Lemmas 3.1, 3.2 and 3.5.

Lemma 3.6. *Let φ be a surjective additive rank-1 preserver on $H_n(\mathbb{F})$ satisfying (i) of Lemma 3.1. Then the followings hold.*

(i) There exist bijective additive maps $g_1, \ldots, g_n$ and $x_1, \ldots, x_n \in M_{n1}(\mathbb{F})$ such that $g_i : V_{i-1}^t \to V_{i-1}^t$ where $V_0 = V_1$ and $x_i \in \begin{cases} U_2, & \text{if } i = 1 \\ U_i, & \text{if } i \neq 1. \end{cases}$
Moreover, such $x_1, \ldots, x_n$ are linearly independent.

(ii) There exist bijective additive maps $c_1, \ldots, c_n$ and $v_1, \ldots, v_n \in M_{n1}(\mathbb{F})$ such that $c_i : U_{i+1} \to U_{i+1}$ where $U_{n+1} = U_n$ and $v_i \in \begin{cases} V_i^t, & \text{if } i \neq n \\ V_{n-1}^t, & \text{if } i = n. \end{cases}$
Moreover, such $v_1, \ldots, v_n$ are linearly independent.

Proof. From the assumption, there exist $s, q \in \{1, \ldots, n\}$ with $s \leq 2$ and $q \geq n-1$, nonzero elements $x_1 \in U_s$ and $v_n \in V_q^t$ and injective additive maps g_1, c_n on $M_{n1}(\mathbb{F})$ such that

$$\varphi(e_1 \otimes z) = x_1 \otimes g_1(z) \quad \text{for all} \quad z \in M_{n1}(\mathbb{F}) \tag{1}$$

$$\text{and} \quad \varphi(z \otimes e_n) = c_n(z) \otimes v_n \quad \text{for all} \quad z \in M_{n1}(\mathbb{F}). \tag{2}$$

By Lemmas 3.5 and 3.2, we obtain that for all $2 \leq i \leq n$, there exist $l_i, k_i \in \{1, \ldots, n\}$ with $1 \leq l_i \leq k_i + 1 \leq i$, a nonzero element $x_i \in U_{l_i}$ and an injective additive map $g_i : V_{i-1}^t \to V_{k_i}^t$ such that

$$\varphi(e_i \otimes z) = x_i \otimes g_i(z) \quad \text{for all} \quad z \in V_{i-1}^t \tag{3}$$

and for each $1 \leq i \leq n-1$, there exist $p_i, r_i \in \{1, \ldots, n\}$ with $i+1 \leq p_i \leq r_i + 1 \leq n+1$, a nonzero element $v_i \in V_{r_i}^t$ and an injective additive map $c_i : U_{i+1} \to U_{p_i}$ such that

$$\varphi(z \otimes e_i) = c_i(z) \otimes v_i \quad \text{for all} \quad z \in U_{i+1}. \tag{4}$$

From (1) and (3), it follows that $\varphi(e_i \otimes z) = x_i \otimes g_i(z)$ and is also an element in Ω for all $z \in V_{i-1}^t$ for all $1 \leq i \leq n$ where $V_0^t = M_{n1}(\mathbb{F})$; moreover, $x_1 \in U_2$ and

$x_i \in U_i$ for all $2 \leq i \leq n$. Thus $\operatorname{im} g_1, \operatorname{im} g_2 \in V_1$ and $\operatorname{im} g_i \in V_{i-1}$ for all $i \geq 3$ by making use of Corollary 1.11. In this way,

$$g_i : V_{i-1}^t \to V_{i-1}^t \text{ for all } i \in \{1, \dots, n\} \text{ where } V_0 = V_1. \tag{5}$$

Now, first of all, since φ maps onto $H_n(\mathbb{F})$, for each $A \in H_n(\mathbb{F})$, there exists $B \in H_n(\mathbb{F})$ such that $\varphi(B) = A$; however, B can be written as $\sum_{i=1}^{n}(e_i \otimes z_i^t)$ where $B = \begin{pmatrix} — & z_1 & — \\ & \vdots & \\ — & z_n & — \end{pmatrix}$. It follows that

$$A = \varphi\Big(\sum_{i=1}^{n}(e_i \otimes z_i^t)\Big) = \sum_{i=1}^{n}\varphi\big((e_i \otimes z_i^t)\big) = \sum_{i=1}^{n}\big(x_i \otimes g_i(z_i^t)\big). \tag{6}$$

Consequently, every Hessenberg matrix A is represented by the sum of the form $x_i \otimes g_i(z_i^t)$ where z_i is the i-row of B such that $\varphi(B) = A$.

