อาร์-ไอดีล ๒-แอบซอร์บิงของมอดูลเหนือเนียร์ริง

นางสาวสุธิดา พัฒเลิศสินธุ์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2559

2-ABSORBING R-IDEALS OF MODULES OVER NEAR RINGS

Miss Sutida Patlertsin

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2016

Thesis Title	2-ABSORBING R-IDEALS OF MODULES OVER NEAR RINGS
By	Miss Sutida Patlertsin
Field of Study	Mathematics
Thesis Advisor	Assistant Professor Sajee Pianskool, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

.... Dean of the Faculty of Science

(Associate Professor Polkit Sangvani, Ph.D.)

THESIS COMMITTEE

.... Chairman

(Associate Professor Amorn Wasanawichit, Ph.D.)

.... Thesis Advisor

(Assistant Professor Sajee Pianskool, Ph.D.)

.... Examiner

(Teeraphong Phongpattanacharoen, Ph.D.)

.... External Examiner

(Assistant Professor Ruangvarin Intarawong Sararnrakskul, Ph.D.)

สุธิดา พัฒเลิศสินธุ์ : อาร์-ไอดีล ๒-แอบซอร์บิงของมอดูลเหนือเนียร์ริง. (2-ABSORBING R-IDEALS OF MODULES OVER NEAR RINGS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : ผศ.ดร. ศจี เพียรสกุล, 40 หน้า.

เป็นที่ทราบกันว่าเนียร์ริงและมอดูลเหนือเนียร์ริงเป็นพีชคณิตที่มีโครงสร้างวางนัยทั่วไปของริงและมอดูลเหนือริงตามลำดับ ในงานวิจัยนี้เราแนะนำแนวคิดของ อาร์-ไอดีล ๒-แอบซอร์บิงของมอดูลเหนือเนียร์ริงและไอดีล ๒-แอบซอร์บิงของเนียร์ริงซึ่งเป็นการวางนัยทั่วไปของอาร์-ไอดีลเฉพาะของมอดูลเหนือเนียร์ริง และไอดีลเฉพาะของเนียร์ริงตามลำดับ ยิ่งไปกว่านั้น เรานำเสนอผลวิจัยบางอย่างในทำนองเดียวกับผลวิจัยเหล่านั้นในทฤษฎีบนริง เราให้แนวคิดของ อาร์-ไอดีล ๒-แอบซอร์บิงอย่างแข็งของมอดูลเหนือเนียร์ริง และไอดีล ๒-แอบซอร์บิงอย่างแข็งของเนียร์ริงพร้อมกับศึกษาสมบัติบางประการ ท้ายสุดแล้ว เรายังมุ่งเน้นผลวิจัยบางประการของ อาร์-ไอดีล ๒-แอบซอร์บิงของมอดูลเหนือเนียร์ริงที่สามารถแยกได้ และไอดีล ๒-แอบซอร์บิงของเนียร์ริงที่สามารถแยกได้

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์	ลายมือชื่อนิสิต
สาขาวิชา คณิตศาสตร์	ลายมือชื่อ อ.ที่ปรึกษาหลัก
ปีการศึกษา 2559	

5772186023 : MAJOR MATHEMATICS
KEYWORDS : 2-ABSORBING R-IDEALS / NEAR RINGS / MODULES OVER NEAR RINGS

SUTIDA PATLERTSIN : 2-ABSORBING R-IDEALS OF MODULES OVER NEAR RINGS. ADVISOR : ASST. PROF. SAJEE PIANSKOOL, Ph.D., 40 pp.

It is known that near rings and modules over near rings are generalized algebraic structures of rings and modules over rings, respectively. In this thesis, we introduce the concepts of 2-absorbing R-ideals of modules over near rings and 2-absorbing ideals of near rings which are generalization of prime R-ideals of modules over near rings and prime ideals of near rings, respectively. Moreover, we present some results analogous to those in ring theory. We provide the notions of strongly 2-absorbing R-ideals of modules over near rings and strongly 2-absorbing ideals of near rings and study some of their properties. Finally, we also focus on some results of 2-absorbing R-ideals of modules over decomposable near rings and 2-absorbing ideals of decomposable near rings.

Department : ...Mathematics and..... ...Computer Science...	Student's Signature : ....
Field of Study : ......Mathematics......	Advisor's Signature : ....
Academic Year : ....2016.............	

ACKNOWLEDGEMENTS

Foremost, I would like to express my sincere gratitude to my advisor, Assistant Professor Dr. Sajee Pianskool, for the continuous support of my Master study and research, motivation, enthusiasm and immense knowledge. Her guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisor and mentor for my Masters study.

Besides my advisor, I would like to thank my thesis committee: Associate Professor Dr. Amorn Wasanawichit, Dr. Teeraphong Phongpattanacharoen and Assistant Professor Dr. Ruangvarin Intarawong Sararnraskul for their encouragement, insightful comments, and challenge questions. I would also like to thank Science Achievement Scholarship of Thailand (SAST) for providing me financial support through my Undergrad and Master studies. I am also thankful to my friends for their support, help and motivation. They always encourage me to do better.

CONTENTS

page

CHAPTER I
INTRODUCTION

In ring theory, there is one special kind of ideals which is quite important called prime ideals. A proper ideal P of a commutative ring R with nonzero identity is called a **prime ideal** if whenever $a, b \in R, ab \in P$ implies $a \in P$ or $b \in P$. A **2-absorbing ideal** P of a commutative ring R with nonzero identity was introduced by Badawi [1] in 2007 and was defined to be a proper ideal of R and if whenever $a, b, c \in R, abc \in P$ implies $ab \in P$ or $bc \in P$ or $ac \in P$. He also proved that every prime ideal of a commutative ring with nonzero identity is a 2-absorbing ideal but its converse does not hold. As a result, the notion of 2-absorbing ideals are a generalization of prime ideals.

It is known that rings and modules over rings are related algebraic structures. In fact, every ring is a module over itself and for this case any left ideal of this ring is also a submodule of that module. So, it is natural that many research works related to prime ideals and 2-absorbing ideals of rings are extended to prime submodules and 2-absorbing submodules of modules over rings. In 2011, Darani and Soheilnia [3] introduced the concept of 2-absorbing submodules of unitary modules over commutative rings with identities. A proper submodule N of a unitary module M over a commutative ring R with identity is said to be a **2-absorbing submodule** of M if whenever $a, b \in R$ and $m \in M, abm \in N$ implies $abM \subseteq N$ or $am \in N$ or $bm \in N$. Moreover, every prime submodule of a unitary module over a commutative ring with identity is a 2-absorbing submodule but not vice versa. Hence 2-absorbing submodules are a generalization of prime submodules. In addition, it is obvious that 2-absorbing ideals are a special case of 2-absorbing submodules.

In 1905, Dickson [4] showed that there exists a near field which is an algebraic structure similar to a field except that its multiplication is not necessarily commutative and at least one distributive law holds. Some years later, the notion of near rings was introduced. Many research have been carried out on this structure, for examples, the works of Gunter Pilz, Yuen Fong, Alan Oswald and K.C. Smith. In this thesis, many of the definitions we refer to are from Gunter Pilz's book [10].

There are four chapters in this thesis. In Chapter I, we collect definitions of near rings and modules over near rings as well as present some results which are used in this thesis. In Chapter II, we study some properties of prime R-ideals and 2-absorbing R-ideals of modules over near rings. Besides, we introduce strongly 2-absorbing R-ideals of modules over near rings and strongly 2-absorbing ideals of near rings and discuss some properties. Moreover, in Chapter III, we investigate some properties of prime R-ideals and 2-absorbing R-ideals of modules over decomposable near rings as well as prime ideals and 2-absorbing ideals of decomposable near rings. This thesis is completed by Chapter IV which is the conclusion of our work.

1.1 Near Rings

A near ring is a generalization of a ring whose two axioms are omitted, namely, the addition is not necessarily abelian and the multiplication distributes over the addition is applied either on left or right side.

Definition 1.1. [10] A **near ring** is a set R together with two operations, called the addition $+$ and the multiplication $\cdot$, satisfying the following conditions:

(i) $(R, +)$ is a group where the additive identity of $(R, +)$ is denoted by 0,

(ii) $(R, \cdot)$ is a semigroup, and

(iii) the right distributive law holds, i.e., $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in R$.

This near ring is also called a **right near ring**. If the condition (iii) is replaced by the "left distributive law", then this is called a **left near-ring**.

From here on, all near rings are right near rings. Moreover, for any a, b in a near ring R, we may write ab instead of $a \cdot b$ and $-a$ means the additive inverse of a.

Example 1.2. [10] Let G be a group and $M(G) = \{f : f$ is a function from G into $G\}$ with addition $+$ and multiplication $\circ$ on $M(G)$ given by

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (f \circ g)(x) = f(g(x))$$

for all $f, g \in M(G)$ and all $x \in G$. Then $(M(G), +)$ is a group. It is easy to see that $(M(G), \circ)$ is a semigroup and $(f+g) \circ h = f \circ h + g \circ h$ for all $f, g, h \in M(G)$. Hence $(M(G), +, \circ)$ is a near ring which is not a ring because $h \circ (f+g)$ is not necessary equal to $h \circ f + h \circ g$. For example, consider the group $(\mathbb{R}^+, \cdot)$, let $f, g, h : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $f(x) = 2x, g(x) = x$ and $h(x) = \sqrt{x}$ for all $x \in \mathbb{R}^+$. Then $(h \circ (f+g))(x) = \sqrt{3x}$ and $(h \circ f + h \circ g)(x) = \sqrt{2x} + \sqrt{x}$. If $x = 1$, then $(h \circ (f+g))(1) = \sqrt{3} \neq \sqrt{2} + 1 = (h \circ f)(1) + (h \circ g)(1)$.

Definition 1.3. [10] A near ring R is called a **near ring with identity** if there is an element $b \in R$ such that $ab = a = ba$ for all $a \in R$; we say that b is the **(multiplicative) identity** of the near ring R.

Example 1.4. Let $R = \{0, 1, a, b\}$ be the set with addition $+$ and multiplication $\cdot$ on R given by the following tables:

$+$	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	b
b	0	b	0	0

Then $(R, +, \cdot)$ is a near ring with identity 1 which is not a ring because $b(a+b) = b(1) = b \neq 0 = 0 + 0 = ba + bb$.

If R is a near ring, then it is always true that $0r = 0$ for all $r \in R$ because

$0r = (r-r)r = rr - rr = 0$ for each $r \in R$. However, the following example shows that $r0$ is not necessarily equal to 0.

Example 1.5. Let $R = \{0, 1\}$ be the set with addition $+$ and multiplication $\cdot$ given by the following tables:

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	1	1

Then $(R, +, \cdot)$ is a near ring without identity and $1 \cdot 0 = 1 \neq 0$. Note also that $(R, +, \cdot)$ is not a ring.

Definition 1.6. [10] A near ring R is called a **zero symmetric near ring** if $r0 = 0$ for all $r \in R$.

The near ring given in Example 1.4 is a zero symmetric near ring with identity 1. While, the near ring given in Example 1.5 is not a zero symmetric near ring because $1 \cdot 0 = 1 \neq 0$.

It is known that the multiplication of nonempty subsets A and B of a ring is defined as $AB = \{\sum_{i=1}^{n} a_i b_i : a_i \in A \text{ and } b_i \in B \text{ for all } i\}$. But, in near ring, if we defined AB in the same way, then there would have been a problem that $(AB)C$ is not necessary equal to $A(BC)$ because the distributive property of near rings may not be applied on the left side and the addition is not necessarily commutative.

