CHAPTER III

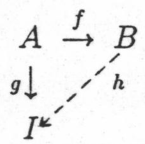
$\mathfrak{C}_S$-INJECTIVE SEMIMODULES

This chapter is primarily about injectivity in the category $\mathfrak{C}_S$ of cancellative semimodules over a semiring S. However, some of the principal definitions and results are stated for an arbitrary category of semimodules. This chapter is divided into four parts. The first introduces injectivity in a category $\mathfrak{M}$ of semimodules. The second is about properties of $\mathfrak{C}_S$-injective semimodules; in particular, it answers the important question of the existence of nonzero $\mathfrak{C}_S$-injective semimodules. The two last contain very important results of this research, namely the connection between essential extensions and $\mathfrak{C}_S$-injectivity, and the existence and uniqueness of $\mathfrak{C}_S$-injective hulls.

3.1. $\mathfrak{M}$-injective semimodules

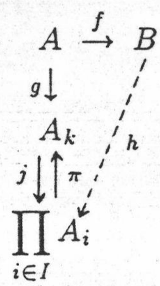
The most obvious way to define injectivity for semimodules is to copy the definition for modules over a ring almost verbatim, simply changing "ring" to "semiring" and "module" to "semimodule". Indeed, this is the approach taken in *[1]*. However, it has not met with much success. Almost none of the standard results about injective modules over a ring have been proved, and Proposition 15.17 of *[1]* implies that even the apparently "nice" semiring $\mathbb{Z}_0^+$ has no nonzero injective semimodules, if this definition is used. An examination of the proof of the proposition just cited shows that the trouble is caused by the existence of non-cancellative semimodules, even over a cancellative semiring. This immediately suggests a solution: only allow cancellative semimodules to be used. I will formalize this idea by using categories.

Definition 3.1.1. *Let $\mathfrak{M}$ be a category of S-semimodules. An element I of $\mathfrak{M}$ will be called* **$\mathfrak{M}$-injective** *iff for each pair of elements A and B of $\mathfrak{M}$, each S-monomorphism $f: A \to B$, and each S-homomorphism $g: A \to I$, there exists an S-homomorphism $h: B \to I$ such that $g = h \circ f$ (i.e., the following diagram commutes).*



Proposition 3.1.2. *Let $\mathfrak{M}$ be a category of S-semimodules and $(A_i)_{i \in I}$ a family of elements of $\mathfrak{M}$ such that $\prod_{i \in I} A_i$ is in $\mathfrak{M}$. Then $\prod_{i \in I} A_i$ is $\mathfrak{M}$-injective iff A_i is $\mathfrak{M}$-injective for all $i \in I$.*

Proof. Assume that $\prod_{i \in I} A_i$ is $\mathfrak{M}$-injective. Fix $k \in I$. Let A, $B \in \mathfrak{M}$, and $f: A \to B$ and $g: A \to A_k$ be S-homomorphisms such that f is injective. Let $\pi: \prod_{i \in I} A_i \to A_k$ be the canonical projection and $j: A_k \to \prod_{i \in I} A_i$ the canonical injection. Since $\prod_{i \in I} A_i$ is $\mathfrak{M}$-injective, there exists an S-homomorphism $h: B \to \prod_{i \in I} A_i$ such that $j \circ g = h \circ f$. Then $\pi \circ h: B \to A_k$ and $(\pi \circ h) \circ f = \pi \circ (h \circ f) = \pi \circ (j \circ g) = (\pi \circ j) \circ g = g$. Hence A_k is $\mathfrak{M}$-injective.



Conversely, assume that A_i is $\mathfrak{M}$-injective for all $i \in I$. Let A, $B \in \mathfrak{M}$, and $f: A \to B$ and $g: A \to \prod_{i \in I} A_i$ be S-homomorphisms such that f is injective. Fix $i \in I$, let $\pi_i: \prod_{j \in I} A_j \to A_i$ be the canonical projection, and set $g_i = \pi_i \circ g$. Since A_i is $\mathfrak{M}$-injective, there exists an S-homomorphism $h_i: B \to A_i$ such that $g_i = h_i \circ f$. By the universal mapping property of direct products, there exists a unique S-homomorphism $h: B \to \prod_{j \in I} A_j$ such that $h_i = \pi_i \circ h$ for all $i \in I$. Since g is the unique S-homomorphism mapping A to $\prod_{j \in I} A_j$ such that $g_i = \pi_i \circ g$ for all $i \in I$, to show $g = h \circ f$ it suffices to show $\pi_i \circ (h \circ f) = g_i$ for all $i \in I$. Fix $i \in I$; then $\pi_i \circ (h \circ f) = h_i \circ f = g_i$. Hence $g = h \circ f$, so $\prod_{j \in I} A_j$ is $\mathfrak{M}$-injective. #

Let $\mathfrak{C}_S$ denote the category of all cancellative S-semimodules.

