## Chapter IV

## The results

In this chapter, we show the results of the shortest arc of the scaled considered set. The results are obtained from programming by using Mathematica.

### 4.1 Equilateral Triangle

The aim is to show that an equilateral ciangle of unit side can cover every arc of length $\ell_{1}=\sqrt{\frac{27}{28}} \approx 0.981981$. To stroy this we suppose that an arc $\gamma$ cannot be covered. Then we will show that the leogth of $\gamma$ is greater than $\ell_{1}$. Thus, $\gamma$ must not be covered by the triangle in the standing position and vis yeflection.


Figure 4.1 : An equilateral triangle $A_{1} B_{1} C_{1}$ in standing position and is its reflection $\mathrm{A}_{2} \mathrm{~B}_{2} \mathrm{C}_{2}$

Let $P_{1}\left(x_{1}, y_{1}\right), P_{2}\left(x_{2}, y_{2}\right), P_{4}\left(x_{4}, y_{4}\right)$, and $P_{5}\left(x_{5}, y_{5}\right)$ be points on the side $\overline{A_{1} C_{1}}, \overline{A_{1} B_{1}}$ $\overline{\mathrm{A}_{2} C_{2}}$ and $\overline{\mathrm{A}_{2} \mathrm{~B}_{2}}$ of the triangle that touches $\gamma$, respectively.

Let $\mathrm{P}_{3}\left(\mathrm{x}_{3}, \mathrm{y}_{3}\right)$ and $\mathrm{P}_{6}\left(\mathrm{x}_{6}, \mathrm{y}_{6}\right)$ be points which are not in the triangles or on the side $\overline{\mathrm{B}_{1} C_{1}}$ and $\overline{\mathrm{C}_{2} \mathrm{~B}_{2}}$, respectively.

As $\gamma$ touches all the 6 points, then $\gamma$ is not shorter than the polysegment connecting those points in the order on $\gamma$. There are $\frac{6!}{2}=360$ ways to make it differently as a path and its reverse are the same. This means 360 cases have to be checked but we can reduce the cases by using the following lemma.

It is clear that there exists a worm with equal or greater convex hall that does not intersect itself. Moreover, the considered set is symmetric. Its reflection does not have to be checked. Thus, the cases can be reduced to the following.

Define $\Pi=P_{1} P_{2} P_{3} P_{4} \ldots P_{n}$ is a polysegment that is formed by joining $\mathrm{P}_{1} \mathrm{P}_{2} \mathrm{P}_{3} \mathrm{P}_{4} \ldots \mathrm{P}_{n}$ respectively.

Case $1 \mathrm{P}_{2}$ and $\mathrm{P}_{5}$ are connected.


There will be the following cases

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## Subcase $1.1 \quad \Pi=P_{6} P_{1} P_{5} P_{2} P_{4} P_{3}$.

By using numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.


Subcase $1.2 \quad \Pi=P_{6} P_{1} P_{5} P_{2} P_{3} P_{4}$.
By using numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.


Subcase $\left.1.3 \quad \Pi=P_{1} P_{6} P_{5} P_{2} P_{3} P_{1}\right)$
By using numerical minimization theshortest polysegment is approximately 1 unit long. It is shown as the figure berowy.


Subcase 1.4 โุ $\mathrm{P}_{1} \mathrm{P}_{6} \mathrm{P}_{2} \mathrm{P}_{5} \mathrm{P}_{3} \mathrm{P}_{4} \mathrm{P}_{4}$ าวิทยาลัย
By using numerical/mbimzation, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.


Obviously, the given polysegment can be covered by the equilateral triangle though it satisfies all given conditions. So, more conditions must be added in order to find the shortest polysegment that can't be covered. We found that the thickness is the sufficient condition that we need.

Definition The thickness of an arc is the least width of a strip cover.
We can suppose that we lay down the polysegment on the cover by letting its thickness to the vertical side. This means it is set as flat as possible. The thickness can be form by 2 following conditions.

Condition T1 The points $P$ and $\mathbb{B}$ must be on the same vertical line. This means $x_{1}=x_{4}$.

Condition T2 Without loss of generality, we can say that there is another point $P_{1 b}$ which lays on the same horizontal oval 68 P such that $y_{1}<y_{4}<y_{1 b}$.

Subbase 1.4.1 $\Pi=P_{1} P_{6} P_{1} P_{5} \frac{D_{3}-P_{4} \text { and }}{}$ the condition $T 1$ is satisfied. By using the numerical minimization the shortest polysegment is approximately 1.50213 units long. It is shown as the figure below.

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Subcase 1.4.2 $\Pi=P_{1} P_{6} P_{2} P_{5} P_{3} P_{4}$ and the condition T2 is satisfied.
There is $P_{1 b}$ which is on the same horizontal of $P_{1}$. Thus, this subcase can be split into the following subcases.

## Subcase 1.4.2.1 $\Pi=P_{1} P_{6} P_{1 b} P_{2} P_{5} P_{3} P_{4}$

By using the numerical minimization, the shortest polysegment is approximately 1.541 units long. It is shown as the figure below.


Subcase $1.5 \quad \Pi=P_{1} P_{6} P_{2} P_{5} B_{1} P_{3}$ $\longrightarrow$

By using numerical minimization, the shortest polysegment is approximately 0.866025 unit long. It is showh as the figure below


Subcase 1.5.1 $H \# P_{1} \mid P_{6} P_{2}, P_{5} \mathrm{P}_{4} \mathrm{P}_{3}$ and line condition T 1 is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 1.541 units long. It is shown as the figure below.


Subcase 1.5.2 $\Pi=P_{1} P_{6} P_{2} P_{5} P_{4} P_{3}$ and the condition $T 2$ is satisfied.
There is a $\mathrm{P}_{1 \mathrm{~b}}$ which is on the same vertical of $\mathrm{P}_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 1.5.2.1 $\Pi=P_{1} P_{6} P_{1 b} P_{2} P_{5} P_{4} P_{3}$
By using the numerical minimization, the shortest polysegment is approximately 1.79561 units long. It is shown as the figure below.


By using numeriagt minimization the shonest polysegment is approximately 0.866025 unit long. It is shown as the figure betow.

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Subcase 1.6.1 $\Pi=P_{6} P_{1} P_{2} P_{5} P_{4} P_{3}$ and the condition $T 1$ is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 1.73205 units long. It is shown as the figure below.


There is a $\mathrm{P}_{16}$ which is on the same ryatical of $\mathrm{P}_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 1.6.2.1 14 P P R R P $2=P, P_{4} P_{3}$
By using bis numericat minimization the shortest polysegment is approximately 1 unit tong. It is shown as the figure below.

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Case $2 \mathrm{P}_{1}$ and $\mathrm{P}_{4}$ are connected.