Example 1.7. Let R be the near ring given in Example 1.4. Moreover, for any nonempty subsets X and Y of R, assume that $XY = \{\sum_{i=1}^{n} x_i y_i : x_i \in X$ and $y_i \in Y$ for all $i\}$. Next, let $A = \{b\}, B = \{a, b\}$ and $C = \{1\}$. Since $1 \notin B$, it follows that $AB = \{0\}$ and then $(AB)C = \{0\}$. However, $a1 + b1 \in BC$ so that $b(a1 + b1) \in A(BC)$ with $b(a1 + b1) = b(a + b) = b1 = b \neq 0$. This shows that $(AB)C \neq A(BC)$.

Consequently, in our work, for any nonempty subsets A and B of a near ring, the set AB has to be defined as follows.

Definition 1.8. [10] Let A and B be nonempty subsets of a near ring R. Then the set AB is defined to be $\{ab : a \in A \text{ and } b \in B\}$. For any nonempty subset A of a near ring R and $r \in R$, we write Ar instead of $A\{r\}$ and rA instead of $\{r\}A$.

Proposition 1.9. *If A, B and C are nonempty subsets of a near ring, then $(AB)C = A(BC)$.*

Proof. This is clear from the definition. □

In ring theory, ideals are special subsets of a ring. It is natural to extend ideals of a ring to ideals of a near ring. It turns out that there are two types of such those special subset objects in a near ring which are closed to ideals of rings, called R-subgroups and ideals of near rings.

Definition 1.10. [10] A subset H of a near ring R is called an R**-subgroup** of R if

(i) $(H, +)$ is a subgroup of $(R, +)$,

(ii) $HR \subseteq H$, and

(iii) $RH \subseteq H$.

However, if the conditions (i) and (ii) are satisfied, then H is called a **right** R**-subgroup**. If the conditions (i) and (iii) are satisfied, then H is called a **left** R**-subgroup**.

If we consider rings as near rings, then every ideal of rings is an R-subgroup of near rings.

Example 1.11. Let R be the near ring given in Example 1.5. It is easy to check that R is the only one R-subgroup of R. Note that $\{0\}$ is a right R-subgroup but not an R-subgroup of R because $\{0\}R \subseteq \{0\}$ but $1 = 1 \cdot 0 \in R\{0\}$, i.e., $R\{0\} \nsubseteq \{0\}$.

Example 1.12. Let R be the near ring given in Example 1.4. Then all R-subgroups of R are $\{0\}, \{0, b\}$ and R. Moreover, $\{0, a\}$ is a left R-subgroup but not an R-subgroup of R because $b = ab \in \{0, a\}R$, i.e., $\{0, a\}R \nsubseteq \{0, a\}$.

Proposition 1.13. *Let R be a near ring. If $a \in R$, then Ra is a left R-subgroup of R.*

Proof. Let $r_1 a, r_2 a \in Ra$ where $r_1, r_2 \in R$. We obtain that $r_1 a - r_2 a = (r_1 - r_2)a \in Ra$. Moreover, $R(Ra) = (RR)a \subseteq Ra$. Therefore, Ra is a left R-subgroup of R. □

Let consider the near ring $R = \{0, 1, a, b\}$ given in Example 1.4. Notice that $Ra = \{0, a\}$. However, $(Ra)R = \{0, a, b\} \nsubseteq Ra$. Then Ra is not a right R-subgroup of R. This shows that, in general, Ra is not necessarily a right R-subgroup of R. Moreover, aR is not necessarily a right R-subgroup of R. For example, $aR = \{0, a, b\}$ but $a + b = 1 \notin aR$. Then aR is not closed under addition.

Definition 1.14. [10] A subset I of a near ring R is called an **ideal** of R if

(i) $(I, +)$ is a normal subgroup of $(R, +)$,

(ii) $IR \subseteq I$, and

(iii) $r_1 (r_2 + k) - r_1 r_2 \in I$ for all $r_1, r_2 \in R$ and $k \in I$.

However, if I satisfies the conditions (i) and (ii), then I is called a **right ideal** of R, while I is called a **left ideal** of R if the conditions (i) and (iii) are satisfied.

In the same way as R-subgroups, if rings are considered as near rings, then every ideal of rings is an ideal of near rings.

Example 1.15. (i) Let R be the near ring given in Example 1.5. Then all ideals of R are $\{0\}$ and R.
(ii) Let R be the near ring given in Example 1.4. Then all ideals of R are $\{0\}$, $\{0, b\}$ and R. Note that $\{0, a\}$ is not an ideal because $a(b + a) - ab = a(1) - b = a + b = 1 \notin \{0, a\}$ (in fact, $\{0, a\}$ is not a left ideal; moreover, $\{0, a\}$ is not a right ideal because $b = ab \in \{0, a\}R$).

In general, R-subgroups and ideals of near rings may not be related. However, if $(R, +)$ is an abelian group, then left R-subgroups and left ideals of R are identical. Although R is a near ring such that $(R, +)$ is an abelian group, right R-subgroups are not necessary right ideals and vice versa. This is because the near ring R may satisfy only one distributive law. For example, $\{0\}$ is always an ideal of any near ring R so that $\{0\}$ is a right ideal of R but $\{0\}$ may not be a right R-subgroup of R, see Example 1.11.

The next proposition provides the condition that makes each ideal an R-subgroup.

Proposition 1.16. *Let R be a zero symmetric near ring and I be an ideal of R. Then*

(1) $RI \subseteq I$; and

(2) I is an R-subgroup of R.

Proof. Assume that R is a zero symmetric near ring. Let I be an ideal of R. Since I is an ideal of R, it follows that $(I, +)$ is a normal subgroup of $(R, +)$ and $IR \subseteq I$. Next, we show that $rk \in I$ for all $r \in R$ and $k \in I$. Let $r \in R$ and $k \in I$. Since R is a zero symmetric near ring, $r0 = 0$. And we have $rk = r(0+k) - r0 \in I$ because I is an ideal of R and $k \in I$. Therefore, $RI \subseteq I$ and then I is an R-subgroup of R. □

However, even a near ring is a zero symmetric near ring, an R-subgroup may not be an ideal.

Example 1.17. Let $K = \{e, a, b, c\}$ be the Klein-4-group. Define the multiplication $\cdot$ on K by $r \cdot c = r$ and $r \cdot y = e$ for all $r \in K$ and $y \in \{e, a, b\}$. We illustrate these in the following tables:

$+$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\cdot$	e	a	b	c
e	e	e	e	e
a	e	e	e	a
b	e	e	e	b
c	e	e	e	c

Then K is a zero symmetric near ring, $\{e\}$ is the only one ideal of K and $\{e, b\}$ is an R-subgroup of K, see [9]. Therefore, $\{e, b\}$ is an R-subgroup of K but is not an ideal of K even K is a zero symmetric near ring.

The last result is another advantage of being a zero symmetric near ring.

Proposition 1.18. *Let R be a zero symmetric near ring and $I_1, I_2, \ldots, I_n$ be ideals of R. Then $I_1 I_2 \cdots I_n \subseteq I_1 \cap I_2 \cap \cdots \cap I_n$.*

Proof. Let $x_1 x_2 \cdots x_n \in I_1 I_2 \cdots I_n$ where $x_i \in I_i$ for all i. Since each I_i is an ideal of R and R is a zero symmetric near ring, $RI_i \subseteq I_i$ and $I_i R \subseteq I_i$ for all i so that $x_1 x_2 \ldots x_i \ldots x_n \in I_i$ for all i. Therefore, $I_1 I_2 \cdots I_n \subseteq I_1 \cap I_2 \cap \cdots \cap I_n$. □

1.2 Modules over Near Rings

Now, it is time to introduce modules over near rings which are a generalization of near rings. In fact, modules over near rings also are a generalization of modules over rings.

Definition 1.19. [10] Let R be a near ring and $(M, +)$ a group. Then M is called a **module over near ring** R (or an R**-module**) if there exists a **scalar multiplication** $\cdot : R \times M \to M$ such that for all $r_1, r_2 \in R$ and $m \in M$,

(i) $(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$, and

(ii) $(r_1 r_2) \cdot m = r_1 (r_2 \cdot m)$.

For any $r \in R$ and $m \in M$, we may write rm instead of $r \cdot m$. It is obvious that every near ring is a module over itself and every module over a ring R is a module over R where R is considered as a near ring.

Example 1.20. Let $(R = \{0, 1\}, +, \cdot)$ be the near ring which is not a ring given in Example 1.5 and $M = \{0, a\}$. Define the addition $+$ on M and the scalar multiplication $\odot : R \times M \to M$ by the following tables:

$+$	0	a
0	0	a
a	a	0

$\odot$	0	a
0	0	a
1	0	a

Then M is a module over the near ring R.

Definition 1.21. [10] Let R be a near ring with identity 1. An R-module M is called a **unitary R-module** if $1m = m$ for all $m \in M$.

Example 1.22. Let R be the near ring with identity 1 given in Example 1.4. Then R is an R-module. We can see that $1m = m$ for all $m \in R$. Then R is a unitary R-module.

Definition 1.23. [10] Let A be a nonempty subset of a near ring R and N be a nonempty subset of an R-module. Then define the set AN as $AN = \{an : a \in A$ and $n \in N\}$.

Proposition 1.24. *If A, B are nonempty subsets of a near ring R and N is a nonempty subset of an R-module, then $(AB)N = A(BN)$.*

Proof. Assume that A, B are nonempty subsets of a near ring R and N is a nonempty subset of an R-module M. First, let $(ab)n \in (AB)N$ where $a \in A$, $b \in B$ and $n \in N$. Since M is an R-module, $(ab)n = a(bn) \in A(BN)$. Then $(AB)N \subseteq A(BN)$. Similarly, $A(BN) \subseteq (AB)N$ is obtained. Therefore, $(AB)N = A(BN)$. □

Submodules of modules over rings are naturally extended to submodules of modules over near rings. Since some axioms of modules over rings are omitted, there are at least two types of such those special subset objects called R-submodules and R-ideals of modules over near rings.

Definition 1.25. [10] Let R be a near ring. A subgroup N of an R-module M is called an **R-submodule** of M if $rn \in N$ for all $r \in R$ and $n \in N$.

Definition 1.26. [10] Let R be a near ring. A normal subgroup N of an R-module M is called an **R-ideal** of M if $r(m + n) - rm \in N$ for all $r \in R$, $m \in M$ and $n \in N$.

Let R be a near ring. Then R is also an R-module. Thus R-ideals of the R-module R are the same as left ideals of the near ring R and R-submodules of the R-module R are left R-subgroups of the near ring R. The following examples show that R-submodules and R-ideals do not imply each other.

Example 1.27. Let R be the near ring given in Example 1.5. Then $\{0\}$ is an R-ideal of the R-module R but $\{0\}$ is not an R-submodule of the R-module R because $1 \cdot 0 = 1 \notin \{0\}$.

Example 1.28. Let R be the near ring given in Example 1.4. Then all R-submodules of the R-module R are $\{0\}, \{0, a\}, \{0, b\}$ and R. Moreover, all R-ideals of the R-module R are $\{0\}, \{0, b\}$ and R. Note that $\{0, a\}$ is not an R-ideal because $\{0, a\}$ is not a left ideal of the near ring R (see Example 1.15 (ii)). Thus $\{0, a\}$ is an R-submodule but not an R-ideal of the R-module R.