Corollary 3.1.3. *Let $(A_i)_{i \in I}$ be a family of elements of the category $\mathfrak{C}_S$. Then $\prod_{i \in I} A_i$ is an element of $\mathfrak{C}_S$, and thus $\prod_{i \in I} A_i$ is $\mathfrak{C}_S$-injective iff A_i is $\mathfrak{C}_S$-injective for all $i \in I$.*

Proof. Note that $\prod_{i \in I} A_i \in \mathfrak{C}_S$ by *[1], Proposition 13.53.* The rest is obvious. #

3.2. $\mathfrak{C}_S$-injective semimodules

The main result of this section is Theorem 3.2.7, which says that every cancellative semimodule may be embedded in a $\mathfrak{C}_S$-injective semimodule. Proposition 3.2.6 plays a major role in its proof, and also provides a class of specific examples of $\mathfrak{C}_S$-injective semimodules. As we shall see in this section and the next, the key to many of these proofs is the fact that a cancellative semimodule may be embedded in its group of differences.

Before discussing $\mathfrak{C}_S$-injective semimodules, I need to say a little about injective $\mathbb{Z}$-modules.

Definition 3.2.1. *The abelian group D is said to be* **divisible** *iff for every element x of D and every nonzero integer n, there exists an element y of D such that $ny = x$.*

Theorem 3.2.2. *([2], Chapter 1, Theorem 6) An abelian group D is an injective $\mathbb{Z}$-module if and only if D is a divisible group.*

Lemma 3.2.3. *The relation $\approx \; = \{(q_1, q_2) \mid q_1, q_2 \in \mathbb{Q}_0^+ \text{ and } |q_1 - q_2| \in \mathbb{Z}_0^+\}$ is a $\mathbb{Z}_0^+$-congruence on $\mathbb{Q}_0^+$.*

Proof. Clear. #

Notation. Denote $\mathbb{Q}_0^+/\approx$ by $\mathbb{Q}_0^+/\mathbb{Z}_0^+$.

Proposition 3.2.4. *$\mathbb{Q}_0^+/\mathbb{Z}_0^+$ is a divisible group.*

Proof. First, I must show that $\mathbb{Q}_0^+/\mathbb{Z}_0^+$ is an abelian group. Since $\mathbb{Q}_0^+$ is a $\mathbb{Z}_0^+$-semimodule, $\mathbb{Q}_0^+/\mathbb{Z}_0^+$ is also a $\mathbb{Z}_0^+$-semimodule. Thus it only remains to show that every element of $\mathbb{Q}_0^+/\mathbb{Z}_0^+$ has an inverse. Let $[q]_\approx$ be an element of $\mathbb{Q}_0^+/\mathbb{Z}_0^+$. Then there is a positive integer m_q such that $m_q - 1 \leq q < m_q$. Thus $m_q - q \in \mathbb{Q}_0^+$ and $[q]_\approx + [m_q - q]_\approx = [q + m_q - q]_\approx = [m_q]_\approx = [0]_\approx$. Hence $[m_q - q]_\approx = -[q]_\approx$. Therefore, $\mathbb{Q}_0^+/\mathbb{Z}_0^+$ is an abelian group.

Next, let $[q]_\approx$ be an element of $\mathbb{Q}_0^+/\mathbb{Z}_0^+$ and n a nonzero integer. If $n > 0$, then $[q/n]_\approx \in \mathbb{Q}_0^+/\mathbb{Z}_0^+$ and $n[q/n]_\approx = [q]_\approx$. If $n < 0$, then $-[q/(-n)]_\approx \in \mathbb{Q}_0^+/\mathbb{Z}_0^+$ and $n(-[q/(-n)]_\approx) = [q]_\approx$. Therefore, $\mathbb{Q}_0^+/\mathbb{Z}_0^+$ is divisible. #

Lemma 3.2.5. *Let D be a divisible group and S a semiring. Then $\mathrm{Hom}_{\mathbb{Z}_0^+}(S, D)$ is a $\mathfrak{C}_S$-injective left group semimodule.*

Proof. Note that by Proposition 2.1.12 $\mathrm{Hom}_{\mathbb{Z}_0^+}(S, D)$ is an abelian group, and in addition is a left S-semimodule using the scalar multiplication defined on it as follows: if $s \in S$ and $f \in \mathrm{Hom}_{\mathbb{Z}_0^+}(S, D)$, then $sf: S \to D$ is given by

$$(sf)(s') = f(s's) \qquad \text{for every} \quad s' \in S.$$

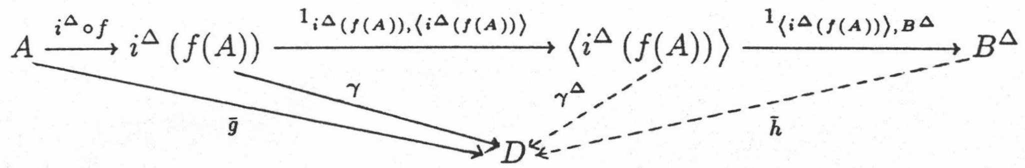
Observe that since $\mathrm{Hom}_{\mathbb{Z}_0^+}(S, D)$ is a group, it is certainly cancellative.

To show that $\mathrm{Hom}_{\mathbb{Z}_0^+}(S, D)$ is $\mathfrak{C}_S$-injective, consider the following diagram of S-homomorphisms and elements of $\mathfrak{C}_S$ with f injective:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \\ \mathrm{Hom}_{\mathbb{Z}_0^+}(S, D) & & \end{array}$$

I must show the existence of $h: B \to \mathrm{Hom}_{\mathbb{Z}_0^+}(S, D)$ such that $h \circ f = g$.

