CHAPTER II

THEORY AND FORMULATION

Bending Theory of Thin , Isotropic Elastic Plates

The bending equations of thin elastic plates are based on the assumption that " plane sections remain plane " during bending and the deflection is small comparing with the thickness of the plate [7].

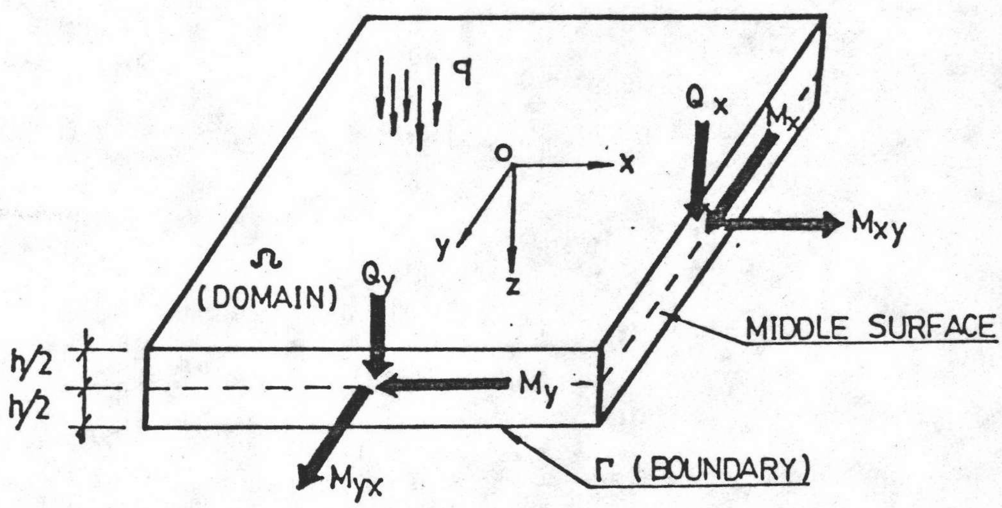

FIG.1 Co-Ordinate System and Sign Convention

Figure 1 shows an element of the thin plate of thickness " h " together with the sign convention that will be used. The material has the modulus of elasticity " E " and poisson's ratio " ν ". The deflection " W " of the middle surface of the plate subjected to transverse force of intensity " q " is related to slope " N " , corner forces " R " arised from the jump of twisting moments at each corner , bending moments and shear force per unit length . The relationships among these quantities can be written in cartesian co-ordinates as follows [7] :

$$M_x = -D \left[\frac{\partial^2 W}{\partial X^2} + \nu \frac{\partial^2 W}{\partial Y^2} \right] \tag{1}$$

$$M_Y = -D \left[\frac{\partial^2 W}{\partial Y^2} + \nu \frac{\partial^2 W}{\partial X^2} \right] \tag{2}$$

$$M_{XY} = -M_{YX} = D\,(1-\nu) \frac{\partial^2 W}{\partial X \partial Y} \tag{3}$$

$$Q_x = -D \left[\frac{\partial^3 W}{\partial X^3} + \frac{\partial^3 W}{\partial X \partial Y^2} \right] \tag{4}$$

$$Q_Y = -D \left[\frac{\partial^3 W}{\partial Y^3} + \frac{\partial^3 W}{\partial Y \partial X^2} \right] \tag{5}$$

$$R = M_{XY} - M_{YX} = 2M_{XY} \tag{6}$$

$$R = 2D(1-\nu) \frac{\partial^2 W}{\partial X \partial Y} \tag{7}$$

In which $D = Eh^3/\{12(1-\nu^2)\}$ denotes the plate rigidity. Also , on the boundary , the twisting moment can be combined with the shear force to produce a resultant boundary shear force or " Kirchhoff's shear " per unit length " V " as given by [7] :

$$V_x = -D \left[\frac{\partial^3 W}{\partial X^3} + (2-\nu) \frac{\partial^3 W}{\partial X \partial Y^2} \right] \tag{8}$$

$$V_Y = -D \left[\frac{\partial^3 W}{\partial Y^3} + (2-\nu) \frac{\partial^3 W}{\partial Y \partial X^2} \right] \tag{9}$$