The next proposition yields that Rm is an R-submodule of an R-module M for any $m \in M$.

Proposition 1.29. *Let M be an R-module. If $m \in M$, then Rm is an R-submodule of M.*

Proof. This is similar to the proof of Proposition 1.13. □

Next, a condition that makes each R-ideal an R-submodule is given.

Proposition 1.30. *Let R be a zero symmetric near ring and N be an R-ideal of an R-module M. Then*

(1) $RN \subseteq N$; and

(2) N is an R-submodule of M.

Proof. This is similar to the proof of Proposition 1.16. □

Notation 1.31. *Let N and K be nonempty subsets of an R-module. Set $(N : K)$ $= \{r \in R : rK \subseteq N\}$.*

In module theory, for a submodule N of a module M over a ring R, $(N : M)$ is an ideal of R. We also obtain this similar result.

Proposition 1.32. *Let N be an R-ideal of an R-module M and K be an R-submodule of M. Then $(N : K)$ is an ideal of R.*

Proof. First, we show that $(N : K)$ is a normal subgroup of R. Since $0K \subseteq N$, it follows that $0 \in (N : K)$. Let $x, y \in (N : K)$. That is $xK \subseteq N$ and $yK \subseteq N$. Let $k \in K$. Then $(x - y)\,k = xk - yk \in N$. Thus $(x-y)K \subseteq N$ so that $x-y \in (N : K)$. Moreover, let $r \in R$. We show that $r + x - r \in (N : K)$. Since $xk \in xK \subseteq N$ and $(N, +)$ is a normal subgroup of $(M, +)$, it follows that $(r + x - r)\,k = rk + xk - rk \in N$. That is $(r + x - r)K \subseteq N$. So $r + x - r \in (N : K)$. Hence $(N : K)$ is a normal subgroup of R. Next, we obtain that $xrK \subseteq xRK \subseteq xK \subseteq N$ because K is an R-submodule of M and $xK \subseteq N$. This means that $xr \in (N : K)$. Hence $(N : K)R \subseteq (N : K)$. To show the rest, let $r_1 \in R$ and verify that $r(r_1 + x) - rr_1 \in (N : K)$. Note that $(r\,(r_1 + x) - rr_1)\,k = r\,(r_1 + x)\,k - (rr_1)k = r\,(r_1k + xk) - r\,(r_1k) \in N$ because $xk \in N$ and N is an R-ideal of M. This shows that $(r\,(r_1 + x) - rr_1)\,K \subseteq N$. That is $(r\,(r_1 + x) - rr_1) \in (N : K)$. Therefore, $(N : K)$ is an ideal of R. □

Proposition 1.33. *Let N and K be R-submodules of an R-module M. Then $(N : K)$ is an R-subgroup of R.*

Proof. Note that $(N : K)$ is a subgroup of R similarly to the proof of Proposition 1.32. Next, let $r \in R$ and $x \in (N : K)$. Then $xK \subseteq N$. Since N and K are R-submodules of M, it follows that $xrK \subseteq xRK \subseteq xK \subseteq N$ and $rxK \subseteq rN \subseteq N$. Then $(N : K)R \subseteq (N : K)$ and $R(N : K) \subseteq (N : K)$. Therefore, $(N : K)$ is an R-subgroup of R. □

It is known that intersections of submodules of modules over rings are submodules. In next propositions, we consider intersections of R-ideals, and of R-submodules of modules over near rings, respectively.

Proposition 1.34. *Let N_i be an R-ideal of an R-module M for all $i \in I$. Then $\bigcap\limits_{i\in I} N_i$ is an R-ideal of M.*

Proof. We obtain that N_i is a normal subgroup of M because N_i is an R-ideal of M for all $i \in I$. Then $\bigcap\limits_{i\in I} N_i$ is a normal subgroup of M. Let $r \in R$, $n \in \bigcap\limits_{i\in I} N_i$ and $m \in M$. Then $r(m+n) - rm \in N_i$ for all $i \in I$ because each N_i is an R-ideal of M. That is $r(m+n) - rm \in \bigcap\limits_{i\in I} N_i$. Therefore, $\bigcap\limits_{i\in I} N_i$ is an R-ideal of M. □

Proposition 1.35. *Let N_i be an R-submodule of an R-module M for all $i \in I$. Then $\bigcap\limits_{i\in I} N_i$ is an R-submodule of M.*

Proof. Since $(N_i, +)$ is a subgroup of $(M_i, +)$ for all $i \in I$, it follows that $(\bigcap\limits_{i\in I} N_i, +)$ is a subgroup of $(M, +)$. Next, let $r \in R$ and $n \in \bigcap\limits_{i\in I} N_i$. Then $rn \in RN_i \subseteq N_i$ for all $i \in I$ because each N_i is an R-submodule of M. Thus $rn \in \bigcap\limits_{i\in I} N_i$ so that $R(\bigcap\limits_{i\in I} N_i) \subseteq \bigcap\limits_{i\in I} N_i$. Therefore, $\bigcap\limits_{i\in I} N_i$ is an R-submodule of M. □

One can see from the definitions of R-submodules and R-ideals that R-ideals are normal subgroups but R-submodules are not necessarily. Consequently, these allow us to define quotient modules over near rings via R-ideals.

Theorem 1.36. *Let N be an R-ideal of an R-module M. Moreover, let $M/N = \{m + N : m \in M\}$. Define the addition $+$ on M/N and the scalar multiplication $\cdot$ by*

$$(m+N) + (n+N) = (m+n) + N \quad \text{and} \quad r \cdot (m+N) = rm + N$$

for all $r \in R$ and for all $m, n \in M$. Then $(M/N, +, \cdot)$ is an R-module.

Proof. Since N is an R-ideal of M, it follows that $(N, +)$ is a normal subgroup of $(M, +)$. Thus $(M/N, +)$ is a group. Now, we show that the scalar multiplication is well-defined. Let $x, y \in M$ and $r \in R$. Assume that $x + N = y + N$. Then $-y + x \in N$. That is $rx - ry = r(y + (-y + x)) - ry \in N$ because N is an R-ideal of M. Since N is a normal subgroup of M and $rx - ry \in N$, it follows that $-ry + rx = -ry + (rx - ry) + ry \in N$. Then we can conclude that the scalar multiplication is well-defined. Next, we show that $(r_1 + r_2) \cdot (x + N) =$

$r_1 \cdot (x+N) + r_2 \cdot (x+N)$ and $(r_1 r_2) \cdot (x+N) = r_1 \cdot (r_2 \cdot (x+N))$ for all $r_1, r_2 \in R$. Let $r_1, r_2 \in R$. Then,

$$\begin{aligned}(r_1 + r_2) \cdot (x+N) &= (r_1 + r_2)\, x + N \\ &= (r_1 x + r_2 x) + N \\ &= (r_1 x + N) + (r_2 x + N) \\ &= r_1 \cdot (x+N) + r_2 \cdot (x+N)\end{aligned}$$

and

$$(r_1 r_2) \cdot (x+N) = (r_1 r_2)x + N = r_1(r_2 x) + N = r_1 \cdot ((r_2 x) + N) = r_1 \cdot (r_2 \cdot (x+N)).$$

Therefore, $(M/N, +, \cdot)$ is an R-module. □

Definition 1.37. Let N be an R-ideal of an R-module M. Then $(M/N, +, \cdot)$ given in Theorem 1.36 is called the **quotient module over the near ring** R.

Proposition 1.38. *If N and K are R-ideals of an R-module M with $K \subseteq N$, then N/K is an R-ideal of M/K.*

Proof. First, we show that N/K is a normal subgroup of M/K. Since N and K are R-ideals of M containing K, it follows that N and K are normal subgroups of M so that K is also a normal subgroup of N and then N/K is a subgroup of M/K. Let $m \in M$ and $x \in N$. Then $(m+K)+(x+K)-(m+K) = (m+x-m)+K \in N/K$ because $x \in N$ and N is a normal subgroup of M. Hence N/K is a normal subgroup of M/K. Let $r \in R$. Then we obtain that

$$\begin{aligned}r \cdot ((m+K) + (x+K)) - r \cdot (m+K) &= r \cdot ((m+x) + K) - r \cdot (m+K) \\ &= (r\,(m+x) + K) - (rm + K) \\ &= (r\,(m+x) - rm) + K \in N/K\end{aligned}$$

because N is an R-ideal of M and $x \in N$. Therefore, N/K is an R-ideal of M/K. □

In order to obtain the results of Proposition 1.38, being R-ideals of N and K is crucial because if N and K were R-submodules of M, then N and K may not be normal subgroups of M.

Example 1.39. Let R be the near ring given in Example 1.4. Then R is an R-module. Next, let $N = \{0, b\}$ and $K = \{0\}$. Then $K \subseteq N$. Moreover, we obtain from Example 1.28 that N and K are R-ideals of R. Therefore, N/K is an R-ideal of M/K.

It is known that a near ring is an algebraic structure similar to a ring. In 1970, Holcombe [8] extended the definition of prime ideals of rings to prime ideals of near rings. However, in [8], there are three types of such those prime objects, namely, 0-prime ideals, 1-prime ideals and 2-prime ideals. Moreover, Groenewald [5] introduced in 1991 two more types of prime ideals of near rings, namely, 3-prime ideals and completely prime ideals. Recently, in 2010, Groenewald, Juglal and Lee [7] extended prime ideals of near rings to prime R-ideals of modules over near rings.

In our work, inspired by the above, we aim to study the notions that generalize prime ideals of near rings and prime R-ideals of modules over near rings in the same way as prime ideals of rings and prime submodules of modules over rings were extended, called 2-absorbing ideals and 2-absorbing R-ideals, respectively. We investigate properties of prime R-ideals and 2-absorbing R-ideals of modules over near rings in general. Then we focus on 2-absorbing R-ideals of modules over decomposable near rings and 2-absorbing ideals of decomposable near rings.

CHAPTER II

PRIME R-IDEALS AND 2-ABSORBING R-IDEALS OF MODULES OVER NEAR RINGS

As a near ring is an algebraic structure relative to a ring, it is natural to extend prime ideals of rings to prime ideals of near rings. In 1970, Holcombe [8] extended the definition of prime ideals of rings to prime ideals of near rings. He introduced three types of such those prime objects which he called 0-prime ideals, 1-prime ideals and 2-prime ideals. Moreover, Groenewald [5], in 1991, introduced two more types of prime ideals of near rings which he called 3-prime ideals and completely prime ideals.

Definition 2.1. [5, 8] Let P be a proper ideal of a near ring R. Then P is called

(1) a **0-prime ideal** of R if for all ideals A and B of R, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$;

(2) a **1-prime ideal** of R if for all left ideals A and B of R, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$;

(3) a **2-prime ideal** of R if for all left R-subgroups A and B of R, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$;

(4) a **3-prime ideal** of R if for all $a, b \in R$, $aRb \subseteq P$ implies $a \in P$ or $b \in P$; and

(5) a **completely prime ideal** of R if for all $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$.

Next, relationships between these five types of prime ideals of near rings are given.

