CONSTRUCTION OF DESIGNS FROM SYMPLECTIC GRAPHS AND ORTHOGONAL GRAPHS OVER FINITE LOCAL RINGS

Mr. Wachirapong Jirakitpuwapat

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2015

การสร้างดีไซน์จากกราฟซิมเพล็กติกและกราฟเชิงตั้งฉากเหนือริงเฉพาะที่จำกัด

นายวชิรพงศ์ จิรกิจภูวพัฒน์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2558

Thesis Title	CONSTRUCTION OF DESIGNS FROM SYMPLECTIC GRAPHS AND ORTHOGONAL GRAPHS OVER FINITE LOCAL RINGS
By	Mr. Wachirapong Jirakitpuwapat
Field of Study	Mathematics
Thesis Advisor	Associate Professor Yotsanan Meemark, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

....................................Dean of the Faculty of Science

(Associate Professor Polkit Sangvanich, Ph.D.)

THESIS COMMITTEE

....................................Chairman

(Assistant Professor Chariya Uiyyasathian, Ph.D.)

....................................Thesis Advisor

(Associate Professor Yotsanan Meemark, Ph.D.)

....................................Examiner

(Teeraphong Phongpattanacharoen, Ph.D.)

....................................External Examiner

(Associate Professor Nittiya Pabhapote, Ph.D.)

วชิรพงศ์ จิรกิจภูวพัฒน์ : การสร้างดีไซน์จากกราฟซิมเพล็กติกและกราฟเชิงตั้งฉากเหนือริงเฉพาะที่จำกัด (CONSTRUCTION OF DESIGNS FROM SYMPLECTIC GRAPHS AND ORTHOGONAL GRAPHS OVER FINITE LOCAL RINGS) อ. ที่ปรึกษาวิทยานิพนธ์หลัก: รศ.ดร.ยศนันต์ มีมาก, 53หน้า.

ในวิทยานิพนธ์นี้ เราสร้างดีไซน์จากกราฟซิมเพล็กติกและกราฟเชิงตั้งฉากเหนือริงเฉพาะที่จำกัด เราใช้วิธีการซ้ำจุดและบล็อก และใช้พารามิเตอร์ของกราฟส่วนประกอบย่อย นอกจากนี้เรายังได้พารามิเตอร์ของกราฟมีทิศทางจากดีไซน์เหล่านี้ด้วย

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต..

ลายมือชื่อ อ.ที่ปรึกษาหลัก..

สาขาวิชา คณิตศาสตร์

ปีการศึกษา 2558

5772248223 : MAJOR MATHEMATICS

KEYWORDS :

WACHIRAPONG JIRAKITPUWAPAT : CONSTRUCTION OF DESIGNS FROM SYMPLECTIC GRAPHS AND ORTHOGONNAL GRAPHS OVER FINITE LOCAL RING.

ADVISOR : ASSOC. PROF. YOTSANAN MEEMARK, Ph.D.,

53pp.

In this thesis, we construct designs from symplectic graphs and orthgonal graphs over finite local rings. We use duplications of points and blocks and parameters of subconstiuents of graphs. Moreover, we obtain the parameters of directed graphs from these designs.

Department : ...Mathematics and.....	Student's Signature : ....
...Computer Science...	Advisor's Signature : ....
Field of Study : ......Mathematics......	
Academic Year : ....2015.............	

ACKNOWLEDGEMENTS

In the completion of my Master Thesis, I am deeply indebted to my thesis advisor, Associate Professor Dr. Yotsanan Meemark, not only for coaching my research but also for broadening my academic vision. I would like to express my special thanks to my thesis committee: Assistant Professor Dr. Chariya Uiyyasathian, Dr. Teeraphong Phongpattanacharoen and Associate Professor Dr. Nittiya Pabhapote. Their suggestions and comments are my sincere appreciation. Moreover, I feel very thankful to all of my teachers who have taught me for my knowledge and skills. Also, I wish to express my thankfulness to my family and my friends for their encouragement throughout my study.

CONTENTS

CHAPTER I
INTRODUCTION

1.1 Finite incidence structures

A **finite incidence structure** $(\mathscr{P}, \mathscr{B}, \varepsilon)$ consists of a finite set $\mathscr{P}$ of **points**, a set $\mathscr{B}$ of **blocks**, and an **incidence relation** ε between points and blocks. An incident point-block pair is called a **flag**, and a non-incident point-block pair is called an **antiflag**. The **dual** structure of $(\mathscr{P}, \mathscr{B}, \varepsilon)$ is a $(\mathscr{B}, \mathscr{P}, \varepsilon^d)$ with $\varepsilon^d = \{(B, x) : (x, B) \in \varepsilon\}$.

A **tactical configuration** with parameters (v, b, k, r) or 1**-design** is a finite incidence structure $\mathscr{T} = (\mathscr{P}, \mathscr{B}, \varepsilon)$ with $|\mathscr{P}| = v$ and $|\mathscr{B}| = b$ such that every block contains k points and every point belongs to exactly r blocks satisfy $vr = bk$. For every point $x \in \mathscr{P}$ and every block $B \in \mathscr{B}$, the number of flags (y, C) such that $y \varepsilon B$, $x \varepsilon C$, $y \neq x$, and $C \neq B$ is denoted by $s(x, B)$. For two distinct points $x, y \in \mathscr{P}$ and blocks $B, C \in \mathscr{B}$, denote λ_{xy} the number of blocks containing both x and y, and μ_{BC} the number of common points of B and C. Then for every point $x \in \mathscr{P}$ and every block $B \in \mathscr{B}$ we have

$$s(x,B) = \begin{cases} \sum\limits_{y \varepsilon B} \lambda_{xy} = \sum\limits_{x \varepsilon C} \mu_{BC} & \text{if } (x,B) \notin \varepsilon, \\ \sum\limits_{y \varepsilon B, y \neq x} (\lambda_{xy} - 1) = \sum\limits_{x \varepsilon C, C \neq B} (\mu_{BC} - 1) & \text{if } (x,B) \in \varepsilon. \end{cases}$$

Let $\{s(x, B) : x \in \mathscr{P},\ B \in \mathscr{B} \text{ and } (x, B) \notin \varepsilon\} = \{\alpha_1, \alpha_2, \ldots, \alpha_a\}$ and $\{s(x, B) : x \in \mathscr{P},\ B \in \mathscr{B} \text{ and } (x, B) \in \varepsilon\} = \{\beta_1, \beta_2, \ldots, \beta_b\}$. We may write parameters as $(v, b, k, r; \alpha_1, \ldots, \alpha_a; \beta_1, \ldots, \beta_b)$. If $\alpha_1, \ldots, \alpha_a \geq 1, 3 \leq k \leq v-3$ and $3 \leq r \leq b-3$, then we call $\mathscr{T}$ **proper**. If $a = 1$, then $\mathscr{T}$ is called an α**-strongly tactical configuration** and if $b = 1$, then $\mathscr{T}$ is called a β**-strongly tactical configuration**. In particular, if $a = b = 1$, then $\mathscr{T}$ is called a $1\frac{1}{2}$**-design**.

Example 1.1.1. *Let $\mathscr{P}$ be a finite set and let $\mathscr{B}$ be a collection of subsets of $\mathscr{P}$. For $x \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $x\varepsilon B$ if and only if $x \in B$. Then $\mathscr{T} = (\mathscr{P}, \mathscr{B}, \varepsilon)$ is a finite incidence structure.*

Example 1.1.2. *Let $\mathscr{P}$ be a finite set of size $n < 2$. Let $\mathscr{B} = \{B : B \subseteq \mathscr{P}$ and $|B| = t\}$ for $1 < t < n$. For $x \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $x\varepsilon B$ if and only if $x \in B$. Then the incidence structure $\mathscr{T} = (\mathscr{P}, \mathscr{B}, \varepsilon)$ is a $1\frac{1}{2}$-design with parameters $\big(n, \binom{n}{t}, t, \binom{n-1}{t-1}; t\binom{n-2}{t-2}; (t-1)(\binom{n-2}{t-2} - 1)\big)$.*

Example 1.1.3. *Let $\mathscr{P} = \{1, 2, 3, 4, 5\}$. Let $\mathscr{B} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\}$. For $x \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $x\varepsilon B$ if and only if $x \in B$. Then the incidence structure $\mathscr{T} = (\mathscr{P}, \mathscr{B}, \varepsilon)$ is a β-strongly tactical configuration with parameters* $(5, 5, 2, 2; 1, 0; 0)$.

Example 1.1.4. *Let $\mathscr{P} = \{1, 2, \ldots, 2n\}$ for some positive integer $n \geq 2$. Let $\mathscr{B} = \{\{i, j\} \in \mathscr{P} \times \mathscr{P} : i$ is a odd and j is a even $\}$. For $x \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $x\varepsilon B$ if and only if $x \in B$. Then the incidence structure $\mathscr{T} = (\mathscr{P}, \mathscr{B}, \varepsilon)$ is a $1\frac{1}{2}$-design with parameters $(2n, n^2, 2, n; 1; 0)$.*

Example 1.1.5. *This can be rephrased in the graph way as follows. Let $K_{n,n}$ be the complete bipartite graph and $n \geq 2$. Let $\mathscr{P} = \mathcal{V}(K_{n,n})$ and $\mathscr{B} = \mathcal{E}(K_{n,n})$. For $x \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $x\varepsilon B$ if and only if x is a B.*

1.2 Duplications

In this section, we construct new tactical configurations from a tactical configuration by using duplications of points and blocks.

Proposition 1.2.1. *Let $(\mathscr{P}, \mathscr{B}, \varepsilon)$ be a tactical configuration with parameters $(v, b, k, r; \alpha_1, \ldots, \alpha_a; \beta_1, \ldots, \beta_b)$. Then the dual of $(\mathscr{P}, \mathscr{B}, \varepsilon)$ is a tactical configuration with parameters $(b, v, r, k; \alpha_1, \ldots, \alpha_a; \beta_1, \ldots, \beta_b)$.*

Proof. Recall that for every $x \in \mathscr{P}$ and $B \in \mathscr{B}$, $x\varepsilon B$ if and only if $B\varepsilon^d x$. Then the dual of $(\mathscr{P}, \mathscr{B}, \varepsilon)$ is a tactical configuration $(\mathscr{B}, \mathscr{P}, \varepsilon^d)$ with parameters $(b, v, r, k; \alpha_1, \ldots, \alpha_a; \beta_1, \ldots, \beta_b)$. $\square$

Lemma 1.2.2. [Duplication of points] *Let* $(\mathscr{P}, \mathscr{B}, \varepsilon)$ *be a tactical configuration with parameters* $(v, b, k, r; \alpha_1, \ldots, \alpha_a; \beta_1, \ldots, \beta_b)$. *Let* $n \in \mathbb{N}$. *Let* $\mathscr{P}' = \{(x, i) : x \in \mathscr{P}$ *and* $i \in \{1, 2, \ldots, n\}\}$. *For* $(x, i) \in \mathscr{P}'$ *and* $B \in \mathscr{B}$, *we define* $(x, i)\varepsilon_{DP}B$ *if and only if* $x\varepsilon B$. *Then the incidence structure* $(\mathscr{P}', \mathscr{B}, \varepsilon_{DP})$ *is a tactical configuration with parameters* $(vn, b, kn, r; n\alpha_1, \ldots, n\alpha_a; n\beta_1+(n-1)(r-1), \ldots, n\beta_b+(n-1)(r-1))$.

Proof. It is clear that the incidence structure $(\mathscr{P}', \mathscr{B}, \varepsilon_{DP})$ is a tactical configuration with parameters (vn, b, kn, r). Let $(x, i_0) \in \mathscr{P}'$ and $B \in \mathscr{B}$. We count the number of flags $((y, j), C)$ such that $(y, j)\varepsilon_{DP}B, (x, i_0)\varepsilon_{DP}C, (y, j) \neq (x, i_0)$ and $C \neq B$.

Case 1: $((x, i_0), B)$ is an antiflag. Then (x, B) is an antiflag of $(\mathscr{P}, \mathscr{B}, \varepsilon)$ and $s(x, B) = \alpha_{a'}$ for some $a' \in \{1, \ldots, a\}$. Since $(y, j)\varepsilon_{DP}B$, $(y, j) \neq (x, i_0)$ for all $j \in \{1, 2, \ldots, n\}$. Thus, the number of flags $((y, j), C)$ such that $(y, j)\varepsilon_{DP}B, (x, i_0)\varepsilon_{DP}C$, $(y, j) \neq (x, i_0)$, and $C \neq B$ is $n\alpha_{a'}$.

Case 2: $((x, i_0), B)$ is a flag. Then $x\varepsilon B$ and $s(x, B) = \beta_{b'}$ for some $b' \in \{1, \ldots, b\}$. Thus, the number of flags $((y, j), C)$ such that $(y, j)\varepsilon_{DP}B, (y, j) \neq (x, i_0)\varepsilon_{DP}C, C \neq B$ and $y \neq x$ is $n\beta_{b'}$. Hence, the number of flags $((y, j), C)$ such that $(y, j)\varepsilon_{DP}B, (y, j) \neq (x, i_0)\varepsilon_{DP}C, C \neq B$ and $y = x$ is $(n-1)(r-1)$. Therefore, the number of flags $((y, j), C)$ such that $(y, j)\varepsilon_{DP}B, (x, i_0)\varepsilon_{DP}C, (y, j) \neq (x, i_0)$, and $C \neq B$ is $n\beta_{b'} + (n-1)(r-1)$. □

Remark. If $\alpha_1, \ldots, \alpha_a \geq 1$ and $1 \leq k \leq v-1$, then we can choose an $n \in \mathbb{N}$, such that $(\mathscr{P}', \mathscr{B}, \varepsilon_{DP})$ is a is a tactical configuration with $n\alpha_1, \ldots, n\alpha_a \geq 1$ and $3 \leq kn \leq vn - 3$.

Lemma 1.2.3. [Duplication of blocks] *Let* $(\mathscr{P}, \mathscr{B}, \varepsilon)$ *be a tactical configuration with parameters* $(v, b, k, r; \alpha_1, \ldots, \alpha_a; \beta_1, \ldots, \beta_b)$. *Let* $m \in \mathbb{N}$. *Let* $\mathscr{B}' = \{(B, i) : B \in \mathscr{B}$ *and* $i \in \{1, 2, \ldots, m\}\}$. *For* $x \in \mathscr{P}$ *and* $(B, i) \in \mathscr{B}'$, *we define* $x\varepsilon_{DB}(B, i)$ *if and only if* $x\varepsilon B$. *Then the incidence structure* $(\mathscr{P}, \mathscr{B}', \varepsilon_{DB})$ *is a tactical configuration with parameters* $(v, bm, k, rm; m\alpha_1, \ldots, m\alpha_a; m\beta_1+(m-1)(k-1), \ldots, m\beta_b+(m-1)(k-1))$.

Proof. Let $(\mathscr{B}, \mathscr{P}, \varepsilon^d)$ be the dual of $(\mathscr{P}, \mathscr{B}, \varepsilon)$. It is a tactical configuration with parameters $(b, v, r, k; \alpha_1, \ldots, \alpha_a; \beta_1, \ldots, \beta_b)$ as in Proposition 1.2.1. Let $(\mathscr{B}', \mathscr{P}, \varepsilon^{d'})$ be the incidence structure obtained from duplicating m points of $(\mathscr{B}, \mathscr{P}, \varepsilon^d)$ as in Lemma 1.2.2. It is a tactical configuration with parameters $(mb, v, mr, k; m\alpha_1, \ldots, m\alpha_a; m\beta_1 + (m-1)(k-1), \ldots, m\beta_b + (m-1)(k-1))$. Let $(\mathscr{P}, \mathscr{B}', (\varepsilon^{d'})^d)$ be the dual of $(\mathscr{B}', \mathscr{P}, \varepsilon^{d'})$. Then $(\mathscr{P}, \mathscr{B}', (\varepsilon^{d'})^d)$ is the incidence structure obtained from duplicating m blocks of $(\mathscr{P}, \mathscr{B}, \varepsilon)$, so $(\mathscr{P}, \mathscr{B}', (\varepsilon^{d'})^d) = (\mathscr{P}, \mathscr{B}', \varepsilon_{DB})$ is a tactical configuration with parameters $(v, bm, k, rm; m\alpha_1, \ldots, m\alpha_a; m\beta_1 + (m-1)(k-1), \ldots, m\beta_b + (m-1)(k-1))$. $\square$

Remark. If $\alpha_1, \ldots, \alpha_a \geq 1$ and $1 \leq r \leq b-1$, then we can choose an $m \in \mathbb{N}$, such that $(\mathscr{P}', \mathscr{B}', \varepsilon_{DB})$ is a tactical configuration with $m\alpha_1, \ldots, m\alpha_a \geq 1$ and $3 \leq rm \leq bm - 3$.

Let $(\mathscr{P}, \mathscr{B}, \varepsilon)$ be a tactical configuration with parameters $(v, b, k, r; \alpha_1, \ldots, \alpha_a; \beta_1, \ldots, \beta_b)$. The **duplication of points and blocks** is a incidence structure $(\mathscr{P}', \mathscr{B}', \varepsilon_{DPB})$ such that $n, m \in \mathbb{N}$, $\mathscr{P}' = \{(x, i) : x \in \mathscr{P} \text{ and } i \in \{1, 2, \ldots, n\}\}$, $\mathscr{B}' = \{(B, j) : B \in \mathscr{B} \text{ and } j \in \{1, 2, \ldots, m\}\}$ and for $(x, i) \in \mathscr{P}'$ and $(B, j) \in \mathscr{B}'$, we define $(x, i)\varepsilon_{DPB}(B, j)$ if and only if $x \varepsilon B$.

Theorem 1.2.4. [Duplication of points and blocks] *Let $(\mathscr{P}, \mathscr{B}, \varepsilon)$ be a tactical configuration with parameters $(v, b, k, r; \alpha_1, \ldots, \alpha_a; \beta_1, \ldots, \beta_b)$. Then the duplication of points and blocks is the incidence structure such that it is a tactical configuration with parameters $(vn, bm, kn, rm; nm\alpha_1, \ldots, nm\alpha_a; nm(\beta_1 + r + k - 1) - mr - kn + 1, \ldots, nm(\beta_b + r + k - 1) - mr - kn + 1)$.*

Proof. Let $(\mathscr{P}_{DP}, \mathscr{B}_{DP}, \varepsilon_{DP})$ be the incidence structure obtained from duplicating n points of $(\mathscr{P}, \mathscr{B}, \varepsilon)$ as in Lemma 1.2.2. Then it is a tactical configuration with parameters $(vn, b, kn, r; n\alpha_1, \ldots, n\alpha_a; n\beta + (n-1)(r-1), \ldots, n\beta_b + (n-1)(r-1))$. Let $(\mathscr{P}', \mathscr{B}', \varepsilon_{DPB})$ be the incidence structure obtained from duplicating m blocks of $(\mathscr{P}_{DP}, \mathscr{B}_{DP}, \varepsilon_{DP})$ as in Lemma 1.2.3. Then it is a $1\frac{1}{2}$-design with parameters $(vn, bm, kn, rm; nm\alpha_1, \ldots, nm\alpha_a; nm(\beta_1 + r + k - 1) - mr - kn + 1, \ldots, nm(\beta_b + r + k - 1) - mr - kn + 1)$. Hence, $(\mathscr{P}', \mathscr{B}', \varepsilon_{DPB})$ is the incidence structure obtained from duplicating n points and m blocks of $(\mathscr{P}, \mathscr{B}, \varepsilon)$ with parameters $(vn, bm, kn, rm; nm\alpha_1, \ldots, nm\alpha_a; nm(\beta_1 + r + k - 1) - mr - kn + 1, \ldots, nm(\beta_b + r + k - 1) - mr - kn + 1)$. □

Theorem 1.2.5. *Let $(\mathscr{P}, \mathscr{B}, \varepsilon)$ be a tactical configuration with parameters $(v, b, k, r; \alpha_1, \ldots, \alpha_a; \beta_1, \ldots, \beta_b)$ such that $\alpha_1, \ldots, \alpha_a \geq 1$, $1 \leq k \leq v - 1$, and $1 \leq r \leq b - 1$. Then we can construct a proper tactical configuration from $(\mathscr{P}, \mathscr{B}, \varepsilon)$.*

Proof. It follows from combining the two remarks mentioned earlier. □

Example 1.2.1. *Let $\mathscr{T}$ be not a proper $1\frac{1}{2}$-design with parameters $(2n, n^2, 2, n; 1; 0)$ in Example 1.1.4. Let $\mathscr{T}'$ be the incidence structure obtained from duplicating c points and d blocks of $\mathscr{T}$ with parameters $(2nc, n^2d, 2c, nd; cd; cd(n+1)-nd-2c+1)$ such that $c, d \geq 3$. Then $\mathscr{T}'$ is a proper $1\frac{1}{2}$-design.*

We shall apply this duplication techniques to obtain $1\frac{1}{2}$-designs from symplectic geometry and orthogonal geometry over finite local rings in Chapters II and III, respectively.

1.3 Undirected graphs

A **graph** is an ordered pair $G = (V, E)$ comprising a set V of vertices with a set E of edges, consisting of 2-element subsets of V.

A **k-regular graph** is a graph such that for every vertices there are k adjacent vertices.

A **strongly regular graph** with parameters (v, k, λ, μ) is a k-regular graph on v vertices such that for every pair of adjacent vertices there are λ vertices adjacent to both, and every pair of non-adjacent vertices there are μ vertices adjacent to both.

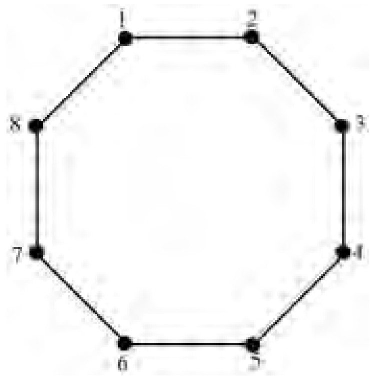
A **quasi-strongly regular graph** with parameters $(v, k, \lambda, c_1, c_2)$ is a k-regular graph on v vertices such that for every pair of adjacent vertices there are λ vertices adjacent to both, and every pair of non-adjacent vertices there are c_1 or c_2 vertices adjacent to both.

For a graph G, we write $\mathcal{V}(G)$ for its vertex set and $\mathcal{E}(G)$ for its edge set. Let G and H be graphs. A function f from $\mathcal{V}(G)$ to $\mathcal{V}(H)$ is a **homomorphism** from G to H if $f(g_1)$ and $f(g_2)$ are adjacent in H whenever g_1 and g_2 are adjacent in G. It is called an **isomorphism** if it is a bijection and f^{-1} is a homomorphism from H onto G. Moverover, an isomorphism on G is called an **automorphism**. The set of all isomorphisms of a graph G is denoted by $\mathrm{Aut}(G)$. It is a group under composition, called the **automorphism group** of G.

A graph G is **vertex transitive** if its automorphism group acts transitively on the vertex set. That is, for any two vertices of G, there is an automorphism carrying one to the other. An **arc** in G is a ordered pair of adjacent vertices, and G is **arc transitive** if its automorphism group acts transitively on its arcs. Note that an arc transitive graph is necessarily vertex and edge transitive.

A set I of vertices of a graph $\mathcal{G}$ is called an **independent set** if no two distinct vertices of I are adjacent.

Example 1.3.1. *Let* $G = (V, E)$ *be a graph such that* $V = \{1, \ldots, 8\}$ *and* $E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{6,7\}, \{7,8\}, \{8,1\}\}$



1. *G is a quasi-strongly regular graph with parameters* $(8, 2, 0, 1, 0)$.

2. *G is an arc transitive graph.*

3. $\{1, 3, 5\}$ *is an independent set but it is not a maximal independent set.*

4. $\{1, 3, 5, 7\}$ *is a maximal independent set.*

5. $\{1, 4, 6\}$ *is a maximal independent set.*

The thesis is organized as follows. Chapter II works on tacical configurations arising from symplectic graphs over finite local rings. We construct parallel tacical configurations from orthogonal graphs over finite local rings in Chapter III. They provide an applications of duplication of points and blocks. Also the results on subconstiuens studied in [5, 6, 7, 8, 9] allow as to compute the parameters of the new configurations explicitly. The final chapter studies the parameters of directed graphs arising from tacical configurations constructed in Chapters II and III. The definition of these directed graphs are from Brouwer, Olmez and Song [1].

CHAPTER II

TACTICAL CONFIGURATIONS FROM SYMPLECTIC GRAPHS OVER FINITE LOCAL RINGS

In this chapter, we discuss symplectic graphs over finite local rings [7, 8, 10] and construct tactical configurations from symplectic graph over finite local rings.

