Chapter II

Preliminary

In this chapter, we discuss about the 3 parts that lead us to this research. They are about convex programming, convex functions, and some conjectures about the shortest arc for each considered cover of this research.

1. Convex Programming

Convex programming studies the case when the objective function is convex and the constraints, if any, form a convex set. This can be viewed as a particular case of nonlinear programming or as generalization of linear or convex quadratic programming.

Convex optimization is a subfield of mathematical optimization. Given a real vector space V together with a convex, real-valued function $f: X \to \mathbb{R}$ defined on a convex subset X of V, the problem is to find the point x^* in X for which the number $f(x)$ is smallest, i.e., the point x^* such that $f(x^*) \le f(x)$ for all $x \in X$.

The following statements are true in convex optimization.

- If a local minimum exists, it's a global minimum.
- The set of all global minima is convex.
- If the function is strictly convex, there exists at most one minimum

2. Convex Functions

Definition Let A be a vector space. A function $f: A \to \mathbb{R}$ is **convex** if for all $x_1, x_2 \in A$,

then we have $f(\frac{x_1+x_2}{2}) \le \frac{f(x_1)+f(x_2)}{2}$

Lemma 1 Let f_1 and f_2 be convex functions from a vector space A to the set of real numbers. Then $f_1 + f_2$ is also a convex function.

Proof Let x_1 and x_2 be in A.

$$
\begin{aligned}
(f_1+f_2)(\tfrac{x_1+x_2}{2}) &\le f_1(\tfrac{x_1+x_2}{2})+f_2(\tfrac{x_1+x_2}{2}) \\
&= \frac{f_1(x_1)+f_1(x_2)}{2}+\frac{f_2(x_1)+f_2(x_2)}{2} \\
&= \frac{f_1(x_1)+f_1(x_2)+f_2(x_1)+f_2(x_2)}{2} \\
&= \frac{f_1(x_1)+f_2(x_1)+f_1(x_2)+f_2(x_2)}{2} \\
&= \frac{(f_1+f_2)(x_1)+(f_1+f_2)(x_2)}{2}
\end{aligned}
$$

Thus, f_1+f_2 is a convex function. □

Lemma 2 Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined that $f(x,y)=\sqrt{x^2+y^2}$. Then f is convex.

Proof

$$
\begin{aligned}
0 &\le (x_1y_2-x_2y_1)^2 \\
0 &\le x_1^2y_2^2-2x_1y_2x_2y_1+x_2^2y_1^2 \\
2x_1y_2x_2y_1 &\le x_1^2y_2^2+x_2^2y_1^2 \\
x_1^2x_2^2+2x_1y_2x_2y_1+y_1^2y_2^2 &\le x_1^2x_2^2+y_1^2y_2^2+x_1^2y_2^2+x_2^2y_1^2 \\
\left(x_1x_2+y_1y_2\right)^2 &\le (x_1^2+y_1^2)(x_2^2+y_2^2) \\
x_1x_2+y_1y_2 &\le \sqrt{(x_1^2+y_1^2)}\sqrt{(x_2^2+y_2^2)} \\
2x_1x_2+2y_1y_2 &\le 2\sqrt{(x_1^2+y_1^2)}\sqrt{(x_2^2+y_2^2)} \\
x_1^2+y_1^2+2x_1x_2+2y_1y_2+x_2^2+y_2^2 &\le x_1^2+y_1^2+2\sqrt{(x_1^2+y_1^2)}\sqrt{(x_2^2+y_2^2)}+x_2^2+y_2^2 \\
\left(x_1+x_2\right)^2+\left(y_1+y_2\right)^2 &\le \left(\sqrt{(x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}\right)^2 \\
\frac{\left(x_1+x_2\right)^2+\left(y_1+y_2\right)^2}{4} &\le \frac{\left(\sqrt{(x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}\right)^2}{4} \\
\sqrt{\left(\frac{x_1+x_2}{2}\right)^2+\left(\frac{y_1+y_2}{2}\right)^2} &\le \frac{\sqrt{(x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}}{2}
\end{aligned}
$$

Thus, $\sqrt{x^2+y^2}$ is a convex on x and y. □

According to Lemma 1 and 2, they lead us to the corollary bellows.