Proposition 2.2. *[5] Let P be a proper ideal of a near ring R. Then P is a completely prime ideal $\rightarrow$ P is a 3-prime ideal $\rightarrow$ P is a 2-prime ideal $\rightarrow$ P is a 1-prime ideal $\rightarrow$ P is a 0-prime ideal.*

Example 2.3. Let K be the near ring given in Example 1.17. Then $\{e\}$ is a 0-prime ideal but is not 3-prime ideal of K. Note that $\{e\}$ is a 0-prime ideal of K because $\{e\}$ and K are the only ideals of K and $KK \neq \{e\}$. Moreover, $aRa = \{e, a\}a = \{e\}$ and $a \notin \{e\}$. Therefore, $\{e\}$ is not a prime ideal of K.

Recall that modules over near rings are a generalization of near rings. In 2010, Groenewald, Juglal and Lee [7] extended prime ideals of near rings to prime R-ideals of modules over near rings.

Definition 2.4. [7] Let N be a proper R-ideal of an R-module M. Then N is called

(1) a **0-prime R-ideal** of M if for all ideal A of R and R-ideal K of M, $AK \subseteq N$ implies $AM \subseteq N$ or $K \subseteq N$;

(2) a **1-prime R-ideal** of M if for all left ideal A of R and R-ideal K of M, $AK \subseteq N$ implies $AM \subseteq N$ or $K \subseteq N$;

(3) a **2-prime R-ideal** of M if for all left R-subgroup A of R and R-submodule K of M, $AK \subseteq N$ implies $AM \subseteq N$ or $K \subseteq N$;

(4) a **3-prime R-ideal** of M if for all $r \in R, m \in M, rRm \subseteq N$ implies $rM \subseteq N$ or $m \in N$; and

(5) a **completely prime R-ideal** of M if for all $r \in R, m \in M, rm \in N$ implies $rM \subseteq N$ or $m \in N$.

Similary, Groenewald, Juglal and Lee showed relationships between the five types of prime R-ideals of modules over near rings.

Proposition 2.5. *[7] Let N be a proper R-ideal of an R-module M. Then N is a completely prime R-ideal $\rightarrow$ N is a 3-prime R-ideal $\rightarrow$ N is a 2-prime R-ideal $\rightarrow$ N is a 1-prime R-ideal $\rightarrow$ N is a 0-prime R-ideal.*

In 2007, Badawi extended the notion of prime ideals to 2-absorbing ideals of commutative rings with identities, see [1]. In 2011, Darani and Soheilnia extended the notion of prime submodules to 2-absorbing submodules of modules over commutative rings with identities, see [3]. In this thesis, we extend the concept of prime ideals of near rings and prime R-ideals of modules over near rings to 2-absorbing ideals of near rings and 2-absorbing R-ideals of modules over near rings, respectively. This extension is done in the same way as prime ideals of commutative rings with identities, and prime submodules of modules over commutative rings with identities were extended. First, let see how Darani and Soheilnia extended the definition of prime submodules to 2-absorbing submodules of modules over commutative rings with identities. Recall that a proper submodule N of a unitary module M over a commutative ring R with identity is said to be a **prime submodule** of M if for any $a \in R, m \in M$, $am \in N$ implies $aM \subseteq N$ or $m \in N$. And they extended this to 2-absorbing submodules. A proper submodule N of a unitary module M over a commutative ring R with identity is said to be a **2-absorbing submodule** of M if for any $a, b \in R, m \in M, abm \in N$ implies $abM \subseteq N$ or $am \in N$ or $bm \in N$.

Similarly, we would like to extend the idea of various prime R-ideals of modules over near rings to 2-absorbing R-ideals. First, consider 0-prime R-ideals of a module over a near ring. The definition of a 2-absorbing R-ideal N of a module M over a near ring R should be defined to be a proper R-ideal of M and for any ideals A, B of R, any R-ideal C of $M, ABC \subseteq N$ implies $ABM \subseteq N$ or $AC \subseteq N$ or $BC \subseteq N$. With this definition, "every 0-prime R-ideal of a module over a near ring is a 2-absorbing R-ideal" should be obtained. However, this is not the case because for any ideals A and B of a near ring R, it is not always that AB is an ideal of R. Besides, for any ideal A of a near ring R and any R-ideal C of a module M over the near ring R, it is not necessary that AC is an R-ideal of M.

Consequently, 2-absorbing R-ideals should not be extended from 0-prime R-ideals. Analogously, similar problems would have arisen if we extended 2-absorbing R-ideals from 1-prime R-ideals or 2-prime R-ideals. However, these does not occur with extending 3-prime R-ideals and completely prime R-ideals of modules over near rings. In this thesis, we define 2-absorbing ideals and 2-absorbing R-ideals by extending these from 3-prime ideals and 3-prime R-ideals, respectively. As for this, to be convenient, we call 3-prime ideals and 3-prime R-ideals as prime ideals and prime R-ideals, respectively. Let us rephrase these as follows.

Definition 2.6. Let R be a near ring and P be a proper ideal of R. Then P is called a **prime ideal** of R if for all $a, b \in R, aRb \subseteq P$ implies $a \in P$ or $b \in P$.

Definition 2.7. Let R be a near ring, M be an R-module and N be a proper R-ideal of M. Then N is called a **prime R-ideal** of M, if for all $a \in R, m \in M$, $aRm \subseteq N$ implies $aM \subseteq N$ or $m \in N$.

Example 2.8. Let $R = \{0, 1\}$ be the near ring given in Example 1.5 under following operations:

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	1	1

Then $\{0\}$ is the only proper R-ideal of the R-module R, see Example 1.27. And it is easy to verify that $\{0\}$ is a prime R-ideal of the R-module R. To check this, let $x, y \in R$. Assume that $xRy = \{0\}$. Then $x = 0$ because if $x = 1$, then $xRy = 1Ry = \{1\} \neq \{0\}$. Therefore, $\{0\}$ is the only prime R-ideal of the R-module R.

Example 2.9. Let $R = \{0, 1, a, b\}$ be the zero symmetric near ring given in Example 1.4 under the following operations:

$+$	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	b
b	0	b	0	0

Recall from Example 1.28 that $\{0\}$ and $\{0, b\}$ are the only proper R-ideals of the R-module R. One can check that $\{0, b\}$ is the only prime R-ideal of the R-module R. However, $\{0\}$ is not a prime R-ideal of the R-module R because $b \cdot b = 0$ but $b \notin \{0\}$.

The following result shows a relationship between prime submodules and prime R-ideals in some cases.

Proposition 2.10. *Let R be a commutative ring with identity 1 and M be a module over the ring R. We know that the ring R is also a near ring so that M can be considered as a module over the near ring R. To distinguish between these structures, we write M' to interpret the module M over the near ring R. Then every prime submodule of M is a prime R-ideal of M'.*

Proof. By definition of R-ideals of modules over near rings, we can conclude that every submodule of M is an R-ideal of M'. Next, let N be a prime submodule of M. To show that N is a prime R-ideal of M', let $a \in R$ and $m \in M$. Assume that $aRm \subseteq N$. Then $am = (a1)m \in N$. Since N is a prime submodule of M, we obtain that $aM \subseteq N$ or $m \in N$. Therefore, N is a prime R-ideal of M'. $\square$

Example 2.11. Let p be a prime number. Recall that $p\mathbb{Z}$ is a prime submodule of the module $\mathbb{Z}$ over the ring $\mathbb{Z}$. Then $p\mathbb{Z}$ is a prime R-ideal of the module $\mathbb{Z}$ over the near ring $\mathbb{Z}$ by Proposition 2.10.

Now, we are ready to provide the definitions of 2-absorbing ideals of near rings and 2-absorbing R-ideals of modules over near rings.

Definition 2.12. Let R be a near ring and P be a proper ideal of R. Then P is called a **2-absorbing ideal** of R if for all $a, b, c \in R, aRbRc \subseteq P$ implies $ab \in P$ or $bc \in P$ or $ac \in P$.

Definition 2.13. Let R be a near ring, M be an R-module and N be a proper R-ideal of M. Then N is called a **2-absorbing R-ideal** of M if for all $a, b \in R$, $m \in M$, $aRbRm \subseteq N$ implies $abM \subseteq N$ or $am \in N$ or $bm \in N$.

Badawi introduced, in [1], 2-absorbing ideals of rings and showed that every prime ideal of a ring is a 2-absorbing ideal. Later, Darani and Soheilnia provided the notion of 2-absorbing submodules of modules over rings and proved that every prime submodule of a module over a ring is a 2-absorbing submodule, see [3]. Consequently, we expect to obtain the similar result in term of 2-absorbing R-ideals of modules over near rings. Anyhow, the following result is needed.

Proposition 2.14. *Let N be a prime R-ideal of an R-module M. For all $a, b \in R$, $m \in M$, if $aRbRm \subseteq N$ and $am \notin N$, then $bM \subseteq N$.*

Proof. Let $a, b \in R$ and $m \in M$. Assume that $aRbRm \subseteq N$ and $am \notin N$. First, we show that $bRm \subseteq N$. Let $r \in R$. Then $aR(brm) \subseteq aRbRm \subseteq N$. Since N is a prime R-ideal of M, $aM \subseteq N$ or $brm \in N$. Then $brm \in N$ because $am \notin N$. That is $bRm \subseteq N$ as desired. Since N is a prime R-ideal of M, it follows that $bM \subseteq N$ or $m \in N$. Note that $am \notin N$ so $m \notin N$. Therefore, $bM \subseteq N$. □

Proposition 2.15. *Let R be a zero symmetric near ring. If N is a prime R-ideal of an R-module M, then N is a 2-absorbing R-ideal of M.*

Proof. Assume that N is a prime R-ideal of an R-module M. Let $a, b \in R$ and $m \in M$. Assume that $aRbRm \subseteq N$ but $am \notin N$. Thus $bM \subseteq N$ by Proposition 2.14. Then $bm \in N$ and $abM \subseteq aN \subseteq RN \subseteq N$ by Proposition 1.30 (i). Therefore, N is a 2-absorbing R-ideal of M. □

Proposition 2.15 guarantees that every prime R-ideal is a 2-absorbing R-ideal provide that R is a zero symmetric near ring. But the converse does not necessarily hold.

Example 2.16. Let $R = \{0, 1, a, b\}$ be the R-module considered in Example 2.9. Note also that R is a zero symmetric near ring and $\{0\}$ is not a prime R-ideal of R. Moreover, $\{0\}$ is a 2-absorbing R-ideal of R. To see this, let $x, y, z \in R$. Assume that $xRyRz = \{0\}$. If $x = 0$ or $y = 0$ or $z = 0$, then $xy = 0$ or $xz = 0$ or $yz = 0$ because R is a zero symmetric near ring. Next, suppose that each of x, y and z is not zero. There are 2 cases to be considered:

(i) at least two of x, y and z are 1, and

(ii) at most one of x, y and z are 1.

First, we consider Case(i). Without loss of generality, it suffices to assume that x and y are 1. It follows that $1R1Rz \neq \{0\}$ which is a contradiction. Thus Case(i) does not occur. Next, Case(ii) is considered. Then there is exactly one of x, y and z such that it is 1. We claim that $x = b$ or $y = b$ or $z = b$. Suppose not, i.e., $x \neq b$ and $y \neq b$ and $z \neq b$. First, assume that $x = 1$. This leads to $y = z = a$. Thus $a = 1 \cdot 1 \cdot a \cdot 1 \cdot a \in 1RyRz$, i.e. , $1RyRz \neq \{0\}$. Similarly, if $y = 1$ or $z = 1$, then $xRyRz \neq \{0\}$. This is a contradiction. Hence the claim is proved. As a result, there are 3 possible choices of $xRyRz$, namely, $bRyRz, xRbRz$, or $xRyRb$. We obtain from the table of multiplication in Example 2.9 that $xRyRb \neq \{0\}$. Moreover, if $\{0\} = xRyRz = bRyRz$, then $by = 0$. Or, if $\{0\} = xRyRz = xRbRz$, then $bz = 0$. This shows that whenever $xRyRz = \{0\}$, then $xy = 0$ or $xz = 0$ or $yz = 0$. Therefore, $\{0\}$ is a 2-absorbing R-ideal of M.

