CHAPTER IV

TREATMENT OF SINGULARITIES

In order to avoid the problem of singularities of the integrals which contain the terms $\ln(r)$, $1/r$ and $1/r^2$, two methods for the calculation of the integrals are introduced in this chapter for the cases of the plate with smooth boundary and that with the corners.

Treatment of Singularities by Equilibrium Equations on the Boundary

In considering a semi-circular plate , it may be more convenient to use polar co-ordinates. The deflection ($\overset{*}{W}$) of the middle surface of the plate subjected to unit load (P) is related to stress resultants , bending and twisting moments and shears force per unit length of the section , as follows :

$$\overset{*}{M}_r = -D\left[\frac{\partial^2 \overset{*}{W}}{\partial r^2} + \nu\left\{\frac{1}{r}\frac{\partial \overset{*}{W}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \overset{*}{W}}{\partial \theta^2}\right\}\right] \quad (43)$$

$$\overset{*}{M}_{r\theta} = D(1-\nu)\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial \overset{*}{W}}{\partial \theta}\right] \quad (44)$$

$$Q_r = -D \frac{\partial(\nabla^2 \overset{*}{W})}{\partial r} \tag{45}$$

$$V_r = Q_r + \frac{1}{r}\frac{\partial M_{\theta r}}{\partial \theta} \tag{46}$$

$$\nabla^2 \overset{*}{W} = \frac{\partial^2 \overset{*}{W}}{\partial r^2} + \frac{1}{r}\frac{\partial \overset{*}{W}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \overset{*}{W}}{\partial \theta^2} \tag{47}$$

From the fundamental solution $\overset{*}{W} = \{ r^2 \ln(r/Z) \}/8\pi D$, the expressions of bending moment , twisting moment , sheer force and concentrated force can be determined by equation (43) through (47) as follows :

$$\overset{*}{M}_r = - [3+\nu+2(1+\nu)\ln(r/Z)]/8\pi \tag{48}$$

$$\overset{*}{M}_{r\theta} = 0 \tag{49}$$

$$\overset{*}{Q}_r = -1/2\pi r \tag{50}$$

$$\overset{*}{V}_r = -1/2\pi r \tag{51}$$

$$\overset{*}{R} = 2M_{r\theta} = 0 \tag{52}$$

Consider the equilibrium of a semi-circular plate of radius E in which a unit load is applied at the center of the circle (0,0) as shown in FIG.7

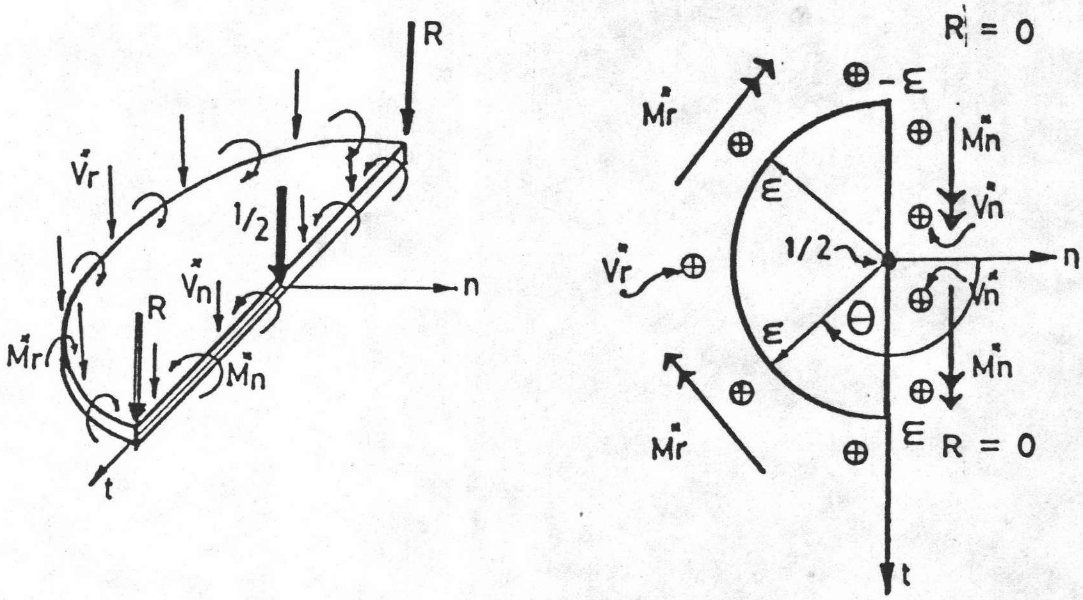

FIG.7 Equilibrium of a Semi-Circular Plate with Unit Load

The total vertical shear along the straight edge can be found from the condition of zero vertical force [$\sum F_z = 0$] as :

$$\int_{-\epsilon}^{\epsilon} \overset{*}{V}_n\,dt + 1/2 + \int_{\pi/2}^{3\pi/2} \overset{*}{V}_r \Big|_{r=\epsilon} \epsilon\,d\theta + \overset{*}{R}\Big|_{\substack{t=\epsilon\\ n=0}} + \overset{*}{R}\Big|_{\substack{t=-\epsilon\\ n=0}} = 0$$

$$\int_{-\epsilon}^{\epsilon} \overset{*}{V}_n\,dt = -1/2 + \int_{\pi/2}^{3\pi/2} 1/2\pi\,d\theta = 0 \qquad (53)$$