First, we are to show that $\{x_1, \dots, x_n\}$ is linearly independent. In fact, $E_{nn} \in H_n(\mathbb{F})$, $x_1 \in U_2$ and $x_i \in U_i$ for all $i \geq 2$, it follows that

$$\begin{aligned} E_{nn} &= \sum_{i=1}^{n-1}\big(x_i \otimes g_i(z_i^t)\big) + x_n \otimes g_n(z_n^t) \quad \text{where } z_i \text{ is the } i\text{-row of } B \text{ with } \varphi(B) = E_{nn} \\ &= \begin{pmatrix} & & \\ & & \\ & 0 & 0 \end{pmatrix} + \begin{pmatrix} & & \\ & & \\ & * & * \end{pmatrix} \qquad \text{where each } * \text{ is an element of } \mathbb{F}, \end{aligned}$$

accordingly, the n-position of x_n must not be zero. Moreover, with the same argument, $E_{n-1,n-2}$ is an element of $H_n(\mathbb{F})$ which forces the $(n-1)$-position of x_{n-1} must not be zero either. Similarly, the i-position of x_i must not be zero for all $i \geq 3$. Besides, $E_{21} \in H_n(\mathbb{F})$ and $x_1, x_2 \in U_2$, it follows that, without loss of generality, the 2-position of x_2 must not be zero. Hence $\{x_2, \dots, x_n\}$ is linearly independent. It remains to show that $\{x_1, \dots, x_n\}$ is linearly independent. By the above reason and properties that $x_1, x_2 \in U_2$ and $x_i \in U_i$ for all $i \geq 3$, we obtain that $\{x_1, x_i\}$ is certainly linearly independent for all $i \geq 3$. Suppose that $\{x_1, x_2\}$ is linearly dependent. Put $x_1 = \begin{pmatrix} a_1 & a_2 & 0 & \cdots & 0 \end{pmatrix}^t$ and $x_2 = \begin{pmatrix} b_1 & b_2 & 0 & \cdots & 0 \end{pmatrix}^t$ with $b_2 \neq 0$. Then $a_2 \neq 0$. If $a_1 = 0$, then $b_1 = 0$ so E_{11} would not be represented by φ, which is impossible. Thus $a_1 \neq 0$. Now, a_1 and a_2 are not zero and $x_2 = \gamma x_1$

for some nonzero γ of $\mathbb{F}$. It follows that E_{2j} would not be represented by φ for all j, hence this case does not occur. As a result, $\{x_1, x_2\}$ is linearly independent.

Next, we are to show that g_i is a bijective additive map on V_{i-1}^t for all $i \in \{1, \dots, n\}$. Fix i. Then, from (1), (3) and (5), g_i is an injective additive map on V_{i-1}^t. It remains to show that g_i is onto. Let $\overline{a} \in V_{i-1}^t$. Since $x_1 \in U_2$ and $x_k \in U_k$ for $k \geq 2$, we obtain that $x_i \otimes \overline{a} \in H_n(\mathbb{F})$. By applying (6), $x_i \otimes \overline{a} = \sum_{k=1}^n \big(x_k \otimes g_k(r_k^t)\big)$ where $r_k \in V_{k-1}$ is the k-row of B such that $\varphi(B) = x_i \otimes \overline{a}$. Then $\sum_{\substack{k=1 \\ k\neq i}}^n \big(x_k \otimes g_k(r_k^t)\big) + x_i \otimes \big(g_i(r_i^t) - \overline{a}\big) = 0$. Write $g_k(r_k^t) = \begin{pmatrix} b_{k1} & \cdots & b_{kn}\end{pmatrix}^t$ and $\overline{a} = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}^t$ with $a_j = 0$ for all $j < i-1$. Thus $\sum_{\substack{k=1 \\ k\neq i}}^n b_{kl} x_k + \big(b_{il} - a_l\big) x_i = 0$ and hence $b_{kl} = 0$ and $b_{il} = a_l$ for each $l \in \{1, \dots, n\}$ since $\{x_1, \dots, x_n\}$ is linearly independent. Then $g_k(r_k^t) = 0$ for all $k \neq i$ and $g_i(r_i^t) = \overline{a}$, respectively. As a result, g_i is onto for all i.