In 2011, Darani and Soheilnia showed in [3] that intersections of each pair of prime submodules of modules over rings are 2-absorbing submodules. It is reasonable to generalize this result to intersections of each pair of prime R-ideals of modules over near rings.

Theorem 2.17. *Let M be an R-module. Then intersections of each pair of prime R-ideals of M are 2-absorbing R-ideals of M.*

Proof. Let N and K be two prime R-ideals of M. If $N = K$, then $N \cap K$ is a prime R-ideal of M so that $N \cap K$ is a 2-absorbing R-ideal of M. Assume that N and K are distinct. Since N and K are proper R-ideals of M, it follows that $N \cap K$ is a proper R-ideal of M. Next, let $a, b \in R$ and $m \in M$ be such that $aRbRm \subseteq N \cap K$ but $am \notin N \cap K$ and $abM \nsubseteq N \cap K$. Then, we can conclude that (a) $am \notin N$ or $am \notin K$, and (b) $abM \nsubseteq N$ or $abM \nsubseteq K$. These reach to 4 cases:

(i) $am \notin N$ and $abM \nsubseteq N$

(ii) $am \notin N$ and $abM \nsubseteq K$

(iii) $am \notin K$ and $abM \nsubseteq N$

(iv) $am \notin K$ and $abM \nsubseteq K$.

First, we consider Case(i). Since $aRbRm \subseteq N \cap K \subseteq N$ and $am \notin N$, it follows from Proposition 2.14 that $bM \subseteq N$. This is a contradiction because $abM \nsubseteq N$. Hence Case(i) does not occur. Similarly, Case(iv) is not possible.

Next, Case(ii) is considered. Again, we obtain that $bM \subseteq N$ and then $bm \in N$. Let $r \in R$. Since $aRbRm \subseteq N \cap K \subseteq K$, it follows that $aR(brm) \subseteq aR(bRm) \subseteq K$. Hence $aM \subseteq K$ or $brm \in K$ because K is a prime R-ideal of M. If $aM \subseteq K$, then $abM \subseteq aM \subseteq K$ contradicts $abM \nsubseteq K$. Thus $brm \in K$. That is $bRm \subseteq K$. Since K is a prime R-ideal, $bM \subseteq K$ or $m \in K$. If $bM \subseteq K$, then $abM \subseteq K$ leading to the same contradiction. Hence, $m \in K$ and then $bm \in K$. As a result, $bm \in N \cap K$.

The proof of Case(iii) is similar to that of Case(ii).

Therefore, intersections of each pair of prime R-ideals of M are 2-absorbing R-ideals of M. □

Example 2.18. It follows from Example 2.11 that $2\mathbb{Z}$ and $3\mathbb{Z}$ are prime $\mathbb{Z}$-ideals of the $\mathbb{Z}$-module $\mathbb{Z}$. Then $6\mathbb{Z} = 2\mathbb{Z} \cap 3\mathbb{Z}$ is a 2-absorbing $\mathbb{Z}$-ideal of $\mathbb{Z}$ by Theorem 2.17.

Since intersections of each pair of prime R-ideals are 2-absorbing R-ideals and prime R-ideals are 2-absorbing R-ideals, is it true that intersections of each pair of 2-absorbing R-ideals are 2-absorbing R-ideals? The following example shows that intersections of each pair of 2-absorbing R-ideals are not necessary 2-absorbing R-ideals.

Example 2.19. Since $5\mathbb{Z}$ is a prime $\mathbb{Z}$-ideal of the $\mathbb{Z}$-module $\mathbb{Z}$ by Example 2.11 so that $5\mathbb{Z}$ is also a 2-absorbing $\mathbb{Z}$-ideal. Moreover, $6\mathbb{Z}$ is a 2-absorbing $\mathbb{Z}$-ideal of $\mathbb{Z}$ from Example 2.18. Note that $6\mathbb{Z} \cap 5\mathbb{Z} = 30\mathbb{Z}$ but $30\mathbb{Z}$ is not a 2-absorbing $\mathbb{Z}$-ideal of $\mathbb{Z}$ because $2 \cdot 3 \cdot 5 \in 30\mathbb{Z}$ but $2 \cdot 3, 2 \cdot 5, 3 \cdot 5 \notin 30\mathbb{Z}$.

The following proposition shows results of intersections of an R-ideal and a prime (2-absorbing) R-ideal.

Proposition 2.20. *Let N and K be R-ideals of an R-module M with $K \nsubseteq N$.*

(1) If N is a prime R-ideal of M, then $K \cap N$ is a prime R-ideal of K.

(2) If N is a 2-absorbing R-ideal of M, then $K \cap N$ is a 2-absorbing R-ideal of K.

Proof. We proof only (2) because the proof of (1) can be obtained similarly. Since N and K are R-ideals of M and $K \nsubseteq N$, it follows that $K \cap N$ is a proper R-ideal of K. Assume that N is a 2-absorbing R-ideal of M. Let $a, b \in R$ and $x \in K$ be such that $aRbRx \subseteq K \cap N$. Since K ia an R-ideal of M, we obtain that $abK \subseteq K$ and $ax, bx \in K$. Moreover, since $aRbRx \subseteq K \cap N \subseteq N$ and N is a 2-absorbing R-ideal of M, it follows that $abM \subseteq N$ or $ax \in N$ or $bx \in N$. Thus $abK \subseteq abK \cap abM \subseteq K \cap N$ or $ax \in K \cap N$ or $bx \in K \cap N$. Therefore, $K \cap N$ is a 2-absorbing R-ideal of K. □

In a module M over a commutative ring with identity, N is a prime (2-absorbing) submodule of M if and only if N/K is a prime (2-absorbing) submodule of M/K. Next, we aim to consider prime R-ideals and 2-absorbing R-ideals of quotient modules over near rings.

Theorem 2.21. *Let N and K be R-ideals of an R-module M with $K \subseteq N$. Then*

(1) N is a prime R-ideal of M if and only if N/K is a prime R-ideal of M/K; and

(2) N is a 2-absorbing R-ideal of M if and only if N/K is a 2-absorbing R-ideal of M/K.

Proof. It suffices to proof only (2). First, assume that N is a 2-absorbing R-ideal of M. Then N/K is a proper R-ideal of M/K. Let $a, b \in R$ and $m \in M$ be such that $aRbR \cdot (m + K) \subseteq N/K$. Let $s, t \in R$. Then $asbtm + K = asbt \cdot (m + K) \in aRbR \cdot (m + K) \subseteq N/K$. Thus there exists $n \in N$ such that $asbtm + K = n + K$ so that $-n + asbtm \in K \subseteq N$ and then $asbtm \in N$. This shows that $aRbRm \subseteq N$. As a result, $am \in N$ or $bm \in N$ or $abM \subseteq N$ because N is a 2-absorbing R-ideal of M. Hence $a \cdot (m + K) \in N/K$ or $b \cdot (m + K) \in N/K$ or $ab \cdot (M/K) \subseteq N/K$. Therefore, N/K is a 2-absorbing R-ideal of M/K.

Conversely, assume that N/K is a 2-absorbing R-ideal of M/K. Then N is a proper R-ideal of M. Let $a, b \in R$ and $m \in M$ be such that $aRbRm \subseteq N$. Then $aRbR \cdot (m + K) \subseteq N/K$. Since N/K is a 2-absorbing R-ideal of M/K, we obtain that $a \cdot (m + K) \in N/K$ or $b \cdot (m + K) \in N/K$ or $ab \cdot (M/K) \subseteq N/K$. That is $am \in N$ or $bm \in N$ or $abM \subseteq N$. This implies that N is a 2-absorbing R-ideal of M. □

Now, we know that 2-absorbing ideals (R-ideals) are generalization of prime ideals (R-ideals) of near rings (modules over near rings). Moreover, in ring theory, many research work are studied on prime ideals of rings regarding special properties called a strongly prime ideals. Moreover, strongly prime ideals of rings are extended to strongly prime submodules of modules over rings. Groenewald and Juglal extended strongly prime submodules of modules over rings to strongly prime R-ideals of modules over near rings in 2011, see [6]. At this point, we introduce the notion of strongly 2-absorbing R-ideals of modules over near rings and strongly 2-absorbing ideals of near rings and study some properties.

Definition 2.22. [6] Let M be an R-module. A proper R-ideal N of M is called a **strongly prime R-ideal** of M if for all $m \in M \setminus N$, there exists a finite subset F of R such that for all $a \in R, aFm \subseteq N$ implies $aM \subseteq N$.

Example 2.23. Recall from Example 1.27 that $\{0\}$ is an R-ideal of the R-module $R = \{0, 1\}$. Now, we show that $\{0\}$ is a strongly prime R-ideal of the R-module R. Let $F = \{0\}$. Since $0F1 = 0\{0\}1 = \{0\}$ and $1F1 = 1\{0\}1 = \{1\}$, if for any $a \in R$, $aF1 \subseteq \{0\}$, then $a = 0$ and so that $aR = 0R = \{0\}$. Therefore, $\{0\}$ is a strongly prime R-ideal of the R-module R.

Theorem 2.24. *[6] If N is a strongly prime R-ideal of an R-module M, then N is a prime R-ideal of M.*

We give the definition of strongly 2-absorbing R-ideals.

Definition 2.25. Let M be an R-module. A proper R-ideal N of M is called a **strongly 2-absorbing R-ideal** of M if for all $m \in M \setminus N$, there exists a finite subset F of R such that for all $a, b \in R, aFbFm \subseteq N$ implies $abM \subseteq N$.

Theorem 2.26. *If N is a strongly 2-absorbing R-ideal of an R-module M, then N is a 2-absorbing R-ideal of M.*

Proof. Assume that N is a strongly 2-absorbing R-ideal of M. Let $a, b \in R$ and $m \in M$. Assume that $aRbRm \subseteq N$ but $am \notin N$ and $bm \notin N$. Since $bm \notin N$ and N is a strongly 2-absorbing R-ideal of M, there exists a finite subset F of R such that for all $x, y \in R$, $xFyF(bm) \subseteq N$ implies $xyM \subseteq N$. Since $aFbF(bm) \subseteq aRbRbm = aRb(Rb)m \subseteq aRb(R)m = aRbRm \subseteq N$, we obtain that $abM \subseteq N$. Therefore, N is 2-absorbing R-ideal of M. □

From the definition of strongly prime R-ideals of modules over near rings, we define a strongly prime ideal of a near ring by considering a near ring as a module over itself.

Definition 2.27. Let R be a near ring. A proper ideal P of R is called a **strongly 2-absorbing ideal** of R if for all $r \in R \setminus P$, there exists a finite subset F of R such that for all $a, b \in R, aFbFr \subseteq P$ implies $ab \in P$.

In modules over rings, there are some interesting results of 2-absorbing submodules of modules over rings. If N is a 2-absorbing submodule of a module M over a ring R, then $(N : M)$ is a 2-absorbing ideal of R. Now, we provide a condition that makes $(N : M)$ a 2-absorbing ideal of near rings. Anyhow, the following result is needed.