The total normal bending moment along the same edge can be obtained by considering the moment equilibrium along t-axis [$\sum M = 0$] as :

$$\int_{-\varepsilon}^{\varepsilon} \overset{*}{M}_n \, dt + \int_{\pi/2}^{3\pi/2} \overset{*}{M}_r \Big|_{r=\varepsilon} \varepsilon\cos(\theta)d\theta - \int_{\pi/2}^{3\pi/2} \overset{*}{V}_r \Big|_{r=\varepsilon} \varepsilon^2\cos(\theta)\, d\theta = 0$$

$$\int_{-\varepsilon}^{\varepsilon} \overset{*}{M}_n \, dt = 2\varepsilon\overset{*}{M}_r - 2\varepsilon^2\overset{*}{V}_r$$

$$\int_{-\varepsilon}^{\varepsilon} \overset{*}{M}_n \, dt = \varepsilon\,[\, 1-\nu-2(1+\nu)\ln(\varepsilon/Z)\,]/4\pi \tag{54}$$

Similarly , for the case of a unit couple applied at center of the circle (0,0) , the expression of the deflection [$\overset{**}{W}$] and the corresponding stress resultants can be obtained by substitute [$\overset{**}{W}$] into the equation (43) through (46) as follows:

$$\overset{**}{W} = \partial\overset{*}{W}/\partial n = -\, r\cos(\theta)[2\ln(r/Z)+1]/8\pi d \tag{55}$$

$$\overset{**}{M}_r = (1+\nu)\cos(\theta)/4\pi r \tag{56}$$

$$\overset{**}{M}_{r\theta} = (1-\nu)\sin(\theta)/4\pi r \tag{57}$$

$$\overset{**}{Q}_r = -\cos(\theta)/2\pi r^2 \tag{58}$$

$$\overset{**}{V}_r = -(3-\nu)\cos(\theta)/4\pi r^2 \tag{59}$$

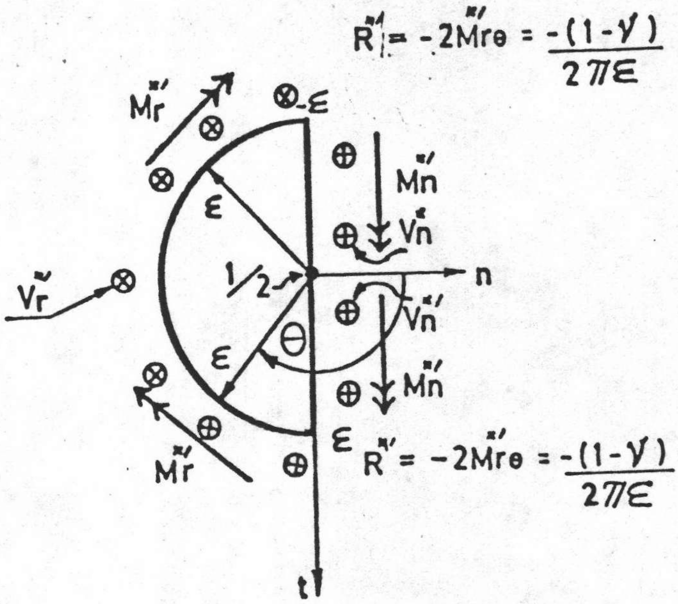

FIG.8 Equilibrium of a Semi-Circular Plate with Unit Couple

In the same manner as before , the total vertical shear forces along the straight edge can be found from the condition of zero vertical shear force [$\sum F_z = 0$]

$$\int_{-\epsilon}^{\epsilon} \overset{*\prime}{V}_n \, dt + \int_{\pi/2}^{3\pi/2} \overset{*\prime}{V}_r \Big|_{r=\epsilon} \epsilon \, d\theta + \overset{*\prime}{R} \Big|_{\substack{t=\epsilon \\ n=0}} + \overset{*\prime}{R} \Big|_{\substack{t=-\epsilon \\ n=0}} = 0$$

$$\int_{-\epsilon}^{\epsilon} \overset{*\prime}{V}_n \, dt + (3-\nu)/2\pi\epsilon - (1-\nu)/2\pi\epsilon - (1-\nu)/2\pi\epsilon = 0$$

$$\int_{-\epsilon}^{\epsilon} \overset{**}{V}_n \, dt = -(1+\nu)/2\pi\epsilon \tag{60}$$