There will be the following cases

## Subcase $2.1 \quad \Pi=P_{6} P_{2} \quad P_{1} P_{4} P_{3}$

By using numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below


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Subcase 2.1.1. $\boldsymbol{H}_{\#} \mathrm{P}_{6} \mathrm{P}_{0} \mathrm{P}_{1} \in \mathrm{P}_{4} \cdot P_{3} \mathrm{P}_{5}$ and the condition $T 1$ is satisfied
By using the numerical minimization, the shortest polysegment is approximately 0.999999 units long. It is shown as the figure below.


Subcase 2.1.2 $\Pi=P_{6} P_{2} P_{1} P_{4} P_{3} P_{5}$ and the condition T2 is satisfied.
There is a $P_{1 b}$ which is on the same vertical of $P_{1}$. Thus, this subcase can be split into the following subcases.

## Subcase 2.1.2.1 $\Pi=P_{1} P_{6} P_{2} P_{1 b} P_{4} P_{3} P_{5}$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.


Subcase 2.1.2.2 4 H P P $P_{2} P_{3} P_{5} P_{1 b}$
By using the numerical minimization the shortest polysegment is approximately 0.981981 units tong. It is shownlas the figure below.

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Subcase 2.1.2.3 $\Pi=P_{6} P_{2} P_{1} P_{4} P_{3} P_{1 b} P_{5}$
By using the numerical minimization, the shortest polysegment is approximately 0.981981 units long. It is shown as the figure below.


Subcase 2.1.2.4 $\pi=P_{0} P_{4} P_{1} P_{4} P_{16} \cdot P_{3} P_{5}$
By using the numpericar (Minimization, the shortest polysegment is approximately 0.981981 unitstonith stown as the figure below.


Subcase 2.2 $\quad \Pi=P_{6} P_{2} P_{1} P_{4} P_{5} P_{3}$.
By using numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below


Subcase 2.2.1 $\Pi=P_{b} P_{2} P_{1} \quad P / P, P_{3}$ and the condition $T 1$ is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure belowen


Subcase 2.2.2 $\Pi=P_{6} P_{2} P_{1} P_{4} P_{5} P_{3}$ and the condition $T 2$ is satisfied.
There is a $P_{16}$ which is on the same vertical of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 2.2.2.1 $\Pi=P_{1} P_{6} P_{2} P_{1 b} P_{4} P_{5} P_{3}$
By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.


Subcase 2.2.2. $2 \mathrm{M}=\mathrm{P}_{6} \mathrm{P}_{2}$ R $\mathrm{P}_{4} \mathrm{P}_{\mathrm{tb}} \mathrm{P}_{5} \mathrm{P}_{3}$
By using the numenizatathimization, the shortest polysegment is approximately 0.981981 ~


## Subcase 2.2.2.3 $\Pi=P_{6} P_{1} P_{2} P_{1 b} P_{4} P_{5} P_{3}$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.


By using numerical phimimation, the shotest polysegment is approximately 0.866025 units long. It is shown as phe figure below


Subcase 2.3.1 $\Pi=P_{2} P_{6} P_{1} P_{4} P_{3} P_{5}$ and the condition T1 is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.


Subcase 2.3.2 $\Pi=P_{2} P_{6} P_{1} P_{4} P_{3} P_{5}$ and the condition T2 is satisfied.
There is a $P_{1 b}$ which is on the same vertical of $P_{1}$. Thus, this subcase can be split into the following subcases.

## Subcase 2.3.2.1 $\Pi=P_{2} P_{6} P_{1} P_{4} P_{3} P_{1 b} P_{5}$

By using the numerical minimization, the shortest polysegment is approximately 0.981981 units long. It is shown as the figure below.


By using the numerigal midmization, the shortest polysegment is approximately 0.98 g


Subcase 2.4 $\quad \Pi=P_{6} P_{2} P_{4} P_{1} P_{5} P_{3}$
By using numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below


Subcase 2.4.1 $\Pi=P_{6} \frac{P_{2} P_{1} P_{1}}{1} P_{5} P_{3}$ and the condition $T 1$ is satisfied.
By using the numerical mimimitation the shortest polysegment is approximately 1 unit long. It is shown as the figure gelow


Subcase 2.4.2 $\Pi=P_{6} P_{2} P_{4} P_{1} P_{5} P_{3}$ and the condition $T 2$ is satisfied.
There is a $P_{1 b}$ which is on the same vertical of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 2.4.2.1 $\Pi=P_{1} P_{6} P_{2} P_{4} P_{1 b} P_{5} P_{3}$
By using the numerical minimization, the shortest polysegment is approximately 0.981981 units long. It is shown as the figure below.


By using the nurnefica minimization, the shortest polysegment is approximately 0.981981 units long- tys shownas the figure below.


Subcase 2.5 $\quad \Pi=P_{2} P_{6} P_{4} P_{1} P_{3} P_{5}$
By using numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below


Subcase 2.5.1 $\Pi=P_{2} P_{6} P_{4} P_{1} P_{3} P_{5}$ and the condition T1 is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 1.00054 units long. It is shown as the figure below.


Subcase 2.5.1 $\Pi=P_{2} P_{6} P . P, P_{2} P_{5}$ and the condition $T 2$ is satisfied.
There is a $P_{1 b}$ which is on the same yertical of $P_{1}$. Thus, this subcase can be split into the following subcases.

By using the numendenmimization, the shortest polysegment is approximately 0.981981 units


Case $3 \mathrm{P}_{5}$ and $\mathrm{P}_{6}$ are connected.


There will be the following cases

Subcase $3.1 \quad \Pi=P_{1} P_{5} P_{0} \cdot P_{2} P_{4} P_{f}$
By using numerical minimization the strotest polysegment is approximately 1 unit long. It is shown as the figure belowy.


Subcase 3.2 กิฬ $\mathrm{P}_{1} \mathrm{P}_{5} \mathrm{P}_{6} \mathrm{P}_{2} \mathrm{P}_{3} \mathrm{P}_{4}$ าวิทยาลัย
By using numericat minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.


Case 4 Convex arc.


There will be the following cases

Subcase 4.1 $\quad \Pi=P_{6} P_{2} P_{4} P_{3} P_{5} P_{1}$
By using numerical mininizamon, the shortest polysegment is approximately 0.866025 units long. It is shown as ribe figires below.


Subcase 4.1. $\because \Pi \Perp P_{6} P_{2} P_{4} P_{3} P_{5} P_{1}$ and the condition T1 is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.


Subcase 4.1.2 $\Pi=P_{6} P_{2} P_{4} P_{3} P_{5} P_{1}$ and the condition T2 is satisfied.
There is a $P_{1 b}$ which is on the same vertical of $P_{1}$. Thus, this subcase can be split into the following subcases.

## Subcase 4.1.2.1 $\Pi=P_{1} P_{6} P_{2} P_{4} P_{3} P_{5} P_{1 b}$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.