Let $\bar{g}: A \to D$ be defined by $\bar{g}(x) = [g(x)](1)$ for every $x \in A$. Then $\bar{g} \in \mathrm{Hom}_{\mathbb{Z}_0^+}(A, D)$. Let $i^\Delta: B \to B^\Delta$ be the standard embedding. Then $\gamma = \bar{g} \circ f^{-1} \circ (i^\Delta)^{-1}: i^\Delta(f(A)) \to D$ is a homomorphism and $A \cong i^\Delta(f(A))$, which is a submonoid of B^Δ. Thus, we have the following diagram, where $\langle i^\Delta(f(A))\rangle$ denotes the subgroup of B^Δ generated by $i^\Delta(f(A))$.



Note that B^{Δ}, $\langle i^{\Delta}(f(A))\rangle$, and D are groups. By Proposition 2.3.6, there exists a unique group homomorphism $\gamma^{\Delta}\colon \langle i^{\Delta}(f(A))\rangle \to D$ such that $\gamma^{\Delta}|_{i^{\Delta}(f(A))} = \gamma$. Since D is divisible, D is an injective $\mathbb{Z}$-module by Theorem 3.2.2, so there exists a group homomorphism $\bar{h}\colon B^{\Delta} \to D$ such that $\bar{h} \circ 1_{\langle i^{\Delta}(f(A))\rangle, B^{\Delta}} = \gamma^{\Delta}$.

I claim that $\bar{h} \circ i^{\Delta} \circ f = \bar{g}$. Let a be an element of A. Note that $(i^{\Delta} \circ f)(a) \in \langle i^{\Delta}(f(A))\rangle$ and $\gamma^{\Delta}|_{i^{\Delta}(f(A))}(i^{\Delta}(f(a))) = (\gamma \circ i^{\Delta} \circ f)(a)$, which equals $\bar{g}(a)$ from the diagram. Thus $(\bar{h} \circ i^{\Delta} \circ f)(a) = \bar{h} \circ 1_{\langle i^{\Delta}(f(a))\rangle, B^{\Delta}}(i^{\Delta}(f(a))) = \gamma^{\Delta}(i^{\Delta}(f(a))) = \bar{g}(a)$, so the claim is proved.

Now, define $h\colon B \to \mathrm{Hom}_{\mathbb{Z}_0^+}(S, D)$ by $h(y)(s) = \bar{h} \circ i^{\Delta}(sy)$ for all $y \in B$ and all $s \in S$. Clearly $h(y) \in \mathrm{Hom}_{\mathbb{Z}_0^+}(S, D)$ for all $y \in B$. Let x, y be elements of B and t, s elements of S. Then $h(x+y)(s) = \bar{h} \circ i^{\Delta}(s(x+y)) = \bar{h} \circ i^{\Delta}(sx+sy) = h(x)(s) + h(y)(s)$, and $h(tx)(s) = \bar{h} \circ i^{\Delta}(s(tx)) = \bar{h} \circ i^{\Delta}((st)x) = h(x)(st) = (th(x))(s)$. Hence h is an S-homomorphism.

Finally, I will show that $g = h \circ f$. Let a be an element of A and s an element of S. Then $[(h \circ f)(a)](s) = [h(f(a))](s) = \bar{h} \circ i^{\Delta}(f(sa)) = \bar{g}(sa) = [g(sa)](1) = [sg(a)](1) = [g(a)](1s) = [g(a)](s)$. Hence $h \circ f = g$.

Therefore, $\mathrm{Hom}_{\mathbb{Z}_0^+}(S, D)$ is a $\mathfrak{C}_S$-injective left group semimodule. #

Proposition 3.2.6. *Let F be a free right S-semimodule and D a divisible abelian group. Then $\mathrm{Hom}_{\mathbb{Z}_0^+}(F, D)$ is a $\mathfrak{C}_S$-injective left group semimodule.*

Proof. By Proposition 2.1.12, $\mathrm{Hom}_{\mathbb{Z}_0^+}(F, D)$ is a left group S-semimodule, so it only remains to show it is $\mathfrak{C}_S$-injective. Let B be a subset of F such that every element a of F can be written uniquely as $a = \sum_{b \in B} bs_b$, where $s_b \in S$ for all $b \in B$ and only finitely many of the s_b's are nonzero. Let $M_b = S$ for all $b \in B$.