By considering the equilibrium of forces in Z-direction (vertical direction) and moments about X and Y axes , the following governing biharmonic equation for plates is obtained . [7]

$$\nabla^2 \nabla^2 W = \nabla^4 W = q/D \tag{10}$$

In which $\nabla^2(.) = \frac{\partial^2(.)}{\partial X^2} + \frac{\partial^2(.)}{\partial Y^2}$ is defines as " laplacian operator "

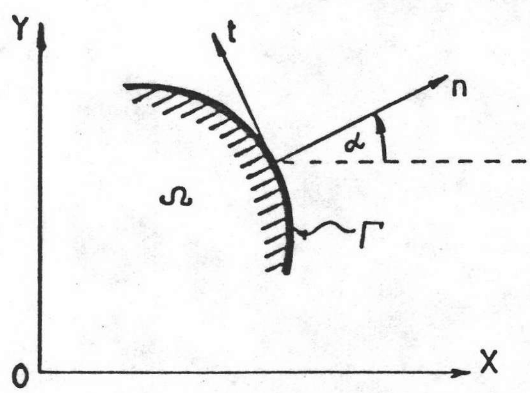

FIG.2 Normal Co-Ordinate

In normal co-ordinates (FIG.2) these relationships become [7]:

$$M_n = -D \left[\frac{\partial^2 W}{\partial n^2} + \nu \frac{\partial^2 W}{\partial t^2} \right] \tag{11}$$

$$M_t = -D \left[\frac{\partial^2 W}{\partial t^2} + \nu \frac{\partial^2 W}{\partial n^2} \right] \tag{12}$$

$$M_{nt} = D\,(1-\nu) \frac{\partial^2 W}{\partial n \partial t} \tag{13}$$

$$Q_n = -D \left[\frac{\partial^3 W}{\partial n^3} + \frac{\partial^3 W}{\partial n \partial t^2} \right] \tag{14}$$

$$V_n = -D \left[\frac{\partial^3 W}{\partial n^3} + (2-\nu) \frac{\partial^3 W}{\partial n \partial t^2} \right] \tag{15}$$

$$R = M_{nt1} - M_{nt2} = D(1-\nu) \left[\left(\frac{\partial^2 W}{\partial n \partial t} \right)_1 - \left(\frac{\partial^2 W}{\partial n \partial t} \right)_2 \right] \tag{16}$$

In which the subscript (1) is referred to the first side of the corner and (2) is the other side.

In considering a skew plates , it may be more convenient to refer these relationships to a skew or oblique co-ordinate system (see appendix A for more detail) as follows :

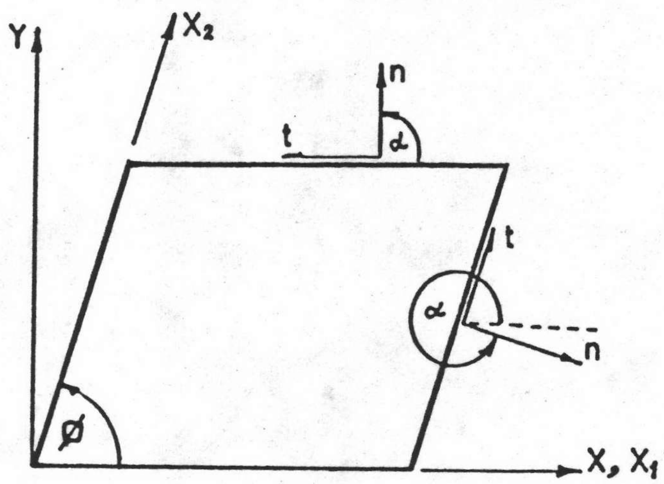

FIG.3 Skew or Oblique Co-Ordinate System

$$N_n = \frac{1}{\sin(\phi)} \left\{ \frac{\partial W}{\partial X_1} \sin(\phi-\alpha) + \frac{\partial W}{\partial X_2} \sin(\alpha) \right\} \quad (17)$$