By using numericah minimization the shortest polysegment is approximately 0.997131 units long. Itis shown as the figure below ERSITY


Subcase $4.3 \quad \Pi=P_{4} P_{3} P_{5} P_{1} P_{6} P_{2}$
By using numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.


Subcase 4.3.1 $\Pi=P_{4} P_{3} P_{5} P_{P} P_{2}$ and the condition $T 1$ is satisfied.
By using the numerical mimmotation, the shortest polysegment is approximately 1 unit long. It is shown as the figure below. A


Subcase 4.3.2 $\Pi=P_{4} \quad P_{3} \quad P_{5} \quad P_{1} \quad P_{6} P_{2}$ and the condition T2 is satisfled.
There is a $P_{16}$ which is on the same vertical of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 4.3.2.1 $\Pi=P_{4} P_{3} P_{5} P_{1 b} P_{1} P_{6} P_{2}$
By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.


## 4.2 isosceles Right angled Triangle

The aim is to show that an isosceles right angled triangle with 1 unit hypotenuse can cover every arc of length $\ell_{1}=0.948683$. To show this we suppose that an arc $\gamma$ cannot be covered. Then we will show that the length of $\gamma$ is greater than $\ell_{1}$. Thus, $\gamma$ must not be covered by the triangle in the standing position and its reflection.


Figure 4.2 : An isosceles ingh-angled triangle $\mathrm{A}_{1} \mathrm{~B}_{1} \mathrm{C}_{1}$ in standing positior and is reflection $\mathrm{A}_{2} \mathrm{~B}_{2} \mathrm{C}_{2}$

Let $P_{1}\left(x_{1}, y_{1}\right), P_{2}\left(x_{2}, y_{2}\right) \overline{\left(x_{4}, y_{4}\right) \text {, and } P_{5}\left(x_{5}, y_{5}\right)}$ be points on the side $\overline{A_{1} C_{1}}, \overline{A_{1} B_{1}}$ $\overline{\mathrm{A}_{2} C_{2}}$ and $\overline{\mathrm{A}_{2} \mathrm{~B}_{2}}$ of the triangle that touches $\gamma$, respectively.
Let $\mathrm{P}_{3}\left(\mathrm{x}_{3}, \mathrm{y}_{3}\right)$ and $\mathrm{P}_{6}$ ( $\mathrm{x}_{6}, \mathrm{y}_{6}$ ) be points which are not in the triangles or on the side $\overline{\mathrm{B}_{1} C_{1}}$ and $\overline{\mathrm{C}_{2} \mathrm{~B}_{2}}$, respectively.

Case $1 \mathrm{P}_{2}$ and $\mathrm{P}_{5}$ are connected.


There will be the following cases

Subcase $1.1 \quad \Pi=P_{6} P_{1} P_{5} P_{2} P_{4} P_{3}$.
By using numerical minimization, the shortest polysegment is approximately 0.541196 units long. It is shown as the figure below.


Subcase 1.1.1 $\Pi=P_{6} P_{1} P_{3} P_{2}, P_{4} P_{6}$ ald the condition T1 is satisfied. By using the numerical minimization, shortest polysegment is approximately 0.667673 units long. It is showd as the fingure below.


Subcase 1.1.2 $H=P_{6} P_{1} \mathrm{P}_{5} \mathrm{P}_{2} \mathrm{P}_{4}^{R} \mathrm{P}_{3}$ and the condition T2 is satisfied.
There is $P_{1 b}$ which is on the same horizontal of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 1.1.2.1 $\Pi=P_{6} P_{1} P_{5} P_{2} P_{1 b} P_{4} P_{3}$
By using the numerical minimization, the shortest polysegment is approximately 1.36705 units long. It is shown as the figure below.


Subcase 1.1.2.2 $\Pi=P_{6} P_{1} P_{1 b} P_{5} P_{2} P_{4} P_{3}$
By using the numerical minimization, the shortest polysegment is approximately 0.754716 units long, It is shown as the figure below.


Subcase $\left.1.2 \quad \Pi=P_{6} P_{1} P_{5} P_{2} P_{3} P_{f}\right)$
By using numerical minimization, theshortest polysegment is approximately 0.541196 units long. It is shown as the figure below


Subcase 1.2.1 ก2 ${ }^{2} P_{6} \widetilde{P}_{1}{ }^{ง} P_{5} \Psi_{2} P_{3}^{\sigma} P_{4}$ and he coñdition $T 1$ is satisfied. By using the numerical minimization, Rithe shortest Spolysegment is approximately 0.667673 units long. It is shown as the figure below.


Subcase 1.2.2 $\Pi=P_{6} P_{1} P_{5} P_{2} P_{3} P_{4}$ and the condition T2 is satisfied.
There is $P_{1 b}$ which is on the same horizontal of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 1.2.2.1 $\Pi=P_{6} P_{1} P_{1 b} P_{5} P_{2} P_{3} P_{4}$
By using the numerical minimization, the shortest polysegment is approximately 0.667673 units long. It is shown as the figure below.


Subcase $1.3 \quad \Pi=P_{7} P_{6} D_{5} / P / P_{4} P_{4}$
By using numerical phinimzatione, the shorest polysegment is approximately 0.541196 units long. It is shown as the figut ebelow.


Subcase 1.3.1 $1 / \# \mathrm{P}_{4} / \mathrm{P}_{6}\left(\mathrm{P}_{5} / \mathrm{P}_{2} / \mathbb{P}_{3}, \mathrm{P}_{4} /\right.$ and 1 he condition $T 1$ is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 0.667673 units long. It is shown as the figure below.


Subcase 1.3.2 $I I=P_{1} P_{6} P_{5} P_{2} P_{3} P_{4}$ and the condition $T 2$ is satisfied.
There is $P_{1 b}$ which is on the same horizontal of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 1.3.2.1 $\Pi=P_{1} P_{6} P_{1 b} P_{5} P_{2} P_{3} P_{4}$
By using the numerical minimization, the shortest polysegment is approximately 0.667673 units long. It is shown as the figure below.


By using numerical mhinnization the shortest polysegment is approximately 0.765367 units long. It is shown th the ingere belay.


Subcase 1.4.1 $\Pi=P_{1} P_{6} P_{2} P_{5} P_{3} P_{4}$ and the condition $T 1$ is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 1.24259 units long. It is shown as the figure below.


Subcase 1.4.2 $\Pi=P_{1} P_{6} P_{2} P_{5} P_{3} P_{4}$ and the condition $T 2$ is satisfied.
There is $P_{1 b}$ which is on the same horizontal of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 1.4.2.1 $\Pi=P_{1} P_{6} P_{1 b} P_{2} P_{5} P_{3} P_{4}$
By using the numerical minimization, the shortest polysegment is approximately 1.19617 units long. It is shown as the figure below.


Subcase $1.5 \quad \Pi=\mathrm{P} P$ P P/ P/ Pras
By using numerical minimizatront, the shartest polysegment is approximately 0.765367 units long. It is shown as the that fe below.