I claim that $\mathrm{Hom}_{\mathbb{Z}_0^+}(F, D) \cong \prod_{b \in B} \mathrm{Hom}_{\mathbb{Z}_0^+}(M_b, D)$. (As motivation, note that by Proposition 2.5.2(iii), $F \cong \oplus\sum_{b \in B} M_b$.) Define a map $\phi\colon \mathrm{Hom}_{\mathbb{Z}_0^+}(F, D) \to \prod_{b \in B} \mathrm{Hom}_{\mathbb{Z}_0^+}(M_b, D)$ as follows: for each $\lambda \in \mathrm{Hom}_{\mathbb{Z}_0^+}(F, D)$, let

$$\phi(\lambda) = (\lambda_b)_{b \in B},$$

where $\lambda_b \in \mathrm{Hom}_{\mathbb{Z}_0^+}(M_b, D)$ is given by $\lambda_b(s) = \lambda(bs)$ for all $s \in S$. Then it is easily checked that ϕ is a group homomorphism and ϕ is injective. I will show that ϕ is surjective. Let $(\gamma_b)_{b \in B} \in \prod_{b \in B} \mathrm{Hom}_{\mathbb{Z}_0^+}(M_b, D)$. Define $\lambda\colon F \to D$ by $\lambda\left(\sum_{b \in B} bs_b\right) = \sum_{b \in B} \gamma_b(s_b)$. Then λ is well-defined and $\lambda \in \mathrm{Hom}_{\mathbb{Z}_0^+}(F, D)$. It is easy to show that $\lambda_b = \gamma_b$ for all $b \in B$. Hence $\phi(\lambda) = (\gamma_b)_{b \in B}$ and ϕ is surjective. Thus $\mathrm{Hom}_{\mathbb{Z}_0^+}(F, D) \cong \prod_{b \in B} \mathrm{Hom}_{\mathbb{Z}_0^+}(M_b, D)$. By Lemma 3.2.5, $\mathrm{Hom}_{\mathbb{Z}_0^+}(M_b, D) = \mathrm{Hom}_{\mathbb{Z}_0^+}(S, D)$ is $\mathfrak{C}_S$-injective and by Corollary 3.1.3, $\prod_{b \in B} \mathrm{Hom}_{\mathbb{Z}_0^+}(M_b, D)$ is $\mathfrak{C}_S$-injective. Therefore $\mathrm{Hom}_{\mathbb{Z}_0^+}(F, D)$ is also $\mathfrak{C}_S$-injective. #

Theorem 3.2.7. *Every cancellative S-semimodule A is a subsemimodule of a $\mathfrak{C}_S$-injective group semimodule.*

Proof. By Proposition 2.1.13, it suffices to show that A is embedded in a $\mathfrak{C}_S$-injective group semimodule. Let $D = \mathbb{Q}_0^+/\mathbb{Z}_0^+$. Then D is a divisible group, all of its elements are of finite order, and for each positive integer n, D has an element of order n. Let $\tilde{A} = \mathrm{Hom}_{\mathbb{Z}_0^+}(A, D)$. Note that $\tilde{A}$ is a right S-semimodule by the note following Proposition 2.1.12, where for each $s \in S$ and $\lambda \in \mathrm{Hom}_{\mathbb{Z}_0^+}(A, D)$ λs is defined by $(\lambda s)(a) = \lambda(sa)$ (for all $a \in A$).

By Proposition 2.5.2(iv), there is a free right S-semimodule F and a surjective map $\alpha: F \to \tilde{A}$. Define an S-homomorphism $\alpha^*: \mathrm{Hom}_{\mathbb{Z}_0^+}(\tilde{A}, D) \to \mathrm{Hom}_{\mathbb{Z}_0^+}(F, D)$ by $\alpha^*(\lambda) = \lambda \circ \alpha$ for all $\lambda \in \mathrm{Hom}_{\mathbb{Z}_0^+}(\tilde{A}, D)$. It is easy to show that α^* is injective. Hence $\mathrm{Hom}_{\mathbb{Z}_0^+}(\tilde{A}, D)$ can be embedded in $\mathrm{Hom}_{\mathbb{Z}_0^+}(F, D)$, which is a $\mathfrak{C}_S$-injective left group semimodule by Proposition 3.2.6. To finish the proof, it suffices to show A can be embedded in $\mathrm{Hom}_{\mathbb{Z}_0^+}(\tilde{A}, D)$.

Define $\theta: A \to \mathrm{Hom}_{\mathbb{Z}_0^+}(\tilde{A}, D)$ by $\theta(a)(\tilde{a}) = \tilde{a}(a)$ for all $a \in A$ and all $\tilde{a} \in \tilde{A}$. Then θ is an S-homomorphism, so it only remains to show that θ is injective. By Proposition 2.3.5, there exists a unique S-homomorphism $\bar{\theta}: A^{\Delta} \to \mathrm{Hom}_{\mathbb{Z}_0^+}(\tilde{A}, D)$ such that $\bar{\theta} \circ i^{\Delta} = \theta$, where $i^{\Delta}: A \to A^{\Delta}$ is the standard embedding. Because i^{Δ} is injective, it is enough to show that $\bar{\theta}$ is injective, and since A^{Δ} and $\mathrm{Hom}_{\mathbb{Z}_0^+}(\tilde{A}, D)$ are both groups, it suffices to show $\bar{\theta}$ maps nonzero elements to nonzero elements.

Let a be a nonzero element of A^{Δ}. I claim there exists a $\mathbb{Z}$-homomorphism $\bar{\lambda}: A^{\Delta} \to D$ such that $d = \bar{\lambda}(a)$ is nonzero. There are two cases:

Case I. a has infinite order.