From (2) and (4), it implies that (ii) holds. $\square$

Theorem 3.7. *Let φ be a surjective additive map on $H_n(\mathbb{F})$. Then φ preserves rank-1 matrices if and only if there exist a field automorphism θ on $\mathbb{F}$ and non-singular upper Hessenberg matrices P and Q such that $\varphi(A) = PA^\theta Q$ for all $A \in H_n(\mathbb{F})$ or $\varphi(A) = P(A^\theta)^\sim Q$ for all $A = (a_{ij}) \in H_n(\mathbb{F})$ where $A^\theta = \big(\theta(a_{ij})\big)$.*

Proof. Assume that φ preserves rank-1 matrices. By Lemma 3.1,

(i) there exist $s, q \in \{1, \dots, n\}$ with $s \leq 2$ and $q \geq n-1$, nonzero elements $x_1 \in U_s$ and $v_n \in V_q^t$ and injective additive maps g_1, c_n on $M_{n1}(\mathbb{F})$ such that

$$\varphi(e_1 \otimes z) = x_1 \otimes g_1(z) \quad \text{for all} \quad z \in M_{n1}(\mathbb{F})$$
$$\text{and} \quad \varphi(z \otimes e_n) = c_n(z) \otimes v_n \quad \text{for all} \quad z \in M_{n1}(\mathbb{F}), \text{ or}$$

(ii) there exist $s, q \in \{1, \dots, n\}$ with $s \leq 2$ and $q \geq n-1$, nonzero elements $u_n \in U_s$ and $y_1 \in V_q^t$ and injective additive maps d_1, h_n on $M_{n1}(\mathbb{F})$ such that

$$\varphi(e_1 \otimes z) = d_1(z) \otimes y_1 \quad \text{for all} \quad z \in M_{n1}(\mathbb{F})$$
$$\text{and} \quad \varphi(z \otimes e_n) = u_n \otimes h_n(z) \quad \text{for all} \quad z \in M_{n1}(\mathbb{F}).$$

Case 1: Assume that (i) holds. Then by Lemma 3.6, we obtain that $\{x_1, \ldots, x_n\}$ and $\{v_1, \ldots, v_n\}$ are linearly independent where $x_1 \in U_2$, $x_i \in U_i$ for all $i \in \{2, \ldots, n\}$, $v_i \in V_i^t$ for all $i \in \{1, \ldots, n-1\}$ and $v_n \in V_{n-1}^t$. Furthermore, g_i and c_i are also bijective additive maps on V_{i-1}^t and on U_{i+1}, respectively, for all $i \in \{1, \ldots, n\}$.

Let $X = \begin{pmatrix} | & & | \\ x_1 & \ldots & x_n \\ | & & | \end{pmatrix}$ and $Y = \begin{pmatrix} - & v_1^t & - \\ & \vdots & \\ - & v_n^t & - \end{pmatrix}$. Then $X \in H_n^1(\mathbb{F}) \cup T_n(\mathbb{F})$ which is nonsingular and $Xe_i = x_i$ for all i. Put $P_1 = X^{-1}$. Then $e_i = P_1 x_i$ for all i and $P_1 \in H_n^1(\mathbb{F}) \cup T_n(\mathbb{F})$.

Similarly, $Y \in H_n^2(\mathbb{F}) \cup T_n(\mathbb{F})$ which is nonsingular and $e_i Y = v_i^t$ for all i. Put $Q_1 = Y^{-1}$. Then $e_i = v_i^t Q_1$ for all i and $Q_1 \in H_n^2(\mathbb{F}) \cup T_n(\mathbb{F})$.