Subcase 1.5.1 $\Pi=P_{1} P_{6} P_{2} P_{5} P_{4} P_{3}$ and the condition T1 is satisfied. By using the numerical minimization, the shortest polysegment is approximately 1.24259 units long. It is shown as the figure below,


Subcase 1.5.2 $\Pi=P_{1} P_{6} P_{2} P_{5} P_{4} P_{3}$ and the condition T2 is satisfied.
There is $P_{1 b}$ which is on the same horizontal of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 1.5.2.1 $\Pi=P_{1} P_{6} P_{1 \mathrm{~b}} P_{2} P_{5} P_{4} P_{3}$
By using the numerical minimization, the shortest polysegment is approximately 1.26769 units long. It is shown as the figure below.


Subcase $1.6 \quad \Pi=P_{6} P_{y} \mathrm{P}_{2} \mathrm{P} / 5 \mathrm{P}_{5} \mathrm{P}_{3}$
By using numerical minimization the shorest polysegment is approximately 0.765367 units long. It is shown as the figmere below.


Subcase 1.6.1 $\Pi=P_{1} P_{6} P_{2} P_{5} P_{4} P_{3}$ and the condition $T 1$ is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 1.24259 units long. It is shown as the figure below.


Subcase 1.6.2 $\Pi=P_{6} P_{1} P_{2} P_{5} P_{4} P_{3}$ and the condition $T 2$ is satisfied.
There is $P_{1 b}$ which is on the same horizontal of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 1.6.2.1 $\Pi=P_{6} P_{1} P_{1 b} P_{2} P_{5} P_{4} P_{3}$
By using the numerical minimization, the shortest polysegment is approximately 1.26770 units long. It is shown as the figure below.


There will be the following cases

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Subcase 2.1 $\quad \Pi=P_{6} P_{2} P_{1} P_{4} P_{3} P_{5}$.
By using numerical minimization, the shortest polysegment is approximately 0.765367 units long. It is shown as the figure below.


Subcase 2.1.1 $\Pi=P_{6} P_{2} P_{1} P_{4} P_{3} P_{5}$ and the condition T1 is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 0.999999 units long. It is shown as the figure below.

Subcase 2.1.2 $\Pi=P_{6} P_{2} P \quad P / 8 / / P_{3} \vec{P}_{5}$ and the condition $T 2$ is satisfied.
There is a $P_{1 b}$ whichis on the sarmel verticak of $P_{1}$. Thus, this subcase can be split into the following subcases

Subcase 3.1.2.1 $\pi / P P P$
By using the pupnerica minmization, the shortest polysegment is approximately 1 unit long feissormas the figure below.


By using the numerical minimization, the shortest polysegment is approximately 0.948683 units long. It is shown as the figure below.


Subcase 2.1.2.3 $\Pi=P_{6} P_{2} P_{1} P_{4} P_{3} P_{1 b} P_{5}$
By using the numerical minimization, the shortest polysegment is approximately 0.948683 units long. It is shown as the figure below.


Subcase 2.1.2.4 \# $=P_{6} P_{2} P_{4} P_{\text {b }}, P_{3} P_{5}$
By using the numerical minmization, the shortest polysegment is approximately 0.948683 unni $\$$ long. It is shown as the figure below.


By using numerical minimization, the shorest polysegment is approximately
0.765367 units long. It is shown as the figure below


Subcase 2.2.1 $\Pi=P_{6} P_{2} P_{1} P_{4} P_{5} P_{3}$ and the condition T1 is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.


Subcase 2.2.2 $\Pi=P$ P. P. P P P P
There is a $P_{1 b}$ which is on the same vertical of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 2.2.2.1 $1=P P_{1} P_{6} P_{2} P_{b} P_{4} P_{5} P_{3}$
By using (the numerical ininimization the shortest polysegment is approximately 1 unitipigg, It is shown as the figure below.

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Subcase 2.2.2.2 $\Pi=P_{6} P_{2} P_{1} P_{4} P_{1 b} P_{5} P_{3}$
By using the numerical minimization, the shortest polysegment is approximately 0.948683 units long. It is shown as the figure below.


Subcase 2.2.2.3 II= $P_{6} P_{0} P_{2} P_{40} P_{4} P_{5} P_{3}$
By using the nymeryal minimization, the shortest polysegment is approximately 1 unitlone. 11 is shown as the figure below.


Subcase $2.3 \quad \Pi=P_{2} P_{6} P_{1} P_{4} P_{3} P_{5}$.
By using numerical minimization, the shortest polysegment is approximately
0.866025 units long. It is shown as the figure below


Subcase 2.3.1 $\Pi=P_{2} P_{6} P_{1} P_{4} P_{3} P_{5}$ and the condition $T 1$ is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.


Subcase 2.3.2 $\Pi=\mathrm{P}_{2} \mathrm{P}_{6} P_{1} X_{1} / / \mathrm{P}_{3} \mathrm{P}_{5}$ and the condition T 2 is satisfied.
There is a $P_{16}$ whichis on the samaerverticat of $P_{1}$. Thus, this subcase can be split into the following subcases

Subcase 2.3.2.1 $\Gamma / \square, \mathrm{P}, \mathrm{P} \quad Q_{16} \mathrm{P}_{4} \mathrm{P}_{3} \mathrm{P}_{5}$
By using the puphericas minimization, the shortest polysegment is approximately 1 unit long


Subcase 2.3.2.2 $\Pi=P_{2} P_{6} P_{1} P_{4} P_{3} P_{1 b} P_{5}$
By using the numerical minimization, the shortest polysegment is approximately 0.948683 units long. It is shown as the figure below.


Subcase 2.3.2.3 $\prod=P_{2} P_{9} P_{1}{\underset{4}{ } P_{3} P_{5} P_{1 b}}^{l}$
By using the numerigal mimimization, the shortest polysegment is approximately 0.948638 units ong it is shown as the figure below.


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## Subcase 2.4 CH\#P $P_{2} P_{6} \mathrm{P}_{4} \mathrm{P}_{1} \mathrm{P}_{3} \mathrm{P}_{5} \mathrm{U}_{5}$ NIVERSITY

By using numerical minimization, the shortest polysegment is approximately 0.707107 units long. It is shown as the figure below


Subcase 3.9.1 II $=P_{2} P_{6} P_{4} P_{1} P_{3} P_{5}$ and the condition $T 1$ is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.


Subcase 2.4.2 $\Pi=P_{2} \xlongequal[\mathrm{P}_{6} \mathrm{P}_{1} \mathrm{P}_{2}]{ } \mathrm{P}_{3} \mathrm{P}_{\mathrm{a}}$ and he condition T 2 is satisfied.
There is a $P_{16}$ which is on the same vertical of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 2.4.2. $M=P_{2} \mathrm{R}_{0} \mathrm{P}_{6} \mathrm{P}_{4} \mathrm{P}_{16} \in \mathrm{P}_{5}$
By using the numeriaet winimization, the shortest polysegment is approximately 0.948683 units lond. It is shown as the figure below.