A **local ring** is a commutative ring which unique maximal ideal M consisting of all non-unit elements. We call the field R/M, the **residue field** of a local ring R. For example, every field is a local ring with maximal ideal $\{0\}$ and $\mathbb{Z}_{p^n}$, p a prime and $n \in \mathbb{N}$, is a local ring with maximal ideal $p\mathbb{Z}_{p^n}$ and residue field $\mathbb{Z}_{p^n}/p\mathbb{Z}_{p^n}$.

2.1 Symplectic graphs over finite local rings

Let R be a finite local ring with unique maximal ideal M and let (V, β) be a **symplectic space** of rank 2ν, where $\nu \geq 1$. That is, V is a free R-module of rank 2ν and possesses a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_{2\nu}\}$ in which

$$[\beta]_{\mathcal{B}} = K_{2\nu} = \begin{pmatrix} 0 & I_\nu \\ -I_\nu & 0 \end{pmatrix}$$

Therefore, if $\vec{x} = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_{2\nu}\vec{b}_{2\nu}$ and $\vec{y} = y_1\vec{b}_1 + y_2\vec{b}_2 + \cdots + y_{2\nu}\vec{b}_{2\nu}$ are vectors in V, then

$$\beta(\vec{x}, \vec{y}) = \begin{pmatrix} x_1 & x_2 & \cdots & x_{2\nu} \end{pmatrix} K_{2\nu} \begin{pmatrix} y_1 & y_2 & \cdots & y_{2\nu} \end{pmatrix}^T.$$

If $\vec{x} = \alpha_1\vec{b}_1 + \alpha_2\vec{b}_2 + \cdots + \alpha_{2\nu}\vec{b}_{2\nu}$ and α_i is a unit in R for some i, then $\vec{x}$ is called a **unimodular vector**.

Example 2.1.1. *Let p be a prime number and let R be the ring of integers module p^n,$\mathbb{Z}_{p^n}$ or the field of p^n elements, $\mathbb{F}$, where $n \in \mathbb{N}$. For $\nu \geq 1$, let V denote the set of 2ν-tuples $(x_1, \ldots, x_{2\nu})$ of elements in R. Define $\beta : V \times V \to R$ by*

$$\beta\Big((x_1, \ldots, x_{2\nu}), (y_1, \ldots, y_{2\nu})\Big) = (x_1, \ldots, x_{2\nu})K_{2\nu}(y_1, \ldots, y_{2\nu})^T,$$

where $K_{2\nu} = \begin{pmatrix} 0 & I_\nu \\ -I_\nu & 0 \end{pmatrix}$ and I_ν is the $\nu \times \nu$ identity matrix, for all vector $(x_1, \ldots, x_{2\nu}), (y_1, \ldots, y_{2\nu}) \in V$. Then (V, β) is a symplectic space, and unimodular vectors in V are those $(x_1, \ldots, x_{2\nu})$ of elements in R such that $x_i \in R^\times$ for some $i \in \{1, \ldots, 2\nu\}$.

Define the graph $\mathcal{G}_{Sp_R(V)}$ whose vertex set $\mathcal{V}(\mathcal{G}_{Sp_R(V)})$ is the set of lines (rank one submodules) of unimodular vector, namely,

$$\{R\vec{x} : \vec{x} \text{ is a unimodular vector in } V\}$$

and its adjacency condition is given by

$$R\vec{x} \text{ is adjacent to } R\vec{y} \Longleftrightarrow \beta(\vec{x}, \vec{y}) \text{ is a unit in } R.$$

We call $\mathcal{G}_{Sp_R(V)}$ the **symplectic graph** of (V, β) over R.

Let R be a finite local ring with unique maximal ideal M and residue field $\Bbbk = R/M$. Let (V, β) be a symplectic space of rank 2ν, where $\nu \geq 1$. This symplectic space induces a 2ν dimensional vector space (V', β'), where β' is given via the canonical map $\pi : R \to \Bbbk$ sending $a \mapsto a + M$ by

$$\beta'(\pi(\vec{a}, \vec{b})) = \pi(\beta(\vec{a}, \vec{b}))$$

for all $\vec{a}, \vec{b} \in V_\delta$. Here, we write $\pi(\vec{a}) = (\pi(a_1), \ldots, \pi(a_{2\nu}))$ for all $\vec{a} = (a_1, \ldots, a_{2\nu}) \in V$. It also follows that

$$\beta'(\pi(\vec{a}), \pi(\vec{b})) \in \Bbbk^\times \Leftrightarrow \beta(\vec{a}, \vec{b}) \in R^\times$$

for all $\vec{a}, \vec{b} \in V$, where $\Bbbk^\times = \Bbbk \smallsetminus \{0\}$ and $R^\times = R \smallsetminus M$ are the unit groups of $\Bbbk$ and of R, respectively.

The next theorem presents the relationship of the symplectic graphs over a finite local rings and over its residue field.

Theorem 2.1.1. [Lifting Theorem] *[8] Under the above set up, let* $\kappa = \frac{|\Bbbk|^{2\nu}-1}{|\Bbbk|-1}$ *and let* $\vec{x}_1, \ldots, \vec{x}_\kappa$ *be a unimodular vectors in* V *such that* $\mathcal{V}(\mathcal{G}_{Sp_{\Bbbk}(V')}) = \{\Bbbk\pi(\vec{x}_i) : i = 1, \ldots, \kappa\}$*. Then the following statements hold.*

1. *The set* $\Pi = \{R(\vec{x}_1 + M^{2\nu}), \ldots, R(\vec{x}_\kappa + M^{2\nu})\}$ *is a partition of the vertex set* $\mathcal{V}(\mathcal{G}_{Sp_R(V)})$*, where* $R(\vec{x}_i + M^{2\nu}) = \{R(\vec{x}_i + \vec{m}) : \vec{m} \in M^{2\nu}\}$ *for all* $i \in \{1, \ldots, \kappa\}$*. Moreover, for each* $i \in \{1, \ldots, \kappa\}$*, any two distinct vertices in* $R(\vec{x}_i + M^{2\nu})$ *are non-adjacent vertices.*

$\Bbbk\pi(\vec{x}_i)$		$R(\vec{x}_i + M^{2\nu})$
$\bullet$ $\Bbbk\pi(\vec{x}_1)$	$\longrightarrow$	$\bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet$ $R(\vec{x}_1 + \vec{m}),\ \vec{m} \in M^{2\nu}$
$\bullet$ $\Bbbk\pi(\vec{x}_2)$	$\longrightarrow$	$\bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet$ $R(\vec{x}_2 + \vec{m}),\ \vec{m} \in M^{2\nu}$
$\vdots$	$\vdots$	$\vdots$
$\bullet$ $\Bbbk\pi(\vec{x}_\kappa)$	$\longrightarrow$	$\bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet$ $R(\vec{x}_\kappa + \vec{m}),\ \vec{m} \in M^{2\nu}$

2. $|R(\vec{x}_i + M^{2\nu})| = |M|^{2\nu-1}$ *for all* $i \in \{1, \ldots, \kappa\}$*.*

3. *For unimodular vectors* $\vec{a}, \vec{b} \in V$*, we have* $R\vec{a}$ *and* $R\vec{b}$ *are adjacent vertices in* $\mathcal{G}_{Sp_R(V)}$ *if and only if* $\Bbbk\pi(\vec{a})$ *and* $\Bbbk\pi(\vec{b})$ *are adjacent vertices in* $\mathcal{G}_{Sp_{\Bbbk}(V')}$*.*

4. *The symplectic graph* $\mathcal{G}_{Sp_R(V)}$ *is vertex and arc transitive.*

The lifting theorem gives the following parameters.

Theorem 2.1.2. *[8, 10] Let* R *be a finite local ring with maximal ideal* M*. Let* (V, β) *be a symplectic space of rank* 2ν*, where* $\nu \geq 1$*. Then:*

1. *If R is a field, then $\mathcal{G}_{Sp_R(V)}$ is $|R|^{2\nu-1}$ -regular on*

$$\frac{|R|^{2\nu}-1}{|R|-1}$$

many vertices. Moreover, it is a strongly regular graph with parameters

$$\lambda = |R|^{2\nu-2}(|R|-1) \text{ and } \mu = |R|^{2\nu-2}(|R|-1)$$

2. *If R is a local ring which is not a field, then $\mathcal{G}_{Sp_R(V)}$ is $|R|^{2\nu-1}$ -regular on*

$$\frac{|R|^{2\nu}-|M|^{2\nu}}{|R|-|M|}$$

many vertices. Moreover, it is a quasi-strongly regular graph with parameters

$$\lambda = |R|^{2\nu-2}(|R|-|M|),\ c_1 = |R|^{2\nu-2}(|R|-|M|) \text{ and } c_2 = |R|^{2\nu-1}$$

Let R be a finite local ring with unique maximal ideal M and residue field $\Bbbk = R/M$. Let (V, β) be a symplectic space of rank 2ν, where $\nu \geq 1$. For unimodolar vectors $\vec{x}_1, \ldots, \vec{x}_\ell, \vec{y}_1, \ldots, \vec{y}_\ell$ in V and $\ell \geq 1$, we write $(\vec{x}_1, \ldots, \vec{x}_\ell) \approx (\vec{y}_1, \ldots, \vec{y}_\ell)$ if there exists an automorphism σ of $\mathcal{G}_{Sp_R(V)}$ such that $\sigma(R\vec{x}_i) = R\vec{y}_i$ for all $i \in \{1, \ldots, \ell\}$. Write $\vec{e}_i$ for all row vector with 1 at i th row and 0 otherwise for all $i \in \{1, 2, \ldots, 2\nu\}$. Li , Wang and Zhou [7] proved the following results.

Theorem 2.1.3. *[7] Let $\mathbb{F}$ be a finite field of order q and let (V, β) be a symplectic space of dimension 2ν, where $\nu \geq 2$. For any distinct vertices $\mathbb{F}\vec{x}, \mathbb{F}\vec{y}, \mathbb{F}\vec{z} \in \mathcal{V}(\mathcal{G}_{Sp_{\mathbb{F}}(V)})$, we have the following statements.*

1. *If $\mathbb{F}\vec{x}$ is adjacent to $\mathbb{F}\vec{y}$, then $(\vec{x}, \vec{y}) \approx (\vec{e}_1, \vec{e}_{\nu+1})$.*

2. *If $\mathbb{F}\vec{x}$ is non-adjacent to $\mathbb{F}\vec{y}$, then $(\vec{x}, \vec{y}) \approx (\vec{e}_1, \vec{e}_2)$.*

3. *If $\mathbb{F}\vec{x}$ is non-adjacent to $\mathbb{F}\vec{y}$, $\mathbb{F}\vec{x}$ is non-adjacent to $\mathbb{F}\vec{z}$ and $\mathbb{F}\vec{y}$ is adjacent to $\mathbb{F}\vec{z}$, then $(\vec{x}, \vec{y}, \vec{z}) \approx (\vec{e}_1, \vec{e}_2, \vec{e}_{\nu+2})$.*

4. *If $\mathbb{F}\vec{x}$ is adjacent to $\mathbb{F}\vec{y}$, $\mathbb{F}\vec{x}$ is adjacent to $\mathbb{F}\vec{z}$ and $\mathbb{F}\vec{y}$ is non-adjacent to $\mathbb{F}\vec{z}$, then $(\vec{x}, \vec{y}, \vec{z}) \approx (\vec{e}_1, \vec{e}_{\nu+1}, \vec{e}_2 + \vec{e}_{\nu+1})$ or $(\vec{e}_1, \vec{e}_{\nu+1}, \vec{e}_{\nu+1} + \vec{e}_{\nu+2})$.*

5. *If $\mathbb{F}\vec{x}$ is adjacent to $\mathbb{F}\vec{y}$, $\mathbb{F}\vec{x}$ is adjacent to $\mathbb{F}\vec{z}$ and $\mathbb{F}\vec{y}$ is adjacent to $\mathbb{F}\vec{z}$, then $(\vec{x},\vec{y},\vec{z}) \approx (\vec{e}_1, \vec{e}_{\nu+1}, \vec{e}_1 + a_{\nu+1}\vec{e}_{\nu+1})$, $(\vec{e}_1, \vec{e}_{\nu+1}, \vec{e}_1 + \vec{e}_2 + a_{\nu+1}\vec{e}_{\nu+1})$ or $(\vec{e}_1, \vec{e}_{\nu+1}, \vec{e}_1 + a_{\nu+1}\vec{e}_{\nu+1} + \vec{e}_{\nu+2})$ where $a_{\nu+1} \in \mathbb{F}^\times$.*

Let R be a finite local ring of with unique maximal ideal M and residue field $\Bbbk = R/M$. Let (V,β) be a symplectic space of rank 2ν over R, where $\nu \geq 2$. Next, we consider the **subconstituents** $\mathscr{G}^{(i)}_{Sp_R(V)}$, $i = 1, 2$, defined to be the induced subgraphs of $\mathcal{G}_{Sp_R(V)}$ on the vertex sets

$$\begin{aligned}
\mathcal{V}_1 &= \{R\vec{x} \in \mathcal{V}(\mathcal{G}_{Sp_R(V_\delta)}) : R\vec{x},\ \text{is adjacent to } R\vec{e}_1\} \\
\mathcal{V}_2 &= \{R\vec{x} \in \mathcal{V}(\mathcal{G}_{Sp_R(V_\delta)}) : R\vec{x},\ \text{is non-adjacent to } R\vec{e}_1 \text{ and } R\vec{x} \neq R\vec{e}_1\}
\end{aligned}$$

$i = 1, 2$, respectively.

Theorem 2.1.4. *[7, 8] Let R be a finite local ring and let (V,β) be a symplectic space of rank 2ν, where $\nu \geq 2$.*

1. *If R is a field, then $\mathscr{G}^{(1)}_{Sp_R(V)}$ is $|R|^{2\nu-2}(|R|-1)$-regular on $|R|^{2\nu-1}$ vertices. Moreover,*

   (a) *every two adjacent vertices of $\mathscr{G}^{(1)}_{Sp_R(V)}$ has $(|R|-2)|R|^{2\nu-2}$ or $(|R|-1)^2|R|^{2\nu-3}$ common neighbors,*

   (b) *every two non-adjacent vertices of $\mathscr{G}^{(1)}_{Sp_R(V)}$ has $(|R|-1)^2|R|^{2\nu-3}$ common neighbors,*

   (c) *there are $|R|-1$ vertices in $\mathscr{G}^{(1)}_{Sp_R(V)}$ adjacent to $R\vec{e}_{\nu+1}$ such that the number of their common neighbors is $(|R|-2)|R|^{2\nu-2}$, and*

   (d) *there are $(|R|-1)(|R|^{2\nu-2}-1)$ vertices in $\mathscr{G}^{(1)}_{Sp_R(V)}$ adjacent to $R\vec{e}_{\nu+1}$ such that the number of their common neighbors is $(|R|-1)^2|R|^{2\nu-3}$.*

2. *If R is a local ring which is not a field, then $\mathscr{G}^{(1)}_{Sp_R(V)}$ is $|R|^{2\nu-2}(|R|-|M|)$-regular on $|R|^{2\nu-1}$ vertices. Moreover,*

   (a) *every two adjacent vertices of $\mathscr{G}^{(1)}_{Sp_R(V)}$ has $(|R|-2|M|)|R|^{2\nu-2}$ or $(|R|-|M|)^2|R|^{2\nu-3}$ common neighbors,*

(b) *every two non-adjacent vertices of* $\mathscr{G}^{(1)}_{Sp_R(V)}$ *has* $(|R|-|M|)^2|R|^{2\nu-3}$ *or* $|R|^{2\nu-2}(|R|-|M|)$ *common neighbors,*

(c) *there are* $(|R|-|M|)|M|^{2\nu-2}$ *vertices in* $\mathscr{G}^{(1)}_{Sp_R(V)}$ *adjacent to* $R\vec{e}_{\nu+1}$ *such that the number of their common neighbors is* $(|R|-2|M|)|R|^{2\nu-2}$*, and*

(d) *there are* $(|R|-|M|)(|R|^{2\nu-2}-|M|^{2\nu-2})$ *vertices in* $\mathscr{G}^{(1)}_{Sp_R(V)}$ *adjacent to* $R\vec{e}_{\nu+1}$ *such that the number of their common neighbors is* $(|R|-|M|)^2|R|^{2\nu-3}$.

Theorem 2.1.5. *[7, 8] Let* R *be a finite local ring and let* (V,β) *be a symplectic space of rank* 2ν*, where* $\nu \geq 2$.

1. *If* R *is a field, then* $\mathscr{G}^{(2)}_{Sp_R(V)}$ *is* $|R|^{2\nu-2}$*-regular on* $\frac{|R|^{2\nu-1}-|R|}{|R|-1}$ *vertices. Moreover,*

   (a) *every two adjacent vertices of* $\mathscr{G}^{(2)}_{Sp_R(V)}$ *has* $|R|^{2\nu-3}(|R|-1)$ *common neighbors, and*

   (b) *every two non-adjacent vertices of* $\mathscr{G}^{(2)}_{Sp_R(V)}$ *has* $|R|^{2\nu-3}(|R|-1)$ *or* $|R|^{2\nu-2}$ *common neighbors.*

2. *If* R *is a local ring which is not a field, then* $\mathscr{G}^{(2)}_{Sp_R(V)}$ *is* $|R|^{2\nu-2}|M|$*-regular on* $\frac{(|R|^{2\nu-1}-|R|)|M|}{|R|-|M|}$ *vertices. Moreover,*

   (a) *every two adjacent vertices of* $\mathscr{G}^{(2)}_{Sp_R(V)}$ *has* $|R|^{2\nu-3}(|R|-|M|)|M|$ *common neighbors, and*

   (b) *every two non-adjacenct vertices of* $\mathscr{G}^{(2)}_{Sp_R(V)}$ *has* $|R|^{2\nu-3}(|R|-|M|)|M|$ *or* $|R|^{2\nu-2}|M|$ *common neighbors.*

2.2 Construction of tactical configurations

Let R be a finite local ring of with unique maximal ideal M and residue field $\Bbbk = R/M$ and let (V,β) be a symplectic space of rank 2ν, $\nu \geq 2$.

2.2.1 $1\frac{1}{2}$-designs from symplectic graphs over finite local rings

Let $\mathscr{P}$ be the set of vertices of symplectic graph $\mathcal{G}_{Sp_R(V)}$ and let $\mathscr{B}$ be collection of maximal independent sets of the graph. For $x \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $x \varepsilon B$ if and only if $x \in B$. The next theorem was stuided in [2] by Chai, Feng and Zeng. Its results show that if R is a field, then the incidence structure $\mathscr{T}_1 = (\mathscr{P}, \mathscr{B}, \varepsilon)$ is a $1\frac{1}{2}$-design.

Theorem 2.2.1. *[2] Let $\mathbb{F}$ be a finite field of order q and let (V, β) be a symplectic space of dimension 2ν, $\nu \geq 2$. Let $\mathscr{P} = \mathcal{V}(\mathcal{G}_{Sp_{\mathbb{F}}(V)})$ and let $\mathscr{B}$ be a collection of maximal independent sets of $\mathcal{G}_{Sp_{\mathbb{F}}(V)}$. For $x \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $x \varepsilon B$ if and only if $x \in B$. Then the incidence structure $\mathscr{T}_1' = (\mathscr{P}, \mathscr{B}, \varepsilon)$ is a $1\frac{1}{2}$-design. Furthermore, the parameters of $\mathscr{T}_1'$ is*

$$v = \tfrac{q^{2\nu}-1}{q-1}, \quad b = \prod_{i=1}^{\nu}(q^i + 1)$$
$$k = \tfrac{q^{\nu}-1}{q-1}, \quad r = \prod_{i=1}^{\nu-1}(q^i + 1),$$

with $\alpha_1 = 1$ and $\beta_1 = 0$ if $\nu = 2$, and with

$$\alpha_1 = \tfrac{q^{\nu-1}-1}{q-1} \cdot \prod_{i=1}^{\nu-2}(q^i + 1),\ \beta_1 = \tfrac{q(q^{\nu-1}-1)}{q-1} \cdot (\prod_{i=1}^{\nu-2}(q^i + 1) - 1)$$

otherwise.

By Theorem 2.1.1 (2) and (3) (Lifting Theorem), $\mathscr{T}_1$ is the incidence structure obtained from duplicating $|M|^{2\nu-1}$ points and 1 blocks of $\mathscr{T}_1'$ in Theorem 2.2.1. Hence, by Theorem 1.2.4, it is a $1\frac{1}{2}$-design with parameters recorded in the following theorem.

Theorem 2.2.2. *Let R be a finite local ring with maximal ideal M and let (V, β) be a symplectic space of rank 2ν, $\nu \geq 2$. Let $\mathscr{P} = \mathcal{V}(\mathcal{G}_{Sp_R(V)})$ and let $\mathscr{B}$ be a collection of maximal independent sets of $\mathcal{G}_{Sp_R(V)}$. For $x \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $x \varepsilon B$ if and only if $x \in B$. Then the incidence structure $\mathscr{T}_1 = (\mathscr{P}, \mathscr{B}, \varepsilon)$ is a $1\frac{1}{2}$-design. Furthermore, the parameters of $\mathscr{T}_1$ is*

$$v = \frac{|R|^{2\nu}-|M|^{2\nu}}{|R|-|M|}, \qquad b = \frac{\prod\limits_{i=1}^{\nu}(|R|^i+|M|^i)}{|M|^{\frac{(\nu+1)\nu}{2}}},$$

$$k = \frac{|R|^{\nu}-|M|^{\nu}}{|R|-|M|}|M|^{\nu}, \quad r = \frac{\prod\limits_{i=1}^{\nu-1}(|R|^i+|M|^i)}{|M|^{\frac{(\nu-1)\nu}{2}}},$$

with $\alpha_1 = |M|^3$ *and* $\beta_1 = (2|R|+|M|)|M|^2-(|R|+|M|)|M|^2-\frac{|R|+|M|}{|M|}+1$ *if* $\nu = 2$, *and with*

$$\alpha_1 = \frac{(|R|^{\nu-1}-|M|^{\nu-1})\prod\limits_{i=1}^{\nu-2}(|R|^i+|M|^i)}{(|R|-|M|)|M|^{\frac{(\nu-1)(\nu-2)-2(\nu+1)}{2}}},$$

$$\beta_1 = \frac{(2|R|^{\nu}-|R|^{\nu-1}|M|-|M|^{\nu})\prod\limits_{i=1}^{\nu-2}(|R|^i+|M|^i)}{(|R|-|M|)|M|^{\frac{(\nu-1)(\nu-2)-2\nu}{2}}} - \frac{\prod\limits_{i=1}^{\nu-1}(|R|^i+|M|^i)}{|M|^{\frac{(\nu-1)\nu}{2}}} - \frac{|R|^{\nu}-|R|^{\nu}}{|R|-|M|}|M|^{\nu}+1$$

otherwise.

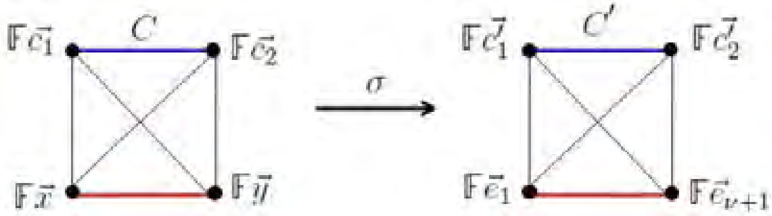
2.2.2 Other tactical configurations

In this section, we apply results on subconstituents, namely, Theorems 2.1.4 and 2.1.5, in construction other tactical configurations. They are not $1\frac{1}{2}$-designs. However, we can compute the parameters α's and β's.

Lemma 2.2.3. *Let* $\mathbb{F}$ *be a finite field of order* q *and let* (V, β) *be a symplectic space of dimension* 2ν, *where* $\nu \geq 2$. *Let* $\mathbb{F}\vec{x}$ *and* $\mathbb{F}\vec{y}$ *be adjacenct vertices in* $\mathcal{G}_{Sp_{\mathbb{F}}(V)}$. *Then the number of edges whose both vertices are common neighbors of* $\mathbb{F}\vec{x}$ *and* $\mathbb{F}\vec{y}$ *is given by*

$$A_{Sp} = \frac{(q-1)^3(q^{2\nu-2}-1)q^{2\nu-3}+(q-1)(q-2)q^{2\nu-2}}{2}.$$

Proof. Let C be an edge such that both vertices are common neighbors of $\mathbb{F}\vec{x}$ and $\mathbb{F}\vec{y}$. Since $\mathbb{F}\vec{x}$ is adjacent to $\mathbb{F}\vec{y}$, there exists σ automorphism carries $\mathbb{F}\vec{x}$ to $\mathbb{F}\vec{e}_1$, $\mathbb{F}\vec{y}$ to $\mathbb{F}\vec{e}_{\nu+1}$, $\mathbb{F}\vec{c_1}$ to $\mathbb{F}\vec{c_1'}$ and $\mathbb{F}\vec{c_2}$ to $\mathbb{F}\vec{c_2'}$ where $\mathbb{F}\vec{c_1}$ and $\mathbb{F}\vec{c_2}$ are both vertices of C by Theorem 2.1.3 (1).