Let $\varphi_1 : H_n(\mathbb{F}) \to M_n(\mathbb{F})$ be defined by $\varphi_1(X) = P_1\varphi(X)Q_1$ for all $X \in H_n(\mathbb{F})$. Then $P_1\varphi(X)Q_1 \in H_n(\mathbb{F})$ for all $X \in H_n(\mathbb{F})$, i.e., $\varphi_1 : H_n(\mathbb{F}) \to H_n(\mathbb{F})$ from Proposition 1.15. In fact, φ_1 is a surjective additive rank-1 preserver resulted from φ. Fix $i \in \{1, \ldots, n\}$. For each $z \in M_{n1}(\mathbb{F})$ with $e_i \otimes z \in H_n(\mathbb{F})$, by applying (3) in the proof of Lemma 3.6, we get that

$$\varphi_1(e_i \otimes z) = P_1\varphi(e_i \otimes z)Q_1 = P_1\big(x_i \otimes g_i(z)\big)Q_1 = e_i \otimes Q_1^t g_i(z),$$

similarly,

$$\varphi_1(z \otimes e_i) = P_1\varphi(z \otimes e_i)Q_1 = P_1\big(c_i(z) \otimes v_i\big)Q_1 = P_1\big(c_i(z)\big) \otimes e_i.$$

Let $\psi_i(z) = Q_1^t g_i(z)$ where $z \in V_{i-1}^t$ when $V_0^t = M_{n1}(\mathbb{F})$ and $\phi_i(z) = P_1\big(c_i(z)\big)$ where $z \in V_{i+1}$ when $U_{n+1} = M_{1n}(\mathbb{F})$. Then ψ_i and ϕ_i are bijective additive maps on V_{i-1}^t and on U_{i+1}, respectively, for all i by the virtue of g_i and c_i, respectively.

Let $c \in \mathbb{F}$ and $i, j \in \{1, \ldots, n\}$ with $i \leq j+1$. Since

$$e_i \otimes \psi_i(ce_j) = \varphi_1(e_i \otimes ce_j) = \varphi_1(ce_i \otimes e_j) = \phi_j(ce_i) \otimes e_j,$$

it follows that there exists $\alpha_{ij}(c) \in \mathbb{F} \smallsetminus \{0\}$ such that $\psi_i(ce_j) = \alpha_{ij}(c)e_j$ owing to (ii) of Proposition 1.8. Then we are to show that $\alpha_{ij} : \mathbb{F} \to \mathbb{F}$ is a bijective additive map.

Suppose that $\alpha_{ij}(a) = \alpha_{ij}(b)$ for any $a, b \in \mathbb{F}$. Then $\psi_i(ae_j) = \alpha_{ij}(a)e_j = \alpha_{ij}(b)e_j = \psi_i(be_j)$. Since ψ_i is a 1-1 additive map, α_{ij} is additive and $ae_j - be_j = 0$ and hence $a = b$. Thus α_{ij} is a 1-1 additive map. To show that α_{ij} is onto, let $a \in \mathbb{F}$. Then $(Q_1^t)^{-1}ae_j \in V_{i-1}^t$ owing to $i \leq j+1$. Since g_i is onto V_{i-1}^t, it follows that $g_i(\overline{z}) = (Q_1^t)^{-1}ae_j$ for some $\overline{z} \in V_{i-1}^t$. Hence

$$\psi_i(\overline{z}) = Q_1^t g_i(\overline{z}) = Q_1^t(Q_1^t)^{-1}ae_j = ae_j.$$

However,

$$\psi_i(\overline{z}) = \psi_i\big(\sum_{k=i-1}^{n} b_k e_k\big) = \sum_{k=i-1}^{n} \alpha_{ik}(b_k)e_k$$

where $\overline{z} = \sum_{k=i-1}^{n} b_k e_k$ with $b_k \in \mathbb{F}$ for all k. It follows that $a = \alpha_{ij}(b_j)$ thus α_{ij} is onto $\mathbb{F}$.