$$M_n = \frac{-D}{\sin^2(\phi)} \left[C_1 \frac{\partial^2 W}{\partial X_1^2} + C_2 \frac{\partial^2 W}{\partial X_1 \partial X_2} + \sin^2(\alpha) \frac{\partial^2 W}{\partial X_2^2} + \nu \left\{ C_3 \frac{\partial^2 W}{\partial X_1^2} + C_4 \frac{\partial^2 W}{\partial X_1 \partial X_2} + \cos^2(\alpha) \frac{\partial^2 W}{\partial X_2^2} \right\} \right] \quad (18)$$

$$V_n = \frac{-D}{\sin^2(\phi)} \left[C_5 \frac{\partial^3 W}{\partial X_1^3} + C_6 \frac{\partial^3 W}{\partial X_1^2 \partial X_2} + C_7 \frac{\partial^3 W}{\partial X_1 \partial X_2^2} + C_8 \frac{\partial^3 W}{\partial X_2^3} \right] \quad (19)$$

$$M_{nt} = \frac{D(1-\nu)}{\sin^2(\phi)} \left[\frac{\sin(2\alpha-2\phi)}{2} \frac{\partial^2 W}{\partial X_1^2} + \sin(\phi-2\alpha) \frac{\partial^2 W}{\partial X_1 \partial X_2} + \frac{\sin(2\alpha)}{2} \frac{\partial^2 W}{\partial X_2^2} \right] \quad (20)$$

$$R = (M_{nt})_1 - (M_{nt})_2 \quad (21)$$

in which

$$C_1 = \cos^2(\alpha)\sin^2(\phi) - \frac{\sin(2\alpha)\sin(2\phi)}{2} + \sin^2(\alpha)\cos^2(\phi)$$

$$C_2 = \sin(2\alpha)\sin(\phi) - 2\sin^2(\alpha)\cos(\phi)$$

$$C_3 = \sin^2(\alpha)\sin^2(\phi) + \frac{\sin(2\alpha)\sin(2\phi)}{2} + \cos^2(\alpha)\cos^2(\phi)$$

$$C_4 = -\{\sin(2\alpha)\sin(\phi) + 2\cos^2(\alpha)\cos(\phi)\}$$

$$C_5 = \sin(\phi-\alpha) - \frac{(1-\nu)\cos(\phi-\alpha)\sin(2\alpha-2\phi)}{2}$$

$$C_6 = \sin(\alpha) - 2\cos(\phi)\sin(\phi-\alpha) + (1-\nu)\{\frac{\cos(\alpha)\sin(2\alpha-2\phi)}{2}$$

$$- \cos(\phi-\alpha)\sin(\phi-2\alpha)\}$$

$$C_7 = \sin(\phi-\alpha) - 2\cos(\phi)\sin(\alpha) + (1-\nu)\{\cos(\alpha)\sin(\phi-2\alpha)$$

$$- \frac{\cos(\phi-\alpha)\sin(2\alpha)}{2}\}$$

$$C_8 = \sin(\alpha) + (1-\nu)\frac{\cos(\alpha)\sin(2\alpha)}{2}$$

Integral Boundary Equation for Direct Methods

The direct methods of boundary element analysis are those that make use of the generalised green's theorem. In the case of the bending of elastic plates , this theorem is the same as the Betti-Maxwell reciprocal theorem [6] which states that for any two equilibrium states , (A) and (B) , of an elastic body " The work that would be done by the forces (A) if given the displacements (B) is equal to the work that would be done by the forces (B) if given the displacements (A). " or " The displacement at co-ordinate (i) due to a unit force at co-ordinate (j) is equal to the displacement at (j) due to a unit force acting at (i). "

This means that for any elastic plate [6]

$$\int_{\Omega} q_A W_B \, d\Omega + \int_{\Gamma} \{V_A W_B - M_A N_B\} d\Gamma = \int_{\Omega} q_B W_A \, d\Omega + \int_{\Gamma} \{V_B W_A - M_B N_A\} d\Gamma \qquad (22)$$