Then $\langle a \rangle$, the subgroup of A^{Δ} generated by a, is a free $\mathbb{Z}$-module. Let d be a nonzero element of D and define a group homomorphism $\lambda: \langle a \rangle \to D$

by $\lambda(ka) = kd$ for all $k \in \mathbb{Z}$. Since D is divisible, D is an injective $\mathbb{Z}$-module. Consider the following diagram:

$$\begin{array}{ccc} \langle a \rangle & \xrightarrow{\lambda} & D \\ {\scriptstyle 1_{\langle a \rangle, A^{\Delta}}} \downarrow & \nearrow {\scriptstyle \bar{\lambda}} & \\ A^{\Delta} & & \end{array}$$

Then there exists a $\mathbb{Z}$-homomorphism $\bar{\lambda}\colon A^{\Delta} \to D$ such that $\lambda = \bar{\lambda} \circ 1_{\langle a \rangle, A^{\Delta}} = \bar{\lambda}|_{\langle a \rangle}$. In particular, $\bar{\lambda}(a) = \bar{\lambda}|_{\langle a \rangle}(a) = \lambda(a) = d$, which is nonzero.

Case II. a has finite order.

Then there is an element d of D such that $o(d) = o(a)$. Hence $\langle d \rangle \cong \langle a \rangle$. Let $\lambda\colon \langle a \rangle \to \langle d \rangle$ be the isomorphism such that $\lambda(a) = d$. Clearly $d \neq 0$. As in Case I, there exists a $\mathbb{Z}$-homomorphism $\bar{\lambda}\colon A^{\Delta} \to D$ such that $\lambda = \bar{\lambda}|_{\langle a \rangle}$, and consequently $d = \bar{\lambda}(a) \neq 0$.

To show $\bar{\theta}(a) \neq 0$, let $\gamma = \bar{\lambda} \circ i^{\Delta}$. Then $\gamma\colon A \to D$, so $\gamma \in \tilde{A}$. Note that because $a \in A^{\Delta}$, there exist elements a_1, a_2 of A such that $a = i^{\Delta}(a_1) - i^{\Delta}(a_2)$. Thus $\bar{\theta}(a) = \bar{\theta} \circ i^{\Delta}(a_1) - \bar{\theta} \circ i^{\Delta}(a_2) = \theta(a_1) - \theta(a_2)$, so $\bar{\theta}(a)(\gamma) = \theta(a_1)(\bar{\lambda} \circ i^{\Delta}) - \theta(a_2)(\bar{\lambda} \circ i^{\Delta}) = \bar{\lambda} \circ i^{\Delta}(a_1) - \bar{\lambda} \circ i^{\Delta}(a_2) = \bar{\lambda}(a) = d \neq 0$. Hence $\bar{\theta}(a) \neq 0$, so $\bar{\theta}$ is injective.

Therefore, A can be embedded in $\mathrm{Hom}_{\mathbb{Z}_0^+}(\tilde{A}, D)$, which can be embedded in the $\mathfrak{C}_S$-injective group semimodule $\mathrm{Hom}_{\mathbb{Z}_0^+}(F, D)$, so A is embedded in a $\mathfrak{C}_S$-injective group semimodule. #

Corollary 3.2.8. *If there exists a nonzero cancellative S-semimodule, then there exists a nonzero $\mathfrak{C}_S$-injective semimodule.*

Proof. This follows from Theorem 3.2.7. #

The hypothesis that there exists a nonzero cancellative S-semimodule in the above corollary is necessary, because there are semirings with no nonzero cancellative semimodules, as the following example shows.

Example. Consider $S = \mathbb{Z}_0^+ \cup \{\infty\}$ as mentioned in Chapter I. Then the only cancellative S-semimodule is $\{0\}$.

Proof. Let A be a cancellative S-semimodule. If $x \in A$, then $\infty \cdot x = (\infty + 1) \cdot x = \infty \cdot x + x$ which implies that $x = 0$. Hence $A = \{0\}$. #

Corollary 3.2.9. *Let $D = \mathbb{Q}_0^+/\mathbb{Z}_0^+$. If there exists a nonzero cancellative S-semimodule, then $\mathrm{Hom}_{\mathbb{Z}_0^+}(S, D)$ is nonzero.*

Proof. This follows from the proofs of Theorem 3.2.7 and Proposition 3.2.6. #

3.3. Essential extensions of semimodules

The important theorem here is Theorem 3.3.4. The two last results of this section are needed in some of the proofs concerning $\mathfrak{C}_S$-injective hulls.

Definition 3.3.1. *An S-semimodule B is an* **essential extension** *of a subsemimodule A iff for every S-semimodule C and every S-homomorphism $f: B \to C$, $f|_A$ is injective implies f is injective.*

Note. To say that B is an essential extension of A is the same as saying that A is large in B in the terminology of *[1]* (see page 179).