As a result, $\varphi_1(cE_{ij}) = \varphi_1(ce_i \otimes e_j) = e_i \otimes \psi_i(ce_j) = e_i \otimes \alpha_{ij}(c)e_j = \alpha_{ij}(c)E_{ij}$. In another word, $\varphi_1(cE_{ij}) = \alpha_{ij}(c)E_{ij}$ for any $c \in \mathbb{F}$ and i, j such that $i \leq j+1$ where $\alpha_{ij}(c) \in \mathbb{F} \smallsetminus \{0\}$.

Let $\varphi_2 : H_n(\mathbb{F}) \to H_n(\mathbb{F})$ be defined by $\varphi_2(X) = P_2\varphi_1(X)Q_2$ for all $X \in H_n(\mathbb{F})$ where $P_2 = \begin{pmatrix} \alpha_{1n}(1)^{-1} & & \\ & \ddots & \\ & & \alpha_{nn}(1)^{-1} \end{pmatrix}$, $Q_2 = \begin{pmatrix} \alpha_{1n}(1)\alpha_{11}(1)^{-1} & & \\ & \ddots & \\ & & \alpha_{1n}(1)\alpha_{1n}(1)^{-1} \end{pmatrix}$ and $\alpha_{ij}(1)^{-1}$ is the inverse of $\alpha_{ij}(1)$ for all i, j. Then φ_2 is a surjective additive rank-1 preserver on $H_n(\mathbb{F})$. Furthermore,

$$\begin{aligned}
\varphi_2(cE_{ij}) &= P_2\varphi_1(cE_{ij})Q_2 = P_2\alpha_{ij}(c)E_{ij}Q_2 \\
&= P_2\alpha_{ij}(c)\big(e_i \otimes e_j\big)Q_2 = \alpha_{ij}(c)\big(P_2e_i\big)\big(e_j^tQ_2\big) \\
&= \beta_{ij}(c)E_{ij} \quad \text{where} \quad \beta_{ij}(c) = \alpha_{ij}(c)\alpha_{in}(1)^{-1}\alpha_{1n}(1)\alpha_{1j}(1)^{-1}.
\end{aligned}$$

For each $k \in \{1, \ldots, n\}$, we obtain that

$$\begin{aligned}
&\varphi_2(E_{1k}) = \alpha_{1k}(1)\alpha_{1n}(1)^{-1}\alpha_{1n}(1)\alpha_{1k}(1)^{-1}E_{1k} = E_{1k} \\
\text{and} \quad &\varphi_2(E_{kn}) = \alpha_{kn}(1)\alpha_{kn}(1)^{-1}\alpha_{1n}(1)\alpha_{1n}(1)^{-1}E_{kn} = E_{kn}.
\end{aligned}$$

Hence

$$\beta_{1k}(1) = 1 = \beta_{kn}(1) \quad \text{for all } k. \tag{1}$$

Similarly, it can be shown that β_{ij} is a bijective additive map on $\mathbb{F}$.

Next, let $c \in \mathbb{F}$. We are showing that $\beta_{1j}(c) = \beta_{1n}(c) = \beta_{in}(c)$ for all $i, j \in \{1, \ldots, n\}$. Without loss of generality, since

$$\begin{array}{cc} & \begin{array}{cc} j^{\text{th}} & n^{\text{th}} \\ \downarrow & \downarrow \end{array} \\ \begin{array}{c} 1^{\text{st}}\to \\ \\ i^{\text{th}}\to \\ \\ \end{array} & \begin{pmatrix} c & c \\ & \\ 1 & 1 \\ & \end{pmatrix} \end{array} \in \Omega$$

and φ_2 is an additive rank-1 preserver, we get that

$$\begin{array}{cc} & \begin{array}{cc} j^{\text{th}} & n^{\text{th}} \\ \downarrow & \downarrow \end{array} \\ \begin{array}{c} 1^{\text{st}}\to \\ \\ i^{\text{th}}\to \\ \\ \end{array} & \begin{pmatrix} \beta_{1j}(c) & \beta_{1n}(c) \\ & \\ \beta_{ij}(1) & \beta_{in}(1) \\ & \end{pmatrix} \end{array} \in \Omega.$$

By Theorem 1.6 and (1), we obtain $\beta_{1n}(c)\beta_{ij}(1) = \beta_{1j}(c)\beta_{in}(1) = \beta_{1j}(c)$. In particular, letting $c = 1$ implies $\beta_{ij}(1) = 1$. Hence $\beta_{1n}(c) = \beta_{1j}(c)$ for all j.