Corollary 1 Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined that $f(x_1,x_2)=\sqrt{{x_1}^2+{x_2}^2}$, and x,y be linear functions from $\mathbb{R}^n$ to $\mathbb{R}$ .Then the composition $f(x,y)$ is convex.

3. Conjectures

In this research, we consider 3 sets nominating to be covers – equilateral triangle, isosceles right-angled triangle, and $30^\circ - 60^\circ - 90^\circ$ triangle. In order to consider them easier, we scales the considered set to make the longest side of length 1 unit. This means if the scaled considered set can cover every arc of length ℓ, then the considered set with the longest side of length $\frac{1}{\ell}$ can cover any unit arc. The scaled considered sets are an equilateral triangle with unit side, an isosceles right-angled triangle with unit hypotenuse, a $30^\circ - 60^\circ - 90^\circ$ triangle with unit hypotenuse.

3.1 An equilateral triangle with unit side

Figure 1.7 : Besicovitch Z-arc of length $\sqrt{\frac{27}{28}} \simeq 0.981981$ in a unit equilateral triangle

In 1963, R. Graham stated a question "what is the shortest arc in the plane that does not fit in an open equilateral triangle of side 1?" Shortly after that, Besicovitch described that a three-segment polygonal arc of length $\sqrt{\frac{27}{28}} < 1$, called Besicovitch Z-arc, doesn't fit in an open equilateral triangle of side 1. Moreover, he conjectured that every equal of shorter arc can be covered by the closed equilateral triangle of side 1. This conjecture has been unable to be proved until 2006. It is proved by P. Coulton and Y. Movshovich [6]

3.2 An isosceles right-angled triangle with unit hypotenuse

In 1970's, John E. Wetzel [Wetzel] proved that an isosceles right-angled triangle with unit hypotenuse (area 0.25) can cover every convex unit arc. That is the beginning of the problem whether it is still a cover for every unit arc,

Figure 1.8 : Z- shaped three-segment polygonal arc of length $\frac{\sqrt{9+\cot^2 45^\circ}}{3} = 0.948683$ in an isosceles right-angled triangle with unit hypotenuse

absolutely "No". Then the problem "What is the shortest arc in the plane that does not fit in an open isosceles right triangle with unit hypotenuse?" follows. In private discussion with Wetzel [7], we have a conjecture that a Z- shaped three-segment polygonal arc of length $\frac{\sqrt{9+\cot^2 45^\circ}}{3} = 0.948683$ may be the solution.

3.3 A 30°- 60°- 90° triangle with unit hypotenuse

According to the classical worm problem-a status report by John E. Wetzel [7], there is a conjecture that the $30^\circ - 60^\circ - 90^\circ$ triangle with legs $\frac{1}{6}(4+\sqrt{3})$ and $\frac{1}{18}(4+\sqrt{3})$, and hypotenuse $\frac{1}{9}(4+\sqrt{3})$ just large enough to contain a square of side $\frac{1}{3}$ resting on it's hypotenuse is a cover.

Hence, we can conjecture that a staple shaped three-segment polygonal arc of length $\frac{9}{(3+4\sqrt{3})} \approx 0.906508$ may be the shortest arc that does not fit in an open $30^\circ - 60^\circ - 90^\circ$ triangle with unit hypotenuse. The interesting thing in this set is that it has no symmetric lines.

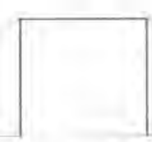

Figure 1.9 : A staple shaped three-segment polygonal arc of length $\frac{9}{(3+4\sqrt{3})} \approx 0.906508$ in a 30°- 60°- 90° triangle with unit hypotenuse