Theorem 2.28. *Let R be a near ring with identity 1. If N is a strongly 2-absorbing R-ideal of a unitary R-module M, then $(N : M)$ is a strongly 2-absorbing ideal of R.*

Proof. If $(N : M) = R$, then $1 \in (N : M)$, i.e., $M = 1M \subseteq N$ contradicts the fact that N is a proper R-ideal of M. Then $(N : M)$ is a proper ideal of R. Let

$r \in R \setminus (N : M)$. Then $rM \subsetneq N$. Thus there exists $m \in M$ such that $rm \notin N$. Since $rm \notin N$ and N is a strongly 2-absorbing R-ideal of M, there exists a finite subset F of R such that for all $x, y \in R, xFyFrm \subseteq N$ implies $xyM \subseteq N$. Let $a, b \in R$ be such that $aFbFrm \subseteq N$. Then $abM \subseteq N$, i.e., $ab \in (N : M)$. Therefore, $(N : M)$ is a strongly 2-absorbing ideal of R. □

Corollary 2.29. *Let R be a near ring with identity 1. If N is a strongly 2-absorbing R-ideal of a unitary R-module M, then $(N : M)$ is a 2-absorbing ideal of R.*

Next, we pay attention to the converse of Theorem 2.28. In order to make the converse of Theorem 2.28 hold, some definitions used for this result are introduced.

Definition 2.30. [6] Let M be an R-module.

(i) A nonempty subset C of M is called a **multiplication set** if $(C : M)M = C$.

(ii) An element $m \in M$ is called a **multiplication element** if the singleton set $\{m\}$ is a multiplication set.

Definition 2.31. [6] An R-module M is called a **completely multiplication module** if every $m \in M$ is a multiplication element.

Theorem 2.32. *Let R be a near ring with identity 1 and M be a unitary R-module. Moreover, assume that M is a completely multiplication module and N is a proper R-ideal of M. If $(N : M)$ is a strongly 2-absorbing ideal of R, then N is a strongly 2-absorbing R-ideal of M.*

As a result, N is a strongly 2-absorbing R-ideal of M if and only if $(N : M)$ is a strongly 2-absorbing ideal of R.

Proof. Assume that $(N : M)$ is a strongly 2-absorbing ideal of R. Let $m \in M \setminus N$. Since M is completely multiplication module, $(\{m\} : M)M = \{m\} \nsubseteq N$. Then $(\{m\} : M) \nsubseteq (N : M)$. Thus there exists $r \in (\{m\} : M) \setminus (N : M)$. Since $(N : M)$ is a strongly 2-absorbing ideal of R and $r \in (\{m\} : M) \setminus (N : M)$, it follows that there exists a finite subset F of R such that for all $x, y \in R$,

$xFyFr \subseteq (N : M)$ implies $xy \in (N : M)$. Let $a, b \in R$ be such that $aFbFm \subseteq N$. Since $rM \subseteq (\{m\} : M)M = \{m\}$, we obtain that $aFbFrM \subseteq aFbF\{m\} \subseteq N$. Then $aFbFr \subseteq (N : M)$ implies that $ab \in (N : M)$, i.e., $abM \subseteq N$. Therefore, N is a strongly 2-absorbing R-ideal of M.

For another result, assume that N is a strongly 2-absorbing R-ideal of M. Then $(N : M)$ is a strongly 2-absorbing ideal of R by Theorem 2.28. □

Theorem 2.33. *Let N be an R-ideal of a completely multiplication module M. If $(N : M)$ is a 2-absorbing ideal of R, then N is a 2-absorbing R-ideal of M.*

Proof. Assume that $(N : M)$ is a 2-absorbing ideal of R. Then N is a proper ideal of M because if $N = M$, then $(N : M) = \{r \in R : rM \subseteq N = M\} = R$ contradicts the fact that $(N : M)$ is a proper ideal of R. Let $a, b \in R$ and $m \in M$ be such that $aRbRm \subseteq N$. Since $m \in M$ and M is a completely multiplication module, $(\{m\} : M) = \{m\}$. Then $aRbR((\{m\} : M)M) = aRbR\{m\} \subseteq N$. Then $aRbR(\{m\} : M) \subseteq (N : M)$. Let $r \in (\{m\} : M)$. Then $rM = \{m\}$ and $aRbRr \subseteq aRbR(\{m\} : M) \subseteq (N : M)$. Since $(N : M)$ is a 2-absorbing ideal of R, we obtain that $ab \in (N : M)$ or $br \in (N : M)$ or $ar \in (N : M)$, i.e., $abM \subseteq N$ or $brM \subseteq N$ or $arM \subseteq N$. Since $rM = \{m\}$, we obtain that $abM \subseteq N$ or $b\{m\} \subseteq N$ or $a\{m\} \subseteq N$. Therefore, N is a 2-absorbing R-ideal of M. □

Finally, in this chapter, we aim to consider module homomorphisms between modules over the same near rings.

Definition 2.34. Let M and M' be R-modules. A function $f : M \to M'$ is called a **module homomorphism** if for all $r \in R$ and $x, y \in M$.

(i) $f(x + y) = f(x) + f(y)$,

(ii) $f(rx) = rf(x)$.

Module epimorphisms can be defined in a similar way for epimorphisms of modules over rings.

Theorem 2.35. *Let M, M' be R-modules and $\varphi : M \to M'$ be a module homomorphism.*

(1) If N' is a prime R-ideal of M', then $\varphi^{-1}(N') = M$ or $\varphi^{-1}(N')$ is a prime R-ideal of M.

(2) If N' is a 2-absorbing R-ideal of M', then $\varphi^{-1}(N') = M$ or $\varphi^{-1}(N')$ is a 2-absorbing R-ideal of M.

Proof. We proof only (2). Assume that N' is a 2-absorbing R-ideal of M' and $\varphi^{-1}(N') \neq M$. Let $a, b \in R$ and $m \in M$. Suppose that $aRbRm \subseteq \varphi^{-1}(N')$. Then $\varphi(aRbRm) \subseteq N'$. Thus $aRbR\varphi(m) = \varphi(aRbRm) \subseteq N'$ because φ is a module homomorphism. Moreover, since N' is a 2-absorbing R-ideal of M', we obtain that $\varphi(abM) = ab\varphi(M) \subseteq abM' \subseteq N'$ or $\varphi(am) = a\varphi(m) \in N'$ or $\varphi(bm) = b\varphi(m) \in N'$. Hence $abM \subseteq \varphi^{-1}(N')$ or $am \in \varphi^{-1}(N')$ or $bm \in \varphi^{-1}(N')$. Therefore, $\varphi^{-1}(N')$ is a 2-absorbing R-ideal of M. □

Moreover, if $\varphi : M \to M'$ is a module epimorphism, then the converses of statements in Theorem 2.35 hold.

Theorem 2.36. *Let M, M' be R-modules and $\varphi : M \to M'$ be a module epimorphism. Moreover, let N' be a proper R-ideal of M' such that $\varphi^{-1}(N')$ is proper.*

(1) N' is a prime R-ideal of M' if and only if $\varphi^{-1}(N')$ is a prime R-ideal of M; and

(2) N' is a 2-absorbing R-ideal of M' if and only if $\varphi^{-1}(N')$ is a 2-absorbing R-ideal of M.

Proof. It suffices to proof only (2). From Theorem 2.35, it is enough to show the sufficient part. Let $a, b \in R$ and $m' \in M'$. Suppose that $aRbRm' \subseteq N'$. Since φ is a module epimorphism and $m' \in M'$, there exists $m \in M$ such that $\varphi(m) = m'$. Then $\varphi(aRbRm) = aRbR\varphi(m) = aRbRm' \subseteq N'$. It follows that $aRbRm \subseteq \varphi^{-1}(N')$. Since $\varphi^{-1}(N')$ is a 2-absorbing R-ideal of M, we obtain that $abM \subseteq \varphi^{-1}(N')$ or $am \in \varphi^{-1}(N')$ or $bm \in \varphi^{-1}(N')$. Since φ is a module epimorphism, $abM' = ab\varphi(M) = \varphi(abM) \subseteq N'$ or $am' = a\varphi(m) = \varphi(am) \in N'$ or $bm' = b\varphi(m) = \varphi(bm) \in N'$. Therefore, N' is a 2-absorbing ideal of M'. □

Theorem 2.37. *Let M, M' be R-modules and $\varphi : M \to M'$ be a module epimorphism. Moreover, let N be an R-ideal of M with $Ker\,\varphi \subseteq N$.*

(1) If N is a prime R-ideal of M, then $\varphi(N)$ is a prime R-ideal of M'.

(2) If N is a 2-absorbing R-ideal of M, then $\varphi(N)$ is a 2-absorbing R-ideal of M'.

Proof. The proof of (1) and (2) are similar so we proof only (2). First, assume that N is a 2-absorbing R-ideal of M. Let $a, b \in R$ and $m' \in M'$. Assume that $aRbRm' \subseteq \varphi(N)$. Since φ is a module epimorphism and $m' \in M'$, there exists $m \in M$ such that $\varphi(m) = m'$. Then $\varphi(aRbRm) = aRbR\varphi(m) = aRbRm' \subseteq \varphi(N)$. Let $s, t \in R$ be such that $asbtm \in aRbRm$. Then there exists $x \in N$ such that $\varphi(asbtm) = \varphi(x)$. Thus $\varphi(asbtm - x) = \varphi(asbtm) - \varphi(x) = 0$. That is $asbtm - x \in \operatorname{Ker}\varphi \subseteq N$ so that $asbtm \in N$. Thus $aRbRm \subseteq N$. Since N is a 2-absorbing R-ideal of M, we obtain that $abM \subseteq N$ or $am \in N$ or $bm \in N$. Then $abM' = ab\varphi(M) = \varphi(abM) \subseteq \varphi(N)$ or $am' = a\varphi(m) = \varphi(am) \in \varphi(N)$ or $bm' = b\varphi(m) = \varphi(bm) \in \varphi(N)$. Therefore, $\varphi(N)$ is a 2-absorbing R-ideal of M'.

□

Since a near ring R is also an R-module, 2-absorbing ideals of a near ring R are special cases of 2-absorbing R-ideals of the R-module R. Then all properties of 2-absorbing R-ideals of R-modules in this chapter can be applied to 2-absorbing R-ideals of the near ring R. For example, we also obtain that "intersections of each pair of prime ideals of a near ring R are 2-absorbing ideals of R" as a corollary of Theorem 2.17.

CHAPTER III

2-ABSORBING R-IDEALS OF MODULES OVER DECOMPOSABLE NEAR RINGS

In 2015, Chinwarakorn and Pianskool [2] introduced almost generalized 2-absorbing ideals of commutative rings with identities which are a generalization of 2-absorbing ideals of commutative rings with identities and investigated some of their properties on decomposable rings. A commutative ring R is said to be a **decomposable commutative ring** if it can be written as a product of commutative nonzero rings, i.e., $R = R_1 \times R_2 \times \cdots \times R_n$ for some commutative nonzero rings $R_1, R_2, \ldots, R_n$. This leads us to study some properties of 2-absorbing R-ideals of modules over decomposable near rings and 2-absorbing ideals of decomposable near rings.

Definition 3.1. A near ring R is said to be a **decomposable near ring** if it is a product of nonzero near rings equipped by componentwise on both addition and multiplication.