Subcase 2.4.2.2 $\Pi=P_{2} P_{6} P_{1} P_{4} P_{16} P_{3} P_{5}$
By using the numerical minimization, the shortest polysegment is approximately 0.981981 units long. It is shown as the figure below.


Case $3 \mathrm{P}_{5}$ and $\mathrm{P}_{6}$ are connected.


There will be the following cases

## Subcase $3.1 \quad \Pi=P_{1} P_{5} P_{6} P_{2} P_{4} P_{3}$.

By using numerical minimizations fyle) shortest polysegment is approximately 0.541196 units long. It is shown as the figure bew.


Subcase 3.1.1 $\Pi=\mathscr{L}_{2} \mathrm{P}_{5} \mathrm{P}_{6} \mathrm{P}_{2} \mathrm{P}_{4} \mathrm{P}_{3}$ and the cand dition T 1 is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 0.676617 units long. It is shown as the figure below. ยา ลั

Phulalongkorin University


Subcase 3.1.2 $\Pi=P_{1} P_{5} P_{6} P_{2} P_{4} P_{3}$ and the condition $T 2$ is satisfied.
There is $P_{1 b}$ which is on the same horizontal of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 3.1.2.1 $\Pi=P_{1 b} P_{5} P_{6} P_{1} P_{2} P_{4} P_{3}$
By using the numerical minimization, the shortest polysegment is approximately 0.648746 units long. It is shown as the figure below.


Subcase 3.1.2.2 $\Pi=P_{1 b} P_{6} P_{6} P_{2} P_{3} P_{4}$
By using the numerical minimization, the shortest polysegment is approximately 0.648746 unis 10 of lit is shown as the figure below.


By using numeriqakกnลimization Nthel sิhortestãpolysegment is approximately 0.541196 units long. It is showalas the figure below.IVERSITY


Subcase 3.2.1 $\Pi=P_{1} P_{5} P_{6} P_{2} P_{3} P_{4}$ and the condition $T 1$ is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 0.676617 units long. It is shown as the figure below.


Subcase 3.2.2 $\Pi=P_{1} P_{5} P_{6} P_{2} P_{3} P_{4}$ and the condition $T 2$ is satisfied.
There is $P_{1 b}$ which is on the same horizontal of $P_{1}$ Thus, this subcase can be split into the following subcases.

Subcase 3.2.2.1 $\pi=P_{16} P_{(\Omega)} P_{6}^{6} P_{1} P_{2} P_{3} P_{4}$
By using the numerical minimization the shortest polysegment is approximately 0.648747 units lone hisishown as the figure below.


Case 4 Convex arc.


There will be the following cases

Subcase 4.1 $\quad \Pi=P_{6} \quad P_{2} P_{4} P_{3} P_{5} P_{1}$.
By using numerical minimization, the shortest polysegment is approximately 0.707107 units long. It is shown as the figure below.


Subcase 4.1.1 $\Pi=P_{6} \quad P_{2} \quad P_{4} \quad P_{3} \quad P_{5} \quad P_{1}$ and the condition $T 1$ is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 0.859389 units long. It is shown as he y, gure berow:


Subcase 4.1.2 $\left.\Pi=P_{6}\right) P_{2} P_{4} P_{3} P_{5} P_{4}$ anet the condition $T 2$ is satisfied.
There is a $P_{16}$ whichsis onsthesamemertical on Pri.thus, this subcase can be split into the following subdases:LALONGKORN UNIVERSITY

Subcase 4.1.2.1 $\Pi=P_{1} P_{6} P_{2} P_{4} P_{3} P_{5} P_{1 b}$
By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.


Subcase 4.1.2.2 $\Pi=P_{1} P_{6} P_{2} P_{4} P_{3} P_{5} P_{1 b}$
By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.


Subcase $4.2 \quad \Pi=P_{2} P_{4} P_{3} P_{5} P_{4} P_{6}$
By using numerical minimizarion, the shortest polysegment is approximately 0.541196 units long. It is shown as the figure belaw:


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Subcase 4.2.1 $\Pi=P_{2} P_{4} P_{3} P_{5} P_{1} P_{6}$ and the condition $T 1$ is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 0.621262 units long. It is shown as the figure below.


Subcase 4.2.2 $\Pi=P_{2} P_{4} P_{3} P_{5} P_{1} P_{6}$ and the condition T2 is satisfied.
There is a $P_{1 b}$ which is on the same vertical of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 4.2.2.1 $\Pi=P_{2} P_{4} P_{3} P_{5} P_{1 b} \quad P_{1} P_{6}$
By using the numerical minimization, the shortest polysegment is approximately 0.621262 unit long. It is shown as the figure below.


Subcase $4.3 \quad \Pi=P_{4} P_{3} P_{5}$ P P. PA P O A A A
By using numerical rafinimization shertest polysegment is approximately 0.765367 units long. It is shown as the limolo below-


Subcase 4.3.1 $\Pi=P_{4} P_{3} P_{5} P_{1} P_{6} P_{2}$ and the condition $T 1$ is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 0.845299 units long. It is shown as the figure below.


Subcase 4.3.2 $\Pi=P_{4} P_{3} P_{5} P_{1} P_{6} P_{2}$ and the condition T2 is satisfied.
There is a $\mathrm{P}_{1 \mathrm{~b}}$ which is on the same vertical of $\mathrm{P}_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 4.3.2.1 $\Pi=P_{4} P_{3} P_{5} P_{1 b} P_{1} P_{6} P_{2}$
By using the numerical minimization, the shortest polysegment is approximately 0.845299 units long. It is shown as the figure below.


By using the numerical fimmazation, the shortest polysegment is approximately 0.876003 unitslory N N s jown as the figure below.


### 3.3 A $30^{\circ}-60^{\circ}-90^{\circ}$ triangle

The objective is to show that a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle with 1 unit hypotenuse can cover every arc of length $\ell_{1}=\frac{9}{(3+4 \sqrt{3})} \approx 0.906508$. To show this we suppose that an arc $\gamma$ cannot be covered. Then we will show that the length of $\gamma$ is greater than $\ell_{1}$, Thus, $\gamma$ must not be covered by the triangle in the standing position and its reflection.