Moreover, since $E_{1j} + cE_{1n} + E_{ij} + cE_{in}$ has rank one, $\beta_{1j}(1)E_{1j} + \beta_{1n}(c)E_{1n} + \beta_{ij}(1)E_{ij} + \beta_{in}(c)E_{in}$ must have rank one and then $\beta_{in}(c) = \beta_{1n}(c)\beta_{ij}(1)$ by (1). In addition, $\beta_{ij}(1) = 1$ as $c = 1$. Hence $\beta_{in}(c) = \beta_{1n}(c)$ for all i.

In order to prove that $\beta_{pq}(c) = \beta_{1n}(c)$ for all $p, q \in \{1, \ldots, n\}$, given $p, q \in \{1, \ldots, n\}$ and use the same argument on $cE_{1q} + E_{1n} + cE_{pq} + E_{pn}$. Hence $\beta_{pq}(c) = \beta_{1n}(c)$ for all $p, q \in \{1, \ldots, n\}$. Put $\theta = \beta_{1n}$. Then θ is a bijective additive map on $\mathbb{F}$ such that for all $i, j \in \{1, \ldots, n\}$, we get

$$\varphi_2(cE_{ij}) = \beta_{ij}(c)E_{ij} = \beta_{1n}(c)E_{ij} = \theta(c)E_{ij}.$$

Besides, $\theta(ab) = \theta(a)\theta(b)$ for all $a, b \in \mathbb{F}$ by using the same manner as proving that $\beta_{in}(c) = \beta_{1n}(c)$ for all i on $E_{11} + aE_{1n} + bE_{21} + abE_{2n}$ and the fact that $\theta(1) = 1$ because $\beta_{1n}(1) = 1$. Thereby, θ is a field automorphism on $\mathbb{F}$.

Now, for each $i, j \in \{1, \ldots, n\}$, we know that

$$\begin{aligned} \varphi(cE_{ij}) &= P_1^{-1}\varphi_1(cE_{ij})Q_1^{-1} \\ &= X\varphi_1(cE_{ij})Y \\ &= XP_2^{-1}\varphi_2(cE_{ij})Q_2^{-1}Y \\ &= XP_2^{-1}\theta(c)E_{ij}Q_2^{-1}Y \end{aligned}$$

$$= P\theta(c)E_{ij}Q \quad \text{where} \quad P = XP_2^{-1} \text{ and } Q = Q_2^{-1}Y.$$

Hence for each $A \in H_n(\mathbb{F})$, we obtain that

$$\begin{aligned}
\varphi(A) &= \varphi\Big(\sum c_{ij}E_{ij}\Big) \\
&= \sum \varphi(c_{ij}E_{ij}) \\
&= \sum P\theta(c_{ij})E_{ij}Q \\
&= P\Big(\sum \theta(c_{ij})E_{ij}\Big)Q = PA^\theta Q \quad \text{where} \quad A^\theta = \big(\theta(a_{ij})\big).
\end{aligned}$$

Case 2: Assume that (ii) holds. Then there exist $s, q \in \{1, \ldots, n\}$ such that $s \leq 2$ and $q \geq n-1$, nonzero elements $u_n \in U_s$ and $y_1 \in V_q^t$ and injective additive maps d_1, h_n on $M_{n1}(\mathbb{F})$ such that

$$\begin{aligned}
&\varphi(e_1 \otimes z) = d_1(z) \otimes y_1 \quad \text{for all} \quad z \in M_{n1}(\mathbb{F}) \\
\text{and} \quad &\varphi(z \otimes e_n) = u_n \otimes h_n(z) \quad \text{for all} \quad z \in M_{n1}(\mathbb{F}).
\end{aligned}$$