Example 3.2. Let $R_1 = (\{0, 1, a, b\}, +, \cdot)$ and $R_2 = (\{0, 1\}, +, \cdot)$ be the near rings given in Example 1.4 and Example 1.5, respectively. Then $R_1 \times R_2$ is a decomposable near ring which is not zero symmetric.

Let $R = R_1 \times R_2 \times \cdots \times R_k$ be a decomposable near ring and let M_i be an R_i-module for all $i = 1, 2, \ldots, k$. It is clear that the product of $M_1, M_2, \ldots, M_k$ is an R-module, i.e., $M_1 \times M_2 \times \cdots \times M_k$ is an R-module.

Proposition 3.3. *Let N_i be an R_i-ideal of an R_i-module M_i for all $i = 1, 2, \ldots, k$. Then $N_1 \times N_2 \times \cdots \times N_k$ is an R-ideal of M where $R = R_1 \times R_2 \times \cdots \times R_k$ and $M = M_1 \times M_2 \times \cdots \times M_k$.*

Proof. For each i, since N_i is an R_i-ideal of M_i, it follows that N_i is a normal subgroup of M_i and then $N_1 \times N_2 \times \cdots \times N_k$ is a normal subgroup of M. Let

$(r_1, r_2, \ldots, r_k) \in R$, $(m_1, m_2, \ldots, m_k) \in M$ and $(n_1, n_2, \ldots, n_k) \in N_1 \times N_2 \times \cdots \times N_k$. Then

$$\begin{aligned}
&(r_1, r_2, \ldots, r_k)[(m_1, m_2, \ldots, m_k) - (n_1, n_2, \ldots, n_k)] - (r_1, r_2, \ldots, r_k)(m_1, m_2, \ldots, m_k) \\
&\quad = (r_1, r_2, \ldots, r_k)\,(m_1 - n_1, m_2 - n_2, \ldots, m_k - n_k) - (r_1 m_1, r_2 m_2, \ldots, r_k m_k) \\
&\quad = (r_1(m_1 - n_1), r_2(m_2 - n_2), \ldots, r_k(m_k - n_k)) - (r_1 m_1, r_2 m_2, \ldots, r_k m_k) \\
&\quad = (r_1(m_1 - n_1) - r_1 m_1, r_2(m_2 - n_2) - r_2 m_2, \ldots, r_k(m_k - n_k) - r_k m_k) \\
&\quad \in N_1 \times N_2 \times \cdots \times N_k
\end{aligned}$$

because each N_i is an R_i-ideal of M_i. Therefore, $N_1 \times N_2 \times \cdots \times N_k$ is an R-ideal of M. □

Some properties of 2-absorbing R-ideals of modules over decomposable near rings are studied.

Lemma 3.4. *Let M_1 be an R_1-module, M_2 be an R_2-module, $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Then*

(1) N_1 is a 2-absorbing (prime) R_1-ideal of M_1 if and only if $N_1 \times M_2$ is a 2-absorbing (prime) R-ideal of M; and

(2) N_2 is a 2-absorbing (prime) R_2-ideal of M_2 if and only if $M_1 \times N_2$ is a 2-absorbing (prime) R-ideal of M.

Proof. It suffices to prove only (1). First, assume that N_1 is a 2-absorbing R_1-ideal of M_1. Suppose that $(a, b)R(c, d)R(m_1, m_2) \subseteq N_1 \times M_2$ where $(a, b), (c, d) \in R$ and $(m_1, m_2) \in M$. Then $(aR_1cR_1m_1, bR_2dR_2m_2) = (a, b)R(c, d)R(m_1, m_2) \subseteq N_1 \times M_2$, i.e., $aR_1cR_1m_1 \subseteq N_1$ and $bR_2dR_2m_2 \subseteq M_2$. Since N_1 is a 2-absorbing R_1-ideal of M_1, it follows that $acM_1 \subseteq N_1$ or $am_1 \in N_1$ or $cm_1 \in N_1$. That is $(a, b)(c, d)M = (acM_1, bdM_2) \subseteq N_1 \times M_2$ or $(a, b)(m_1, m_2) = (am_1, bm_2) \in N_1 \times M_2$ or $(c, d)(m_1, m_2) = (cm_1, dm_2) \in N_1 \times M_2$. Therefore, $N_1 \times M_2$ is a 2-absorbing R-ideal of M.

Conversely, assume that $N_1 \times M_2$ is a 2-absorbing R-ideal of M. Let $a, b \in R_1$ and $m_1 \in M_1$. Assume that $aR_1bR_1m_1 \subseteq N_1$. Let $x, y \in R_2$ and $m_2 \in M_2$. Then

$(a,x)R(b,y)R(m_1,m_2) = (aR_1bR_1m_1, xR_2yR_2m_2) \subseteq N_1 \times M_2$. Since $N_1 \times M_2$ is a 2-absorbing R-ideal of M, it follows that $(a,x)(b,y)M \subseteq N_1 \times M_2$ or $(a,x)(m_1,m_2) \in N_1 \times M_2$ or $(b,y)(m_1,m_2) \in N_1 \times M_2$. Then $(abM_1, xyM_1) = (a,x)(b,y)M \subseteq N_1 \times M_2$ or $(am_1, xm_2) = (a,x)(m_1,m_2) \in N_1 \times M_2$ or $(bm_1, ym_2) = (b,y)(m_1,m_2) \in N_1 \times M_2$, i.e., $abM_1 \subseteq N_1$ or $am_1 \in N_1$ or $bm_1 \in N_1$. Therefore, N_1 is a 2-absorbing R_1-ideal of M_1. □

Theorem 3.5. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a decomposable near ring and N_i be an R_i-ideal of an R_i-module M_i for all $i = 1, 2, \ldots, k$. Then N_i is a 2-absorbing (prime) R_i-ideal of M_i if and only if $M_1 \times \cdots \times M_{i-1} \times N_i \times M_{i+1} \times \cdots \times M_k$ is a 2-absorbing (prime) R-ideal of $M_1 \times M_2 \times \cdots \times M_k$ for each $i = 1, 2, \ldots, k$.*

Proof. The result follows by applying Lemma 3.4. □

The above theorem shows that the product $N_1 \times N_2 \times \cdots \times N_k$ is a 2-absorbing R-ideal if and only if there is one of $N_i's$ such that N_i is a 2-absorbing R_i-ideal of M_i and the other components must be the whole R_i-module M_i.

Recall that a near ring R is a module over itself. Moreover, if I is a 2-absorbing (prime) ideal of a near ring R, then I is a 2-absorbing (prime) R-ideal of the R-module R and vice versa. Then the result of 2-absorbing (prime) R-ideal of the R-module M in Theorem 3.5 can be rephrased in term of 2-absorbing (prime) ideals of a near ring.

Corollary 3.6. *Let $R = R_1 \times R_2 \times \cdots \times R_k$ be a decomposable near ring and I_i be an ideal of R_i for all $i = 1, 2, \ldots, k$. Then I_i is a 2-absorbing (prime) ideal of R_i if and only if $R_1 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_k$ is a 2-absorbing (prime) ideal of R for each $i = 1, 2, \ldots, k$.*

Next theorem provides conditions that make $N_1 \times N_2$ a 2-absorbing $(R_1 \times R_2)$-ideal of an $(R_1 \times R_2)$-module $M_1 \times M_2$ where each N_i is a proper R_i-ideal of M_i.

Theorem 3.7. *Let R_1 and R_2 be zero symmetric near rings. If N_1 is a prime R_1-ideal of an R_1-module M_1 and N_2 is a prime R_2-ideal of an R_2-module M_2,* *then* $N_1 \times N_2$ *is a 2-absorbing* R*-ideal of the* R*-module* M *where* $R = R_1 \times R_2$ *and* $M = M_1 \times M_2$*.*

Proof. Assume that N_1 is a prime R_1-ideal of an R_1-module M_1 and N_2 is a prime R_2-ideal of an R_2-module M_2. Then $N_1 \times N_2$ is a proper R-ideal of M. Let $(a,b),(c,d) \in R_1 \times R_2$ and $(m_1,m_2) \in M_1 \times M_2$. Assume that $(a,b)R(c,d)R(m_1,m_2) \subseteq N_1 \times N_2$ but $(a,b)(c,d)M \nsubseteq N_1 \times N_2$ and $(a,b)(m_1,m_2) \notin N_1 \times N_2$. We claim $(c,d)(m_1,m_2) \in N_1 \times N_2$. Note that we can conclude from $(a,b)(m_1,m_2) \notin N_1 \times N_2$ that (a) $am_1 \notin N_1$ or $am_2 \notin N_2$, and (b) $acM_1 \nsubseteq N_1$ or $bdM_2 \nsubseteq N_2$. There are 4 cases to be considered:

(i) $am_1 \notin N_1$ and $acM_1 \nsubseteq N_1$

(ii) $am_2 \notin N_2$ and $bdM_2 \nsubseteq N_2$

(iii) $am_1 \notin N_1$ and $bdM_2 \nsubseteq N_2$

(iv) $am_2 \notin N_2$ and $acM_1 \nsubseteq N_1$.

First, we consider Case(i). Note that $aR_1cR_1m_1 \subseteq N_1$ and $bR_2dR_2m_2 \subseteq N_2$ because $(aR_1cR_1m_1, bR_2dR_2m_2) = (a,b)R(c,d)R(m_1,m_2) \subseteq N_1 \times N_2$. Since N_1 is a prime R_1-ideal of M_1 and $am_1 \notin N_1$, we obtain from Proposition 2.14 that $cM_1 \subseteq N_1$. Thus $acM_1 \subseteq aN_1 \subseteq N_1$ because R_1 is a zero symmetric near ring which is a contradiction. Then Case(i) is not possible. In addition, Case(ii) is absurd.

Next, Case (iii) is considered. Similarly, $cM_1 \subseteq N_1$. Thus $cm_1 \in N_1$. Moreover, $bR_2dR_2m_2 \subseteq N_2$. Let $r \in R_2$. Then $bR_2drm_2 \subseteq N_2$. Since N_2 is a prime R_2-ideal of M_2, we have $bM_2 \subseteq N_2$ or $drm_2 \in N_2$. If $bM_2 \subseteq N_2$, then $bdM_2 \subseteq bM_2 \subseteq N_2$ contradicts $bdM_2 \nsubseteq N_2$. Then $drm_2 \in N_2$. That is $dR_2m_2 \subseteq N_2$. And again, since N_2 is a prime R_2-ideal of M_2 and $bdM_2 \nsubseteq N_2$, we obtain that $m_2 \in N_2$ so that $dm_2 \in N_2$. Therefore, $(c,d)(m_1,m_2) = (cm_1, dm_2) \in N_1 \times N_2$.

The proof of Case(iv) is similar to that of Case(iii).

Therefore, $N_1 \times N_2$ is a 2-absorbing R-ideal of M. □

However, it is not necessary true that products of prime R-ideals are prime R-ideals. For example, let $N_1 = \{0\}$ be the prime R_1-ideal of $M_1 = \{0,1\}$ and

$N_2 = \{0, b\}$ be the prime R_2-ideal of $M_2 = \{0, 1, a, b\}$ given in Example 2.8 and Example 2.9, respectively. Let $R = R_1 \times R_2$. Then $N_1 \times N_2 = \{(0, 0), (0, b)\}$ is not a prime R-ideal of $M_1 \times M_2$ because $(0, a)R(1, b) \subseteq \{(0, 0), (0, b)\} = N_1 \times N_2$ but $(0, a), (1, b) \notin N_1 \times N_2$.