Figure 4.3 : $\mathrm{A} 30^{\circ}-60-700$ thangle $A, B, C$, in standing position andits refiection $\mathrm{A}_{2} \mathrm{~B}_{2} \mathrm{C}_{2}$
Let $P_{1}\left(x_{1}, y_{1}\right), P_{2}\left(x_{2}, y_{2}\right), P_{1}\left(x_{1}, y_{1}\right)$, and $P_{2}\left(x_{5}, y_{5}\right)$ ore points on the side $\overline{A_{1} C_{1}}, \overline{A_{1} B_{1}}$ $\overline{\mathrm{A}_{2} \mathrm{C}_{2}}$ and $\overline{\mathrm{A}_{2} \mathrm{~B}_{2}}$ of theltriangle that touches $\mathrm{P}_{1}$ respectively.
Let $P_{3}\left(x_{3}, y_{3}\right)$ and $P_{6}\left(x_{6} y_{\uparrow}\right)$ be points which are not in the triangles or on the side $\overline{\mathrm{B}_{1} \mathrm{C}_{1}}$ and $\overline{\mathrm{C}_{2} \mathrm{~B}_{2}}$ respectively GKORN UNIVERSITY
Unfortunately, the set is not symmetric. Thus each arc and its reflection have to be checked.
Case $1 \mathrm{P}_{2}$ and $\mathrm{P}_{5}$ are connected.


There will be the following cases

Subcase $1.1 \quad \Pi=P_{6} P_{1} P_{5} P_{2} P_{4} P_{3}$.
By using numerical minimization, the shortest polysegment is approximately 0.448288 units long. It is shown as the figure below.


Subcase 1.1.1 $\Pi=P_{6} P_{1} P_{5} P_{2} P_{4} P_{3}$ ) and the condition $T 1$ is satisfied.
By using the numerical minimization, the shorfest polysegment is approximately 0.507214 units long. It is shown as the 年gure belows.


Subcase 1.1.2 $\uparrow=\mathrm{P}_{6}$ ค $\mathrm{P}_{1} \mathrm{P}_{5} \mathrm{P}_{2} \mathrm{P}_{4} \mathrm{P}_{3}$ and the condition T 2 is satisfied. There is $P_{10}$ which is on the same fiorizontal of $P_{1}$. Thus! this subcase can be split into the following subcases.

Subcase 1.1.2.4 $\Pi=P_{6} P_{1} P_{1 b} P_{5} P_{2} P_{4} P_{3}$
By using the numerical minimization, the shortest polysegment is approximately 0.507214 units long. It is shown as the figure below.


Subcase 1.2 $\quad \Pi=P_{6} P_{1} P_{5} P_{2} P_{3} P_{4}$.
By using numerical minimization, the shortest polysegment is approximately 0.448288 units long. It is shown as the figure below.


Subcase 1.2.1 $\Pi=P_{6} \quad P_{1} \quad P_{5} \quad P_{2}, P_{3} P_{A} /$ and 1 the condition $T 1$ is satisfied.
By using the numerical minimizatton, the shortest polysegment is approximately 0.507214 units long. It is shown as the figure below


Subcase 1.2.2 $\Pi=P_{6} P_{1} P_{5} P_{2} P_{3} P_{4}$ and the condition $T 2$ is satisfied. There is $P_{16}$ which is on the same horizontal of $P_{1}$ \&his, this subcase can be split into the following subcases! HIULALONGKORN UNIVERSITY

Subcase 1.2.2.1 $\Pi=P_{6} P_{1} P_{16} P_{5} P_{2} P_{3} P_{4}$
By using the numerical minimization, the shortest polysegment is approximately 0.507214 units long. It is shown as the figure below.


Subcase 1.3 $\quad \Pi=P_{1} P_{6} P_{5} P_{2} P_{3} P_{4}$
By using numerical minimization, the shortest polysegment is approximately 0.448288 units long. It is shown as the figure below.


Subcase 1.3.1 $\Pi=P_{1} P_{6} P_{5} P_{2} P_{3} P_{4}$ and the condition $T 1$ is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 0.507214 units long. It is shown as the figufe betow.:


Subcase 1.3.2 $\Pi=\mathrm{P}_{1} \mathrm{P}_{6} \mathrm{P}_{3} \mathrm{P}_{2} \mathrm{P}_{3} \mathrm{P}_{4}$ and the dondition T 2 is satisfied.
There is $P_{1 b}$ which is on the sकmernerizontaf off $P_{1}$.Thus. this subcase can be split into the following subcases?HULALONGKORN UNIVERSITY

Subcase 1.3.2.1 $\Pi=P_{1} P_{6} P_{16} P_{5} P_{2} P_{3} P_{4}$
By using the numerical minimization, the shortest polysegment is approximately 0.507214 units long. It is shown as the figure below.


Subcase $1.4 \quad \Pi=P_{1} P_{6} P_{2} P_{5} P_{3} P_{4}$
By using numerical minimization, the shortest polysegment is approximately 0.841949 units long. It is shown as the figure below.


Subcase 1.4.1 $\Pi=P_{1} \quad P_{6} \quad P_{2} \quad P_{5} \quad P_{3} P_{4} /$ and the condition $T 1$ is satisfied.
By using the numerical minimization, the shortest potysegment is approximately 1.17298 units long. It is shown as the figure borbyy


Subcase 1.4.2 $\Pi \uparrow=P_{1} P_{6} \mathrm{P}_{2} \mathrm{P}_{5} \mathrm{P}_{3} \mathrm{P}_{4}$ andthecondition T 2 is satisfied.
There is $P_{16}$ which is onfthe same horizontal of $P_{1}$ Thussthis subcase can be split into the following subcases.

Subcase 1.4.2.1 $\Pi=P_{1} P_{6} P_{1 b} P_{2} P_{5} P_{3} P_{4}$
By using the numerical minimization, the shortest polysegment is approximately 1.17298 units long. It is shown as the figure below.


Subcase $1.5 \quad \Pi=P_{1} \quad P_{6} P_{2} P_{5} P_{4} P_{3}$
By using numerical minimization, the shortest polysegment is approximately 0.841949 units long. It is shown as the figure below.


Subcase 1.5.1 $\Pi=P_{1} P_{6} P_{2} P_{5}, P_{4} P_{3} /$ and the condition $T 1$ is satisfied.
By using the numerical minimization, the shorestpolysegment is approximately 1.17298 units long. It is shown as the figure below.

 There is $P_{1 b}$ which is onlthe samelnorizontal of NPMansithis subcase can be split into the following subcases,

Subcase 1.5.2.1 $\Pi=P_{1} P_{6} P_{i b} P_{2} P_{5} P_{4} P_{3}$
By using the numerical minimization, the shortest polysegment is approximately 1.19069 units long. It is shown as the figure below.


Subcase $1.6 \quad I I=P_{6} P_{1} P_{2} P_{5} P_{4} P_{3}$
By using numerical minimization, the shortest polysegment is approximately 0.841948 units long, It is shown as the figure below.


Subcase 1.6.1 $\Pi=P_{1} P_{6} P_{2} P_{2} P_{1} P_{3}$ amethe condition $T 1$ is satisfied.
By using the numerical minimization. Ifo shoitest polysegment is approximately 1.19069 units long. It is shown as the figure berdios


Subcase 1.6.2 Гमุ-P $P_{6} P_{1}, P_{2}, P_{5} \stackrel{\rightharpoonup}{2}_{4} P_{3}$ and the condition $T 2$ is satisfied. There is $P_{1 b}$ which is onithe samel horizonfal of $P$ PUThas ithis subcase can be split into the following subcases.