In which q , V , M , N , W represent distributed or concentrated load , Kirchoff's shear forces , bending moment , slope and deflection respectively. If the boundary (Γ) has corners then it is necessary to add the terms $\sum R_{Ai} W_{Bi}$ and $\sum R_{Bi} W_{Ai}$; $i = 1, 2, \ldots, k$ to the left and right-hand sides of the equation as follows [6] :

$$\int_{\Omega} q_A W_B \, d\Omega + \int_{\Gamma} \{V_A W_B - M_A N_B\} d\Gamma + \sum_{i=1}^{k} R_{Ai} W_{Bi}$$

$$= \int_{\Omega} q_B W_A \, d\Omega + \int_{\Gamma} \{V_B W_A - M_B N_A\} d\Gamma + \sum_{i=1}^{k} R_{Bi} W_{Ai} \qquad (23)$$

Where (R) denotes the effective corner force and summation is made over all corners (k).

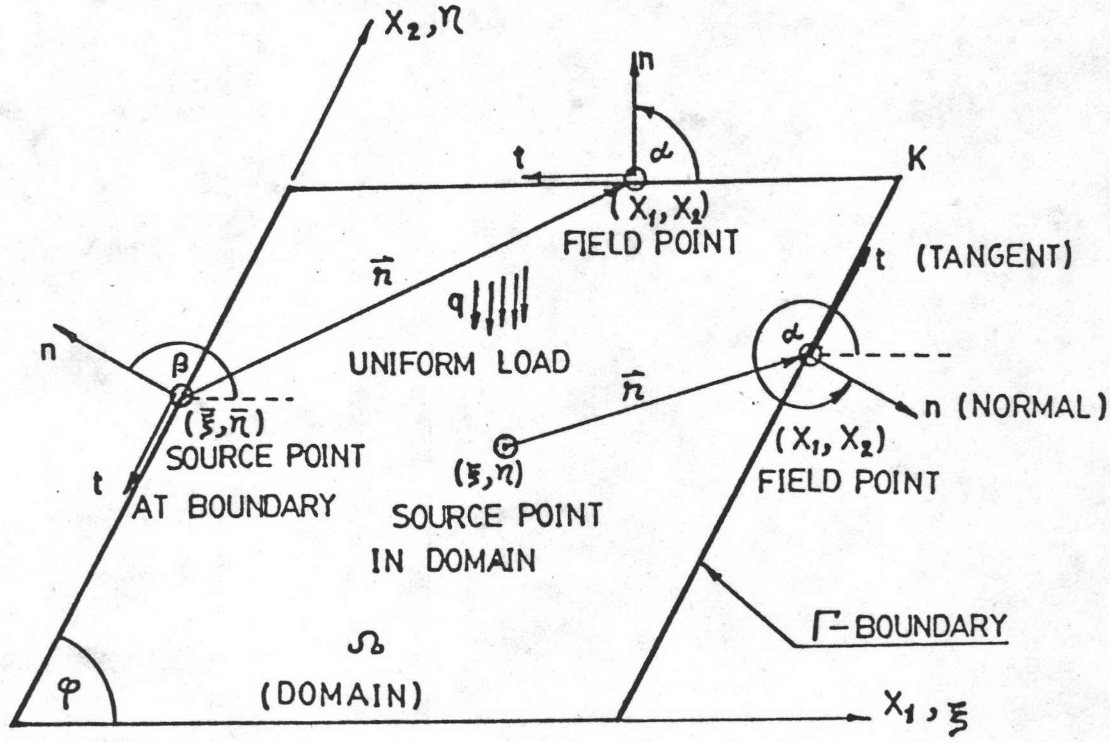

FIG. 4 Co-Ordinate System and Relevant Notation

Now consider two distinct systems of compatible deflections and equilibrium states of stresses as shown in FIG.4, one is a real plate, the problem under consideration , which is the rectilinear plate having arbitrary supports along the boundary (Γ) and loaded by transverse force of intensity (q) , hence all the boundary conditions of the real plate are known . The other is the virtual plate , designated by asterisks , subjected to a unit singular load (dirac delta function " Δ ") acting at a point inside the domain (Ω) which is called a source point (ξ,η) . From eq.(10),the governing equation of the virtual plate is

$$\nabla^4 \overset{*}{W}[X_1,X_2;\xi,\eta] = \Delta[X_1,X_2;\xi,\eta]/D \tag{24}$$