Thus, the number of edges C is the number of 3-cycle at $\mathbb{F}\vec{e}_{\nu+1}$ in $\mathcal{G}^{(1)}_{Sp_{\mathbb{F}}(V)}$. We distinguish two cases.

Case 1: $\mathbb{F}\vec{c_1'}$ and $\mathbb{F}\vec{e}_{\nu+1}$ have $(q-1)^2q^{2\nu-3}$ common neighbors. It follows from Theorem 2.1.4 (1d) that the number of 3-cycles at $\mathbb{F}\vec{e}_{\nu+1}$ is given by product that

$$\tfrac{(q-1)^3(q^{2\nu-2}-1)q^{2\nu-3}}{2}.$$

Case 2: It follows from Theorem 2.1.4 (1c) that the number of 3-cycles at $\mathbb{F}\vec{e}_{\nu+1}$ is given by product that

$$\tfrac{(q-1)(q-2)q^{2\nu-2}}{2}.$$

Hence, the number of edges C is the sum

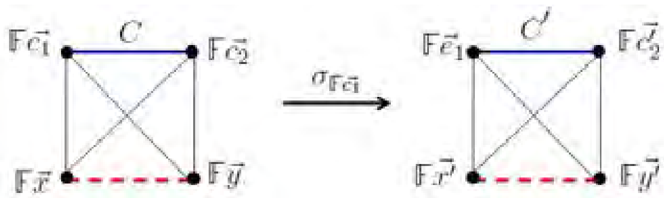
$$A_{Sp} = \tfrac{(q-1)^3(q^{2\nu-2}-1)q^{2\nu-3}}{2} + \tfrac{(q-1)(q-2)q^{2\nu-2}}{2} = \tfrac{(q-1)^3(q^{2\nu-2}-1)q^{2\nu-3}+(q-1)(q-2)q^{2\nu-2}}{2}$$

as desired. $\square$

Lemma 2.2.4. *Let $\mathbb{F}$ be a finite field of order q and let (V, β) be a symplectic space of dimension 2ν, where $\nu \geq 2$. Let $\mathbb{F}\vec{x}$ and $\mathbb{F}\vec{y}$ be non-adjacent vertices in $\mathcal{G}_{Sp_{\mathbb{F}}(V)}$. The number of edges whose both vertices are common neighbors of $\mathbb{F}\vec{x}$ and $\mathbb{F}\vec{y}$ is given by*

$$C_{Sp} = \tfrac{q^{4\nu-5}(q-1)^3}{2}.$$

Proof. Let C be an edge such that both vertices are common neighbors of $\mathbb{F}\vec{x}$ and $\mathbb{F}\vec{y}$. Since $\mathbb{F}\vec{x}$ is non-adjacent to $\mathbb{F}\vec{y}$, there exists $\sigma_{\mathbb{F}\vec{c_1}}$ automorphism carries $\mathbb{F}\vec{x}$ to $\mathbb{F}\vec{x'}$, $\mathbb{F}\vec{y}$ to $\mathbb{F}\vec{y'}$, $\mathbb{F}\vec{c_1}$ to $\mathbb{F}\vec{e}_1$ and $\mathbb{F}\vec{c_2}$ to $\mathbb{F}\vec{c_2'}$ where $\mathbb{F}\vec{c_1}$ and $\mathbb{F}\vec{c_2}$ are both vertices of C by Theorem 2.1.1 (4).



Thus, the number of edges C is the product of the number of the common neighbors $\mathbb{F}\vec{c_1'}$ of $\mathbb{F}\vec{x'}$ and $\mathbb{F}\vec{y'}$ in $\mathcal{G}_{Sp_{\mathbb{F}}(V)}$ and half of the number of common neighbor of $\mathbb{F}\vec{x'}$ and $\mathbb{F}\vec{y'}$ in $\mathcal{G}^{(1)}_{Sp_{\mathbb{F}}(V)}$. By Theorem 2.1.2 (1) the common neighbors of $\mathbb{F}\vec{x'}$ and $\mathbb{F}\vec{y'}$ in $\mathcal{G}_{O_{\mathbb{F}}(V)}$ is $(q-1)q^{2\nu-2}$. The number of common neighbors of $\mathbb{F}\vec{x'}$ and $\mathbb{F}\vec{y'}$ in $\mathcal{G}^{(1)}_{Sp_{\mathbb{F}}(V)}$ is $(q-1)^2q^{2\nu-3}$ by Theorem 2.1.4 (1b). Hence,

$$C_{Sp} = \frac{q^{4\nu-5}(q-1)^3}{2}$$

as desired. □

Let R be a finite local ring with unique maximal ideal M and residue field $\Bbbk = R/M$ and let (V,β) be a symplectic space of rank 2ν, $\nu \geq 2$. Let $\mathscr{P} = \mathcal{V}(\mathcal{G}_{O_R(V)})$ and $\mathscr{B} = \mathcal{E}(\mathcal{G}_{O_R(V)})$. For $x \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $x\varepsilon B$ if and only if x is a common neighbor of B. If R is a field, then the incidence structure $\mathscr{T}_2 = (\mathscr{P},\mathscr{B},\varepsilon)$ was stuided in next theorem.

Theorem 2.2.5. *Let $\mathbb{F}$ be a finite field of order q of and let (V,β) be a symplectic space of dimension 2ν, $\nu \geq 2$. Let $\mathscr{P} = \mathcal{V}(\mathcal{G}_{Sp_{\mathbb{F}}(V)})$ and $\mathscr{B} = \mathcal{E}(\mathcal{G}_{Sp_{\mathbb{F}}(V)})$. For $\mathbb{F}\vec{x} \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $\mathbb{F}\vec{x}\varepsilon B$ if and only if $\mathbb{F}\vec{x}$ is a common neighbor of B. Then the incidence structure $\mathscr{T}_2' = (\mathscr{P},\mathscr{B},\varepsilon)$ is a tactical configuration with parameters*

$$\begin{aligned}
v &= \tfrac{q^{2\nu}-1}{q-1},\\
b &= \tfrac{(q^{2\nu}-1)q^{2\nu-1}}{2(q-1)},\\
k &= q^{2\nu-2}(q-1),\\
r &= \tfrac{q^{4\nu-3}(q-1)}{2},\\
\alpha_1 &= (q-1)^2q^{2\nu-3}\tfrac{(q-1)^3(q^{2\nu-2}-1)q^{2\nu-3}+(q-1)(q-2)q^{2\nu-2}}{2} + (q-1)q^{2\nu-3}\tfrac{q^{4\nu-5}(q-1)^3}{2},\\
\alpha_2 &= (q-1)q^{2\nu-2}\tfrac{(q-1)^3(q^{2\nu-2}-1)q^{2\nu-3}+(q-1)(q-2)q^{2\nu-2}}{2},\\
\beta_1 &= (q-1)^2q^{2\nu-3}\tfrac{(q-1)^3(q^{2\nu-2}-1)q^{2\nu-3}+(q-1)(q-2)q^{2\nu-2}-2}{2} + ((q-1)q^{2\nu-3}-1)\tfrac{q^{4\nu-5}(q-1)^3-2}{2},\\
\beta_2 &= (q-2)q^{2\nu-2}\tfrac{(q-1)^3(q^{2\nu-2}-1)q^{2\nu-3}+(q-1)(q-2)q^{2\nu-2}-2}{2} + (q^{2\nu-2}-1)\tfrac{q^{4\nu-5}(q-1)^3-2}{2}.
\end{aligned}$$

Proof. Since $\mathcal{G}_{Sp_{\mathbb{F}}(V)}$ is strongly regular and vertex transitive, $\mathscr{T}_2'$ is a tactical configuration with parameters $(\frac{q^{2\nu}-1}{q-1}, \frac{(q^{2\nu}-1)q^{2\nu-1}}{2(q-1)}, q^{2\nu-2}(q-1), \frac{q^{4\nu-3}(q-1)}{2})$. Let A_{Sp} and C_{Sp} be given in Lemmas 2.2.3 and 2.2.4, respectively. Let $\mathbb{F}\vec{x} \in \mathscr{P}$ and $B \in \mathscr{B}$ such that B is with vertices $\mathbb{F}\vec{y_1}$ and $\mathbb{F}\vec{y_2}$.

Case 1: $(\mathbb{F}\vec{x}, B)$ is an antiflag.

Case 1.1: $\mathbb{F}\vec{x}$ is adjacent to $\mathbb{F}\vec{y_1}$ but x is non-adjacent to $\mathbb{F}\vec{y_2}$. Then $(\vec{x},\vec{y_1},\vec{y_2}) \approx$

$(\vec{e}_{\nu+1}, \vec{e}_1, \vec{e}_2 + \vec{e}_{\nu+1})$ or $(\vec{e}_{\nu+1}, \vec{e}_1, \vec{e}_{\nu+1} + \vec{e}_{\nu+2})$ by Theorem 2.1.3 (4). Thus,

$$\begin{aligned} s(\mathbb{F}\vec{x}, B) &= \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{x} \text{ is adjacent to } \mathbb{F}\vec{y}} \lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} + \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{x} \text{ is non-adjacent to } \mathbb{F}\vec{y}} \lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} \\ &= (q-1)^2 q^{2\nu-3} A_{Sp} + (q^{2\nu-2}(q-1) - (q-1)^2 q^{2\nu-3}) C_{Sp} \end{aligned}$$

by Theorem 2.1.4 (1b).

Case 1.2: $\mathbb{F}\vec{x}$ is non-adjacent to $\mathbb{F}\vec{y_1}$ and $\mathbb{F}\vec{y_2}$. Then $(\vec{x}, \vec{y_1}, \vec{y_2}) \approx (\vec{e}_1, \vec{e}_2, \vec{e}_{\nu+2})$ by Theorem 2.1.3 (3). Thus,

$$\begin{aligned} s(\mathbb{F}\vec{x}, B) &= \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{x} \text{ is adjacent to } \mathbb{F}\vec{y}} \lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} + \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{x} \text{ is non-adjacent to } \mathbb{F}\vec{y}} \lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} \\ &= (q^{2\nu-2}(q-1) - (q-1) q^{2\nu-3}) A_{Sp} + (q-1) q^{2\nu-3} C_{Sp} \end{aligned}$$

by Theorem 2.1.5 (1a).

Case 1.3: $\mathbb{F}\vec{x} = \mathbb{F}\vec{y_1}$. Then $(\vec{x}, \vec{y_2}) \approx (\vec{e}_1, \vec{e}_{\nu+1})$ by Theorem 2.1.3 (1). Thus,

$$\begin{aligned} s(\mathbb{F}\vec{x}, B) &= \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{x} \text{ is adjacent to } \mathbb{F}\vec{y}} \lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} + \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{x} \text{ is non-adjacent to } \mathbb{F}\vec{y}} \lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} \\ &= q^{2\nu-2}(q-1) A_{Sp} + (q^{2\nu-2}(q-1) - q^{2\nu-2}(q-1)) C_{Sp} \end{aligned}$$

by Theorem 2.1.2 (1).

Case 2: $(\mathbb{F}\vec{x}, B)$ is a flag. Then $(\vec{y_1}, \vec{y_2}, \vec{x}) \approx (\vec{e}_1, \vec{e}_{\nu+1}, \vec{e}_1 + a_{\nu+1}\vec{e}_{\nu+1})$, $(\vec{e}_1, \vec{e}_{\nu+1}, \vec{e}_1 + \vec{e}_2 + a_{\nu+1}\vec{e}_{\nu+1})$ or $(\vec{e}_1, \vec{e}_{\nu+1}, \vec{e}_1 + a_{\nu+1}\vec{e}_{\nu+1} + \vec{e}_{\nu+2})$ where $a_{\nu+1} \in \mathbb{F}^\times$ by Theorem 2.1.3 (5) Thus,

$$\begin{aligned} s(\mathbb{F}\vec{x}, B) &= \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{y} \neq \mathbb{F}\vec{x}, \mathbb{F}\vec{x} \text{ is adjacent to } \mathbb{F}\vec{y}} (\lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} - 1) + \\ &\quad \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{y} \neq \mathbb{F}\vec{x}, \mathbb{F}\vec{x} \text{ is non-adjacent to } \mathbb{F}\vec{y}} (\lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} - 1) \\ &= ((q-1)^2 q^{2\nu-3})(A_{Sp} - 1) + (q^{2\nu-2}(q-1) - 1 - (q-1)^2 q^{2\nu-3})(C_{Sp} - 1) \text{ or} \\ &\quad (q-2) q^{2\nu-2}(A_{Sp} - 1) + (q^{2\nu-2}(q-1) - 1 - (q-2) q^{2\nu-2})(C_{Sp} - 1) \end{aligned}$$

by Theorem 2.1.4 (1a). □

By Theorem 2.1.1 (2) and (3) (Lifting Theorem), $\mathscr{T}_2$ is the incidence structure obtained from duplicating $|M|^{2\nu-1}$ points and $(|M|^{2\nu-1})^2$ blocks of $\mathscr{T}_2'$ in Theorems 2.2.5. Hence, by Theorem 1.2.4, it is a tactical configuration with parameters recorded in the following theorem.

Theorem 2.2.6. *Let R be a finite local ring with maximal ideal M and let (V, β) be a symplectic space of rank 2ν, $\nu \geq 2$. Let $\mathscr{P} = \mathcal{V}(\mathcal{G}_{Sp_R(V)})$ and $\mathscr{B} = \mathcal{E}(\mathcal{G}_{Sp_R(V)})$. For $R\vec{x} \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $R\vec{x}\varepsilon B$ if and only if $R\vec{x}$ is a common neighbor of B. Then the incidence structure $\mathscr{T}_2 = (\mathscr{P}, \mathscr{B}, \varepsilon)$ is a tactical configuration with* *parameters*

$$
\begin{aligned}
v &= \tfrac{|R|^{2\nu}-|M|^{2\nu}}{|R|-|M|},\\
b &= \tfrac{(|R|^{2\nu}-|M|^{2\nu})|R|^{2\nu-1}}{2(|R|-|M|)},\\
k &= |R|^{2\nu-2}(|R|-|M|),\\
r &= \tfrac{|R|^{4\nu-3}(|R|-|M|)}{2},\\
\alpha_1 &= (|R|-|M|)^2|R|^{2\nu-3}\tfrac{(|R|-|M|)^3(|R|^{2\nu-2}-|M|^{2\nu-2})|R|^{2\nu-3}+(|R|-|M|)(|R|-2|M|)|R|^{2\nu-2}|M|^{2\nu-2}}{2}+\\
&\quad (|R|-|M|)|R|^{2\nu-3}|M|\tfrac{|R|^{4\nu-5}(|R|-|M|)^3}{2},\\
\alpha_2 &= (|R|-|M|)|R|^{2\nu-2}\tfrac{(|R|-|M|)^3(|R|^{2\nu-2}-|M|^{2\nu-2})|R|^{2\nu-3}+(|R|-|M|)(|R|-2|M|)|R|^{2\nu-2}|M|^{2\nu-2}}{2},\\
\beta_1 &= (|R|-|M|)^2|R|^{2\nu-3}\Big(\tfrac{(|R|-|M|)^3(|R|^{2\nu-2}-|M|^{2\nu-2})|M|^{2\nu-3}}{2}+\\
&\quad \tfrac{(|R|-|M|)(|R|-2|M|)|R|^{2\nu-2}|M|^{2\nu-2}-2|M|^{4\nu-2}}{2}\Big)+((|R|-|M|)|R|^{2\nu-3}-|M|^{2\nu-3})\\
&\quad \times|M|\tfrac{|R|^{4\nu-5}(|R|-|M|)^3-2|M|^{4\nu-2}}{2}+|R|^{2\nu-2}|M|^{4\nu-2}(|R|-|M|)+\tfrac{|R|^{4\nu-3}|M|^{2\nu-1}(|R|-|M|)}{2}\\
&\quad -(|M|^{2\nu-1})^3-|R|^{2\nu-2}(|R|-|M|)-\tfrac{|R|^{4\nu-3}(|R|-|M|)}{2}+1,\\
\beta_2 &= (|R|-2|M|)|R|^{2\nu-2}\Big(\tfrac{(|R|-|M|)^3(|R|^{2\nu-2}-|M|^{2\nu-2})|M|^{2\nu-3}}{2}+\\
&\quad \tfrac{(|R|-|M|)(|R|-2|M|)|R|^{2\nu-2}|M|^{2\nu-2}-2|M|^{4\nu-2}}{2}\Big)+(|R|^{2\nu-2}-|M|^{2\nu-2})\\
&\quad \times|M|\tfrac{|R|^{4\nu-5}(|R|-|M|)^3-2|M|^{4\nu-2}}{2}+|R|^{2\nu-2}|M|^{4\nu-2}(|R|-|M|)+\\
&\quad \tfrac{|R|^{4\nu-3}|M|^{2\nu-1}(|R|-|M|)}{2}-(|M|^{2\nu-1})^3-|R|^{2\nu-2}(|R|-|M|)-\tfrac{|R|^{4\nu-3}(|R|-|M|)}{2}+1.
\end{aligned}
$$

CHAPTER III

TACTICAL CONFIGURATIONS FROM ORTHOGONAL GRAPHS OVER FINITE LOCAL RINGS

We use orthogonal graphs over finite local rings to construct tactical configurations in this Chapter. Results are parallel with Chapter II.

3.1 Orthogonal graphs over finite local rings

Let R be a finite local ring of odd characteristic with unique maximal ideal M and let (V_δ, β) be an **orthogonal space** of rank $2\nu+\delta$, where $\nu \geq 1, \delta \in \{0,1,2\}$. That is, V_δ is a free R-module of rank $2\nu+\delta$ and possesses a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_{2\nu+\delta}\}$ in which

$$[\beta]_{\mathcal{B}} = S_{2\nu+\delta,\Delta} = \begin{pmatrix} 0 & I_\nu & \\ I_\nu & 0 & \\ & & \Delta \end{pmatrix},$$

where

$$\Delta = \begin{cases} \varnothing(\text{disappear}) & \text{if } \delta = 0, \\ (1) \text{ or } (z) & \text{if } \delta = 1, \\ \operatorname{diag}(1, -z) & \text{if } \delta = 2, \end{cases}$$

and z is a fixed non-square unit in R. Therefore, if $\vec{x} = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_{2\nu+\delta}\vec{b}_{2\nu+\delta}$ and $\vec{y} = y_1\vec{b}_1 + y_2\vec{b}_2 + \cdots + y_{2\nu+\delta}\vec{b}_{2\nu+\delta}$ are vectors in V, then

$$\beta(\vec{x}, \vec{y}) = \begin{pmatrix} x_1 & x_2 & \cdots & x_{2\nu+\delta} \end{pmatrix} S_{2\nu+\delta,\Delta} \begin{pmatrix} y_1 & y_2 & \cdots & y_{2\nu+\delta} \end{pmatrix}^T.$$

If $\vec{x} = \alpha_1\vec{b}_1 + \alpha_2\vec{b}_2 + \cdots + \alpha_{2\nu+\delta}\vec{b}_{2\nu+\delta}$ and α_i is a unit in R for some i, then $\vec{x}$ is called a **unimodular vector**.

Example 3.1.1. *Let p be an odd prime number and let R be the ring of integers modulo p^n,$\mathbb{Z}_{p^n}$ or the field of p^n elements, $\mathbb{F}$, where $n \in \mathbb{N}$. For $\nu \geq 1, \delta \in \{0,1,2\}$, let V denote the set of $2\nu+\delta$-tuples $(x_1,\ldots,x_{2\nu+\delta})$ of elements in R. Define $\beta : V \times V \to R$ by*

$$\beta\Big((x_1,\ldots,x_{2\nu+\delta}),(y_1,\ldots,y_{2\nu+\delta})\Big) = (x_1,\ldots,x_{2\nu+\delta})S_{2\nu+\delta,\Delta}(y_1,\ldots,y_{2\nu+\delta})^T,$$

where $S_{2\nu+\delta,\Delta} = \begin{pmatrix} 0 & I_\nu & \\ I_\nu & 0 & \\ & & \Delta \end{pmatrix}$, *such that* $\Delta = \begin{cases} \varnothing(\textit{disappear}) & \textit{if } \delta = 0, \\ (1) \ \textit{or} \ (z) & \textit{if } \delta = 1, \\ \textit{diag}(1,-z) & \textit{if } \delta = 2, \end{cases}$

z is a fixed non-square unit in R and I_ν is the $\nu \times \nu$ identity matrix, for all vector $(x_1,\ldots,x_{2\nu+\delta}),(y_1,\ldots,y_{2\nu+\delta}) \in V$. Then (V,β) is an orthogonal space, and unimodular vectors in V are those $(x_1,\ldots,x_{2\nu+\delta})$ of elements in R such that $x_i \in R^\times$ for some $i \in \{1,\ldots,2\nu+\delta\}$.

Define the graph $\mathcal{G}_{O_R(V_\delta)}$ whose vertex set $\mathcal{V}(\mathcal{G}_{O_R(V_\delta)})$ is the set of lines (rank one submodules) of unimodular vectors of zero norm, namely,

$$\{R\vec{x} : \vec{x} \text{ is a unimodular vector in } V_\delta \text{ and } \beta(\vec{x},\vec{x}) = 0\}$$

and its adjacency condition is given by

$$R\vec{x} \text{ is adjacent to } R\vec{y} \iff \beta(\vec{x},\vec{y}) \text{ is a unit in } R.$$

We call $\mathcal{G}_{O_R(V_\delta)}$ the **orthogonal graph** of (V_δ,β) over R.

If $\Bbbk$ is a finite field and V'_δ is an orthogonal space over $\Bbbk$ of dimension $2\nu+\delta$, where $\nu \geq 1$ and $\delta \in \{0,1,2\}$, then Gu and Wan [4] showed that $\mathcal{G}_{O_\Bbbk(V'_\delta)}$ is a $|\Bbbk|^{\nu+\delta-1}+1$-partite graph with partite sets $X_1, X_2, \ldots, X_{|\Bbbk|^{\nu+\delta-1}+1}$ such that $|X_i| = \frac{|\Bbbk|^{\nu}-1}{|\Bbbk|-1}$ for all $i \in \{1,2,\ldots,|\Bbbk|^{\nu+\delta-1}+1\}$.

Let R be a finite local ring of odd characteristic with unique maximal ideal M and residue field $\Bbbk = R/M$. Let (V_δ,β) be an orthogonal space of rank $2\nu+\delta$, where $\nu \geq 1$ and $\delta \in \{0,1,2\}$. This orthogonal space induces a $2\nu+\delta$ dimensional vector space (V'_δ, β'), where β' is given via the canonical map $\pi : R \to \Bbbk$ sending $a \mapsto a + M$ by

$$\beta'(\pi(\vec{a}, \vec{b})) = \pi(\beta(\vec{a}, \vec{b}))$$

for all $\vec{a}, \vec{b} \in V_\delta$. Here, we write $\pi(\vec{a}) = (\pi(a_1), \ldots, \pi(a_{2\nu+\delta}))$ for all $\vec{a} = (a_1, \ldots, a_{2\nu+\delta}) \in V_\delta$. It also follows that

$$\beta'(\pi(\vec{a}), \pi(\vec{b})) \in \Bbbk^\times \Leftrightarrow \beta(\vec{a}, \vec{b}) \in R^\times$$

for all $\vec{a}, \vec{b} \in V_\delta$, where $\Bbbk^\times = \Bbbk \smallsetminus \{0\}$ and $R^\times = R \smallsetminus M$ are the unit groups of $\Bbbk$ and of R, respectively.

The next theorem given the relationship of the orthogonal graphs over a finite local rings and over its residue field. It is the lifting theorem for orthogonal graphs.