Subcase 1.6.2.1 $\Pi=P_{6} P_{1} P_{16} P_{2} P_{5} P_{4} P_{3}$
By using the numerical minimization, the shortest polysegment is approximately 1.19069 units long. It is shown as the figure below.


Case $2 \mathrm{P}_{1}$ and $\mathrm{P}_{4}$ are connected.


There will be the following cases

Subcase $2.1 \quad \Pi=P_{6} P_{2} \quad P \quad P_{4} P_{3} P_{5}$
By using numerical minimization the shorlest polysegment is approximately 0.765367 units long. It is shown as the fgure belows


Subcase 2.1.1 $\Pi=P_{6} P_{G} P_{n} P_{\text {g }} P_{i-1} P_{\text {Pa }}$ and thecondition $T 1$ is satisfied.
By using the numericat minimizationt the shortestspolysegment is appoximately 0.908249 units long. It is shown as the figure below.


Subcase 2.1.2 $\Pi=P_{6} P_{2} P_{1} P_{4} P_{3} P_{5}$ and the condition $T 2$ is satisfied.
There is a $P_{1 b}$ which is on the same vertical of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 2.1.2.1 $\Pi=P_{1} P_{6} P_{2} P_{1 b} P_{4} P_{3} P_{5}$
By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.


Subcase 2.1.2.2 $\Pi=P_{0} P_{2} P_{4} P_{4} P_{3} P P_{i b}$
By using the pumerica( maithimization, the shortest polysegment is approximately 0.866025 unistong. In shown as the figure below.


Subcase 2.1.2.3 $\Pi=P_{6} P_{2} P_{1} P_{4} P_{3} P_{1 b} P_{5}$
By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below,


Subcase 2.1.2.4 $\Pi=P_{6} P_{2} P_{1} P_{4} P_{1 b} P_{3} P_{5}$
By using the numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.


Subcase 2.2 $\quad \Pi=P_{6} P_{2} P_{1} P_{4} P_{5} P_{3}$.
By using numerical minimizarion. the stortest polysegment is approximately 0.866025 units long. It is shown as the figyfe below


Subcase 2.2.1 $H_{F=P_{6}} P_{2} P_{1} P_{9} P_{5} P_{1}$ and the condition $T 1$ is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 0.947294 units long. It is shown as the figure below.


Subcase 2.2.2 $\Pi=P_{6} P_{2} P_{1} P_{4} P_{5} P_{3}$ and the condition $T 2$ is satisfied.

There is a $P_{i b}$ which is on the same vertical of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 2.1.2.1 $\Pi=P_{1} P_{6} P_{2} P_{16} P_{4} P_{5} P_{3}$
By using the numerical minimization, the shortest polysegment is approximately 0.998695 units long. It is shown as the figure below.


Subcase 2.2.2.2 $1=P_{6} / P_{2} P_{1} P_{1} P_{1} P_{3} P_{3}$
By using the numericabminimization, the shortest polysegment is approximately 0.866025 onis iong.if is shown as the figure below.


Subcase 3.2.2.3 $\Pi=P_{6} P_{1} P_{2} P_{1 b} P_{4} P_{5} P_{3}$
By using the numerical minimization, the shortest polysegment is approximately 0.998696 units long. It is shown as the figure below.


## Subcase 2.3 $\quad \Pi=P_{6} P_{2} P_{4} P_{1} P_{3} P_{5}$

By using numerical minimization, the shortest polysegment is approximately 0.5 units long. It is shown as the figure below


Subcase 2.3.1 $\Pi=P_{6} P_{2} P_{4} P_{5} P_{3} P_{5} /$ Andd the condition T1 is satisfied.
By using the numerical minimzation, thostrotest polysegment is approximately 1 unit long. It is shown as the figure bobvy/1


Subcase 2.3.2 Пझ्9 $\mathrm{P}_{6} \mathrm{P}_{2} \mathrm{P}_{4} / \mathrm{P}_{7} \mathrm{P}_{3} \mathrm{P}_{5}$ and ${ }^{2}$ dhe condition T 2 is satisfied.
There is a $P_{1 b}$ whibl ist draneisame verticall bfep, Shus, this subcase can be split into the following subcases.

## Subcase 2.3.2.1 $\Pi=P_{1} P_{6} P_{2} P_{4} P_{16} P_{3} P_{5}$

By using the numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.


Subcase 2.3.2.2 $\Pi=P_{6} P_{1} P_{2} P_{4} P_{1 b} P_{3} P_{5}$
By using the numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.


Subcase 2.3.2.3 $\Pi=\mathrm{P}_{6}, \mathrm{P}_{23} \mathrm{P}_{4} /$ P/ $/ P_{3}, \mathrm{P}_{1 \mathrm{~b}} \quad \mathrm{P}_{5}$
By using the nomerical mifflizzation, the shortest polysegment is approximately 1 unit long. Ht shopwn as the figure below.


By using theA nurnecical Rminimization Rther shortest polysegment is approximately 1 unit long. It is shown as the figure below.


Subcase 2.4 $\quad \Pi=P_{6} P_{2} P_{4} P_{1} P_{5} P_{3}$
By using numerical minimization, the shortest polysegment is approximately 0.707107 units long. It is shown as the figure below


Subcase 2.4.1 $\Pi=P_{6}, P, P / D 5 P_{3}$ and the condition $T 1$ is satisfied.
By using the numerical minmpation the shortest polysegment is approximately 1 unit long. It is shown as the figure belpivi $(3)$


Subcase 2.4.2 $\Pi=P_{6} P_{2} P_{4} P_{1} P_{5} P_{3}$ and the condition $T 2$ is satisfied.
There is a $P_{10}$ which is on the same vertical of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 2.4.2.1 $\Pi=P_{1} P_{6} P_{2} P_{4} P_{1 b} P_{5} P_{3}$
By using the numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.


Subcase 2.4.2.2 $\Pi=P_{6} P_{1} P_{2} P_{4} P_{1 b} P_{5} P_{3}$
By using the numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.


Subcase $\left.\left.2.5 \quad \Pi=P_{2} \mathrm{P}_{6} \mathrm{P}_{6} \mathrm{P}_{1} \mathrm{P}_{3} \mathrm{P}_{5}\right)^{\prime}\right)$
By using numerical minimization, the shortest polysegment is approximately 0.5 units long. It is shown as the figure below


Subcase 2.5.1 $\Pi=P_{2} P_{6} P_{4} P_{1} P_{3} P_{5}$ and the condition $T 1$ is satisfied.
By using the numęrical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figute belowDRN UNIVERSITY


Subcase 2.5.2 $\Pi=P_{2} P_{6} P_{4} P_{1} P_{3} P_{5}$ and the condition T2 is satisfied.
There is a $P_{1 b}$ which is on the same vertical of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 2.5.2.1 $\Pi=P_{2} P_{1} P_{6} P_{4} P_{1 b} P_{3} P_{5}$
By using the numerical minimization, the shortest polysegment is approximately 0.891806 units long. It is shown as the figure below.