By capability of S, we get that $S \circ \varphi \begin{pmatrix} \text{———} \\ \\ \\ \end{pmatrix} = S \begin{pmatrix} & \vert & \vert \\ & \vert & \vert \\ & \vert & \vert \end{pmatrix} = \begin{pmatrix} \text{———} \\ \text{———} \\ \\ \end{pmatrix}$ and $S \circ \varphi \begin{pmatrix} & \vert \\ & \vert \\ & \vert \end{pmatrix} = S \begin{pmatrix} \text{———} \\ \text{———} \\ \\ \end{pmatrix} = \begin{pmatrix} & \vert & \vert \\ & \vert & \vert \\ & \vert & \vert \end{pmatrix}$. Besides, $S \circ \varphi$ is a surjective additive rank-1 preserver. Making use of Case 1, there exist $P \in H_n^1(\mathbb{F}) \cup T_n(\mathbb{F})$ and $Q \in H_n^2(\mathbb{F}) \cup T_n(\mathbb{F})$ such that $S \circ \varphi(A) = PA^\theta Q$ for all $A \in H_n(\mathbb{F})$. Hence for each $A \in H_n(\mathbb{F})$, we get that $\varphi(A) = \big(\varphi(A)^\sim\big)^\sim = \big(PA^\theta Q\big)^\sim = Q^\sim (A^\theta)^\sim P^\sim$ where $Q^\sim \in H_n^1(\mathbb{F}) \cup T_n(\mathbb{F})$ and $P^\sim \in H_n^2(\mathbb{F}) \cup T_n(\mathbb{F})$.

For the sufficient part, let $A \in H_n(\mathbb{F})$ such that $\rho(A) = 1$. By the assumption, the property of nonsingular matrices and the property of $\sim$, we obtain that $\rho\big(\varphi(A)\big) = \rho(A^\theta)$. It remains to show that $\rho(A^\theta) = 1$, or equivalently to show that every two rows of A^θ are linearly dependent. Let $A = \begin{pmatrix} — & r_1 & — \\ & \vdots & \\ — & r_n & — \end{pmatrix}$ and

$A^\theta = \begin{pmatrix} — & R_1 & — \\ & \vdots & \\ — & R_n & — \end{pmatrix}$ where $r_i = \begin{pmatrix} a_{i1} & \cdots & a_{in} \end{pmatrix}$ and $R_i = \begin{pmatrix} \theta(a_{i1}) & \cdots & \theta(a_{in}) \end{pmatrix}$

for all i. Fix $i \neq j$ and let $\alpha, \beta \in \mathbb{F}$ be such that $\alpha R_i + \beta R_j = 0$. Since θ is onto, there exist $a, b \in \mathbb{F}$ such that $\theta(a) = \alpha$ and $\theta(b) = \beta$. Then, for each $1 \leq k \leq n$, $\theta(aa_{ik} + ba_{jk}) = \theta(a)\theta(a_{ik}) + \theta(b)\theta(a_{jk}) = 0$ and hence $aa_{ik} + ba_{jk} = 0$ because θ is 1-1. Thus $ar_i + br_j = 0$. Since $\rho(A) = 1$, it forces either $a \neq 0$ or $b \neq 0$. Accordingly, $\alpha \neq 0$ or $\beta \neq 0$ on account of the injectivity of θ and thus R_i and R_j are linearly dependent. $\square$

Corollary 3.8. *Let φ be a surjective additive map on $H_n(\mathbb{F})$. Then φ is a rank preserver if and only if there exist a field automorphism θ on $\mathbb{F}$ and nonsingular upper Hessenberg matrices P and Q such that $\varphi(A) = PA^\theta Q$ for all $A \in H_n(\mathbb{F})$ or $\varphi(A) = P(A^\theta)^\sim Q$ for all $A = (a_{ij}) \in H_n(\mathbb{F})$ where $A^\theta = \big(\theta(a_{ij})\big)$.*

Proof. For the necessary part, by using Theorem 3.7 and the fact that if φ is a rank preserver, then φ is a rank-1 preserver. For the converse, it is done on account of Theorem 1.5 and the property of θ. $\square$

CHAPTER IV
CONCLUSION

This dissertation is motivated by various research, especially, the research of Minc [19] and Chooi and Lim [8]. Notice that these results are quite similar although they are studied in different spaces. Besides, it seems that the $\sim$ of matrices acts instead of the transpose of matrices in the case of triangular matrices.