Theorem 3.1.1. [Lifting Theorem] *[9] Under the above set up, let* $\kappa = |\Bbbk|^{\nu+\delta-1}+1$ *and* $l = \frac{|\Bbbk|^\nu - 1}{|\Bbbk|-1}$. *For each* $i \in \{1, \ldots, \kappa\}$, *let* $X_i = \{\vec{x}_{i_1}, \ldots, \vec{x}_{i_l}\}$ *be the set of unimodular vectors in* V_δ *with zero norm such that* $\{\{\Bbbk\pi(\vec{x}_{i_s}) : s = 1, \ldots, l\} : i = 1, \ldots, \kappa\}$ *is a partition of* $\mathcal{V}(\mathcal{G}_{O_{\Bbbk}(V'_\delta)})$ *satisfying* $\Bbbk\pi(\vec{x}_{i_s})$ *and* $\Bbbk\pi(\vec{x}_{i_t})$ *are non-adjacent vertices for all* $s \neq t$. *Then the following statements hold.*

1. *The set* $\Pi = \{R(X_1 + M^{2\nu+\delta}), \ldots, R(X_\kappa + M^{2\nu+\delta}\}$ *is a partition of the vertex set* $\mathcal{V}(\mathcal{G}_{O_R(V_\delta)})$, *where* $R(X_i + M^{2\nu+\delta}) = \{R(\vec{x}_{i_s} + \vec{m}) : s \in \{1, \ldots, l\}, \vec{m} \in M^{2\nu+\delta}$ *and* $\beta(\vec{x}_{i_s} + \vec{m}, \vec{x}_{i_s} + \vec{m}) = 0\}$ *for all* $i \in \{1, \ldots, \kappa\}$. *For each* i, *the lifting of the vertices corresponding with elements in* X_i *to vertices in* $R(X_i + M^{2\nu+\delta})$ *is demonstrated below.*

$$
\begin{array}{ccc}
X_i & & R(X_i + M^{2\nu+\delta}) \\
\\
\begin{array}{|c|}\hline \bullet \\ \Bbbk\pi(\vec{x}_{i_1}) \\ \hline \bullet \\ \Bbbk\pi(\vec{x}_{i_2}) \\ \hline \vdots \\ \hline \bullet \\ \Bbbk\pi(\vec{x}_{i_l}) \\ \hline \end{array}
&
\begin{array}{c} \longrightarrow \\ \\ \longrightarrow \\ \\ \vdots \\ \longrightarrow \\ \\ \end{array}
&
\begin{array}{|c|}\hline \bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet \\ R(\vec{x}_{i_1} + \vec{m}),\ \vec{m} \in M^{2\nu+\delta} \text{ and } \beta(\vec{x}_{i_1} + \vec{m}, \vec{x}_{i_1} + \vec{m}) = 0 \\ \hline \bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet \\ R(\vec{x}_{i_2} + \vec{m}),\ \vec{m} \in M^{2\nu+\delta} \text{ and } \beta(\vec{x}_{i_2} + \vec{m}, \vec{x}_{i_2} + \vec{m}) = 0 \\ \hline \vdots \\ \hline \bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet \\ R(\vec{x}_{i_l} + \vec{m}),\ \vec{m} \in M^{2\nu+\delta} \text{ and } \beta(\vec{x}_{i_l} + \vec{m}, \vec{x}_{i_l} + \vec{m}) = 0 \\ \hline \end{array}
\end{array}
$$

Moreover, for each $i \in \{1, \ldots, \kappa\}$, any two distinct vertices in $R(X_i + M^{2\nu+\delta})$ are non-adjacent vertices. Hence, $\mathcal{G}_{O_R(V_\delta)}$ is a κ-partite graph.

2. *$|R(X_i + M^{2\nu+\delta})| = l|M|^{2\nu+\delta-2}$ for all $i \in \{1, \ldots, \kappa\}$.*

3. *For unimodular vectors with zero norm $\vec{a}, \vec{b} \in V_\delta$, we have $R\vec{a}$ and $R\vec{b}$ are adjacent vertices in $\mathcal{G}_{O_R(V_\delta)}$ if and only if $\Bbbk\pi(\vec{a})$ and $\Bbbk\pi(\vec{b})$ are adjacent vertices in $\mathcal{G}_{O_{\Bbbk}(V'_\delta)}$.*

4. *For $i, j \in \{1, \ldots, \kappa\}$, $s, t \in \{1, \ldots, l\}$ and $s \neq t$, if $\Bbbk\pi(\vec{x}_{i_s})$ and $\Bbbk\pi(\vec{x}_{j_t})$ are adjacent vertices, then $R(\vec{x}_{i_s} + \vec{m}_1)$ and $R(\vec{x}_{j_t} + \vec{m}_2)$ are adjacent vertices in the graph $\mathcal{G}_{O_R(V_\delta)}$ for all $\vec{m}_1, \vec{m}_2 \in M^{2\nu+\delta}$ such that $\beta(\vec{x}_{i_s} + \vec{m}_1, \vec{x}_{i_s} + \vec{m}_1) = \beta(\vec{x}_{j_t} + \vec{m}_2, \vec{x}_{j_t} + \vec{m}_2) = 0$.*

5. *The orthogonal graph $\mathcal{G}_{O_R(V)}$ is vertex and arc transitive.*

The Lifting Theorem gives the following parameters.

Theorem 3.1.2. *[4, 9] Let R be a finite local ring of odd characteristic with maximal ideal M. Let (V, β) be an orthogonal space of rank $2\nu + \delta$, where $\nu \geq 1$ and $\delta \in \{0, 1, 2\}$.*

1. *If R is a field, then $\mathcal{G}_{O_R(V)}$ is $|R|^{2\nu+\delta-2}$ -regular on*

$$\frac{(|R|^\nu - 1)(|R|^{\nu+\delta-1} + 1)}{|R| - 1}$$

many vertices. Moverover,

(a) *If $\nu = 1$, then it is a strongly regular graph with parameters*

$$\lambda = |R|^{\delta} - 1 \text{ and } \mu = \lceil \delta/2 \rceil |R|^{\delta}$$

(b) *If $\nu \geq 2$, then it is a strongly regular graph with parameters*

$$\lambda = |R|^{2\nu+\delta-2} - |R|^{2\nu+\delta-3} - |R|^{\nu-1} + |R|^{\nu+\delta-2}$$
$$\mu = (|R| - 1)|R|^{2\nu+\delta-3}.$$

2. *If R is a local ring which is not a field, then $\mathcal{G}_{O_R(V)}$ is $|R|^{2\nu+\delta-2}$ -regular on*

$$\frac{(|R|^{\nu} - |M|^{\nu})(|R|^{\nu+\delta-1} + |M|^{\nu+\delta-1})}{|R| - |M|}$$

many vertices. Moverover,

(a) *If $\nu = 1$, then it is a strongly regular graph with parameters*

$$\lambda = |R|^{\delta} - |M|^{\delta} \text{ and } \mu = \lceil \delta/2 \rceil |R|^{\delta}$$

(b) *If $\nu \geq 2$, then it is a quasi-strongly regular graph with parameters*

$$\lambda = |R|^{2\nu+\delta-2} - |R|^{2\nu+\delta-3}|M| - |R|^{\nu-1}|M|^{\nu+\delta-1} + |R|^{\nu+\delta-2}|M|^{\nu}$$
$$c_1 = (|R| - |M|)|R|^{2\nu+\delta-3} \text{ and } c_2 = |R|^{2\nu+\delta-2}$$

Let R be a finite local ring of odd characteristic with unique maximal ideal M and residue field $\Bbbk = R/M$. Let (V_{δ}, β) be an orthogonal space of rank $2\nu + \delta$, where $\nu \geq 1$ and $\delta \in \{0, 1, 2\}$. For unimodolar vectors $\vec{x}_1, \ldots, \vec{x}_{\ell}, \vec{y}_1, \ldots, \vec{y}_{\ell}$ in V_{δ} with zero norm and $\ell \geq 1$, we write $(\vec{x}_1, \ldots, \vec{x}_{\ell}) \approx (\vec{y}_1, \ldots, \vec{y}_{\ell})$ if there exists an automorphism σ of $\mathcal{G}_{O_R(V)}$ such that $\sigma(R\vec{x}_i) = R\vec{y}_i$ for all $i \in \{1, \ldots, \ell\}$. Write $\vec{e}_i$ for all row vector with 1 at i th row and 0 otherwise for all $i \in \{1, 2, \ldots, 2\nu + \delta\}$. Gu and Wan [6] obtained the following results.

Theorem 3.1.3. *[6] Let $\mathbb{F}$ be a finite field of odd order q and let (V, β) be an orthogonal space of dimension $2\nu + \delta$, where $\nu \geq 2$ and $\delta \in \{0, 1, 2\}$. For any distinct vertices $\mathbb{F}\vec{x}, \mathbb{F}\vec{y}, \mathbb{F}\vec{z} \in \mathcal{V}(\mathcal{G}_{O_{\mathbb{F}}(V)})$, we have the following statements.*

1. *If $\mathbb{F}\vec{x}$ is adjacent to $\mathbb{F}\vec{y}$, then* $(\vec{x},\vec{y}) \approx (\vec{e}_1, \vec{e}_{\nu+1})$.

2. *If $\mathbb{F}\vec{x}$ is non-adjacent to $\mathbb{F}\vec{y}$, then* $(\vec{x},\vec{y}) \approx (\vec{e}_1, \vec{e}_2)$.

3. *If $\mathbb{F}\vec{x}$ is non-adjacent to $\mathbb{F}\vec{y}$, $\mathbb{F}\vec{x}$ is non-adjacent to $\mathbb{F}\vec{z}$ and $\mathbb{F}\vec{y}$ is adjacent to $\mathbb{F}\vec{z}$, then* $(\vec{x},\vec{y},\vec{z}) \approx (\vec{e}_1, \vec{e}_2, \vec{e}_{\nu+2})$.

4. *If $\mathbb{F}\vec{x}$ is adjacent to $\mathbb{F}\vec{y}$, $\mathbb{F}\vec{x}$ is adjacent to $\mathbb{F}\vec{z}$ and $\mathbb{F}\vec{y}$ is non-adjacent to $\mathbb{F}\vec{z}$, then* $(\vec{x},\vec{y},\vec{z}) \approx (\vec{e}_1, \vec{e}_{\nu+1}, \vec{e}_2 + \vec{e}_{\nu+1})$.

5. *If $\mathbb{F}\vec{x}$ is adjacent to $\mathbb{F}\vec{y}$, $\mathbb{F}\vec{x}$ is adjacent to $\mathbb{F}\vec{z}$ and $\mathbb{F}\vec{y}$ is adjacent to $\mathbb{F}\vec{z}$, then* $(\vec{x},\vec{y},\vec{z}) \approx (\vec{e}_1, \vec{e}_{\nu+1}, \vec{e}_1 + \vec{e}_2 + a\vec{e}_{\nu+1} - a\vec{e}_{\nu+2})$ *where* $a \in \mathbb{F}^{\times}$.

Let R be a finite local ring of odd characteristic with unique maximal ideal M and residue field $\Bbbk = R/M$. Let (V_δ, β) be an orthogonal space of rank $2\nu+\delta$ over R, where $\nu \geq 2$, $\delta = \{0,1,2\}$. Next, we consider the **subconstituents** $\mathcal{G}^{(i)}_{O_R(V_\delta)}$, $i = 1,2$, defined to be the induced subgraphs of $\mathcal{G}_{O_R(V_\delta)}$ on the vertex sets

$$\begin{aligned} \mathcal{V}_1 &= \{R\vec{x} \in \mathcal{V}(\mathcal{G}_{O_R(V_\delta)}) : R\vec{x}, \text{ is adjacent to } R\vec{e}_1\} \\ \mathcal{V}_2 &= \{R\vec{x} \in \mathcal{V}(\mathcal{G}_{O_R(V_\delta)}) : R\vec{x}, \text{ is non-adjacent to } R\vec{e}_1 \text{ and } R\vec{x} \neq R\vec{e}_1\} \end{aligned}$$

$i = 1,2$, respectively.

Theorem 3.1.4. *[5, 6, 9] Let R be a finite local ring of odd characteristic with maximal ideal M. Let (V,β) be an orthogonal space of rank $2\nu+\delta$, where $\nu \geq 2$ and $\delta \in \{0,1,2\}$.*

1. *If R is a field, then $\mathcal{G}^{(1)}_{O_R(V)}$ is $|R|^{2\nu+\delta-2} - |R|^{2\nu+\delta-3} + |R|^{\nu+\delta-2} - |R|^{\nu-1}$-regular on $|R|^{2\nu+\delta-2}$ many vertices. Moverover,*

   (a) *If $\delta = 0$, then any two adjacent vertices have $(|R|^{\nu} - 2|R|^{\nu-1} + |R|^{\nu-2} - 2|R| + 3)|R|^{\nu-2}$ common neighbors and any two non-adjacent vertices have $(|R|-1)(|R|^{\nu-1} - |R|^{\nu-2} - 1)|R|^{\nu-2}$ common neighbors.*

   (b) *If $\delta = 1$, then any two adjacent vertices have $|R|^{2\nu-3}(|R-1)^2$ or $|R|^{\nu-2}(|R|^{\nu+1} - 2|R|^{\nu} + |R|^{\nu-1} + 2|)$ common neighbors and any two non-adjacent vertices have $|R|^{2\nu-3}(2|R|-3)$ common neighbors.*

(c) *If* $\delta = 2$, *then any two adjacent vertices have* $|R|^{2\nu} - 2|R|^{2\nu-1} + |R|^{2\nu-2} + 2|R|^{\nu} - 3|R|^{\nu-1}$ *common neighbors and any two non-adjacent vertices have* $(|R| - 1)(|R|^{2\nu-1} - |R|^{2\nu-2} + |R|^{\nu-1})$ *common neighbors.*

2. *If R is a local ring which is not a field, then* $\mathcal{G}^{(1)}_{O_R(V)}$ *is* $|R|^{2\nu+\delta-2} - |R|^{2\nu+\delta-3}|M| + |R|^{\nu+\delta-2}|M|^{\nu} - |R|^{\nu-1}|M|^{\nu+\delta-1}$*-regular on* $|R|^{2\nu+\delta-2}$ *many vertices. Moverover,*

   (a) *If* $\delta = 0$, *then any two adjacent vertices have* $(|R|^{\nu} - 2|R|^{\nu-1}|M| + |R|^{\nu-2}|M|^2 - 2|R||M|^{\nu-1} + 3|M|^{\nu})|R|^{\nu-2}$ *common neighbors and any two non-adjacent vertices have* $(|R| - |M|)(|R|^{\nu-1} - |R|^{\nu-2}|M| - |M|^{\nu-1})|R|^{\nu-2}$ *or* $(|R| - |M|)(|R|^{\nu-1} - |M|^{\nu-1})|R|^{\nu-2}$ *common neighbors.*

   (b) *If* $\delta = 1$, *then any two adjacent vertices have* $|R|^{2\nu-3}(|R| - |M|)^2$ *or* $|R|^{\nu-2}(|R|^{\nu+1} - 2|R|^{\nu}|M| + |R|^{\nu-1}|M|^2 + 2|M|^{\nu+1})$ *common neighbors and any two non-adjacent vertices have* $|R|^{2\nu-3}(2|R||M| - 3|M|^2)$ *or* $|R|^{2\nu-2}(|R| - |M|)$ *common neighbors.*

   (c) *If* $\delta = 2$, *then any two adjacent vertices have* $|R|^{2\nu} - 2|R|^{2\nu-1}|M| + |R|^{2\nu-2}|M|^2 + 2|R|^{\nu}|M|^{\nu} - 3|R|^{\nu-1}|M|^{\nu+1}$ *common neighbors and any two non-adjacent vertices have* $(|R| - |M|)(|R|^{2\nu-1} - |R|^{2\nu-2}|M| + |R|^{\nu-1}|M|^{\nu})$ *or* $(|R| - |M|)(|R|^{2\nu-1} + |R|^{\nu-1}|M|^{\nu})$ *common neighbors.*

Theorem 3.1.5. *[5, 6, 9] Let R be a finite local ring of odd characteristic with maximal ideal M. Let* (V_δ, β) *be an orthogonal space of rank* $2\nu + \delta$, *where* $\nu \geq 2$ *and* $\delta \in \{0, 1, 2\}$.

1. *If R is a field, then* $\mathscr{G}^{(2)}_{O_R(V_\delta)}$ *is* $|R|^{2\nu+\delta-3}|M|$*-regular on* $\dfrac{|R|(|R|^{\nu-1} - 1)(|R|^{\nu+\delta-2} + 1)}{|R| - 1}$ *many vertices. Moreover,*

   (a) *If* $\nu = 2$, *then it is a strongly regular graph with parameters*

$$\lambda = |R|^{\delta+1} - |R| \text{ and } \mu = |R|^{\delta+1},$$

   respectively.

(b) If $\nu \geq 3$, then it is a quasi-strongly regular graph with parameters

$$\lambda = |R|^{\nu-2}(|R|^{\nu+\delta-1} - |R|^{\nu+\delta-2} - |R| + |R|^{\delta}),$$
$$c_1 = |R|^{2\nu+\delta-4}(|R| - 1) \text{ and } c_2 = |R|^{2\nu+\delta-3},$$

respectively.

2. *If R is not a field, then $\mathscr{G}^{(2)}_{O_R(V_\delta)}$ is $|R|^{2\nu+\delta-3}|M|$-regular on $\dfrac{|R|(|R|^{\nu-1} - |M|^{\nu-1})(|R|^{\nu+\delta-2} + |M|^{\nu+\delta-2})|M|}{|R| - |M|}$ many vertices. Moreover,*

   (a) If $\nu = 2$, then it is a strongly regular graph with parameters

$$\lambda = (|R|^{\delta+1} - |R||M|^{\delta})|M| \text{ and } \mu = |R|^{\delta+1}|M|,$$

   respectively.

   (b) If $\nu \geq 3$, then it is a quasi-strongly regular graph with parameters

$$\lambda = |R|^{\nu-2}(|R|^{\nu+\delta-1}|M| - |R|^{\nu+\delta-2}|M|^2 - |R||M|^{\nu+\delta-1} + |R|^{\delta}|M|^{\nu}),$$
$$c_1 = |R|^{2\nu+\delta-4}(|R| - |M|)|M| \text{ and } c_2 = |R|^{2\nu+\delta-3}|M|,$$

   respectively.

3.2 Construction of tactical configurations

Let R be a finite local ring of odd characteristic with unique maximal ideal M and residue field $\Bbbk = R/M$ and let (V_δ, β) be an orthogonal space of rank $2\nu+\delta$, $\nu \geq 2$ and $\delta \in \{0, 1, 2\}$.

3.2.1 $1\frac{1}{2}$-designs from orthogonal graphs over finite local rings

Let $\mathscr{P}$ be the set of vertices of orthogonal graph $\mathcal{G}_{O_R(V)}$ and let $\mathscr{B}$ be collection of maximal independent sets of the graph. For $x \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $x\varepsilon B$ if and only if $x \in B$. The next theorem was stuided in [3] by Feng, Zhoa and Zeng. Its results show that if R is a field, then the incidence structure $\mathscr{T}_3 = (\mathscr{P}, \mathscr{B}, \varepsilon)$ is a $1\frac{1}{2}$-design.

Theorem 3.2.1. *[3] Let $\mathbb{F}$ be the finite field of odd order q and let (V, β) be an orthogonal space of rank $2\nu+\delta$, $\nu \geq 2$ and $\delta \in \{0, 1, 2\}$. Let $\mathscr{P} = \mathcal{V}(\mathcal{G}_{O_{\mathbb{F}}(V)})$ and let $\mathscr{B}$ be a collection of maximal independent sets of $\mathcal{G}_{O_{\mathbb{F}}(V)}$. For $x \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $x\varepsilon B$ if and only if $x \in B$. Then the incidence structure $\mathscr{T}_3' = (\mathscr{P}, \mathscr{B}, \varepsilon)$ is a $1\frac{1}{2}$-design with parameters*

$$
\begin{array}{ll}
v = \frac{(q^\nu-1)(q^{\nu+\delta-1}+1)}{q-1}, & b = \prod_{i=1}^{\nu}(q^{i+\delta-1}+1), \\
k = \frac{q^\nu-1}{q-1}, & r = \prod_{i=1}^{\nu-1}(q^{i+\delta-1}+1),
\end{array}
$$

with $\alpha_1 = 1$ and $\beta_1 = 0$ if $\nu = 2$, and with

$$\alpha_1 = \frac{q^{\nu-1}-1}{q-1} \cdot \prod_{i=1}^{\nu-2}(q^{i+\delta-1}+1),\ \beta_1 = \frac{q(q^{\nu-1}-1)}{q-1} \cdot \left(\prod_{i=1}^{\nu-2}(q^{i+\delta-1}+1) - 1\right)$$

otherwise.

By Theorem 3.1.1 (2) and (3) (Lifting Theorem), $\mathscr{T}_3$ is the incidence structure obtained from duplicating $|M|^{2\nu+\delta-2}$ points and 1 blocks of $\mathscr{T}_3'$ in Theorem 3.2.1. Hence, by Theorem 1.2.4, it is a $1\frac{1}{2}$-design with parameters recorded in the following theorem.

Theorem 3.2.2. *Let R be a finite local ring of odd characteristic with maximal ideal M and let (V, β) be an orthogonal space of rank $2\nu+\delta$, $\nu \geq 2$ and $\delta \in \{0, 1, 2\}$. Let $\mathscr{P} = \mathcal{V}(\mathcal{G}_{O_R(V)})$ and let $\mathscr{B}$ be a collection of maximal independent sets of $\mathcal{G}_{O_R(V)}$. For $x \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $x\varepsilon B$ if and only if $x \in B$. Then the incidence structure $\mathscr{T}_3 = (\mathscr{P}, \mathscr{B}, \varepsilon)$ is a $1\frac{1}{2}$-design with parameters*

$$v = \frac{(|R|^{\nu}-|M|^{\nu})(|R|^{\nu+\delta-1}+|M|^{\nu+\delta-1})}{|R|-|M|}, \quad b = \frac{\prod_{i=1}^{\nu}(|R|^{i+\delta-1}+|M|^{i+\delta-1})}{|M|^{\frac{(\nu+2\delta-1)\nu}{2}}},$$

$$k = \frac{|R|^{\nu}-|M|^{\nu}}{|R|-|M|}|M|^{\nu+\delta-1}, \qquad r = \frac{\prod_{i=1}^{\nu-1}(|R|^{i+\delta-1}+|M|^{i+\delta-1})}{|M|^{\frac{(\nu+2\delta-2)(\nu-1)}{2}}},$$

with $\alpha_1 = |M|^{2+\delta}$ *and* $\beta_1 = |R|^{\delta+1}|M|^{-\delta} + |R| - |M|^{2+\delta} - \frac{|R|^2-|M|^2}{|R|-|M|}|M|^{\delta+1} - \frac{(|R|^{\delta+1}+|M|^{\delta+1})}{|M|^{(\delta+1)}} + 1$ *if* $\nu = 2$, *and*

$$\alpha_1 = \frac{(|R|^{\nu-1}-|M|^{\nu-1})\prod_{i=1}^{\nu-2}(|R|^{i+\delta-1}+|M|^{i+\delta-1})}{(|R|-|M|)|M|^{\frac{(\nu+2\delta-3)(\nu-2)-2(\nu+\delta)}{2}}},$$

$$\beta_1 = \frac{|R|^{\nu}|M|^{\delta-2}+|R|^{\nu+\delta-1}|M|^{-1}-|R|^{\nu+\delta-2}-|M|^{\nu+\delta-2}}{|R|-|M|}\frac{\prod_{i=1}^{\nu-2}(|R|^{i+\delta-1}+|M|^{i+\delta-1})}{|M|^{\frac{(\nu+2\delta-3)(\nu-2)-2(\nu+1)}{2}}}$$
$$- \frac{\prod_{i=1}^{\nu-1}(|R|^{i+\delta-1}+|M|^{i+\delta-1})}{|M|^{\frac{(\nu+2\delta-2)(\nu-1)}{2}}} - \frac{|R|^{\nu}-|M|^{\nu}}{|R|-|M|}|M|^{\nu+\delta-1} + 1$$

otherwise.

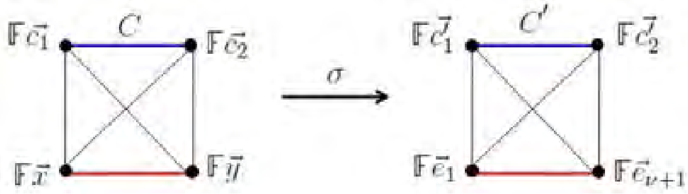
3.2.2 Other tactical configurations

In this section, we apply result on subconstituents, namely, Theorems 3.1.4 and 3.1.5, in construction other tactical configurations. They are not $1\frac{1}{2}$-designs. However, we can compute the parameters α's and β's for $\delta = 0$ or 2.

Lemma 3.2.3. *Let* $\mathbb{F}$ *be a finite field of odd order* q *and let* (V, β) *be an orthogonal space of dimension* $2\nu + \delta$, *where* $\nu \geq 2$ *and* $\delta \in \{0, 2\}$. *Let* $\mathbb{F}\vec{x}$ *and* $\mathbb{F}\vec{y}$ *be adjacent vertices in* $\mathcal{G}_{O_{\mathbb{F}}(V)}$. *Then the number of edges whose both vertices of* C *are common neighbors of* $\mathbb{F}\vec{x}$ *and* $\mathbb{F}\vec{y}$ *is given by*

$$A_{O,\delta} = \begin{cases} \frac{(q-1)(q^{\nu-1}-1)(q^{\nu}-2q^{\nu-1}+q^{\nu-2}-2q+3)q^{2\nu-4}}{2} & \text{if } \delta = 0, \\ \frac{(q-1)(q^{\nu}+1)(q^{\nu+1}-2q^{\nu}+q^{\nu-1}+2q-3)q^{2\nu-2}}{2} & \text{if } \delta = 2. \end{cases}$$

Proof. Let C be an edge such that both vertices are common neighbors of $\mathbb{F}\vec{x}$ and $\mathbb{F}\vec{y}$. Since $\mathbb{F}\vec{x}$ is adjacent to $\mathbb{F}\vec{y}$, there exists σ automorphism carries $\mathbb{F}\vec{x}$ to $\mathbb{F}\vec{e}_1$, $\mathbb{F}\vec{y}$ to $\mathbb{F}\vec{e}_{\nu+1}$, $\mathbb{F}\vec{c_1}$ to $\mathbb{F}\vec{c_1'}$ and $\mathbb{F}\vec{c_2}$ to $\mathbb{F}\vec{c_2'}$ where $\mathbb{F}\vec{c_1}$ and $\mathbb{F}\vec{c_2}$ are both vertices of C by Theorem 3.1.3 (1).