Considering the moment equilibrium [$\sum \overset{**}{M}_n = 0$]

$$\int_{-\epsilon}^{\epsilon} \overset{**}{M}_n \, dt + \int_{\pi/2}^{3\pi/2} \overset{**}{M}_r \Big|_{r=\epsilon} \epsilon\cos(\theta) d\theta - \int_{\pi/2}^{3\pi/2} \overset{**}{V}_r \Big|_{r=\epsilon} \epsilon^2 \cos(\theta)\, d\theta - 1/2 = 0$$

$$\int_{-\epsilon}^{\epsilon} \overset{**}{M}_n \, dt + \frac{(1+\nu)}{4\pi} \int_{\pi/2}^{3\pi/2} \cos^2(\theta) d\theta + \frac{(3-\nu)}{4\pi} \int_{\pi/2}^{3\pi/2} \cos^2(\theta) d\theta - 1/2 = 0$$

$$\int_{-\epsilon}^{\epsilon} \overset{**}{M}_n dt + (1+\nu)/8 + (3-\nu)/8 - 1/2 = 0$$

$$\int_{-\epsilon}^{\epsilon} \overset{**}{M}_n dt = 0 \tag{61}$$

From equation (53), (54), (60) and (61), by substituting $\epsilon = (A-B)/2$ on X_1-axis and $\epsilon = L(A-B)/2$ on X_2-axis.

where A = upper integral limit

B = lower integral limit

L = dimension ratio (b/a)

One can evaluate singularity terms in skew co-ordinate as follows :

$$\int_B^A \overset{*}{M}_n dX_1 = (A-B)[\,1-\nu-2(1+\nu)\ln((A-B)/2Z)\,]/8\pi \tag{62}$$

$$\int_B^A \overset{*}{M}_n dX_2 = L(A-B)[\,1-\nu-2(1+\nu)\ln(L(A-B)/2Z)\,]/8\pi \tag{63}$$

$$\int_B^A \overset{*}{V}_n dX_1 = \int_B^A \overset{*}{V}_n dX_2 = 0 \tag{64}$$

$$\int_B^A \overset{**}{M}_n dX_1 = \int_B^A \overset{**}{M}_n dX_2 = 0 \tag{65}$$

$$\int_B^A \overset{**}{V}_n dX_1 = -(1+\nu)/(A-B)\pi \tag{66}$$

$$\int_B^A \overset{**}{V}_n dX_2 = -(1+\nu)/(A-B)\pi L \tag{67}$$

Singularities at Corner

Consider the unit force acting on the corner of the plate and investigate the equilibrium condition of the free-body of the circular sector element of the plate corner as shown in FIG.9 .

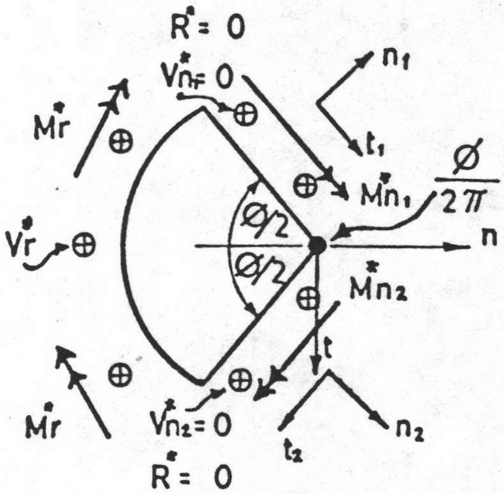

FIG.9 Equilibrium of a Circular Sector with Unit Load

From equation (49) , (52) and the relation of

$$\overset{*}{V}_n = -\frac{D}{r}\frac{\partial(\nabla^2 \overset{*}{W})}{\partial \theta} + \frac{\partial \overset{*}{M}_{r\theta}}{\partial r} = 0$$

It can be concluded that the corner reaction [$\overset{*}{R}$] and Kirchoff's shear force are always zero. Thus $\int \overset{*}{V}_n dt = 0$ the normal bending moment along the corner edges can be obtained by considering the moment equilibrium about t-axis and n-axis as follows :