In this work, the space of Hessenberg matrices is chosen among various types of matrices. For the first reason, Hessenberg matrices are full matrices and triangular matrices also are Hessenberg. Another reason is that Hessenberg matrices are applied in many areas such as applied mathematics and quantum theory of Physics. From whole reasons, it make us investigate linear rank-1 preservers on Hessenberg matrices. Theorem 2.29 provides the standard form in the sense of Hessenberg matrices. Like the spaces of upper triangular matrices, the $\sim$ is needed for Hessenberg matrices since the $\sim$ of upper Hessenberg matrices are still upper Hessenberg matrices but the transpose of upper Hessenberg matrices become lower Hessenberg matrices.

This result leads to study a linear preserving determinants and a linear preserving eigenvalues. Observingly, the pattern of linear maps preserving determinants on $H_n(\mathbb{F})$ and on $M_n(\mathbb{F})$ are more similar than that of linear maps preserving determinants on $T_n(\mathbb{F})$. Unsurprisingly, the determinant of each upper triangular matrices is the product of all elements in its main diagonal, hence the pattern of linear maps preserving determinants on $T_n(\mathbb{F})$ relates to only entries on its main diagonal. However, a Hessenberg matrix has one subdiagonal added, thus its determinant should not be considered in the same way as the determinant of a triangular matrix.

Furthermore, Theorem 2.29 can be generalized by replacing linear maps with surjective additive maps as Theorem 3.7.

Let T be a surjective linear map on $H_n(\mathbb{F})$ preserving rank one. Then both of Theorems 2.29 and 3.7 can be applied on this T. In another word, there are nonsingular upper Hessenberg matrices P and Q such that

$$\begin{aligned} T(A) &= PAQ \qquad \text{for all} \quad A \in H_n(\mathbb{F}) \\ \text{or} \quad T(A) &= PA^{\sim}Q \qquad \text{for all} \quad A \in H_n(\mathbb{F}) \end{aligned}$$

and there are nonsingular upper Hessenberg matrices X and Y and an injective additive map θ on $\mathbb{F}$ such that

$$\begin{aligned} T(A) &= XA^{\theta}Y \qquad \text{for all} \quad A \in H_n(\mathbb{F}) \\ \text{or} \quad T(A) &= X(A^{\theta})^{\sim}Y \qquad \text{for all} \quad A \in H_n(\mathbb{F}) \end{aligned}$$

where $A^{\theta} = \big(\theta(a_{ij})\big)$ for $A = (a_{ij})$. To be certain that these results are the same, it is enough to show that $A^{\theta} = A$ for any A in $H_n(\mathbb{F})$. Thus, it is adequate to prove that θ is the identity map.

Let $a \in \mathbb{F}$. From the proof of Theorem 3.7, $T(a_{ij}E_{ij}) = \alpha(a_{ij})E_{ij}$ for all i, j with $i \leq j + 1$ when $A = (a_{ij})$. In particular, $T(aE_{11}) = \alpha(a)E_{11}$ for $A = aE_{11}$. However, $T(aE_{11}) = aT(E_{11})$ because T is a linear map. Since $\alpha(1) = 1$, we get that

$$\alpha(a)E_{11} = T(aE_{11}) = aT(E_{11}) = a\alpha(1)E_{11} = aE_{11}.$$

Hence α is the identity map.

In my opinion, this work can be generalized by neglecting the surjectivity condition and may still have the same result since there is a research of Zhang and Sze [25] concerning additive rank-1 preservers between spaces of rectangular matrices. Furthermore, the result of linear maps preserving rank-k matrices are convinced to be similar to Theorem 2.29 provided that we can show that linear rank-k preservers are linear rank-1 preservers because this property holds on $M_{mn}(\mathbb{F})$. Moreover, the results of additive maps preserving rank-k, additive maps preserving determinants and additive maps preserving eigenvalues should have the same type.

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