Thus, the number of edges C is the number of 3-cycle at $\mathbb{F}\vec{e}_{\nu+1}$ in $\mathcal{G}^{(1)}_{O_{\mathbb{F}}(V)}$. It is the product of the number of the common neighbors c_1' of $\mathbb{F}\vec{e}_1$ and $\mathbb{F}\vec{e}_{\nu+1}$ in $\mathcal{G}_{O_{\mathbb{F}}(V)}$ and half of the number of common neighbors of $\mathbb{F}\vec{c_1'}$ and $\mathbb{F}\vec{e}_{\nu+1}$ in $\mathcal{G}^{(1)}_{O_{\mathbb{F}}(V)}$. By Theorem 3.1.2 (1) the common neighbors of $\mathbb{F}\vec{e}_1$ and $\mathbb{F}\vec{e}_{\nu+1}$ is

$$\begin{cases} (q-1)(q^{\nu-1}-1)q^{\nu-2} & \text{if } \delta = 0, \\ (q-1)(q^{\nu}+1)q^{\nu-1} & \text{if } \delta = 2. \end{cases}$$

The number of common neighbor of $\mathbb{F}\vec{c_1'}$ and $\mathbb{F}\vec{e}_{\nu+1}$ is

$$\begin{cases} (q^{\nu}-2q^{\nu-1}+q^{\nu-2}-2q+3)q^{\nu-2} & \text{if } \delta = 0, \\ (q^{\nu+1}-2q^{\nu}+q^{\nu-1}+2q-3)q^{\nu-1} & \text{if } \delta = 2. \end{cases}$$

by Theorem 3.1.4 (1a) and (1c), respectively. Hence,

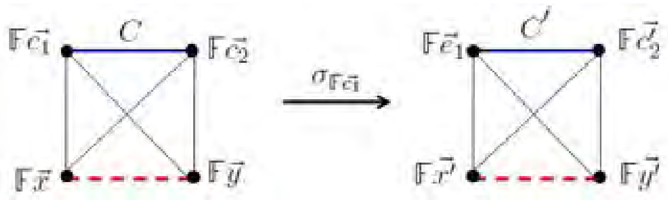
$$A_{O,\delta} = \begin{cases} \frac{(q-1)(q^{\nu-1}-1)(q^{\nu}-2q^{\nu-1}+q^{\nu-2}-2q+3)q^{2\nu-4}}{2} & \text{if } \delta = 0, \\ \frac{(q-1)(q^{\nu}+1)(q^{\nu+1}-2q^{\nu}+q^{\nu-1}+2q-3)q^{2\nu-2}}{2} & \text{if } \delta = 2. \end{cases}$$

as desired. $\square$

Lemma 3.2.4. *Let $\mathbb{F}$ be a finite field of odd order q and let (V, β) be an orthogonal space of dimension $2\nu+\delta$, where $\nu \geq 2$ and $\delta \in \{0, 1, 2\}$. Let $\mathbb{F}\vec{x}$ and $\mathbb{F}\vec{y}$ be distinct non-adjacent vertices in $\mathcal{G}_{O_{\mathbb{F}}(V)}$. The number of edge whose both vertices of C are* *common neighbor of* $\mathbb{F}\vec{x}$ *and* $\mathbb{F}\vec{y}$ *is given by*

$$C_{O,\delta} = \begin{cases} \frac{(q-1)q^{2\nu-3}(q-1)(q^{\nu-1}-q^{\nu-2}-1)q^{\nu-2}}{2} & \textit{if } \delta = 0, \\ \frac{(q-1)q^{2\nu-2}q^{2\nu-3}(2q-3)}{2} & \textit{if } \delta = 1, \\ \frac{(q-1)q^{2\nu-1}(q-1)(q^{2\nu-1}-q^{2\nu-2}+q^{\nu-1})}{2} & \textit{if } \delta = 2. \end{cases}$$

Proof. Let C be an edge such that both vertices are common neighbors of $\mathbb{F}\vec{x}$ and $\mathbb{F}\vec{y}$. Since $\mathbb{F}\vec{x}$ is non-adjacent to $\mathbb{F}\vec{y}$, there exists $\sigma_{\mathbb{F}\vec{c_1}}$ automorphism carries $\mathbb{F}\vec{x}$ to $\mathbb{F}\vec{x'}$, $\mathbb{F}\vec{y}$ to $\mathbb{F}\vec{y'}$, $\mathbb{F}\vec{c_1}$ to $\mathbb{F}\vec{e}_1$ and $\mathbb{F}\vec{c_2}$ to $\mathbb{F}\vec{c_2'}$ where $\mathbb{F}\vec{c_1}$ and $\mathbb{F}\vec{c_2}$ are both vertices of C by Theorem 3.1.1 (5).



Thus, the number of edges C is the product of the number of the common neighbors $\mathbb{F}\vec{c_1'}$ of $\mathbb{F}\vec{x'}$ and $\mathbb{F}\vec{y'}$ in $\mathcal{G}_{O_{\mathbb{F}}(V)}$ and half of the number of common neighbor of $\mathbb{F}\vec{x'}$ and $\mathbb{F}\vec{y'}$ in $\mathcal{G}^{(1)}_{O_{\mathbb{F}}(V)}$. By Theorem 3.1.2 (1) the common neighbors of $\mathbb{F}\vec{x'}$ and $\mathbb{F}\vec{y'}$ in $\mathcal{G}_{O_{\mathbb{F}}(V)}$ is

$$\begin{cases} (q-1)q^{2\nu-3} & \text{if } \delta = 0, \\ (q-1)q^{2\nu-2} & \text{if } \delta = 1, \\ (q-1)q^{2\nu-1} & \text{if } \delta = 2. \end{cases}$$

The number of common neighbors of $\mathbb{F}\vec{x'}$ and $\mathbb{F}\vec{y'}$ in $\mathcal{G}^{(1)}_{O_{\mathbb{F}}(V)}$ is

$$\begin{cases} (q-1)(q^{\nu-1}-q^{\nu-2}-1)q^{\nu-2} & \text{if } \delta = 0, \\ q^{2\nu-3}(2q-3) & \text{if } \delta = 1, \\ (q-1)(q^{2\nu-1}-q^{2\nu-2}+q^{\nu-1}) & \text{if } \delta = 2. \end{cases}$$

by Theorem 3.1.4 (1a), (1b) and (1c), respectively. Hence,

$$C_{O,\delta} = \begin{cases} \frac{(q-1)q^{2\nu-3}(q-1)(q^{\nu-1}-q^{\nu-2}-1)q^{\nu-2}}{2} & \text{if } \delta = 0, \\ \frac{(q-1)q^{2\nu-2}q^{2\nu-3}(2q-3)}{2} & \text{if } \delta = 1, \\ \frac{(q-1)q^{2\nu-1}(q-1)(q^{2\nu-1}-q^{2\nu-2}+q^{\nu-1})}{2} & \text{if } \delta = 2. \end{cases}$$

as desired. □

Remark. The case $\delta = 1$ involves solving a more complicated equation in order to determine $A_{O,\delta}$. Therefore, we can compute α's and β's only for $\delta = 0$ or 2.

Let R be a finite local ring of odd characteristic with unique maximal ideal M and residue field $\Bbbk = R/M$ and let (V_δ, β) be an orthogonal space of rank $2\nu + \delta$, $\nu \geq 2$ and $\delta \in \{0, 2\}$. Let $\mathscr{P} = \mathcal{V}(\mathcal{G}_{O_R(V)})$ and $\mathscr{B} = \mathcal{E}(\mathcal{G}_{O_R(V)})$. For $x \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $x \varepsilon B$ if and only if x is a common neighbor of B. If R is a field, then the incidence structure $\mathscr{T}_{4,\delta} = (\mathscr{P}, \mathscr{B}, \varepsilon)$ was stuided in the next two theorems.

Theorem 3.2.5. *Let $\mathbb{F}$ be a finite field of odd order q and let (V, β) an orthogonal space of dimension 2ν, $\nu \geq 2$. Let $\mathscr{P} = \mathcal{V}(\mathcal{G}_{O_{\mathbb{F}}(V)})$ and $\mathscr{B} = \mathcal{E}(\mathcal{G}_{O_{\mathbb{F}}(V)})$. For $\mathbb{F}\vec{x} \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $\mathbb{F}\vec{x} \varepsilon B$ if and only if $\mathbb{F}\vec{x}$ is a common neighbor of B. Then the incidence structure $\mathscr{T}'_{4,0} = (\mathscr{P}, \mathscr{B}, \varepsilon)$ is a β-strongly tactical configuration with parameters*

$$\begin{aligned}
v &= \tfrac{(q^\nu-1)(q^{\nu-1}+1)}{q-1},\\
b &= \tfrac{(q^\nu-1)(q^{\nu-1}+1)q^{2\nu-2}}{(q-1)2},\\
k &= q^{2\nu-2}-q^{2\nu-3}-q^{\nu-1}+q^{\nu-2},\\
r &= \tfrac{q^{2\nu-2}(q^{2\nu-2}-q^{2\nu-3}-q^{\nu-1}+q^{\nu-2})}{2},\\
\alpha_1 &= (q-1)(q^{\nu-1}-q^{\nu-2}-1)q^{\nu-2}\tfrac{(q-1)(q^{\nu-1}-1)(q^\nu-2q^{\nu-1}+q^{\nu-2}-2q+3)q^{2\nu-4}}{2}+\\
&\quad (q-1)q^{2\nu-4}\tfrac{(q-1)q^{2\nu-3}(q-1)(q^{\nu-1}-q^{\nu-2}-1)q^{\nu-2}}{2},\\
\alpha_2 &= (q^\nu-2q^{\nu-1}+q^{\nu-2})q^{\nu-2}\tfrac{(q-1)(q^{\nu-1}-1)(q^\nu-2q^{\nu-1}+q^{\nu-2}-2q+3)q^{2\nu-4}}{2}+\\
&\quad (q^{\nu-1}-q^{\nu-2}-q+1)q^{\nu-2}\tfrac{(q-1)q^{2\nu-3}(q-1)(q^{\nu-1}-q^{\nu-2}-1)q^{\nu-2}}{2},\\
\alpha_3 &= (q^{2\nu-2}-q^{2\nu-3}-q^{\nu-1}+q^{\nu-2})\tfrac{(q-1)(q^{\nu-1}-1)(q^\nu-2q^{\nu-1}+q^{\nu-2}-2q+3)q^{2\nu-4}}{2},\\
\beta_1 &= (q^\nu-2q^{\nu-1}+q^{\nu-2}-2q+3)q^{\nu-2}\tfrac{(q-1)(q^{\nu-1}-1)(q^\nu-2q^{\nu-1}+q^{\nu-2}-2q+3)q^{2\nu-4}-2}{2}+\\
&\quad ((q^{\nu-1}-q^{\nu-2}+q-2)q^{\nu-2}-1)\tfrac{(q-1)q^{2\nu-3}(q-1)(q^{\nu-1}-q^{\nu-2}-1)q^{\nu-2}-2}{2}.
\end{aligned}$$

Proof. Since $\mathcal{G}_{O_{\mathbb{F}}(V)}$ is strongly regular and vertex transitive, $\mathscr{T}'_{4,0}$ is a tactical configuration with parameters $(\frac{(q^\nu-1)(q^{\nu-1}+1)}{q-1}, \frac{(q^\nu-1)(q^{\nu-1}+1)q^{2\nu-2}}{(q-1)2}, q^{2\nu-2}-q^{2\nu-3}-q^{\nu-1}+q^{\nu-2}, \frac{q^{2\nu-2}(q^{2\nu-2}-q^{2\nu-3}-q^{\nu-1}+q^{\nu-2})}{2})$. Let $A_{O,0}$ and $C_{O,0}$ be given in Lemmas 3.2.3 and 3.2.4, respectively. Let $R\vec{x} \in \mathscr{P}$ and $B \in \mathscr{B}$ such that B is with vertices $\mathbb{F}\vec{y_1}$ and $\mathbb{F}\vec{y_2}$.

Case 1: $(\mathbb{F}\vec{x}, B)$ is an antiflag.

Case 1.1: $\mathbb{F}\vec{x}$ is adjacent to $\mathbb{F}\vec{y_1}$ but $\mathbb{F}\vec{x}$ is non-adjacent to $\mathbb{F}\vec{y_2}$. Then $(\vec{x}, \vec{y_1}, \vec{y_2}) \approx (\vec{e}_{\nu+1}, \vec{e}_1, \vec{e}_2 + \vec{e}_{\nu+1})$ by Theorem 3.1.3 (4). Thus,

$$\begin{aligned}
s(\mathbb{F}\vec{x}, B) &= \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{x} \text{ is adjacent to } \mathbb{F}\vec{y}} \lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} + \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{x} \text{ is non-adjacent to } \mathbb{F}\vec{y}} \lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}}\\
&= (q-1)(q^{\nu-1}-q^{\nu-2}-1)q^{\nu-2}A_{O,0} + \Big(q^{2\nu-2}-q^{2\nu-3}-q^{\nu-1}+q^{\nu-2}-\\
&\quad (q-1)(q^{\nu-1}-q^{\nu-2}-1)q^{\nu-2}\Big)C_{O,0}
\end{aligned}$$

by Theorem 3.1.4 (1a).

Case 1.2: $\mathbb{F}\vec{x}$ is non-adjacent to $\mathbb{F}\vec{y_1}$ and $\mathbb{F}\vec{y_2}$. Then $(\vec{x}, \vec{y_1}, \vec{y_2}) \approx (\vec{e}_1, \vec{e}_2, \vec{e}_{\nu+2})$ by Theorem 3.1.3 (3). Thus,

$$\begin{aligned}
s(\mathbb{F}\vec{x}, B) &= \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{x} \text{ is adjacent to } \mathbb{F}\vec{y}} \lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} + \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{x} \text{ is non-adjacent to } \mathbb{F}\vec{y}} \lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}}\\
&= (q^{2\nu-2}-q^{2\nu-3}-q^{\nu-1}+q^{\nu-2}-(q-1)(q^{\nu-2}-1)q^{\nu-2})A_{O,0}+\\
&\quad (q^{\nu-1}-q^{\nu-2}-q+1)q^{\nu-2}C_{O,0}
\end{aligned}$$

by Theorem 3.1.5 (1).

Case 1.3: $\mathbb{F}\vec{x} = \mathbb{F}\vec{y_1}$. Then $(\vec{x}, \vec{y_2}) \approx (\vec{e}_1, \vec{e}_{\nu+1})$ by Theorem 3.1.3 (1). Thus,

$$\begin{aligned} s(\mathbb{F}\vec{x},B) &= \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{x} \text{ is adjacent to } \mathbb{F}\vec{y}} \lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} + \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{x} \text{ is non-adjacent to } \mathbb{F}\vec{y}} \lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} \\ &= q^{2\nu-2} - q^{2\nu-3} - q^{\nu-1} + q^{\nu-2} A_{O,0} + \Big(q^{2\nu-2} - q^{2\nu-3} - q^{\nu-1} + q^{\nu-2} - \\ &\quad (q^{2\nu-2} - q^{2\nu-3} - q^{\nu-1} + q^{\nu-2}) \Big) C_{O,0} \end{aligned}$$

by Theorem 3.1.2 (1).

Case 2: $(\mathbb{F}\vec{x}, B)$ is a flag. Then $(\vec{y_1}, \vec{y_2}, \vec{x}) \approx (\vec{e}_1, \vec{e}_{\nu+1}, \vec{e}_1 + \vec{e}_2 + a\vec{e}_{\nu+1} - a\vec{e}_{\nu+2})$ where $a \in \mathbb{F}^\times$ by Theorem 3.1.3 (5). Thus,

$$\begin{aligned} s(\mathbb{F}\vec{x},B) &= \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{y}\neq\mathbb{F}\vec{x}, \mathbb{F}\vec{x} \text{ is adjacent to } \mathbb{F}\vec{y}} (\lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} - 1) + \\ &\quad \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{y}\neq\mathbb{F}\vec{x}, \mathbb{F}\vec{x} \text{ is non-adjacent to } \mathbb{F}\vec{y}} (\lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} - 1) \\ &= (q^{\nu} - 2q^{\nu-1} + q^{\nu-2} - 2q + 3) q^{\nu-2} (A_{O,0} - 1) + \Big(q^{2\nu-2} - q^{2\nu-3} - q^{\nu-1} + q^{\nu-2} - 1 - \\ &\quad (q^{\nu} - 2q^{\nu-1} + q^{\nu-2} - 2q + 3) q^{\nu-2} \Big) (C_{O,0} - 1) \end{aligned}$$

by Theorem 3.1.4 (1a). $\square$

Theorem 3.2.6. *Let $\mathbb{F}$ be a finite field of odd order q and let (V, β) an orthogonal space of dimension $2\nu + 2$, $\nu \geq 2$. Let $\mathscr{P} = \mathcal{V}(\mathcal{G}_{O_{\mathbb{F}}(V)})$ and $\mathscr{B} = \mathcal{E}(\mathcal{G}_{O_{\mathbb{F}}(V)})$. For $\mathbb{F}\vec{x} \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $\mathbb{F}\vec{x}\varepsilon B$ if and only if $\mathbb{F}\vec{x}$ is a common neighbor of B. Then the incidence structure $\mathscr{T}'_{4,2} = (\mathscr{P}, \mathscr{B}, \varepsilon)$ is a β-strongly tactical configuration with parameters*

$$\begin{aligned} v &= \tfrac{(q^{\nu}-1)(q^{\nu+1}+1)}{q-1}, \\ b &= \tfrac{q^{2\nu}(q^{\nu}-1)(q^{\nu+1}+1)}{2(q-1)}, \\ k &= q^{2\nu} - q^{2\nu-1} - q^{\nu-1} + q^{\nu}, \\ r &= \tfrac{q^{2\nu}(q^{2\nu}-q^{2\nu-1}-q^{\nu-1}+q^{\nu})}{2}, \\ \alpha_1 &= (q^{2\nu} - 2q^{2\nu-1} + q^{2\nu-2} + q^{\nu} + q^{\nu-1}) \tfrac{(q-1)(q^{\nu}+1)(q^{\nu+1}-2q^{\nu}+q^{\nu-1}+2q-3)q^{2\nu-2}}{2} + \\ &\quad (q^{2\nu-1} - q^{2\nu-2} - 2q^{\nu-1}) \tfrac{(q-1)q^{2\nu-1}(q-1)(q^{2\nu-1}-q^{2\nu-2}+q^{\nu-1})}{2}, \\ \alpha_2 &= (q^{2\nu} - 2q^{2\nu-1} + q^{2\nu-2} - q^{\nu-1} + q^{\nu-2}) \tfrac{(q-1)q^{2\nu-1}(q-1)(q^{2\nu-1}-q^{2\nu-2}+q^{\nu-1})}{2} + \\ &\quad (q^{2\nu-1} - q^{2\nu-2} + q^{\nu} - q^{\nu-2}) \tfrac{(q-1)(q^{\nu}+1)(q^{\nu+1}-2q^{\nu}+q^{\nu-1}+2q-3)q^{2\nu-2}}{2}, \\ \alpha_3 &= (q^{2\nu} - q^{2\nu-1} - q^{\nu-1} + q^{\nu}) \tfrac{(q-1)(q^{\nu}+1)(q^{\nu+1}-2q^{\nu}+q^{\nu-1}+2q-3)q^{2\nu-2}}{2}, \\ \beta_1 &= (q^{2\nu} - 2q^{2\nu-1} + q^{2\nu-2} + 2q^{\nu} - 3q^{\nu-1}) \tfrac{(q-1)(q^{\nu}+1)(q^{\nu+1}-2q^{\nu}+q^{\nu-1}+2q-3)q^{2\nu-2}-2}{2} + \\ &\quad (q^{2\nu-1} - q^{2\nu-2} - q^{\nu} + 2q^{\nu-1}) \tfrac{(q-1)q^{2\nu-1}(q-1)(q^{2\nu-1}-q^{2\nu-2}+q^{\nu-1})-2}{2}. \end{aligned}$$

Proof. Since $\mathcal{G}_{O_{\mathbb{F}}(V)}$ is strongly regular and vertex transitive, $\mathscr{T}'_{4,2}$ is a tactical configuration with parameters $(\frac{(q^{\nu}-1)(q^{\nu+1}+1)}{q-1}, \frac{q^{2\nu}(q^{\nu}-1)(q^{\nu+1}+1)}{2(q-1)}, q^{2\nu} - q^{2\nu-1} - q^{\nu-1} +$

$q^{\nu}, \frac{q^{2\nu}(q^{2\nu}-q^{2\nu-1}-q^{\nu-1}+q^{\nu})}{2}$). Let $A_{O,2}$ and $C_{O,2}$ be given in Lemmas 3.2.3 and 3.2.4, respectively. Let $\mathbb{F}\vec{x} \in \mathscr{P}$ and $B \in \mathscr{B}$ such that B is with vertices $\mathbb{F}\vec{y_1}$ and $\mathbb{F}\vec{y_2}$.

Case 1: $(\mathbb{F}\vec{x}, B)$ is an antiflag.

Case 1.1: $\mathbb{F}\vec{x}$ is adjacent to $\mathbb{F}\vec{y_1}$ but $\mathbb{F}\vec{x}$ is non-adjacent to $\mathbb{F}\vec{y_2}$. Then $(\vec{x}, \vec{y_1}, \vec{y_2}) \approx (\vec{e}_{\nu+1}, \vec{e}_1, \vec{e}_2 + \vec{e}_{\nu+1})$ by Theorem 3.1.3 (4). Thus,

$$\begin{aligned} s(\mathbb{F}\vec{x}, B) &= \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{x} \text{ is adjacent to } \mathbb{F}\vec{y}} \lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} + \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{x} \text{ is non-adjacent to } \mathbb{F}\vec{y}} \lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} \\ &= q^{\nu-1}(q-1)(q^{\nu}-q^{\nu-1}+1)A_{O,0} + \Big(q^{2\nu}-q^{2\nu-1}-q^{\nu-1}+q^{\nu}- \\ &\quad q^{\nu-1}(q-1)(q^{\nu}-q^{\nu-1}+1)\Big)C_{O,0} \end{aligned}$$

by Theorem 3.1.4 (1c).

Case 1.2: $\mathbb{F}\vec{x}$ is non-adjacent to $\mathbb{F}\vec{y_1}$ and $\mathbb{F}\vec{y_2}$. Then $(\vec{x}, \vec{y_1}, \vec{y_2}) \approx (\vec{e}_1, \vec{e}_2, \vec{e}_{\nu+2})$ by Theorem 3.1.3 (3). Thus,

$$\begin{aligned} s(\mathbb{F}\vec{x}, B) &= \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{x} \text{ is adjacent to } \mathbb{F}\vec{y}} \lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} + \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{x} \text{ is non-adjacent to } \mathbb{F}\vec{y}} \lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} \\ &= (q^{2\nu}-q^{2\nu-1}-q^{\nu-1}+q^{\nu}-(q^{\nu+1}-q^{\nu}+q^2-q)q^{\nu-2})A_{O,0}+ \\ &\quad (q^{\nu+1}-q^{\nu}+q^2-q)q^{\nu-2}C_{O,0} \end{aligned}$$

by Theorem 3.1.5 (1).

Case 1.3: $\mathbb{F}\vec{x} = \mathbb{F}\vec{y_1}$. Then $(\vec{x}, \vec{y_2}) \approx (\vec{e}_1, \vec{e}_{\nu+1})$ by Theorem 3.1.3 (1). Thus,

$$\begin{aligned} s(\mathbb{F}\vec{x}, B) &= \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{x} \text{ is adjacent to } \mathbb{F}\vec{y}} \lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} + \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{x} \text{ is non-adjacent to } \mathbb{F}\vec{y}} \lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}} \\ &= (q^{2\nu}-q^{2\nu-1}-q^{\nu-1}+q^{\nu})A_{O,0} + \Big(q^{2\nu}-q^{2\nu-1}-q^{\nu-1}+q^{\nu}- \\ &\quad (q^{2\nu}-q^{2\nu-1}-q^{\nu-1}+q^{\nu})\Big)C_{O,0} \end{aligned}$$

by Theorem 3.1.2 (1).