In which $\overset{*}{W}[X_1,X_2;\xi,\eta]$ is defines as the displacement of the virtual plate at field point (X_1,X_2) due to the unit singular load (dirac delta function " Δ ") acting at the source point (ξ,η) . Recall that the dirac delta function $\Delta[X_1,X_2;\xi,\eta]$ has the following properties :

$$\Delta[X_1,X_2;\xi,\eta] = 0 \qquad \text{for} \quad (X_1,X_2) \neq (\xi,\eta)$$

$$\Delta[X_1,X_2;\xi,\eta] = 1 \qquad \text{for} \quad (X_1,X_2) = (\xi,\eta)$$

and

$$\int_\Omega W(X_1,X_2)\Delta(X_1,X_2;\xi,\eta)d\Omega(X_1,X_2) = W(\xi,\eta) \tag{25}$$

From equation (23) , consider system "A" as a real plate and "B" as a virtual plate whose fundamental solution $\overset{*}{W}[X_1,X_2;\xi,\eta]$ (as derived in appendix B) is :

$$\overset{*}{W}[X_1,X_2;\xi,\eta] = \{r^2 \ln(r/Z)\}/8\pi D \tag{26}$$

Where $r = \{(X_1-\xi)^2+2(X_1-\xi)(X_2-\eta)\cos(\phi)+(X_2-\eta)^2\}^{1/2}$ is the distance between source point (ξ,η) and field point (X_1,X_2) and Z is the longest diagonal of plate. The expressions of the slope $(\overset{*}{N})$, bending moment ($\overset{*}{M}$) and Kirchoff's shear forces ($\overset{*}{V}$) in the normal direction of virtual plate can be obtained by appropriate differentiation of equation (26).

Substitution of equation (24) and (25) in to (23) yields :

$$W(\xi,\eta) = \int_\Gamma \{ -\overset{*}{V}[X_1,X_2;\xi,\eta]W[X_1,X_2] + \overset{*}{M}[X_1,X_2;\xi,\eta]N[X_1,X_2]$$

$$-\overset{*}{N}[X_1,X_2;\xi,\eta]M[X_1,X_2] + \overset{*}{W}[X_1,X_2;\xi,\eta]V[X_1,X_2] \}d\Gamma[X_1,X_2]$$

$$+ \sum_{i=1}^{k} R_i[X_1,X_2]\overset{*}{W}_i[X_1,X_2;\xi,\eta] - \sum_{i=1}^{k} W_i[X_1,X_2]\overset{*}{R}_i[X_1,X_2;\xi,\eta]$$

$$+ \int_\Omega q\overset{*}{W}[X_1,X_2;\xi,\eta]d\Omega \qquad (27)$$

Taking the limiting process [4] in which the point (ξ,η) approaches from inside the domain (Ω) to an arbitrary boundary point $(\bar{\xi},\bar{\eta})$, we can derive the following integral equation :

$$\frac{\gamma}{2\pi} W(\bar{\xi},\bar{\eta}) = \int_\Gamma \{ -\overset{*}{V}[X_1,X_2;\bar{\xi},\bar{\eta}]W[X_1,X_2] + \overset{*}{M}[X_1,X_2;\bar{\xi},\bar{\eta}]N[X_1,X_2]$$

$$-\overset{*}{N}[X_1,X_2;\bar{\xi},\bar{\eta}]M[X_1,X_2] + \overset{*}{W}[X_1,X_2;\bar{\xi},\bar{\eta}]V[X_1,X_2] \}d\Gamma[X_1,X_2]$$

$$+ \sum_{i=1}^{k} R_i[X_1,X_2]\overset{*}{W}_i[X_1,X_2;\bar{\xi},\bar{\eta}] - \sum_{i=1}^{k} W_i[X_1,X_2]\overset{*}{R}_i[X_1,X_2;\bar{\xi},\bar{\eta}]$$

$$+ \int_{\Omega} q \overset{*}{W}[X_1, X_2; \bar{\xi}, \bar{\eta}] d\Omega \qquad (28)$$

Where " γ " is the interior angle of the boundary point $(\bar{\xi}, \bar{\eta})$ as shown in FIG.5 .