Subcase 2.5.2.2 $11=972 \mathrm{P}_{6} \mathrm{P}_{10} \mathrm{P}_{4} \mathrm{P}_{16} \mathrm{P}_{3} \mathrm{P}_{5}$
By using the numencaterinimization the shortest polysegment is approximately 0.866303 units long thown as the figure below.


Case $3 \mathrm{P}_{5}$ and $\mathrm{P}_{6}$ are connected.


There will be the following cases

Subcase $3.1 \quad \Pi=P_{1} P_{5} P_{6} P_{2} P_{4} P_{3}$.
By using numerical minimization, the shortest polysegment is approximately 0.451684 units long. It is shown as the figure below.


Subcase 3.1.1 $\Pi=P_{1} P_{5} P_{6} P_{2} P_{4} P_{3} /$ and $/$ the condition $T 1$ is satisfied. By using the numerical minimization, the shortest polysegment is approximately 0.509526 units long. It is shown as the fogure below


Subcase 3.1.2 $\Pi=P=P_{5} P_{6} P_{2} P_{4} P_{3}$ and the condition $T 2$ is satisfied.
There is $P_{1}$ which is on the samè horizontal of P $P_{1}$. thus, this subcase can be split into the following subcases CHULALONGKORN UNIVERSITY

Subcase 3.1.2.1 $\Pi=P_{1 b} P_{5} P_{6} P_{1} P_{2} P_{4} P_{3}$
By using the numerical minimization, the shortest polysegment is approximately 0.505962 units long. It is shown as the figure below.


Subcase 3.1.2.2 $\Pi=P_{1 b} P_{5} P_{6} P_{1} P_{2} P_{3} P_{4}$
By using the numerical minimization, the shortest polysegment is approximately 0.508501 units long. It is shown as the figure below.


Subcase $3.2 \quad \Pi=P_{1} P_{5} P_{6} P_{2} P_{3} P^{4}$
By using numerical minifization the shortest polysegment is approximately 0.451684 units long. It is shown as the igqure betow.


Subcase 3.2.1 $\sqcap \stackrel{2}{2} \mathrm{P}_{1} \mathrm{P}_{5} \mathrm{P}_{6} \cap \widetilde{P}_{2} \mathrm{P}_{3} \mathrm{P}_{4}$ and the condition T 1 is satisfied. By using the numericatuminimizafion the shortesth polysegment is approximately 0.509526 units long. It is shown as the figure below.


Subcase 3.2.2 $\Pi=P_{1} P_{5} P_{6} P_{2} P_{3} P_{4}$ and the condition $T 2$ is satisfied.
There is $P_{1 b}$ which is on the same horizontal of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 3.2.2.1 $\Pi=P_{1 b} P_{5} P_{6} P_{1} P_{2} P_{3} P_{4}$
By using the numerical minimization, the shortest polysegment is approximately 0.648747 units long, It is shown as the figure below.


Case 4 Convex arc.


There will be the following eases ONGKORN UNIVERSITY

Subcase 4.1 $\quad \Pi=P_{6} P_{2} P_{4} P_{3} P_{5} P_{1}$.
By using numerical minimization, the shortest polysegment is approximately 0.5 units long. It is shown as the figure below.


Subcase 4.1.1 $\Pi=P_{6} P_{2} P_{4} P_{3} P_{5} P_{1}$ and the condition $T 1$ is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 0.686052 units long. It is shown as the figure below.


Subcase 4.1.2 $\Pi=P_{6} P_{2} P_{4} P_{5} P_{5} B_{1}$ ant the condition $T 2$ is satisfied.
There is a $P_{16}$ which is on the same verticat of $P_{1}$. Thus, this subcase can be split into the following subcases.

Subcase 4.1.2 $1 \mathrm{I}=\mathrm{P} P \mathrm{P}_{\mathrm{F}}^{\mathrm{P}} 2 \mathrm{P}, \mathrm{P}_{3} \mathrm{P}, \mathrm{P}_{\mathrm{TB}}$
By using the numericalomimization the shortest polysegment is approximately 0.866025 units longhnis shown as the figure below.


Subcase 4.1.2.2 $\Pi=P_{1} P_{6} P_{2} P_{4} P_{3} P_{1 b} P_{5}$
By using the numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.


Subcase $4.2 \quad \Pi=P_{2} P_{4} P_{3} P_{5} P_{1} P_{6}$
By using numerical minimization, the shortest polysegment is approximately 0.448288 units long. It is shown as the figure below.


Subcase 4.2.1 $\Pi=P_{2} P_{4} P_{3} P_{5} P_{1} P_{6}$ ane thé condition T1 is satisfied.
By using the numericatmimimaztion the shortest polysegment is approximately 0.484994 units long. It is shown as the figure below


Subcase 4.2.2 $\Pi \neq P_{2} P_{4} P_{3} P_{5} P_{9} P_{6}$ and $d$ he condition $T 2$ is satisfied.
There is a $\mathrm{P}_{\mathrm{ib}}$ which is on the samevertical of $\mathrm{P}_{\mathrm{i}}$ This, this subcase can be split into the following subcases.

$$
\text { Subcase 4.2.2.1 } \Pi=P_{2} P_{4} P_{3} P_{5} P_{1 b} P_{1} P_{6}
$$

By using the numerical minimization, the shortest polysegment is approximately 0.48519 units long. It is shown as the figure below.


Subcase $4.3 \quad \Pi=P_{4} P_{3} P_{5} P_{1} P_{6} P_{2}$
By using numerical minimization, the shortest polysegment is approximately 0.750346 units long. It is shown as the figure below.


Subcase 4.3.1 $\Pi=P_{4} P_{3} \quad P_{3} P_{4} \quad P_{6} \cdot P_{2}$ Ald the condition $T 1$ is satisfied.
By using the numericatminimizaiton, shertest polysegment is approximately 0.782044 units long. It is shown as the plgure below


Subcase 4.3.2 $\Pi=P_{4} P_{P} P_{f} P_{1} P_{r} P_{2}$ and the condition $T 2$ is satisfied.
There is a $\mathrm{P}_{1 \mathrm{~b}}$ which is on the same vertical of $\mathrm{P}_{\text {, }}$ Thus, this subcase can be split into the following subcases.

Subcase 4.3.2.1 $\Pi=P_{4} P_{3} P_{5} P_{1 b} P_{1} P_{6} P_{2}$
By using the numerical minimization, the shortest polysegment is approximately 0.782044 units long. It is shown as the figure below.