Case 2: $(\mathbb{F}\vec{x}, B)$ is a flag. Then $(\vec{y_1}, \vec{y_2}, \vec{x}) \approx (\vec{e}_1, \vec{e}_{\nu+1}, \vec{e}_1 + \vec{e}_2 + a\vec{e}_{\nu+1} - a\vec{e}_{\nu+2})$ where $a \in \mathbb{F}^{\times}$ by Theorem 3.1.3 (5). Thus,

$$\begin{aligned} s(\mathbb{F}\vec{x}, B) &= \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{y}\neq\mathbb{F}\vec{x}, \mathbb{F}\vec{x} \text{ is adjacent to } \mathbb{F}\vec{y}} (\lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}}-1)+ \\ &\quad \sum_{\mathbb{F}\vec{y}\varepsilon B, \mathbb{F}\vec{y}\neq\mathbb{F}\vec{x}, \mathbb{F}\vec{x} \text{ is non-adjacent to } \mathbb{F}\vec{y}} (\lambda_{\mathbb{F}\vec{x}\mathbb{F}\vec{y}}-1) \\ &= q^{\nu-1}(q^{\nu+1}-2q^{\nu}+q^{\nu-1}+2q-3)(A_{O,0}-1) + \Big(q^{2\nu}-q^{2\nu-1}-q^{\nu-1}+q^{\nu}-1- \\ &\quad q^{\nu-1}(q^{\nu+1}-2q^{\nu}+q^{\nu-1}+2q-3)\Big)\Big(C_{O,0}-1) \end{aligned}$$

by Theorem 3.1.4 (1c). □

By Theorem 3.1.1 (2) and (3) (Lifting Theorem), $\mathscr{T}_{4,\delta}$ is the incidence structure obtained from duplicating $|M|^{2\nu+\delta-2}$ points and $(|M|^{2\nu+\delta-2})^2$ blocks of $\mathscr{T}'_{4,\delta}$ in

Theorems 3.2.5 and 3.2.6. Hence, by Theorem 1.2.4, it is a β-strongly tactical configuration with parameters recorded in the following two theorems.

Theorem 3.2.7. *Let R be a finite local ring of odd characteristic with maximal ideal M and let (V,β) be an orthogonal space of rank 2ν, $\nu \geq 2$. Let $\mathscr{P} = \mathcal{V}(\mathcal{G}_{O_R(V)})$ and $\mathscr{B} = \mathcal{E}(\mathcal{G}_{O_R(V)})$. For $R\vec{x} \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $R\vec{x}\varepsilon B$ if and only if $R\vec{x}$ is a common neighbor of B. Then the incidence structure $\mathscr{T}_{4,0} = (\mathscr{P},\mathscr{B},\varepsilon)$ is a β-strongly tactical configuration with parameters*

$$
\begin{aligned}
v &= \tfrac{(|R|^{\nu}-|M|^{\nu})(|R|^{\nu-1}+|M|^{\nu-1})}{|R|-|M|},\\
b &= \tfrac{(|R|^{\nu}-|M|^{\nu})(|R|^{\nu-1}+|M|^{\nu-1})|R|^{2\nu-2}}{(|R|-|M|)2},\\
k &= (|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^{\nu}),\\
r &= \tfrac{|R|^{2\nu-2}(|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^{\nu})}{2},\\
\alpha_1 &= \Big((|R|-|M|)(|R|^{\nu-1}-|R|^{\nu-2}|M|-|M|^{\nu-1})|R|^{\nu-2}\\
&\quad\times\tfrac{(|R|-|M|)(|R|^{\nu-1}-|M|^{\nu-1})(|R|^{\nu}-2|R|^{\nu-1}|M|+|R|^{\nu-2}|M|^{\nu+2}-2|R||M|^{\nu-1}+3|M|^{\nu})|R|^{2\nu-4}}{2}\Big)+\\
&\quad(|R|-|M|)|R|^{2\nu-4}|M|\tfrac{(|R|-|M|)|R|^{2\nu-3}(|R|-|M|)(|R|^{\nu-1}-|R|^{\nu-2}|M|-|M|^{\nu-1})|R|^{\nu-2}}{2},\\
\alpha_2 &= \Big((|R|^{\nu}-2|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^{\nu})|R|^{\nu-2}\\
&\quad\times\tfrac{(|R|-|M|)(|R|^{\nu-1}-|M|^{\nu-1})(|R|^{\nu}-2|R|^{\nu-1}|M|+|R|^{\nu-2}|M|^{\nu+2}-2|R||M|^{\nu-1}+3|M|^{\nu})|R|^{2\nu-4}}{2}\Big)+\\
&\quad\Big((|R|^{\nu-1}-|R|^{\nu-2}|M|-|R||M|^{\nu-2}+|M|^{\nu-1})|R|^{\nu-2}|M|\\
&\quad\times\tfrac{(|R|-|M|)|R|^{2\nu-3}(|R|-|M|)(|R|^{\nu-1}-|R|^{\nu-2}|M|-|M|^{\nu-1})|R|^{\nu-2}}{2}\Big),\\
\alpha_3 &= (|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^{\nu})\\
&\quad\times\tfrac{(|R|-|M|)(|R|^{\nu-1}-|M|^{\nu-1})(|R|^{\nu}-2|R|^{\nu-1}|M|+|R|^{\nu-2}|M|^{\nu+2}-2|R||M|^{\nu-1}+3|M|^{\nu})|R|^{2\nu-4}}{2},\\
\beta_1 &= \Big((|R|^{\nu}-2|R|^{\nu-1}|M|+|R|^{\nu-2}|M|^{2}-2|R||M|^{\nu-1}+3|M|^{\nu})|R|^{\nu-2}\\
&\quad\times\tfrac{(|R|-|M|)(|R|^{\nu-1}-|M|^{\nu-1})(|R|^{\nu}-2|R|^{\nu-1}|M|+|R|^{\nu-2}|M|^{\nu+2}-2|R||M|^{\nu-1}+3|M|^{\nu})|R|^{2\nu-4}-2|M|^{4\nu-4}}{2}\Big)+\\
&\quad\Big(((|R|^{\nu-1}-|R|^{\nu-2}|M|+|R||M|^{\nu-2}-2|M|^{\nu-1})|R|^{\nu-2}|M|-|M|^{2\nu-2})\\
&\quad\times\tfrac{(|R|-|M|)|R|^{2\nu-3}(|R|-|M|)(|R|^{\nu-1}-|R|^{\nu-2}|M|-|M|^{\nu-1})|R|^{\nu-2}-2|M|^{4\nu-4}}{2}\Big)+\\
&\quad\Big((|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^{\nu})|M|^{4\nu-4}+\\
&\quad|M|^{2\nu-2}\tfrac{|R|^{2\nu-2}(|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^{\nu})}{2}-|M|^{6\nu-6}\Big)\\
&\quad-(|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^{\nu})\\
&\quad-\tfrac{|R|^{2\nu-2}(|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^{\nu})}{2}+1.
\end{aligned}
$$

Theorem 3.2.8. *Let R be a finite local ring of odd characteristic with maximal ideal M and let (V,β) be an orthogonal space of rank $2\nu+2$, $\nu \geq 2$. Let $\mathscr{P} = \mathcal{V}(\mathcal{G}_{O_R(V)})$ and $\mathscr{B} = \mathcal{E}(\mathcal{G}_{O_R(V)})$. For $R\vec{x} \in \mathscr{P}$ and $B \in \mathscr{B}$, we define $R\vec{x}\varepsilon B$ if and only if $R\vec{x}$ is a common neighbor of B. Then the incidence structure $\mathscr{T}_{4,2} = (\mathscr{P},\mathscr{B},\varepsilon)$ is a β-strongly tactical configuration with parameters*

$$
\begin{aligned}
v &= \tfrac{(|R|^{\nu}-|M|)(|R|^{\nu+1}+|M|^{\nu+1})}{|R|-|M|},\\
b &= \tfrac{|R|^{2\nu}(|R|^{\nu}-|M|^{\nu})(|R|^{\nu+1}+|M|^{\nu+1})}{2(|R|-|M|)},\\
k &= (|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu}),\\
r &= \tfrac{|R|^{2\nu}(|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu})}{2},\\
\alpha_1 &= \Big((|R|^{2\nu}-2|R|^{2\nu-1}|M|+|R|^{2\nu-2}|M|^{2}+|R|^{\nu}|M|^{\nu}+|R|^{\nu-1}|M|^{\nu+1})\\
&\quad \times\tfrac{(|R|-|M|)(|R|^{\nu}+|M|^{\nu})(|R|^{\nu+1}-2|R|^{\nu}|M|+|R|^{\nu-1}|M|^{2}+2|R||M|^{\nu}-3|M|^{\nu+1})|R|^{2\nu-2}}{2}\Big)+\\
&\quad \Big((|R|^{2\nu-1}|M|-|R|^{2\nu-2}|M|^{2}-2|R|^{\nu-1}|M|^{\nu+1})\\
&\quad \times\tfrac{(|R|-|M|)|R|^{2\nu-1}(|R|-|M|)(|R|^{2\nu-1}-|R|^{2\nu-2}|M|+|R|^{\nu-1})|M|}{2}\Big),\\
\alpha_2 &= \Big((|R|^{2\nu}-2|R|^{2\nu-1}|M|+|R|^{2\nu-2}|M|^{2}-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu-2}|M|^{\nu+2})\\
&\quad \times\tfrac{(|R|-|M|)(|R|^{\nu}+|M|^{\nu})(|R|^{\nu+1}-2|R|^{\nu}|M|+|R|^{\nu-1}|M|^{2}+2|R||M|^{\nu}-3|M|^{\nu+1})|R|^{2\nu-2}}{2}\Big)+\\
&\quad \Big((|R|^{2\nu-1}|M|-|R|^{2\nu-2}|M|^{2}+|R|^{\nu}|M|^{\nu}-|R|^{\nu-2}|M|^{\nu+2})\\
&\quad \times\tfrac{(|R|-|M|)|R|^{2\nu-1}(|R|-|M|)(|R|^{2\nu-1}-|R|^{2\nu-2}|M|+|R|^{\nu-1})|M|}{2}\Big),\\
\alpha_3 &= (|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu})\\
&\quad \times\tfrac{(|R|-|M|)(|R|^{\nu}+|M|^{\nu})(|R|^{\nu+1}-2|R|^{\nu}|M|+|R|^{\nu-1}|M|^{2}+2|R||M|^{\nu}-3|M|^{\nu+1})|R|^{2\nu-2}}{2},\\
\beta_1 &= \Big((|R|^{2\nu}-2|R|^{2\nu-1}|M|+|R|^{2\nu-2}|M|^{2}+2|R|^{\nu}|M|^{\nu}-3|R|^{\nu-1}|M|^{\nu+1})\\
&\quad \times\tfrac{(|R|-|M|)(|R|^{\nu}+|M|^{\nu})(|R|^{\nu+1}-2|R|^{\nu}|M|+|R|^{\nu-1}|M|^{2}+2|R||M|^{\nu}-3|M|^{\nu+1})|R|^{2\nu-2}-2|M|^{4\nu}}{2}\Big)+\\
&\quad \Big((|R|^{2\nu-1}|M|-|R|^{2\nu-2}|M|^{2}-|R|^{\nu}|M|^{\nu}+2|R|^{\nu-1}|M|^{\nu+1})\\
&\quad \times\tfrac{(|R|-|M|)|R|^{2\nu-1}(|R|-|M|)(|R|^{2\nu-1}-|R|^{2\nu-2}|M|+|R|^{\nu-1})|M|-2|M|^{4\nu}}{2}\Big)+\\
&\quad |M|^{4\nu}(|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu})+\\
&\quad |M|^{2\nu}\tfrac{|R|^{2\nu}(|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu})}{2}-|M|^{6\nu}\\
&\quad -(|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu})\\
&\quad -\tfrac{|R|^{2\nu}(|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu})}{2}+1.
\end{aligned}
$$

CHAPTER IV
DIRECTED REGULAR GRAPHS

In this Chapter, we introduce directed graph and construct directed regular graph from tactical configuration.

4.1 Directed graphs

A **directed graph** is a graph, where the edges have a direction associated with them. In formal terms, a directed graph is an ordered pair $\Gamma = (\mathcal{V}(\Gamma), \mathcal{E}(\Gamma))$ where $\mathcal{V}(\Gamma)$ is a set whose elements are called **vertices** and $\mathcal{E}(\Gamma)$ is a set of ordered pairs of vertices are called **directed edges**. For any $x, y \in V(\Gamma)$, we say that x is adjacent to y, denoted by $x \to y$, that there is directed edge from x to y, and x is not adjacent to y otherwise, which denoted by $x \nrightarrow y$. If $x \to y$, then y is **out-neighbor** of x and x is **in-neighbor** of y. A directed graph is called a **simple directed graph** if it has no loops or multiple edges. All graphs considered in this thesis will be finite directed simple graphs.

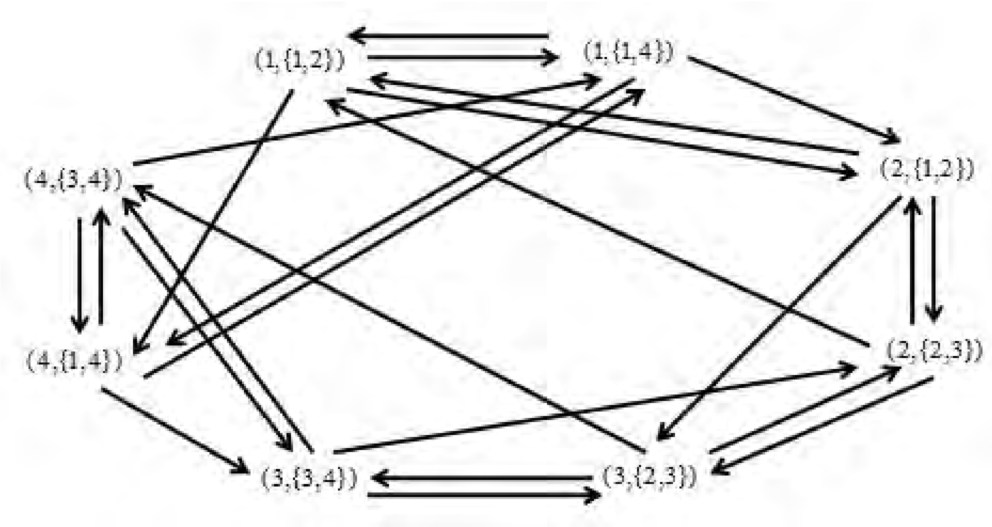
A **directed regular graph** with parameters $(n; k)$ is a finite directed graph $\Gamma = (\mathcal{V}(\Gamma), \mathcal{E}(\Gamma))$ such that $|\mathcal{V}(\Gamma)| = n$ and every vertex has k out-neighbors and k in-neighbors. For every vertices $x, y \in \mathcal{V}(\Gamma)$, the number of vertices z such that $x \to z$ and $z \to y$ is denoted by $t(x, y)$.

Let $\{t(x, x) : x \in \mathcal{V}(\Gamma)\} = \{t_1, \ldots, t_{t'}\}$. Let $\{t(x, y) : x, y \in \mathcal{V}(\Gamma), x \neq y \text{ and } x \to y\} = \{\lambda_1, \ldots, \lambda_{\lambda'}\}$. Let $\{t(x, y) : x, y \in \mathcal{V}(\Gamma), x \neq y \text{ and } x \nrightarrow y\} = \{\mu_1, \ldots, \mu_{\mu'}\}$. We may write parameters as $(n, k; t_1, \ldots, t_{t'}; \lambda_1, \ldots, \lambda_{\lambda'}; \mu_1, \ldots, \mu_{\mu'})$. If $t' = 1$, then Γ is called **directed t-strongly regular graph**. If $\lambda' = 1$, then Γ is called **directed λ-strongly regular graph**. If $\mu' = 1$, then Γ is called **directed μ-strongly regular graph**. If $t' = \lambda' = \mu' = 1$, then Γ is called **directed**

strongly regular graph.

Example 4.1.1. *Let* $\mathscr{P} = \{1,2,3,4\}$ *and* $\mathscr{B} = \{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}$. *For* $x \in \mathscr{P}$ *and* $B \in \mathscr{B}$, *we defined* $x\varepsilon B$ *if and only if* x *is a* B. *Then the incidence structure* $\mathscr{T} = (\mathscr{P},\mathscr{B},\varepsilon)$ *is a* $1\frac{1}{2}$*-design with parameters* $(4,4,2,2;1;0)$. *Let* $\Gamma = \Gamma(\mathscr{T})$ *be the directed graph defined by*

$$V(\Gamma) = \{(x,B) \in \mathscr{P} \times \mathscr{B} : (x,B) \in \varepsilon\} \text{ and}$$
$$(x,B) \to (y,C) \text{ if and only if } (x,B) \neq (y,C) \text{ and } (x,C) \in \varepsilon.$$



Then Γ *is a directed strongly regular graph with parameter* $(8,3;2;1;1)$.

4.2 Directed regular graph from tactical configuration

Let $\mathscr{T} = (\mathscr{P},\mathscr{B},\varepsilon)$ be a tactical configuration. We define two directed graphs.

1. $\Gamma = \Gamma(\mathscr{T})$ is the directed graph defined by $V(\Gamma) = \{(x,B) \in \mathscr{P} \times \mathscr{B} : (x,B) \notin \varepsilon\}$ and $(x,B) \to (y,C)$ if and only if $(x,C) \in \varepsilon$.

2. $\Gamma' = \Gamma'(\mathscr{T})$ is the directed graph defined by $V(\Gamma') = \{(x,B) \in \mathscr{P} \times \mathscr{B} : (x,B) \in \varepsilon\}$ and $(x,B) \to (y,C)$ if and only if $(x,B) \neq (y,C)$ and $(x,C) \in \varepsilon$.

Theorems 4.2.1 and 4.2.2 were proved by Brouwer, Olmez and Song in [1].

Theorem 4.2.1. *[1] Let $\mathscr{T} = (\mathscr{P}, \mathscr{B}, \varepsilon)$ be a tactical configuration with parameters (v, b, k_0, r). Let $\Gamma = \Gamma(\mathscr{T})$ be the directed graph defined by $V(\Gamma) = \{(x, B) \in \mathscr{P} \times \mathscr{B} : (x, B) \notin \varepsilon\}$ and $(x, B) \to (y, C)$ if and only if $(x, C) \in \varepsilon$. Then Γ is a directed strongly regular graph with parameters*

$n = b(v - k_0),$

$k = r(v - k_0),$

$t_1 = \mu_1 = k_0 r - \alpha_1,$

$\lambda_1 = k_0 r - (\beta_1 + r + k_0 - 1)$

if and only if $\mathscr{T}$ is a $1\frac{1}{2}$-design with parameters $(v, b, k_0, r; \alpha_1; \beta_1)$.

Theorem 4.2.2. *[1] Let $\mathscr{T} = (\mathscr{P}, \mathscr{B}, \varepsilon)$ be a tactical configuration with parameters (v, b, k_0, r). Let $\Gamma' = \Gamma'(\mathscr{T})$ be the directed graph defined by $V(\Gamma') = \{(x, B) \in \mathscr{P} \times \mathscr{B} : (x, B) \in \varepsilon\}$ and $(x, B) \to (y, C)$ if and only if $(x, B) \neq (y, C)$ and $(x, C) \in \varepsilon$. Then Γ' is a directed strongly regular graph with parameters*

$n = vr,$

$k = rk_0 - 1,$

$t_1 = \beta_1 + r + k_0 - 2,$

$\lambda_1 = \beta_1 + r + k_0 - 3,$

$\mu_1 = \alpha_1,$

if and only if $\mathscr{T}$ is a $1\frac{1}{2}$-design with parameters $(v, b, k_0, r; \alpha_1; \beta_1)$.

Remark. Let $\mathscr{T} = (\mathscr{P}, \mathscr{B}, \varepsilon)$ be a tactical configuration with parameters (v, b, k_0, r). Γ and Γ' are a directed strongly regular graph in Theorems 4.2.1 and 4.2.2, respectively. Thus, $\varepsilon \neq \varnothing, \mathscr{P} \times \mathscr{B}$. So $1 \leq k_0 \leq v - 1$.

In this section, we apply the definition of Brouwer, Olmez and Song's directed graphs to tactical configurations. We obtain regular directed graphs which may not be directed strongly regular graph. By Theorems 4.2.1 and 4.2.2, it is a directed strongly regular graphs if and only if a tactical configuration is a $1\frac{1}{2}$-design. We use the tactical configuration obtained in the previous chapters to provide some examples of this directed graphs. We can compute all parameters for these graphs.

Lemma 4.2.3. *Let* $\mathscr{T} = (\mathscr{P}, \mathscr{B}, \varepsilon)$ *be a tactical configuration with parameters* (v, b, k, r). *For every point* $x \in \mathscr{P}$ *and every block* $B \in \mathscr{B}$, *the number of flags* (y, C) *such that* $y \varepsilon B$ *and* $x \varepsilon C$ *is denoted by* $s'(x, B)$. *Then for every point* $x \in \mathscr{P}$ *and every block* $B \in \mathscr{B}$, *we have*

$$s'(x,B) = \begin{cases} \text{s(x,B)} & \text{if (x,B)} \notin \varepsilon, \\ \text{s(x,B)+r+k-1} & \text{if (x,B)} \in \varepsilon \end{cases}$$

Proof. Let $x \in \mathscr{P}$ and $B \in \mathscr{B}$. Thus, $s'(x, B) = \sum_{y \varepsilon B} \lambda_{xy}$.
Case 1: (x, B) is an antiflag. Therefore, $s'(x, B) = \sum_{y \varepsilon B} \lambda_{xy} = s(x, B)$.
Case 2: (x, B) is a flag. Therefore, $s'(x, B) = \sum_{y \varepsilon B, y \neq x} \lambda_{xy} + r$
$= s(x, B) + r + k - 1$. □

The relationship between parameters of a tactical configulation $\mathscr{T} = (\mathscr{P}, \mathscr{B}, \varepsilon)$ and parameter of directed graph $\Gamma(\mathscr{T})$ and $\Gamma'(\mathscr{T})$ are as follows.

Theorem 4.2.4. *Let* $\mathscr{T} = (\mathscr{P}, \mathscr{B}, \varepsilon)$ *be a tactical configuration with parameters* (v, b, k_0, r) *such that* $k_0 \leq v - 1$. *Let* $\Gamma = \Gamma(\mathscr{T})$ *be the directed graph defined by* $V(\Gamma) = \{(x, B) \in \mathscr{P} \times \mathscr{B} : (x, B) \notin \varepsilon\}$ *and* $(x, B) \to (y, C)$ *if and only if* $(x, C) \in \varepsilon$. *Then* Γ *is a directed regular graph with parameters*

$n = b(v - k_0)$,

$k = r(v - k_0)$,

$t_i = \mu_i = k_0 r - \alpha_i$, *where* $i = 1, \ldots, a$

$\lambda_j = k_0 r - (\beta_j + r + k_0 - 1)$ *where* $j = 1, \ldots, b$

if and only if $\mathscr{T}$ *is a tactical configuration with parameters* $(v, b, k_0, r; \alpha_1, \ldots, \alpha_a; \beta_1, \ldots, \beta_b)$.

Proof. Assume that Γ is a directed graph with the given parameters. We now proceed count α's and β's. Let $x \in \mathscr{P}$ and $B \in \mathscr{B}$.
Case 1: (x, B) is an antiflag. Since $t((x, B), (x, B))$ is the number of $(y, C) \in V(\Gamma)$ such that $(x, B) \to (y, C)$ and $(y, C) \to (x, B)$, it is the number of $(y, C) \in \mathscr{P} \times \mathscr{B}$ such that $y \varepsilon B$, $x \varepsilon C$ and $(y, C) \notin \varepsilon$. Therefore, the number of $(y, C) \in \mathscr{P} \times \mathscr{B}$ such that $y \varepsilon B$, $x \varepsilon C$ and $(y, C) \in \varepsilon$ is $s'(x, B) = k_0 r - t((x, B), (x, B)) = k_0 r - t_i$

for some $i = 1, \ldots, a$. So $s(x, B) = k_0 r - t_i$ for some $i = 1, \ldots, a$.

Case 2: (x, B) is a flag. Since $k_0 \leq v - 1$, there exist $(x, B_1), (x_2, B) \notin \varepsilon$. Since $t((x, B_1), (x_2, B))$ is the number of $(y, C) \in V(\Gamma)$ such that $(x, B_1) \to (y, C)$ and $(y, C) \to (x_2, B)$, it is the number of $(y, C) \in \mathscr{P} \times \mathscr{B}$ such that $y \varepsilon B$, $x \varepsilon C$ and $(y, C) \notin \varepsilon$. Therefore, the number of $(y, C) \in \mathscr{P} \times \mathscr{B}$ such that $y \varepsilon B$, $x \varepsilon C$ and $(y, C) \in \varepsilon$ is $s'(x, B) = k_0 r - t((x, B_1), (x_2, B)) = k_0 r - \lambda_j$ for some $j = 1, \ldots, b$. So $s(x, B) = k_0 r - \lambda_j - r - k + 1$ for some $j = 1, \ldots, b$.

