AN ANALYTICAL STUDY OF A GENERALIZATION OF THE BINOMIAL COEFFICIENTS USING GAMMA FUNCTIONS

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ABSTRACT

This thesis begins with the binomial coefficients which we can write in the form

$$ {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!} \quad . \qquad \text{.........(1)} $$

When n and r are positive integers, the values of the function ${}^{n}C_{r}$ are defined on the lattice points in the 1st quadrant of the (r, n) plane, and these values form a pattern known as Pascal's triangle. Using the well-known rule for constructing Pascal's triangle, we can find the values of the function on the lattice points in the 2nd quadrant.

We have two ways to extend the function to the 3rd and 4th quadrants.

(a) Using the binomial series

$(1 + a)^n = {}^{n}C_{0}.1 + {}^{n}C_{1}a + {}^{n}C_{2}a^2 + \ldots$, letting n be a negative integer and $|a| < 1$, and using $a^{-x} = \frac{1}{a^x}$, we obtain values of the function on the lattice points in the 3rd and 4th quadrants.

(b) Using the binomial series in (a) and replacing $(1 + a)^n$ by $(a + 1)^n$ with n a negative integer and $|a| > 1$, and using $a^{-x} = \frac{1}{a^x}$, we obtain another set of values of the function on the lattice points in the 3rd and 4th quadrants, that are different from the values in (a).

From (1), replacing factorials by gamma functions, we have

$$^{n}C_{r} = \frac{\Gamma(n + 1)}{\Gamma(r + 1)\ \Gamma(n - r + 1)}, \qquad \ldots\ldots\ldots(2)$$

and replacing $^{n}C_{r}$ in (2) by $f(r, n)$, we have

$$f\ (r, n) = \frac{\Gamma(n + 1)}{\Gamma(r + 1)\ \Gamma(n - r + 1)}. \qquad \ldots\ldots\ldots(3)$$

By using (3), we obtain the values of the function on the other points in the (r, n) plane.

But $\Gamma(n)$ has singularities for $n = 0, -1, -2, -3, \ldots\ldots$ and so on. Therefore, $f(r, n)$ has singularities for $n = -1, -2, -3, \ldots$, or we have singular lines for $n = -1, -2, -3, \ldots\ldots$.

We can remove the singularities on the lattice points of the singular lines

(1) by taking the limit along the line $r = r_1$ to the lattice point (r_1, n_1) from either direction, which gives the same values as in (a) above, and

(2) by taking the limit along the line $n_2 = r_2 + k$, where k is an integer, to the lattice point (r_2, n_2) from either direction, which gives the same values as in (b).

Using (1) and (3) various graphs are drawn illustrating the shape of the function in region $-5 \leq r \leq +5$, $-5 \leq n \leq +5$.

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