As a special case of Theorem 3.7, we obtain that $I_1 \times I_2$ is a 2-absorbing ideal of $R_1 \times R_2$ where I_1 and I_2 are prime ideals of the near rings R_1 and R_2, respectively. In fact, if R_1 and R_2 are zero symmetric near rings with identities, then the converse of Theorem 3.7 for this special case is true.

Theorem 3.8. *Let R_1 and R_2 be zero symmetric near rings with identities, I_1 and I_2 be proper ideals of R_1 and R_2, respectively. Then I_1 is a prime ideal of R_1 and I_2 is a prime ideal of R_2 if and only if $I_1 \times I_2$ is a 2-absorbing ideal of $R_1 \times R_2$.*

Proof. To prove the sufficient part, assume that $I_1 \times I_2$ is a 2-absorbing ideal of $R_1 \times R_2$. Let $a, b \in R_1$ and $x, y \in R_2$. Suppose that $aR_1b \subseteq I_1$ and $xR_2y \subseteq I_2$. Then $aR_11R_1b \subseteq I_1$ and $xy = x1y \in xR_2y \subseteq I_2$. Since I_2 is an ideal of R_2 and $xy \in I_2$, we obtain that $xyR_2 \subseteq I_2$. Moreover, $R_2xyR_2 \subseteq R_2I_2 \subseteq I_2$ because R_2 is a zero symmetric near ring. Note that $(a, 1)R(1, xy)R(b, 1) = (aR_11R_1b, 1R_2xyR_21) \subseteq I_1 \times I_2$. Since $I_1 \times I_2$ is a 2-absorbing ideal of $R_1 \times R_2$, it follows that $(a, 1)(1, xy) \in I_1 \times I_2$ or $(1, xy)(b, 1) \in I_1 \times I_2$ or $(a, 1)(b, 1) \in I_1 \times I_2$, i.e., $(a, xy) \in I_1 \times I_2$ or $(b, xy) \in I_1 \times I_2$ or $(ab, 1_2) \in I_1 \times I_2$. But I_2 is a proper ideal of R_2 so that $(ab, 1) \in I_1 \times I_2$ is not possible. Hence $(a, xy) \in I_1 \times I_2$ or $(b, xy) \in I_1 \times I_2$. Thus $a \in I_1$ or $b \in I_1$. Therefore, I_1 is a 2-absorbing ideal of R_1. Similarly, we obtain that I_2 is a 2-absorbing ideal of R_2. □

The last result provides a characterization of being a 2-absorbing ideal of the ideal $I_1 \times I_2 \times I_3$ of a decomposable near ring where I_1 is proper.

Theorem 3.9. *Let $R = R_1 \times R_2 \times R_3$ where R_1, R_2 and R_3 are zero symmetric near rings with identities, I_1 be a proper ideal of R_1, I_2 and I_3 be ideals of R_2 and R_3, respectively. Then the following statements are equivalent.*

(1) $I_1 \times I_2 \times I_3$ is a 2-absorbing ideal of R.

(2) I_1 *is 2-absorbing ideal of* R_1*,* $I_2 = R_2$ *and* $I_3 = R_3$ *or*

I_1, I_2 *are prime ideals and* $I_3 = R_3$ *or*

I_1, I_3 *are prime ideals and* $I_2 = R_2$*.*

Proof. First, assume that $I := I_1 \times I_2 \times I_3$ is a 2-absorbing ideal of R. Then I is a nonempty subset of R. Let $(a, b, c) \in I$. Note that $(a, 1, 1)R(1, b, 1)R(1, 1, c) = (aR_1, R_2bR_2, R_3c) \subseteq I_1 \times I_2 \times I_3 = I$ because I_1, I_2 and I_3 are ideals of zero symmetric near rings. Since I is a 2-absorbing ideal of R, $(a, 1, 1)(1, b, 1) \in I$ or $(1, b, 1)(1, 1, c) \in I$ or $(a, 1, 1)(1, 1, c) \in I$, i.e., $(a, b, 1) \in I$ or $(1, b, c) \in I$ or $(a, 1, c) \in I$. Then $I_3 = R_3$ or $I_1 = R_1$ or $I_2 = R_2$. But I_1 is a proper ideal of R_1, it follows that $I_3 = R_3$ or $I_2 = R_2$. This reaches to 3 cases:

(i) $I_2 = R_2$ and $I_3 = R_3$,

(ii) $I_2 \neq R_2$ or $I_3 = R_3$,

(iii) $I_2 = R_2$ or $I_3 \neq R_3$.

The first case leads to the result that $I = I_1 \times (R_2 \times R_3)$ where I_1 is 2-absorbing ideal of R_1 by Corollary 3.6. Next, we proof the second case by showing that I_1 and I_2 are prime ideals. Let $a, b \in R_1$ and $x, y \in R_2$. Assume that $aR_1b \subseteq I_1$ and $xR_2y \subseteq I_2$. Then $(a, 1, 1)R(1, xy, 1)R(b, 1, 1) = (aR_1b, R_2xyR_2, R_3) \subseteq I_1 \times I_2 \times I_3 = I$ because I_2 is an ideal of the zero symmetric near ring R_2. Since I ia a 2-absorbing ideal of R, $(a, 1, 1)(1, xy, 1) \in I$ or $(1, xy, 1)(b, 1, 1) \in I$ or $(a, 1, 1)(b, 1, 1) \in I$, i.e., $(a, xy, 1) \in I$ or $(b, xy, 1) \in I$ or $(ab, 1, 1) \in I$. Since $I_2 \neq R_2$, it follows that $(a, xy, 1) \in I$ or $(b, xy, 1) \in I$. That is $a \in I_1$ or $b \in I_1$. Therefore, I_1 is a prime ideal of R_1. Similarly, we obtain that I_2 is a prime ideal of R_2. The proof of Case(iii) is similar to that of Case(ii).

Conversely, if $I = I_1 \times R_2 \times R_3$ and I_1 is a 2-absorbing ideal of R_1, then I is a 2-absorbing ideal of R by Corollary 3.6 because $R_2 \times R_3$ is a near ring. Consider Case(ii), since I_1 and I_2 are prime ideals, $I_1 \times I_2$ is a 2-absorbing ideal by Theorem 3.7. It is easy to verify that I is a 2-absorbing ideal of R by Corollary 3.6 again. The last case is similar to the previous case. □

By the results of Theorem 3.8 and Theorem 3.9, we can see that, in order to obtain these results, being zero symmetric near rings with identities is crucial.

CHAPTER IV
CONCLUSION

By definition of R-subgroups and ideals of near rings, if rings are considered as near rings, then ideals of rings are both R-subgroups and ideals of near rings. Similarly, submodules of modules over rings are both R-submodules and R-ideals of modules over near rings when those modules over rings are considered as modules over near rings. So many results in rings (modules over rings) are extended to those in near rings (modules over near rings). Recall that prime ideals (submodules) of rings (modules over rings) are extended to many types of prime ideals (R-ideals) of near rings (modules over near rings). So far, there have been five types of such those prime objects (0-prime, 1-prime, 2-prime, 3-prime and completely prime). We can see that the definition of 0-prime, 1-prime, 2-prime and 3-prime are equivalent when rings are considered as near rings. Moreover, all of them are equivalent if near rings are commutative.

In this thesis, we introduce and investigate 2-absorbing ideals and 2-absorbing R-ideals which are extended from 3-prime ideals and 3-prime R-ideals, respectively. We obtain that intersections of two prime R-ideals are 2-absorbing R-ideals. Although prime R-ideals are 2-absorbing R-ideals, intersections of two 2-absorbing R-ideals with at least one of them being 2-absorbing but not prime R-ideal may no longer be 2-absorbing R-ideals. For example, $5\mathbb{Z} \cap 6\mathbb{Z}$ is not a 2-absorbing $\mathbb{Z}$-ideal of the $\mathbb{Z}$-module $\mathbb{Z}$ (see Example 2.19).

Moreover, we introduce the concept of strongly 2-absorbing R-ideals in order to extend some results. In modules over rings, if N is a 2-absorbing submodule of an R-module M, then $(N : M)$ is a 2-absorbing ideal of R. In case of near rings, if N is a 2-absorbing R-ideal of an R-module M, then "$(N : M)$ is a 2-absorbing ideal of R" may not hold. However, we show in Chapter II that $(N : M)$ is a

2-absorbing ideal of a near ring R provided that N should be a strongly prime R-ideal of that R-module M. The converse is valid provided that M is a completely multiplication module.

Finally, we focus on decomposable near rings. We prove that products of two prime R-ideals are 2-absorbing R-ideals. Nevertheless, if a product $N \times K$ of R-ideals is a 2-absorbing R-ideal, then it is not necessary that both of N and K are prime R-ideals. However, if we consider the special case of those modules, say the module R over a zero symmetric near ring R with identity, then we can conclude the following: if a product of $I \times J$ of ideals of R is a 2-absorbing ideal, then both of I and J are prime ideals of R. Nevertheless, we obtain that a characterization of being a 2-absorbing ideal of $I_1 \times I_2 \times I_3$ where I_1, I_2 and I_3 are ideals of near rings R_1, R_2 and R_3, respectively.

REFERENCES

[1] Badawi, A.: On 2-absorbing ideals of commutative rings, *Bull. Austral. Math. Soc.* **75**, 417–429 (2007).

[2] Chinwarakorn, S., Pianskool, S.: On almost generalized 2-absorbing and weakly almost generalized 2-absorbing structures, *ScienceAsia* **41**, 64–72 (2015).

[3] Darani, A., Soheilnia, F.: 2-absorbing and weakly 2absorbing submodules, *Thai J. Math.* **9**, 577–584 (2011).

[4] Dickson, L.E.: Definitions of a group and a field by independent pustulates, *Amer. Math.* **6**, 198–204 (1905).

[5] Groenewald, N.J.: Different prime ideals in near rings, *Comm. in Algebra* **19**, 2667–2675 (1991).

[6] Groenewald, N.J., Juglal, S.: Strongly prime near ring modules, *Arab. J. Sci. Eng.* **36**, 985–995 (2011).

[7] Groenewald, N.J., Juglal, S., Lee, K.S.E.: Different prime R-ideals, *Algebra Colloq.* **17**, 887–904 (2010).

[8] Holcombe, W.L.M.: *Primitive Near-Rings*, Ph.D. Thesis, University of Leeds, 1970.

[9] Juglal, S.: *Prime near ring modules and their link with the generalized group near ring*, Ph.D. Thesis, Nelson Mandela Metropolitan University., 2007.

[10] Pilz, G.: *Near Ring*, North Holland, Oxford, 1983.

VITA

Name	Miss Sutida Patlertsin
Date of Birth	20 December 1991
Place of Birth	Bangkok, Thailand
Education	B.Sc. (Mathematics), Srinakrinwirot University, 2014
Scholarship	Science Achievement Scholarship of Thailand (SAST)
Conference	**Presenter** † **2-Absorbing R-ideals of Modules over Near Rings**, at the 22^{nd} Annual Meeting in Mathematics (AMM 2017), 2-4 June 2017 at Lotus Pang Suan Kaew Hotel, Chiang Mai † **Some Properties of 2-Absorbing R-ideals of Modules over Near Rings**, at the Annual Pure and Applied Mathematics Conference 2017 (APAM 2017), 28-30 June 2017 at Chulalongkorn University, Bangkok