Lemma 3.3.2. *Let A be a cancellative S-semimodule. Then A^{Δ} is an essential extension of $i^{\Delta}(A)$.*

Proof. Let C be an S-semimodule and $f: A^{\Delta} \to C$ an S-homomorphism such that $f|_{i^{\Delta}(A)}$ is injective. To show f is injective, let $i^{\Delta}(a_1)-i^{\Delta}(a_2)$ and $i^{\Delta}(b_1)-i^{\Delta}(b_2)$ be elements of A^{Δ} such that $f(i^{\Delta}(a_1)-i^{\Delta}(a_2)) = f(i^{\Delta}(b_1)-i^{\Delta}(b_2))$. Adding $f(i^{\Delta}(a_2))+f(i^{\Delta}(b_2))$ to both sides and simplifying yields the equation $f(i^{\Delta}(a_1)+i^{\Delta}(b_2)) = f(i^{\Delta}(b_1)+i^{\Delta}(a_2))$. But $i^{\Delta}(a_1)+i^{\Delta}(b_2)$ and $i^{\Delta}(b_1)+i^{\Delta}(a_2)$ are elements of $i^{\Delta}(A)$ and $f|_{i^{\Delta}(A)}$ is injective, so we get $i^{\Delta}(a_1)+i^{\Delta}(b_2) = i^{\Delta}(b_1)+i^{\Delta}(a_2)$. Thus $i^{\Delta}(a_1)-i^{\Delta}(a_2) = i^{\Delta}(b_1)-i^{\Delta}(b_2)$, and f is injective. Therefore, A^{Δ} is an essential extension of $i^{\Delta}(A)$. #

Proposition 3.3.3. *Let A be a cancellative S-semimodule. If A has no proper cancellative essential extensions, then A is a group S-semimodule.*

Proof. Let A be a cancellative S-semimodule and assume A has no proper cancellative essential extensions. By Lemma 3.3.2, A^{Δ} is an essential extension of $i^{\Delta}(A)$. Note that $A \cong i^{\Delta}(A)$ by the S-isomorphism i^{Δ}. Let $\varphi = i^{\Delta}$. By Proposition 2.1.13, there exists an S-semimodule A^* with $A \subseteq A^*$ and an S-isomorphism $\varphi^*: A^* \to A^{\Delta}$ such that $\varphi^*|_A = \varphi$.

$$\begin{array}{ccc} A^* & \overset{\varphi^*}{\dashrightarrow} & A^{\Delta} \\ \cup| & & \cup| \\ A & \overset{\varphi}{\longrightarrow} & i^{\Delta}(A) \end{array}$$

I claim that A^* is an essential extension of A. Let C be an S-semimodule and $f: A^* \to C$ an S-homomorphism such that $f|_A$ is injective. Note that $f \circ (\varphi^*)^{-1}: A^{\Delta} \to C$ and $f \circ (\varphi^*)^{-1}|_{i^{\Delta}(A)}$ is injective. Since A^{Δ} is an essential extension of $i^{\Delta}(A)$, $f \circ (\varphi^*)^{-1}$ is injective. Consequently, f is also injective, which implies that A^* is an essential extension of A. Since $A^* \cong A^{\Delta}$, A^* is a group S-semimodule; a fortiori, A^* is cancellative. Thus A^* is a cancellative essential extension of A. By assumption, $A = A^*$. Therefore, A is a group S-semimodule. #

Theorem 3.3.4. *Let A be a cancellative S-semimodule. Then A is $\mathfrak{C}_S$-injective iff A has no proper cancellative essential extensions.*

Proof. Assume that A is a $\mathfrak{C}_S$-injective semimodule. Let B be a cancellative S-semimodule such that B is an essential extension of A. Then $A \subseteq B$. Because A is $\mathfrak{C}_S$-injective, there exists an S-homomorphism $h\colon B \to A$ such that $1_A = h \circ 1_{A,B}$.

$$\begin{array}{ccc} A & \xrightarrow{1_{A,B}} & B \\ {\scriptstyle 1_A}\downarrow & \swarrow {\scriptstyle h} & \\ A & & \end{array}$$

Since $h|_A = 1_A$, $h|_A$ is injective. Thus h itself is injective because B is an essential extension of A. I will now show $B \subseteq A$. Let $b \in B$. Then $h(b) = a$ for some $a \in A$, so that $a = 1_{A,B}(h(b))$. Thus

$$\begin{aligned} h(a) &= h(1_{A,B}(h(b))) \\ &= 1_A(h(b)) \\ &= h(b). \end{aligned}$$

Since h is injective, $b = a$, which implies $b \in A$. Consequently, $A = B$. Hence A has no proper cancellative essential extensions.

Conversely, assume that A has no proper cancellative essential extensions. By Proposition 3.3.3, A is a group S-semimodule. By Theorem 3.2.7, there exists a $\mathfrak{C}_S$-injective group semimodule Q such that $A \subseteq Q$.

Let $\mathcal{P} = \{E \subseteq Q \mid E$ is a group S-semimodule and $E \cap A = \{0\}\}$. Applying Zorn's Lemma, let X be a maximal element of $\mathcal{P}$, i.e., $X \subseteq Q$, X is a group S-semimodule and $X \cap A = \{0\}$. Note that X, Q, and A are groups, and thus $(A \oplus X)/X$ and Q/X are well-defined, and $(A \oplus X)/X \subseteq Q/X$. It is easily checked that $(A \oplus X)/X$ and Q/X are actually S-semimodules, and that the map $\varphi: A \to (A \oplus X)/X$ given by $\varphi(a) = a + X$ for all $a \in A$ is an S-isomorphism. Applying Proposition 2.1.13, let C^* be an S-semimodule with $A \subseteq C^*$ and $\varphi^*: C^* \to Q/X$ an S-isomorphism such that $\varphi^*|_A = \varphi$.

$$\begin{array}{ccc} C^* & \overset{\varphi^*}{\dashrightarrow} & Q/X \\ \cup\!\!| & & \cup\!\!| \\ A & \overset{\varphi}{\to} & (A \oplus X)/X \end{array}$$

I claim that C^* is an essential extension of A.