Conversely, since $\mathscr{T}$ is a tactical configuration, $n = b(v - k_0)$ and $k = r(v - k_0)$. Let $(x, B_1), (x_2, B) \in V(\Gamma')$.

Case 1: $(x, B_1) = (x_2, B)$ is $(x, B) \notin \varepsilon$. Since $t((x, B_1), (x_2, B))$ is the number of $(y, C) \in V(\Gamma)$ such that $(x, B_1) \to (y, C)$ and $(y, C) \to (x_2, B)$, it is the number of $(y, C) \in \mathscr{P} \times \mathscr{B}$ such that $y \varepsilon B$, $x \varepsilon C$ and $(y, C) \notin \varepsilon$. Therefore, the number of $(y, C) \in \mathscr{P} \times \mathscr{B}$ such that $y \varepsilon B$, $x \varepsilon C$ and $(y, C) \notin \varepsilon$ is $t((x, B_1), (x_2, B)) = k_0 r - s'(x, B) = k_0 r - \alpha_i$ for some $i = 1, \ldots, a$.

Case 2: $(x, B_1) \to (x_2, B)$ is $(x, B) \in \varepsilon$. Since $t((x, B_1), (x_2, B))$ is the number of $(y, C) \in V(\Gamma)$ such that $(x, B_1) \to (y, C)$ and $(y, C) \to (x_2, B)$, it is the number of $(y, C) \in \mathscr{P} \times \mathscr{B}$ such that $y \varepsilon B$, $x \varepsilon C$ and $(y, C) \notin \varepsilon$. Therefore, the number of $(y, C) \in \mathscr{P} \times \mathscr{B}$ such that $y \varepsilon B$, $x \varepsilon C$ and $(y, C) \notin \varepsilon$ is $t((x, B_1), (x_2, B)) = k_0 r - s'(x, B) = k_0 r - (\beta_j + r + k - 1)$ for some $j = 1, \ldots, b$.

Case 3: $(x, B_1) \nrightarrow (x_2, B)$ is $(x, B) \notin \varepsilon$. Since $t((x, B_1), (x_2, B))$ is the number of $(y, C) \in V(\Gamma)$ such that $(x, B_1) \to (y, C)$ and $(y, C) \to (x_2, B)$, it is the number of $(y, C) \in \mathscr{P} \times \mathscr{B}$ such that $y \varepsilon B$, $x \varepsilon C$ and $(y, C) \notin \varepsilon$. Therefore, the number of $(y, C) \in \mathscr{P} \times \mathscr{B}$ such that $y \varepsilon B$, $x \varepsilon C$ and $(y, C) \notin \varepsilon$ is $t((x, B_1), (x_2, B)) = k_0 r - s'(x, B) = k_0 r - \alpha_i$ for some $i = 1, \ldots, a$. $\square$

Theorem 4.2.5. *Let $\mathscr{T} = (\mathscr{P}, \mathscr{B}, \varepsilon)$ be a tactical configuration with parameters (v, b, k_0, r) such that $k_0 \geq 1$. Let $\Gamma' = \Gamma'(\mathscr{T})$ be the directed graph defined by $V(\Gamma') = \{(x, B) \in \mathscr{P} \times \mathscr{B} : (x, B) \in \varepsilon\}$ and $(x, B) \to (y, C)$ if and only if $(x, B) \neq (y, C)$ and $(x, C) \in \varepsilon$. Then Γ' is a directed regular graph with parameters*

$n = vr$,

$k = rk_0 - 1$,

$t_j = \beta_j + r + k_0 - 2, \qquad where\ j = 1, \ldots, b$

$\lambda_j = \beta_j + r + k_0 - 3, \qquad where\ j = 1, \ldots, b$

$\mu_i = \alpha_i \qquad where\ i = 1, \ldots, a$

if and only if $\mathscr{T}$ *is a tactical configuration with parameters* $(v, b, k_0, r; \alpha_1, \ldots, \alpha_a; \beta_1, \ldots, \beta_b)$.

Proof. Assume that Γ' is a directed graph with the given parameters. We now proceed count α's and β's. Let $x \in \mathscr{P}$ and $B \in \mathscr{B}$.

Case 1: (x, B) is an antiflag. Since $k_0 \geq 1$, there exist $(x, B_1), (x_2, B) \in \varepsilon$. Since $t((x, B_1), (x_2, B))$ is the number of $(y, C) \in V(\Gamma')$ such that $(x, B_1) \to (y, C)$ and $(y, C) \to (x_2, B)$, it is the number of $(y, C) \in \mathscr{P} \times \mathscr{B}$ such that $y \varepsilon B$, $x \varepsilon C$, $(y, C) \neq (x, B_1), (x_2, B)$ and $(y, C) \in \varepsilon$. Hence, it is the number of $(y, C) \in \mathscr{P} \times \mathscr{B}$ such that $y \varepsilon B$, $x \varepsilon C$ and $(y, C) \in \varepsilon$. Therefore, the number of $(y, C) \in \mathscr{P} \times \mathscr{B}$ such that $y \varepsilon B$, $x \varepsilon C$ and $(y, C) \in \varepsilon$ is $s'(x, B) = t((x, B_1), (x_2, B)) = \mu_i$ for some $i = 1, \ldots, a$. So $s(x, B) = \mu_i$ for some $i = 1, \ldots, a$.

Case 2: (x, B) is a flag. Since $t((x, B), (x, B))$ is the number of $(y, C) \in V(\Gamma')$ such that $(x, B) \to (y, C)$ and $(y, C) \to (x, B)$, it is the number of $(y, C) \in \mathscr{P} \times \mathscr{B}$ such that $y \varepsilon B$, $x \varepsilon C$, $(y, C) \neq (x, B)$ and $(y, C) \in \varepsilon$. Therefore, the number of $(y, C) \in \mathscr{P} \times \mathscr{B}$ such that $y \varepsilon B$, $x \varepsilon C$, $(y, C) \neq (x, B)$ and $(y, C) \in \varepsilon$ is $s'(x, B) - 1 = t((x, B), (x, B)) = t_j$ for some $j = 1, \ldots, b$. So $s(x, B) = t_j - r - k + 2$, for some $j = 1, \ldots, b$.

Conversely, since $\mathscr{T}$ is a tactical configuration, $n = vr$ and $k = rk_0 - 1$. Let $(x, B_1), (x_2, B) \in V(\Gamma')$.

Case 1: $(x, B_1) = (x_2, B)$ is $(x, B) \in \varepsilon$. Since $t((x, B_1), (x_2, B))$ is the number of $(y, C) \in V(\Gamma')$ such that $(x, B_1) \to (y, C)$ and $(y, C) \to (x_2, B)$, it is the number of $(y, C) \in \mathscr{P} \times \mathscr{B}$ such that $y \varepsilon B$, $x \varepsilon C$, $(y, C) \neq (x, B_1)$ and $(y, C) \in \varepsilon$. Therefore, the number of $(y, C) \in \mathscr{P} \times \mathscr{B}$ such that $y \varepsilon B$, $x \varepsilon C$, $(y, C) \neq (x, B_1)$ and $(y, C) \in \varepsilon$ is $t((x, B_1), (x_2, B)) = s'(x, B) - 1 = \beta_j + r + k - 2$ for some $j = 1, \ldots, b$.

Case 2: $(x, B_1) \to (x_2, B)$ is $(x, B) \in \varepsilon$. Since $t((x, B_1), (x_2, B))$ is the number of $(y, C) \in V(\Gamma')$ such that $(x, B_1) \to (y, C)$ and $(y, C) \to (x_2, B)$, it is the number of $(y, C) \in \mathscr{P} \times \mathscr{B}$ such that $y \varepsilon B$, $x \varepsilon C$, $(y, C) \neq (x, B_1), (x_2, B)$ and

$(y,C) \in \varepsilon$. Therefore, the number of $(y,C) \in \mathscr{P} \times \mathscr{B}$ such that $y\varepsilon B$, $x\varepsilon C$, $(y,C) \neq (x,B_1),(x_2,B)$ and $(y,C) \in \varepsilon$ is $t((x,B_1),(x_2,B)) = s'(x,B) - 2 = \beta_j + r + k - 3$ for some $j = 1,\ldots,b$.

Case 3: $(x,B_1) \nrightarrow (x_2,B)$ is $(x,B) \notin \varepsilon$. Since $t((x,B_1),(x_2,B))$ is the number of $(y,C) \in V(\Gamma')$ such that $(x,B_1) \to (y,C)$ and $(y,C) \to (x_2,B)$, it is the number of $(y,C) \in \mathscr{P} \times \mathscr{B}$ such that $y\varepsilon B$, $x\varepsilon C$, $(y,C) \neq (x,B_1),(x_2,B)$ and $(y,C) \in \varepsilon$. Hence, it is the number of $(y,C) \in \mathscr{P} \times \mathscr{B}$ such that $y\varepsilon B$, $x\varepsilon C$ and $(y,C) \in \varepsilon$. Therefore, the number of $(y,C) \in \mathscr{P} \times \mathscr{B}$ such that $y\varepsilon B$, $x\varepsilon C$ and $(y,C) \in \varepsilon$ is $t((x,B_1),(x_2,B)) = s'(x,B) = \alpha_i$ for some $i = 1,\ldots,a$. $\square$

Remark. Let $\mathscr{T} = (\mathscr{P},\mathscr{B},\varepsilon)$ be a tactical configuration with parameters (v,b,k_0,r).

1. If $k_0 = v$, then $V(\Gamma) = \varnothing$ in Theorem 4.2.4.

2. If $k_0 = 0$, then $V(\Gamma') = \varnothing$ in Theorem 4.2.5.

4.3 Directed regular graphs from symplectic graphs and orthogonal graphs

Finally, we apply Theorems 4.2.4 and 4.2.5 to compute the parameters of the directed graphs arising from the tactical structures constructed in Charpters II and III

Theorem 4.3.1. *The directed graph $\Gamma_1 = \Gamma(\mathscr{T}_1)$ is strongly regular with parameters*

$$\begin{array}{rcl} n &=& \dfrac{\prod_{i=1}^{\nu}(|R|^i+|M|^i)}{|M|^{\frac{(\nu+1)\nu}{2}}}\left(\dfrac{|R|^{2\nu}-|M|^{2\nu}}{|R|-|M|} - \dfrac{|R|^{\nu}-|R|^{\nu}}{|R|-|M|}|M|^{\nu}\right), \\ k &=& \left(\dfrac{|R|^{2\nu}-|M|^{2\nu}}{|R|-|M|} - \dfrac{|R|^{\nu}-|M|^{\nu}}{|R|-|M|}|M|^{\nu}\right)\dfrac{\prod_{i=1}^{\nu-1}(|R|^i+|M|^i)}{|M|^{\frac{(\nu-1)\nu}{2}}}, \\ t_1 &=& \mu_1 \\ &=& (|R|+|M|)^2|M| - |M|^3, \\ \lambda_1 &=& (|R|+|M|)^2|M| - (2|R|+|M|)|M|^2. \end{array}$$

if $\nu = 2$, and

$$
\begin{aligned}
t_1 &= \mu_1 \\
&= \frac{|R|^\nu-|R|^\nu}{|R|-|M|}|M|^\nu \frac{\prod_{i=1}^{\nu-1}(|R|^i+|M|^i)}{|M|^{\frac{(\nu-1)\nu}{2}}} - \frac{(|R|^{\nu-1}-|M|^{\nu-1})\prod_{i=1}^{\nu-2}(|R|^i+|M|^i)}{(|R|-|M|)|M|^{\frac{(\nu-1)(\nu-2)-2(\nu+1)}{2}}}, \\
\lambda_1 &= \frac{|R|^\nu-|R|^\nu}{|R|-|M|}|M|^\nu \frac{\prod_{i=1}^{\nu-1}(|R|^i+|M|^i)}{|M|^{\frac{(\nu-1)\nu}{2}}} - \frac{(2|R|^\nu-|R|^{\nu-1}|M|-|M|^\nu)\prod_{i=1}^{\nu-2}(|R|^i+|M|^i)}{(|R|-|M|)|M|^{\frac{(\nu-1)(\nu-2)-2\nu}{2}}}.
\end{aligned}
$$

otherwise.

Theorem 4.3.2. *The directed graph $\Gamma_2 = \Gamma(\mathscr{T}_2)$ is regular with parameters*

$$
\begin{aligned}
n &= \tfrac{(|R|^{2\nu}-|M|^{2\nu})|R|^{2\nu-1}}{2(|R|-|M|)}\left(\tfrac{|R|^{2\nu}-|M|^{2\nu}}{|R|-|M|} - |R|^{2\nu-2}(|R|-|M|)\right), \\
k &= \tfrac{|R|^{4\nu-3}(|R|-|M|)}{2}\left(\tfrac{|R|^{2\nu}-|M|^{2\nu}}{|R|-|M|} - |R|^{2\nu-2}(|R|-|M|)\right), \\
t_1 &= \mu_1 \\
&= |R|^{2\nu-2}(|R|-|M|)^{\frac{|R|^{4\nu-3}(|R|-|M|)}{2}} - \Big((|R|-|M|)^2|R|^{2\nu-3} \\
&\quad \times^{\frac{(|R|-|M|)^3(|R|^{2\nu-2}-|M|^{2\nu-2})|R|^{2\nu-3}+(|R|-|M|)(|R|-2|M|)|R|^{2\nu-2}|M|^{2\nu-2}}{2}} + \\
&\quad (|R|-|M|)|R|^{2\nu-3}|M|^{\frac{|R|^{4\nu-5}(|R|-|M|)^3}{2}}\Big), \\
t_2 &= \mu_2 \\
&= |R|^{2\nu-2}(|R|-|M|)^{\frac{|R|^{4\nu-3}(|R|-|M|)}{2}} - \Big((|R|-|M|)|R|^{2\nu-2} \\
&\quad \times^{\frac{(|R|-|M|)^3(|R|^{2\nu-2}-|M|^{2\nu-2})|R|^{2\nu-3}+(|R|-|M|)(|R|-2|M|)|R|^{2\nu-2}|M|^{2\nu-2}}{2}}\Big), \\
\lambda_1 &= |R|^{2\nu-2}(|R|-|M|)^{\frac{|R|^{4\nu-3}(|R|-|M|)}{2}} - \Big((|R|-|M|)^2|R|^{2\nu-3} \\
&\quad \times^{\frac{(|R|-|M|)^3(|R|^{2\nu-2}-|M|^{2\nu-2})|M|^{2\nu-3}+(|R|-|M|)(|R|-2|M|)|R|^{2\nu-2}|M|^{2\nu-2}-2|M|^{4\nu-2}}{2}} + \\
&\quad ((|R|-|M|)|R|^{2\nu-3}-|M|^{2\nu-3})|M|^{\frac{|R|^{4\nu-5}(|R|-|M|)^3-2|M|^{4\nu-2}}{2}} + \\
&\quad |R|^{2\nu-2}|M|^{4\nu-2}(|R|-|M|) + \tfrac{|R|^{4\nu-3}|M|^{2\nu-1}(|R|-|M|)}{2} - (|M|^{2\nu-1})^3\Big), \\
\lambda_2 &= |R|^{2\nu-2}(|R|-|M|)^{\frac{|R|^{4\nu-3}(|R|-|M|)}{2}} - \Big((|R|-2|M|)|R|^{2\nu-2} \\
&\quad \times^{\frac{(|R|-|M|)^3(|R|^{2\nu-2}-|M|^{2\nu-2})|M|^{2\nu-3}+(|R|-|M|)(|R|-2|M|)|R|^{2\nu-2}|M|^{2\nu-2}-2|M|^{4\nu-2}}{2}} + \\
&\quad (|R|^{2\nu-2}-|M|^{2\nu-2})|M|^{\frac{|R|^{4\nu-5}(|R|-|M|)^3-2|M|^{4\nu-2}}{2}} + \\
&\quad |R|^{2\nu-2}|M|^{4\nu-2}(|R|-|M|) + \tfrac{|R|^{4\nu-3}|M|^{2\nu-1}(|R|-|M|)}{2} - (|M|^{2\nu-1})^3\Big).
\end{aligned}
$$

Theorem 4.3.3. *The directed graph $\Gamma_3 = \Gamma(\mathscr{T}_3)$ is strongly regular with parameters*

$$
\begin{aligned}
n &= \frac{\prod_{i=1}^{\nu}(|R|^{i+\delta-1}+|M|^{i+\delta-1})}{|M|^{\frac{(\nu+2\delta-1)\nu}{2}}}\left(\tfrac{(|R|^\nu-|M|^\nu)(|R|^{\nu+\delta-1}+|M|^{\nu+\delta-1})}{|R|-|M|} - \tfrac{|R|^\nu-|M|^\nu}{|R|-|M|}|M|^{\nu+\delta-1}\right), \\
k &= \frac{\prod_{i=1}^{\nu-1}(|R|^{i+\delta-1}+|M|^{i+\delta-1})}{|M|^{\frac{(\nu+2\delta-2)(\nu-1)}{2}}}\left(\tfrac{(|R|^\nu-|M|^\nu)(|R|^{\nu+\delta-1}+|M|^{\nu+\delta-1})}{|R|-|M|} - \tfrac{|R|^\nu-|M|^\nu}{|R|-|M|}|M|^{\nu+\delta-1}\right),
\end{aligned}
$$

$$\begin{aligned}
t_1 &= \mu_1\\
&= \tfrac{|R|^2-|M|^2}{|R|-|M|}|M|^{\delta+1}\tfrac{(|R|^\delta+|M|^\delta)}{|M|^{\delta)}} - |M|^{2+\delta},\\
\lambda_1 &= \tfrac{|R|^2-|M|^2}{|R|-|M|}|M|^{\delta+1}\tfrac{(|R|^\delta+|M|^\delta)}{|M|^{\delta}} - |R|^{\delta+1}|M|^{-\delta}+|R|-|M|^{2+\delta}
\end{aligned}$$

if $\nu = 2$ *and,*

$$\begin{aligned}
t_1 &= \mu_1\\
&= \tfrac{|R|^\nu-|M|^\nu}{|R|-|M|}|M|^{\nu+\delta-1}\frac{\prod_{i=1}^{\nu-1}(|R|^{i+\delta-1}+|M|^{i+\delta-1})}{|M|^{\frac{(\nu+2\delta-2)(\nu-1)}{2}}} - \frac{(|R|^{\nu-1}-|M|^{\nu-1})\prod_{i=1}^{\nu-2}(|R|^{i+\delta-1}+|M|^{i+\delta-1})}{(|R|-|M|)|M|^{\frac{(\nu+2\delta-3)(\nu-2)-2(\nu+\delta)}{2}}},\\
\lambda_1 &= \tfrac{|R|^\nu-|M|^\nu}{|R|-|M|}|M|^{\nu+\delta-1}\frac{\prod_{i=1}^{\nu-1}(|R|^{i+\delta-1}+|M|^{i+\delta-1})}{|M|^{\frac{(\nu+2\delta-2)(\nu-1)}{2}}} -\\
&\quad \tfrac{|R|^\nu|M|^{\delta-2}+|R|^{\nu+\delta-1}|M|^{-1}-|R|^{\nu+\delta-2}-|M|^{\nu+\delta-2}}{|R|-|M|}\frac{\prod_{i=1}^{\nu-2}(|R|^{i+\delta-1}+|M|^{i+\delta-1})}{|M|^{\frac{(\nu+2\delta-3)(\nu-2)-2(\nu+1)}{2}}}.
\end{aligned}$$

otherwise.

Theorem 4.3.4. *The directed graph* $\Gamma_{4,0} = \Gamma(\mathscr{T}_{4,0})$ *is* λ*-strongly regular with parameters*

$$\begin{aligned}
n &= \tfrac{(|R|^\nu-|M|^\nu)(|R|^{\nu-1}+|M|^{\nu-1})|R|^{2\nu-2}}{(|R|-|M|)2}\Big(\tfrac{(|R|^\nu-|M|^\nu)(|R|^{\nu-1}+|M|^{\nu-1})}{|R|-|M|}-\\
&\quad (|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^\nu)\Big),\\
k &= \tfrac{|R|^{2\nu-2}(|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^\nu)}{2}\Big(\tfrac{(|R|^\nu-|M|^\nu)(|R|^{\nu-1}+|M|^{\nu-1})}{|R|-|M|}-\\
&\quad (|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^\nu)\Big),\\
t_1 &= \mu_1\\
&= (|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^\nu)\\
&\quad \times\tfrac{|R|^{2\nu-2}(|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^\nu)}{2}-\\
&\quad \Big((|R|-|M|)(|R|^{\nu-1}-|R|^{\nu-2}|M|-|M|^{\nu-1})|R|^{\nu-2}\\
&\quad \times\tfrac{(|R|-|M|)(|R|^{\nu-1}-|M|^{\nu-1})(|R|^\nu-2|R|^{\nu-1}|M|+|R|^{\nu-2}|M|^{\nu+2}-2|R||M|^{\nu-1}+3|M|^\nu)|R|^{2\nu-4}}{2}+\\
&\quad (|R|-|M|)|R|^{2\nu-4}|M|\tfrac{(|R|-|M|)|R|^{2\nu-3}(|R|-|M|)(|R|^{\nu-1}-|R|^{\nu-2}|M|-|M|^{\nu-1})|R|^{\nu-2}}{2}\Big),\\
t_2 &= \mu_2\\
&= (|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^\nu)\\
&\quad \times\tfrac{|R|^{2\nu-2}(|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^\nu)}{2}-\\
&\quad \Big((|R|^\nu-2|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^\nu)|R|^{\nu-2}\\
&\quad \times\tfrac{(|R|-|M|)(|R|^{\nu-1}-|M|^{\nu-1})(|R|^\nu-2|R|^{\nu-1}|M|+|R|^{\nu-2}|M|^{\nu+2}-2|R||M|^{\nu-1}+3|M|^\nu)|R|^{2\nu-4}}{2}+\\
&\quad (|R|^{\nu-1}-|R|^{\nu-2}|M|-|R||M|^{\nu-2}+|M|^{\nu-1})|R|^{\nu-2}|M|\\
&\quad \times\tfrac{(|R|-|M|)|R|^{2\nu-3}(|R|-|M|)(|R|^{\nu-1}-|R|^{\nu-2}|M|-|M|^{\nu-1})|R|^{\nu-2}}{2}\Big),
\end{aligned}$$

$$\begin{aligned}
t_3 &= \mu_3 \\
&= (|R|^{2\nu-2} - |R|^{2\nu-3}|M| - |R|^{\nu-1}|M|^{\nu-1} + |R|^{\nu-2}|M|^{\nu}) \\
&\quad \times \tfrac{|R|^{2\nu-2}(|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^{\nu})}{2} - \\
&\quad (|R|^{2\nu-2} - |R|^{2\nu-3}|M| - |R|^{\nu-1}|M|^{\nu-1} + |R|^{\nu-2}|M|^{\nu}) \\
&\quad \times \tfrac{(|R|-|M|)(|R|^{\nu-1}-|M|^{\nu-1})(|R|^{\nu}-2|R|^{\nu-1}|M|+|R|^{\nu-2}|M|^{\nu+2}-2|R||M|^{\nu-1}+3|M|^{\nu})|R|^{2\nu-4}}{2}, \\
\lambda_1 &= (|R|^{2\nu-2} - |R|^{2\nu-3}|M| - |R|^{\nu-1}|M|^{\nu-1} + |R|^{\nu-2}|M|^{\nu}) \\
&\quad \times \tfrac{|R|^{2\nu-2}(|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^{\nu})}{2} - \\
&\quad \Bigg(\bigg((|R|^{\nu} - 2|R|^{\nu-1}|M| + |R|^{\nu-2}|M|^{2} - 2|R||M|^{\nu-1} + 3|M|^{\nu})|R|^{\nu-2} \\
&\quad \times \tfrac{(|R|-|M|)(|R|^{\nu-1}-|M|^{\nu-1})(|R|^{\nu}-2|R|^{\nu-1}|M|+|R|^{\nu-2}|M|^{\nu+2}-2|R||M|^{\nu-1}+3|M|^{\nu})|R|^{2\nu-4}-2|M|^{4\nu-4}}{2}\bigg) + \\
&\quad \bigg(((|R|^{\nu-1} - |R|^{\nu-2}|M| + |R||M|^{\nu-2} - 2|M|^{\nu-1})|R|^{\nu-2}|M| - |M|^{2\nu-2}) \\
&\quad \times \tfrac{(|R|-|M|)|R|^{2\nu-3}(|R|-|M|)(|R|^{\nu-1}-|R|^{\nu-2}|M|-|M|^{\nu-1})|R|^{\nu-2}-2|M|^{4\nu-4}}{2}\bigg) + \\
&\quad \bigg((|R|^{2\nu-2} - |R|^{2\nu-3}|M| - |R|^{\nu-1}|M|^{\nu-1} + |R|^{\nu-2}|M|^{\nu})|M|^{4\nu-4} + \\
&\quad |M|^{2\nu-2}\tfrac{|R|^{2\nu-2}(|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^{\nu})}{2} - |M|^{6\nu-6}\bigg)\Bigg).
\end{aligned}$$