[t-axis $\sum M = 0$]

$$\int_0^{\epsilon} \overset{*}{M}_{n1} \sin(\phi/2)dt_1 + \int_0^{\epsilon} \overset{*}{M}_{n2} \sin(\phi/2)dt_2 + \int_{\pi-\phi/2}^{\pi+\phi/2} \overset{*}{M}_r \epsilon\cos(\theta)d\theta - \int_{\pi-\phi/2}^{\pi+\phi/2} \overset{*}{V}_r \epsilon^2\cos(\theta)d\theta = 0$$

$$\int_0^{\epsilon} \overset{*}{M}_{n1} \sin(\phi/2)dt_1 + \int_0^{\epsilon} \overset{*}{M}_{n2} \sin(\phi/2)dt_2 = \epsilon\sin(\phi/2)[1-\nu-2(1+\nu)\ln(\epsilon/Z)]/4\pi$$

$$\int_0^{\epsilon} \overset{*}{M}_{n1} dt_1 + \int_0^{\epsilon} \overset{*}{M}_{n2} dt_2 = \epsilon[1-\nu-2(1+\nu)\ln(\epsilon/Z)]/4\pi \qquad (68)$$

[n-axis $\sum M = 0$]

$$\int_0^{\epsilon} \overset{*}{M}_{n1} \cos(\phi/2)dt_1 - \int_0^{\epsilon} \overset{*}{M}_{n2} \cos(\phi/2)dt_2 - \int_{\pi-\phi/2}^{\pi+\phi/2} \overset{*}{M}_r \epsilon\sin(\phi)d\theta + \int_{\pi-\phi/2}^{\pi+\phi/2} \overset{*}{V}_r \epsilon^2\sin(\phi)d\theta = 0$$

$$\int_0^{\epsilon} \overset{*}{M}_{n1} dt_1 - \int_0^{\epsilon} \overset{*}{M}_{n2} dt_2 = 0 \qquad (69)$$

$$\int_0^{\epsilon} \overset{*}{M}_{n1} dt_1 + \int_0^{\epsilon} \overset{*}{M}_{n2} dt_2 = \epsilon[1-\nu-2(1+\nu)\ln(\epsilon/Z)]/8\pi \qquad (70)$$

Treatment of Singularities by Analysis on Smooth Boundary

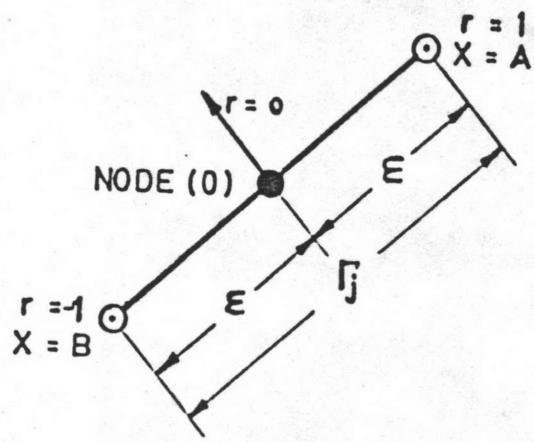

FIG.10 Transforming Co-Crdinates on Smooth Boundary Elements

define :

A,B = upper and lower integral limit

f(X) = integration function

$\varepsilon = (A-B)/2$

The following integrals can be expressed as :

$$\int_B^A f(X)dX = \int_{-\varepsilon}^{\varepsilon} f(X)dX = 2\int_0^{\varepsilon} f(X)dX \qquad (71)$$

$$\int_B^A f(X)dX = 2\varepsilon \int_0^1 f(\varepsilon r)dr \qquad (72)$$

where $X = \varepsilon r$

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Example A)

Let : $f(X) = \ln(X)$

$$\int_B^A \ln(X)dX = 2\varepsilon \int_0^1 \ln(\varepsilon r)dr$$

$$= 2\varepsilon[r\ln(r)-r+r\ln(\varepsilon)]^1_0$$

$$= 2\varepsilon[\ln(\varepsilon)-1] \qquad ; \quad \varepsilon = (A-B)/2$$

Treatment of Singularities by Analysis at Corner

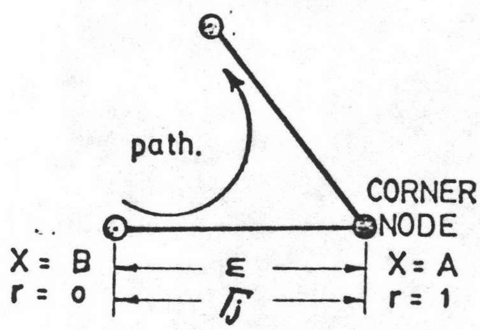

FIG.11 Transforming Co-Ordinate at Corner Elements

The same technique as mentioned before can be used by substituting " ε = A-B " in equation (71)

$$\int_B^A f(X)dX = \int_0^\varepsilon f(X)dX \qquad (72)$$

$$\int_B^A f(X)dX = E\int_0^1 f(Er)dr \tag{74}$$

where $(X = Er)$

Example B)

Let : $f(X) = \ln(X)$

$$\int_B^A \ln(X)dX = E\int_0^1 \ln(Er)dr$$

$$= E[r\ln(r)-r+r\ln(E)]^1_0$$

$$= E[\ln(E)-1] \qquad ; \quad E = (A-B)$$