Let C be an S-semimodule and $f: C^* \to C$ an S-homomorphism such that $f|_A$ is injective. Consider $Q \overset{\pi}{\to} Q/X \xrightarrow{(\varphi^*)^{-1}} C^* \overset{f}{\to} C$, where $\pi(q) = q + X$ for all $q \in Q$. Let $\psi = f \circ (\varphi^*)^{-1} \circ \pi$. Then $X \subseteq \operatorname{zs}\psi$, and $\operatorname{zs}\psi \in \mathcal{P}$. Indeed, the only requirement not obviously satisfied is that $\operatorname{zs}\psi \cap A = \{0\}$. Let $a \in \operatorname{zs}\psi \cap A$. Then $\pi(a) = a + X = \varphi(a) = \varphi^*(a)$, so that $(\varphi^*)^{-1} \circ \pi(a) = a$. It follows that $0 = \psi(a) = f \circ (\varphi^*)^{-1} \circ \pi(a) = f(a)$, and because $f|_A$ is injective, this implies $a = 0$. Hence $\operatorname{zs}\psi \cap A = \{0\}$. Since X is a maximal element of $\mathcal{P}$, $X = \operatorname{zs}\psi$. Now I can show that f is injective by applying Proposition 2.3.7. Let $c \in \operatorname{zs} f$. Since $\varphi^*(c) \in Q/X$ and π is surjective, there exists an element q of Q such that $\pi(q) = \varphi^*(c)$. Consider that $\psi(q) = f \circ (\varphi^*)^{-1} \circ \pi(q) = f(c) = 0$. Thus

$q \in \operatorname{zs}\psi = X$, so $\pi(q) = q + X = X$. Then $\varphi^*(c) = X$, which is the identity of Q/X. Since φ^* is an S-monomorphism, $c = 0^*$, where 0^* is the identity of C^*. Hence $\operatorname{zs} f = \{0^*\}$ which implies f is injective. Consequently, C^* is an essential extension of A.

By assumption, C^* must equal A. Thus $\varphi = \varphi^*$, so $(A \oplus X)/X = Q/X$. Hence $A \oplus X = Q$. Note that $A \oplus X = A \times X$ (direct product). Since Q is $\mathfrak{C}_S$-injective, $A \times X$ is also $\mathfrak{C}_S$-injective. By Corollary 3.1.3, A is $\mathfrak{C}_S$-injective. #

Corollary 3.3.5. *Every $\mathfrak{C}_S$-injective semimodule is a group S-semimodule.*

Proof. This follows from Theorem 3.3.4 and Proposition 3.3.3. #

Lemma 3.3.6. *Let $A \subseteq H \subseteq H'$ be S-semimodules.*

(i) *If H is an essential extension of A and H' is an essential extension of H, then H' is an essential extension of A.*

(ii) *If H' is an essential extension of A, then H' is an essential extension of H.*

Proof. (i) Let C be an S-semimodule and $f: H' \to C$ an S-homomorphism such that $f|_A$ is injective. Note that $f \circ 1_{H,H'}: H \to C$ and $(f \circ 1_{H,H'})|_A = f|_A$ is injective. Thus $f \circ 1_{H,H'}$ is also injective. But then $f|_H = f \circ 1_{H,H'}$ is injective, and so f is injective. Therefore, H' is an essential extension of A.

(ii) Let C be an S-semimodule and $f: H' \to C$ an S-homomorphism such that $f|_H$ is injective. Since $A \subseteq H$, $f|_A$ is also injective. Then f is injective, immediately from the assumption. Therefore, H' is an essential extension of H. #

Lemma 3.3.7. *Let $\varphi: A \to A'$ be an S-monomorphism of cancellative S-semimodules, B a cancellative essential extension of A, and Q a $\mathfrak{C}_S$-injective semimodule containing A'. Then φ can be extended to an S-monomorphism $\psi: B \to Q$.*

$$\begin{array}{ccc} B & \overset{\psi}{\dashrightarrow} & Q \\ {\scriptstyle 1_{A,B}}\uparrow & & \uparrow{\scriptstyle 1_{A',Q}} \\ A & \overset{\varphi}{\longrightarrow} & A' \end{array}$$

Proof. Let $1_{A,B}: A \to B$ and $1_{A',Q}: A' \to Q$ be the inclusion maps. Since Q is $\mathfrak{C}_S$-injective, there exists an S-homomorphism $\psi: B \to Q$ such that $1_{A',Q} \circ \varphi = \psi \circ 1_{A,B}$. Note $\psi|_A = \psi \circ 1_{A,B} = 1_{A',Q} \circ \varphi$ is injective. Since B is an essential extension of A, ψ is injective. It is easy to check that ψ extends φ. #

3.4. $\mathfrak{C}_S$-injective hulls

The main result of this section is that every cancellative S-semimodule has a $\mathfrak{C}_S$-injective hull. Thanks to the work done in the previous section, its proof is almost the same as the proof of the analogous theorem for modules over rings.