Theorem 4.3.5. *The directed graph $\Gamma_{4,2} = \Gamma(\mathscr{T}_{4,2})$ is λ-strongly regular with parameters*

$$\begin{aligned}
n &= \tfrac{|R|^{2\nu}(|R|^{\nu}-|M|^{\nu})(|R|^{\nu+1}+|M|^{\nu+1})}{2(|R|-|M|)}\Big(\tfrac{(|R|^{\nu}-|M|)(|R|^{\nu+1}+|M|^{\nu+1})}{|R|-|M|} - \\
&\quad (|R|^{2\nu} - |R|^{2\nu-1}|M| - |R|^{\nu-1}|M|^{\nu+1} + |R|^{\nu})\Big), \\
k &= \tfrac{|R|^{2\nu}(|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu})}{2}\Big(\tfrac{(|R|^{\nu}-|M|)(|R|^{\nu+1}+|M|^{\nu+1})}{|R|-|M|} - \\
&\quad (|R|^{2\nu} - |R|^{2\nu-1}|M| - |R|^{\nu-1}|M|^{\nu+1} + |R|^{\nu}|M|^{\nu})\Big), \\
t_1 &= \mu_1 \\
&= (|R|^{2\nu} - |R|^{2\nu-1}|M| - |R|^{\nu-1}|M|^{\nu+1} + |R|^{\nu}) \\
&\quad \tfrac{|R|^{2\nu}(|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu})}{2} - \\
&\quad \Big((|R|^{2\nu} - 2|R|^{2\nu-1}|M| + |R|^{2\nu-2}|M|^{2} + |R|^{\nu}|M|^{\nu} + |R|^{\nu-1}|M|^{\nu+1}) \\
&\quad \times \tfrac{(|R|-|M|)(|R|^{\nu}+|M|^{\nu})(|R|^{\nu+1}-2|R|^{\nu}|M|+|R|^{\nu-1}|M|^{2}+2|R||M|^{\nu}-3|M|^{\nu+1})|R|^{2\nu-2}}{2} + \\
&\quad (|R|^{2\nu-1}|M| - |R|^{2\nu-2}|M|^{2} - 2|R|^{\nu-1}|M|^{\nu+1}) \\
&\quad \times \tfrac{(|R|-|M|)|R|^{2\nu-1}(|R|-|M|)(|R|^{2\nu-1}-|R|^{2\nu-2}|M|+|R|^{\nu-1})|M|}{2}\Big),
\end{aligned}$$

$$
\begin{aligned}
t_2 &= \mu_2 \\
&= (|R|^{2\nu} - |R|^{2\nu-1}|M| - |R|^{\nu-1}|M|^{\nu+1} + |R|^{\nu}) \\
&\quad \times \tfrac{|R|^{2\nu}(|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu})}{2} - \\
&\quad \Big((|R|^{2\nu} - 2|R|^{2\nu-1}|M| + |R|^{2\nu-2}|M|^2 - |R|^{\nu-1}|M|^{\nu+1} + |R|^{\nu-2}|M|^{\nu+2}) \\
&\quad \times \tfrac{(|R|-|M|)(|R|^{\nu}+|M|^{\nu})(|R|^{\nu+1}-2|R|^{\nu}|M|+|R|^{\nu-1}|M|^2+2|R||M|^{\nu}-3|M|^{\nu+1})|R|^{2\nu-2}}{2} + \\
&\quad (|R|^{2\nu-1}|M| - |R|^{2\nu-2}|M|^2 + |R|^{\nu}|M|^{\nu} - |R|^{\nu-2}|M|^{\nu+2}) \\
&\quad \times \tfrac{(|R|-|M|)|R|^{2\nu-1}(|R|-|M|)(|R|^{2\nu-1}-|R|^{2\nu-2}|M|+|R|^{\nu-1})|M|}{2}\Big), \\
t_3 &= \mu_3 \\
&= (|R|^{2\nu} - |R|^{2\nu-1}|M| - |R|^{\nu-1}|M|^{\nu+1} + |R|^{\nu}) \\
&\quad \times \tfrac{|R|^{2\nu}(|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu})}{2} - \\
&\quad (|R|^{2\nu} - |R|^{2\nu-1}|M| - |R|^{\nu-1}|M|^{\nu+1} + |R|^{\nu}|M|^{\nu}) \\
&\quad \times \tfrac{(|R|-|M|)(|R|^{\nu}+|M|^{\nu})(|R|^{\nu+1}-2|R|^{\nu}|M|+|R|^{\nu-1}|M|^2+2|R||M|^{\nu}-3|M|^{\nu+1})|R|^{2\nu-2}}{2}, \\
\lambda_1 &= (|R|^{2\nu} - |R|^{2\nu-1}|M| - |R|^{\nu-1}|M|^{\nu+1} + |R|^{\nu}) \\
&\quad \times \tfrac{|R|^{2\nu}(|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu})}{2} -, \\
&\quad \Big(\Big((|R|^{2\nu} - 2|R|^{2\nu-1}|M| + |R|^{2\nu-2}|M|^2 + 2|R|^{\nu}|M|^{\nu} - 3|R|^{\nu-1}|M|^{\nu+1}) \\
&\quad \times \tfrac{(|R|-|M|)(|R|^{\nu}+|M|^{\nu})(|R|^{\nu+1}-2|R|^{\nu}|M|+|R|^{\nu-1}|M|^2+2|R||M|^{\nu}-3|M|^{\nu+1})|R|^{2\nu-2}-2|M|^{4\nu}}{2}\Big) + \\
&\quad \Big((|R|^{2\nu-1}|M| - |R|^{2\nu-2}|M|^2 - |R|^{\nu}|M|^{\nu} + 2|R|^{\nu-1}|M|^{\nu+1}) \\
&\quad \times \tfrac{(|R|-|M|)|R|^{2\nu-1}(|R|-|M|)(|R|^{2\nu-1}-|R|^{2\nu-2}|M|+|R|^{\nu-1})|M|-2|M|^{4\nu}}{2}\Big) + \\
&\quad |M|^{4\nu}(|R|^{2\nu} - |R|^{2\nu-1}|M| - |R|^{\nu-1}|M|^{\nu+1} + |R|^{\nu}|M|^{\nu}) + \\
&\quad |M|^{2\nu}\tfrac{|R|^{2\nu}(|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu})}{2} - |M|^{6\nu}\Big).
\end{aligned}
$$

Theorem 4.3.6. *The directed graph $\Gamma'_1 = \Gamma'(\mathscr{T}_1)$ is strongly regular with parameters*

$$
\begin{aligned}
n &= \tfrac{|R|^{2\nu}-|M|^{2\nu}}{|R|-|M|}\tfrac{\prod_{i=1}^{\nu-1}(|R|^i+|M|^i)}{|M|^{\frac{(\nu-1)\nu}{2}}}, \\
k &= \tfrac{\prod_{i=1}^{\nu-1}(|R|^i+|M|^i)}{|M|^{\frac{(\nu-1)\nu}{2}}}\tfrac{|R|^{\nu}-|M|^{\nu}}{|R|-|M|}|M|^{\nu} - 1, \\
t_1 &= (2|R| + |M|)|M|^2 - 1, \\
\lambda_1 &= t_1 - 1, \\
\mu_1 &= |M|^3
\end{aligned}
$$

if $\nu = 2$, and

$$
\begin{aligned}
t_1 &= \frac{(2|R|^\nu - |R|^{\nu-1}|M| - |M|^\nu)\prod\limits_{i=1}^{\nu-2}(|R|^i+|M|^i)}{(|R|-|M|)|M|^{\frac{(\nu-1)(\nu-2)-2\nu}{2}}} - 1,\\
\lambda_1 &= t_1 - 1,\\
\mu_1 &= \frac{(|R|^{\nu-1} - |M|^{\nu-1})\prod\limits_{i=1}^{\nu-2}(|R|^i+|M|^i)}{(|R|-|M|)|M|^{\frac{(\nu-1)(\nu-2)-2(\nu+1)}{2}}}.
\end{aligned}
$$

otherwise.

Theorem 4.3.7. *The directed graph $\Gamma_2' = \Gamma'(\mathscr{T}_2)$ is regular with parameters*

$$
\begin{aligned}
n &= \tfrac{|R|^{2\nu}-|M|^{2\nu}}{|R|-|M|}\tfrac{|R|^{4\nu-3}(|R|-|M|)}{2},\\
k &= \tfrac{|R|^{4\nu-3}(|R|-|M|)}{2}|R|^{2\nu-2}(|R|-|M|) - 1,\\
t_1 &= (|R|-|M|)^2|R|^{2\nu-3}\Big(\tfrac{(|R|-|M|)^3(|R|^{2\nu-2}-|M|^{2\nu-2})|M|^{2\nu-3}}{2} +\\
&\quad \tfrac{(|R|-|M|)(|R|-2|M|)|R|^{2\nu-2}|M|^{2\nu-2}-2|M|^{4\nu-2}}{2}\Big) + ((|R|-|M|)|R|^{2\nu-3} - |M|^{2\nu-3})|M|\\
&\quad \times\tfrac{|R|^{4\nu-5}(|R|-|M|)^3-2|M|^{4\nu-2}}{2} + |R|^{2\nu-2}|M|^{4\nu-2}(|R|-|M|) + \tfrac{|R|^{4\nu-3}|M|^{2\nu-1}(|R|-|M|)}{2}\\
&\quad -(|M|^{2\nu-1})^3 - 1,\\
t_2 &= (|R|-2|M|)|R|^{2\nu-2}\Big(\tfrac{(|R|-|M|)^3(|R|^{2\nu-2}-|M|^{2\nu-2})|M|^{2\nu-3}}{2} +\\
&\quad \tfrac{(|R|-|M|)(|R|-2|M|)|R|^{2\nu-2}|M|^{2\nu-2}-2|M|^{4\nu-2}}{2}\Big) + (|R|^{2\nu-2} - |M|^{2\nu-2})|M|\\
&\quad \times\tfrac{|R|^{4\nu-5}(|R|-|M|)^3-2|M|^{4\nu-2}}{2} + |R|^{2\nu-2}|M|^{4\nu-2}(|R|-|M|) + \tfrac{|R|^{4\nu-3}|M|^{2\nu-1}(|R|-|M|)}{2}\\
&\quad -(|M|^{2\nu-1})^3 - 1,\\
\lambda_1 &= t_1 - 1,\\
\lambda_2 &= t_2 - 1,\\
\mu_1 &= (|R|-|M|)^2|R|^{2\nu-3}\tfrac{(|R|-|M|)^3(|R|^{2\nu-2}-|M|^{2\nu-2})|R|^{2\nu-3}+(|R|-|M|)(|R|-2|M|)|R|^{2\nu-2}|M|^{2\nu-2}}{2} +\\
&\quad (|R|-|M|)|R|^{2\nu-3}|M|\tfrac{|R|^{4\nu-5}(|R|-|M|)^3}{2},\\
\mu_2 &= (|R|-|M|)|R|^{2\nu-2}\tfrac{(|R|-|M|)^3(|R|^{2\nu-2}-|M|^{2\nu-2})|R|^{2\nu-3}+(|R|-|M|)(|R|-2|M|)|R|^{2\nu-2}|M|^{2\nu-2}}{2}.
\end{aligned}
$$

Theorem 4.3.8. *The directed graph $\Gamma_3' = \Gamma'(\mathscr{T}_3)$ is strongly regular with parameters*

$$
\begin{aligned}
n &= \tfrac{(|R|^2-|M|^2)(|R|^{\delta+1}+|M|^{\delta+1})}{|R|-|M|}\tfrac{(|R|^\delta+|M|^\delta)}{|M|^\delta},\\
k &= \tfrac{(|R|^\delta+|M|^\delta)}{|M|^\delta}\tfrac{|R|^2-|M|^2}{|R|-|M|}|M|^{\delta+1} - 1,\\
t_1 &= |R|^{\delta+1}|M|^{-\delta} + |R| - |M|^{2+\delta} - 1,\\
\lambda_1 &= t_1 - 1,\\
\mu_1 &= |M|^{2+\delta}
\end{aligned}
$$

if $\nu = 2$, and

$$
\begin{aligned}
n &= \frac{(|R|^{\nu}-|M|^{\nu})(|R|^{\nu+\delta-1}+|M|^{\nu+\delta-1})}{|R|-|M|}\frac{\prod_{i=1}^{\nu-1}(|R|^{i+\delta-1}+|M|^{i+\delta-1})}{|M|^{\frac{(\nu+2\delta-2)(\nu-1)}{2}}},\\
k &= \frac{\prod_{i=1}^{\nu-1}(|R|^{i+\delta-1}+|M|^{i+\delta-1})}{|M|^{\frac{(\nu+2\delta-2)(\nu-1)}{2}}}\frac{|R|^{\nu}-|M|^{\nu}}{|R|-|M|}|M|^{\nu+\delta-1}-1,\\
t_1 &= \frac{|R|^{\nu}|M|^{\delta-2}+|R|^{\nu+\delta-1}|M|^{-1}-|R|^{\nu+\delta-2}-|M|^{\nu+\delta-2}}{|R|-|M|}\frac{\prod_{i=1}^{\nu-2}(|R|^{i+\delta-1}+|M|^{i+\delta-1})}{|M|^{\frac{(\nu+2\delta-3)(\nu-2)-2(\nu+1)}{2}}}-1,\\
\lambda_1 &= t_1-1,\\
\mu_1 &= \frac{(|R|^{\nu-1}-|M|^{\nu-1})\prod_{i=1}^{\nu-2}(|R|^{i+\delta-1}+|M|^{i+\delta-1})}{(|R|-|M|)|M|^{\frac{(\nu+2\delta-3)(\nu-2)-2(\nu+\delta)}{2}}}.
\end{aligned}
$$

otherwise.

Theorem 4.3.9. *The directed graph $\Gamma'_{4,0}=\Gamma'(\mathscr{T}_{4,0})$ is t,λ-strongly regular with parameters*

$$
\begin{aligned}
n &= \frac{(|R|^{\nu}-|M|^{\nu})(|R|^{\nu-1}+|M|^{\nu-1})}{|R|-|M|}\frac{|R|^{2\nu-2}(|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^{\nu})}{2},\\
k &= \frac{|R|^{2\nu-2}(|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^{\nu})}{2}\\
&\quad\times(|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^{\nu})-1,\\
t_1 &= \Big(|R|^{\nu}-2|R|^{\nu-1}|M|+|R|^{\nu-2}|M|^{2}-2|R||M|^{\nu-1}+3|M|^{\nu})|R|^{\nu-2}\\
&\quad\times\frac{(|R|-|M|)(|R|^{\nu-1}-|M|^{\nu-1})(|R|^{\nu}-2|R|^{\nu-1}|M|+|R|^{\nu-2}|M|^{\nu+2}-2|R||M|^{\nu-1}+3|M|^{\nu})|R|^{2\nu-4}-2|M|^{4\nu-4}}{2}\Big)\\
&\quad+\Big(((|R|^{\nu-1}-|R|^{\nu-2}|M|+|R||M|^{\nu-2}-2|M|^{\nu-1})|R|^{\nu-2}|M|-|M|^{2\nu-2})\\
&\quad\times\frac{(|R|-|M|)|R|^{2\nu-3}(|R|-|M|)(|R|^{\nu-1}-|R|^{\nu-2}|M|-|M|^{\nu-1})|R|^{\nu-2}-2|M|^{4\nu-4}}{2}\Big)+\\
&\quad\Big((|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^{\nu})|M|^{4\nu-4}+\\
&\quad|M|^{2\nu-2}\frac{|R|^{2\nu-2}(|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^{\nu})}{2}-|M|^{6\nu-6}\Big)-1,\\
\lambda_1 &= t_1-1,\\
\mu_1 &= \Big((|R|-|M|)(|R|^{\nu-1}-|R|^{\nu-2}|M|-|M|^{\nu-1})|R|^{\nu-2}\\
&\quad\times\frac{(|R|-|M|)(|R|^{\nu-1}-|M|^{\nu-1})(|R|^{\nu}-2|R|^{\nu-1}|M|+|R|^{\nu-2}|M|^{\nu+2}-2|R||M|^{\nu-1}+3|M|^{\nu})|R|^{2\nu-4}}{2}\Big)+\\
&\quad(|R|-|M|)|R|^{2\nu-4}|M|\frac{(|R|-|M|)|R|^{2\nu-3}(|R|-|M|)(|R|^{\nu-1}-|R|^{\nu-2}|M|-|M|^{\nu-1})|R|^{\nu-2}}{2},\\
\mu_2 &= \Big((|R|^{\nu}-2|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^{\nu})|R|^{\nu-2}\\
&\quad\times\frac{(|R|-|M|)(|R|^{\nu-1}-|M|^{\nu-1})(|R|^{\nu}-2|R|^{\nu-1}|M|+|R|^{\nu-2}|M|^{\nu+2}-2|R||M|^{\nu-1}+3|M|^{\nu})|R|^{2\nu-4}}{2}\Big)+\\
&\quad\Big((|R|^{\nu-1}-|R|^{\nu-2}|M|-|R||M|^{\nu-2}+|M|^{\nu-1})|R|^{\nu-2}|M|\\
&\quad\times\frac{(|R|-|M|)|R|^{2\nu-3}(|R|-|M|)(|R|^{\nu-1}-|R|^{\nu-2}|M|-|M|^{\nu-1})|R|^{\nu-2}}{2}\Big),\\
\mu_3 &= (|R|^{2\nu-2}-|R|^{2\nu-3}|M|-|R|^{\nu-1}|M|^{\nu-1}+|R|^{\nu-2}|M|^{\nu})\\
&\quad\times\frac{(|R|-|M|)(|R|^{\nu-1}-|M|^{\nu-1})(|R|^{\nu}-2|R|^{\nu-1}|M|+|R|^{\nu-2}|M|^{\nu+2}-2|R||M|^{\nu-1}+3|M|^{\nu})|R|^{2\nu-4}}{2}.
\end{aligned}
$$

Theorem 4.3.10. *The directed graph $\Gamma'_{4,2} = \Gamma'(\mathscr{T}_{4,2})$ is t, λ-strongly regular with parameters*

$$
\begin{aligned}
n &= \tfrac{(|R|^{\nu}-|M|)(|R|^{\nu+1}+|M|^{\nu+1})}{|R|-|M|}\tfrac{|R|^{2\nu}(|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu})}{2},\\
k &= \tfrac{|R|^{2\nu}(|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu})}{2}\\
&\quad \times(|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu})-1,\\
t_1 &= \Big((|R|^{2\nu}-2|R|^{2\nu-1}|M|+|R|^{2\nu-2}|M|^2+2|R|^{\nu}|M|^{\nu}-3|R|^{\nu-1}|M|^{\nu+1})\\
&\quad \times\tfrac{(|R|-|M|)(|R|^{\nu}+|M|^{\nu})(|R|^{\nu+1}-2|R|^{\nu}|M|+|R|^{\nu-1}|M|^2+2|R||M|^{\nu}-3|M|^{\nu+1})|R|^{2\nu-2}-2|M|^{4\nu}}{2}\Big)+\\
&\quad \Big((|R|^{2\nu-1}|M|-|R|^{2\nu-2}|M|^2-|R|^{\nu}|M|^{\nu}+2|R|^{\nu-1}|M|^{\nu+1})\\
&\quad \times\tfrac{(|R|-|M|)|R|^{2\nu-1}(|R|-|M|)(|R|^{2\nu-1}-|R|^{2\nu-2}|M|+|R|^{\nu-1})|M|-2|M|^{4\nu}}{2}\Big)+\\
&\quad |M|^{4\nu}(|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu})+\\
&\quad |M|^{2\nu}\tfrac{|R|^{2\nu}(|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu})}{2}-|M|^{6\nu}-1,\\
\lambda_1 &= t_1-1,\\
\mu_1 &= \Big((|R|^{2\nu}-2|R|^{2\nu-1}|M|+|R|^{2\nu-2}|M|^2+|R|^{\nu}|M|^{\nu}+|R|^{\nu-1}|M|^{\nu+1})\\
&\quad \times\tfrac{(|R|-|M|)(|R|^{\nu}+|M|^{\nu})(|R|^{\nu+1}-2|R|^{\nu}|M|+|R|^{\nu-1}|M|^2+2|R||M|^{\nu}-3|M|^{\nu+1})|R|^{2\nu-2}}{2}\Big)+\\
&\quad \Big((|R|^{2\nu-1}|M|-|R|^{2\nu-2}|M|^2-2|R|^{\nu-1}|M|^{\nu+1})\\
&\quad \times\tfrac{(|R|-|M|)|R|^{2\nu-1}(|R|-|M|)(|R|^{2\nu-1}-|R|^{2\nu-2}|M|+|R|^{\nu-1})|M|}{2}\Big),\\
\mu_2 &= \Big((|R|^{2\nu}-2|R|^{2\nu-1}|M|+|R|^{2\nu-2}|M|^2-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu-2}|M|^{\nu+2})\\
&\quad \times\tfrac{(|R|-|M|)(|R|^{\nu}+|M|^{\nu})(|R|^{\nu+1}-2|R|^{\nu}|M|+|R|^{\nu-1}|M|^2+2|R||M|^{\nu}-3|M|^{\nu+1})|R|^{2\nu-2}}{2}\Big)+\\
&\quad \Big((|R|^{2\nu-1}|M|-|R|^{2\nu-2}|M|^2+|R|^{\nu}|M|^{\nu}-|R|^{\nu-2}|M|^{\nu+2})\\
&\quad \times\tfrac{(|R|-|M|)|R|^{2\nu-1}(|R|-|M|)(|R|^{2\nu-1}-|R|^{2\nu-2}|M|+|R|^{\nu-1})|M|}{2}\Big),\\
\mu_3 &= (|R|^{2\nu}-|R|^{2\nu-1}|M|-|R|^{\nu-1}|M|^{\nu+1}+|R|^{\nu}|M|^{\nu})\\
&\quad \times\tfrac{(|R|-|M|)(|R|^{\nu}+|M|^{\nu})(|R|^{\nu+1}-2|R|^{\nu}|M|+|R|^{\nu-1}|M|^2+2|R||M|^{\nu}-3|M|^{\nu+1})|R|^{2\nu-2}}{2}.
\end{aligned}
$$

REFERENCES

[1] Brouwer, A.E., Olmez, O. and Song, S.Y.: Directed strongly regular graphs from $1\dfrac{1}{2}$-designs, *European J. Combin.* **33**(6), 1174–1177 (2012).

[2] Chai, Z., Feng, R. and Zeng, L.: Constructions of $1\dfrac{1}{2}$-designs from symplectic geometry over finite fields, *Acta Math Sin.* **31**, 1367–1378 (2015).

[3] Feng, R., Zhoa, M. and Zeng, L.: Constructions of $1\frac{1}{2}$-designs from orthogonal geometry over finite fields, *Discrete Math.* **339**, 382–390 (2016).

[4] Gu, Z. and Wan, Z.: Orthogonal graphs of odd characteristic and their automorphisms, *Finite Fields Appl.* **14**, 291–313 (2008).

[5] Gu, Z. and Wan, Z.: Subconsituents of orthogonal graphs of odd characteristic, *Linear Algebra Appl.* **434**, 2430–2447 (2011).

[6] Gu, Z., Wan, Z. and Zhou, K.: Subconsituents of orthogonal graphs of odd characteristic-continued, *Linear Algebra Appl.* **439**, 2861–2855 (2013).

[7] Li, F., Wang, Y. and Zhou, K.: Subconsituents of symplectic graphs, *European J. Combin.* **29**, 1092–1103 (2008).

[8] Meemark, Y. and Puirod, T.: Symplectic graphs over finite local rings, *European J. Combin.* **34**, 1114–1127 (2013).

[9] Meemark, Y. and Sriwongsa S.: Orthogonal graphs over finite commutative rings of odd charcteristic, *Finite Fields Appl.* **40**, 26–45 (2016).

[10] Tang, Z. and Wan, Z.: Symplectic graphs and their automorphisms, *European J. Combin.* **27**, 38–50 (2006).

VITA

Name Mr. Wachirapong Jirakitpuwapat

Date of Birth 6 October 1990

Place of Birth Bangkok, Thailand

Education B.Sc. (Mathematics)(First Class Honours),
King Mongkut's University of Technology Thonburi, 2012

Conference **Speaker**

• *Construction of designs from orthogonal graphs over finite local rings* at the 21^{th} Annual Meeting in Mathematics,
23-25 May 2016 at Chulalongkorn University, Bangkok