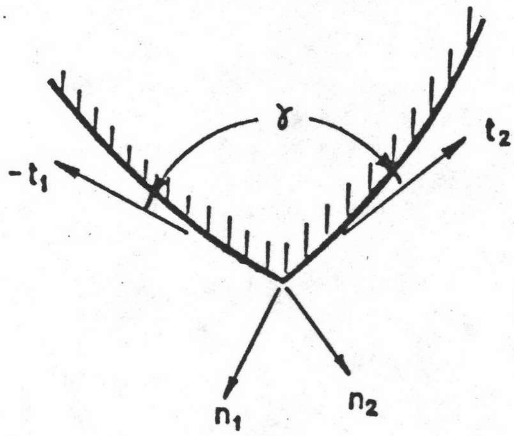

FIG.5 Interior Angle of The Boundary Point

When the boundary is smooth , $\gamma = \pi$, equation (28) leads to

$$\frac{1}{2} W(\bar{\xi}, \bar{\eta}) = \int_{\Gamma} \{ -\overset{*}{V}[X_1, X_2; \bar{\xi}, \bar{\eta}] W[X_1, X_2] + \overset{*}{M}[X_1, X_2; \bar{\xi}, \bar{\eta}] N[X_1, X_2]$$

$$- \overset{*}{N}[X_1, X_2; \bar{\xi}, \bar{\eta}] M[X_1, X_2] + \overset{*}{W}[X_1, X_2; \bar{\xi}, \bar{\eta}] V[X_1, X_2] \} d\Gamma[X_1, X_2]$$

$$+ \sum_{i=1}^{k} R_i[X_1, X_2] \overset{*}{W}_i[X_1, X_2; \bar{\xi}, \bar{\eta}] - \sum_{i=1}^{k} W_i[X_1, X_2] \overset{*}{R}_i[X_1, X_2; \bar{\xi}, \bar{\eta}]$$

$$+ \int_{\Omega} q \overset{*}{W}[X_1, X_2; \bar{\xi}, \bar{\eta}] d\Omega \qquad (29)$$

If the normal derivative of equation (28) is considered , the following integral equation is obtained :

$$\frac{\gamma}{2\pi} N(\xi,\eta) = \int_{\Gamma} \{ -\frac{\partial \overset{*}{V}[X_1,X_2;\xi,\eta]}{\partial n[\xi,\eta]} W[X_1,X_2] + \frac{\partial \overset{*}{M}[X_1,X_2;\xi,\eta]}{\partial n[\xi,\eta]} N[X_1,X_2]$$

$$-\frac{\partial \overset{*}{N}[X_1,X_2;\xi,\eta]}{\partial n[\xi,\eta]} M[X_1,X_2] + \frac{\partial \overset{*}{W}[X_1,X_2;\xi,\eta]}{\partial n[\xi,\eta]} V[X_1,X_2] \} d\Gamma[X_1,X_2]$$

$$+ \sum_{i=1}^{k} R_i[X_1,X_2] \frac{\partial \overset{*}{W}_i[X_1,X_2;\xi,\eta]}{\partial n[\xi,\eta]} - \sum_{i=1}^{k} W_i[X_1,X_2] \frac{\partial \overset{*}{R}_i[X_1,X_2;\xi,\eta]}{\partial n[\xi,\eta]}$$

$$+ \int_{\Omega} q \frac{\partial \overset{*}{W}[X_1,X_2;\xi,\eta]}{\partial n[\xi,\eta]} d\Omega \qquad (30)$$

where

$$N(\xi,\eta) = \frac{\partial W(\xi,\eta)}{\partial n(\xi,\eta)}$$

Equation (28) and (30) are the set of boundary integral equations for the linear bending problems of elastic plates. Since two of the four boundary variables W , N , M and V are prescribed by the boundary conditions , we can determine the remaining unknowns as to be shown in the next chapter.