Definition 3.4.1. *Let $\mathfrak{M}$ be a category of S-semimodules and I, A elements of $\mathfrak{M}$. Then I is an* **$\mathfrak{M}$-injective hull** *of A iff I is $\mathfrak{M}$-injective and I is an essential extension of A.*

Lemma 3.4.2. *Let I be a $\mathfrak{C}_S$-injective hull of a cancellative S-semimodule A. Then*

(i) *I is a maximal cancellative essential extension of A.*

(ii) *I is a minimal $\mathfrak{C}_S$-injective semimodule containing A.*

Proof. (i) This follows from Theorem 3.3.4.

(ii) Let J be a $\mathfrak{C}_S$-injective semimodule such that $A \subseteq J \subseteq I$. Note that I is an essential extension of A. By Lemma 3.3.6(ii), I is an essential extension of J. But by Theorem 3.3.4 J has no proper cancellative essential extensions. Thus $J = I$. #

Theorem 3.4.3. *Every cancellative S-semimodule has a $\mathfrak{C}_S$-injective hull.*

Proof. Let A be a cancellative S-semimodule. By Theorem 3.2.7, there exists a $\mathfrak{C}_S$-injective group semimodule Q such that $A \subseteq Q$. Let

$$\mathcal{P} = \{E \mid E \text{ is a cancellative essential extension of } A \text{ and } A \subseteq E \subseteq Q\}.$$

Then $\mathcal{P}$ satisfies the hypotheses of Zorn's Lemma. Let H be a maximal element of $\mathcal{P}$, i.e., H is a maximal cancellative essential extension of A contained in Q.

I claim that H is a $\mathfrak{C}_S$-injective hull of A. It only remains to show that H is $\mathfrak{C}_S$-injective, which I will do by using Theorem 3.3.4. Let H' be a cancellative essential extension of H. Consider the following diagram:

$$\begin{array}{ccc} H & \xrightarrow{1_{H,H'}} & H' \\ {\scriptstyle 1_{H,Q}}\downarrow & \swarrow {\scriptstyle i^*} & \\ Q & & \end{array}$$

Since Q is $\mathfrak{C}_S$-injective, there exists an S-homomorphism $i^*\colon H' \to Q$ such that $1_{H,Q} = i^* \circ 1_{H,H'} = i^*|_H$. Then i^* is injective, because H' is an essential extension of H and $i^*|_H$ is injective. Let $H'' = i^*(H')$. Then $H \subseteq H''$. Let C be an S-semimodule and $f\colon H'' \to C$ an S-homomorphism such that $f|_H$ is injective. Note that $f \circ i^*: H' \to C$, and $f \circ i^*|_H$ is injective. This implies that $f \circ i^*$ is injective, so f is also, since i^* maps H' onto H''. This proves that H'' is an essential extension of H. By Lemma 3.3.6(i), H'' is an essential extension of A. Note that $H \subseteq H'' \subseteq Q$ and H is a maximal essential extension of A in Q, so $H'' = H$. Now, let $h' \in H'$. Then $i^*(h') \in H'' = H$, so denote $i^*(h')$ by h. Using the above diagram, $i^*(h') = h = 1_{H,Q}(h) = i^* \circ 1_{H,H'}(h) = i^*(h)$. Since i^* is injective, $h' = h$. This shows $H' = H$.

The above work proves that there is no cancellative essential extension of H distinct from H. By Theorem 3.3.4, H is $\mathfrak{C}_S$-injective. Therefore H is a $\mathfrak{C}_S$-injective hull of A. #

Proposition 3.4.4. *If $\varphi: A \to A'$ is an S-isomorphism of cancellative S-semimodules, and H and H' are $\mathfrak{C}_S$-injective hulls of A and A', respectively, then φ can be extended to an S-isomorphism $\psi: H \to H'$.*

In particular, the $\mathfrak{C}_S$-injective hull of A is unique up to an S-isomorphism leaving invariant every element of A.

Proof. Assume that $\varphi: A \to A'$ is an S-isomorphism from one cancellative S-semimodule to another, and H and H' are $\mathfrak{C}_S$-injective hulls of A and A', respectively. Then H is an essential extension of A and H' is a $\mathfrak{C}_S$-injective semimodule. Consider the following diagram:

$$\begin{array}{ccc} H & \overset{\psi}{\dashrightarrow} & H' \\ {\scriptstyle 1_{A,H}}\uparrow & & \uparrow{\scriptstyle 1_{A',H'}} \\ A & \overset{\varphi}{\longrightarrow} & A' \end{array}$$

By Lemma 3.3.7, there exists an S-monomorphism $\psi: H \to H'$ extending φ. Clearly $A' \subseteq \psi(H)$ and $\psi(H)$ is a $\mathfrak{C}_S$-injective semimodule. By Lemma 3.4.2(ii), $\psi(H) = H'$. Therefore, ψ is an S-isomorphism.

To prove the last statement, observe that $1_A: A \to A$ is an S-isomorphism. Thus, for any $\mathfrak{C}_S$-injective hulls H and H' of A, the above work shows that 1_A may be extended to an S-isomorphism $\psi: H \to H'$. #