CHAPTER IV

PRIME SEMIFIELDS

Definition 4.1 Let K be a semifield and $L \subseteq K$. Then L is said to be a subsemifield of K iff L forms a semifield with respect to the same operations on K.

Theorem 4.2 Let K be a semifield. If K is of type I w.r.t.a then there exists a smallest subsemifield contained in K and it is also a semifields of type I w.r.t.a. If K is of type II w.r.t.a they there exists a smallest subsemifield contained in K and it is also a semifield of type II w.r.t.a. If K is of type III then there exists a smallest subsemifield contained in K and it is also a semifield of type III.

Proof Let $a \in K$ be such that $(K \setminus \{a\},\cdot)$ is a group and let e be the identity of $(K \setminus \{a\},\cdot)$. Let $L \subseteq K$ be a subsemifield.

First, we shall show that $a \in L$ and L is a semifield of the same type as K w.r.t.a.

Case 1 K is a semifield of type I w.r.t.a.

Then a and e are the only multiplicative idempotents of K. By Theorem 3.15 and Theorem 3.18, L must be a semifield of type I or II. Then L contains exactly two multiplicative idempotents. Thus $\{a,e\} \subseteq L$. Since a is multiplicative zero of K, we get that a is also multiplicative zero of L. Thus L is a semifield of type I w.r.t.a.

Case 2 K is a semifield of type II w.r.t.a.

Then a and e are the only multiplicative idempotents of K. By Theorem 3.14 and Theorem 3.15, L must be a semifield of type I or II. Then L contains exactly two multiplicative idempotents. Thus $\{a,e\} \subseteq L$. Since a is multiplicative identity of K, we get that a is also multiplicative identity of L. Thus L is a semifield of type II w.r.t.a.

Case 3 K is a semifield of type III.

Then e is the only multiplicative idempotent of K. By Theorem 3.18, L must be a semifield of type III. Then L contains exactly one multiplicative idempotent so $e \in L$. Let $b \in L$ be such that $(L \setminus \{b\}, \cdot)$ is a group. If $b \neq a$, then $b = be$, a contradiction. Hence $b = a$. Therefore L is a semifield of type III and a is the element in L such that $(L \setminus \{a\}, \cdot)$ is a group.

Let $\{L_\alpha\}_{\alpha \in I}$ be the set of all subsemifields of K. By the first part of this proof, we get that $a \in L_\alpha$ and L_α is a semifield of the same types as K w.r.t.a. Let $M = \bigcap_{\alpha \in I} L_\alpha$. Cleary M is a subsemiring of K and $a \in M$. Now $M \setminus \{a\} = (\bigcap_{\alpha \in I} L_\alpha) \setminus \{a\} = \bigcap_{\alpha \in I} (L_\alpha \setminus \{a\})$. Thus $(M \setminus \{a\}, \cdot)$ is a group. Hence M is a subsemifield of K. By the first part of this proof, we get that M is a semifield of the same type as K w.r.t.a. Clearly M is the smallest subsemifield of K. So we obtain that M is the smallest subsemifield of K and M is a semifield of the same type as K w.r.t.a. #

Definition 4.3 Let K be a semifield. Then the prime semifield of K is the smallest subsemifield of K (Which exists by Theorem 4.2) .

Remark 4.4 Let $\mathbb{Q}^+$ with the usual addition and multiplication. Then $(\mathbb{Q}^+,+,\cdot)$ is a ratio semiring. Let a be a symbol not representing any element of $\mathbb{Q}^+$. Extend + and $\cdot$ from $\mathbb{Q}^+$ to $\mathbb{Q}^+ \cup \{a\}$ by $a\cdot x = x\cdot a = x$ $\forall x \in \mathbb{Q}^+ \cup \{a\}$, $a+x = x+a = 1+x \, \forall x \in \mathbb{Q}^+$ and $a+a = 1+1$. Then by Theorem 2.39 we obtain that $\mathbb{Q}^+ \cup \{a\}$ is a semifield of type II.

Theorem 4.5 Let K be a semifield of type II w.r.t. a and K′ the prime semifield of K. Then $K' \cong \{a,1\}$ with $a^2 = a$, $a\cdot 1 = 1\cdot a = 1$, $1\cdot 1 = 1$ and + defined by

(1)

+	a	1
a	a	a
1	a	1

or (2)

+	a	1
a	a	1
1	1	1

or

(3)

+	a	1
a	1	a
1	a	1

or (4)

+	a	1
a	1	1
1	1	1

or $K' \cong \mathbb{Q}^+ \cup \{a\}$ as in remark 4.4.

Proof Let 1 be the identity of $(K \setminus \{a\},\cdot)$.

By Theorem 2.29, $K \setminus \{a\}$ is a ratio semiring. Let D′ be the smallest ratio subsemiring of $K \setminus \{a\}$. By Proposition 1.18, $D' \cong \{1\}$ with $1\cdot 1 = 1$ and $1+1 = 1$ or $D' \cong \mathbb{Q}^+$ with the usual addition and multiplication.

Suppose that $D' \cong \{1\}$. Then $1+1 = 1$. By Theorem 2.30, $a+a = a$ or $a+a = 1+1 = 1$ and $a+1 = a$ or $a+1 = 1+1 = 1$. Therefore we have four cases to consider. They are (1),(2),(3) and (4) above. It is easy to check that they are all semifields. Thus $K' \cong (1)$ or $K' \cong (2)$ or $K' \cong (3)$ or $K' \cong (4)$.

If $D' \cong \mathbb{Q}^+$ with the usual addition and multiplication, then up to isomorphism we can consider $\mathbb{Q}^+ \subseteq K\setminus\{a\}$. Let $D = K\setminus\{a\}$ and $S = \{x \in D \mid a+x = a\}$, by Theorem 2.31, $S \subseteq I_D(1)$. Claim that $\mathbb{Q}^+ \cap S = \Phi$. Let $x \in \mathbb{Q}^+$ and $x \in S$, then $x \in I_D(1)$. So $x+1 = 1$, a contradiction. So we have the claim. Thus $\mathbb{Q}^+ \subseteq D\setminus S$, so $a+x = 1+x \quad \forall x \in \mathbb{Q}^+$. By Theorem 2.30 (1) and (2) we obtain that $a+a = 1+1$. Since $\mathbb{Q}^+ \cup \{a\} \subseteq K$, $ax = x \quad \forall x \in \mathbb{Q}^+ \cup \{a\}$. By Theorem 2.39, $\mathbb{Q}^+ \cup \{a\}$ is a semifield. So up to isomorphism, $K' \subseteq \mathbb{Q}^+ \cup \{a\}$. By Theorem 2.29, $K' \setminus \{a\}$ is a ratio semiring. Then $K' \setminus \{a\}$ is a ratio subsemiring of $K \setminus \{a\}$. Since $\mathbb{Q}^+$ with the usual addition and multiplication is the smallest ratio subsemiring of $K\setminus\{a\}$, we obtain that $\mathbb{Q}^+ \quad K'\setminus\{a\}$. Thus $\mathbb{Q}^+ \cup \{a\} \subseteq K'$. Hence $K' \cong \mathbb{Q}^+ \cup \{a\}$ as in remark 4.4.

<u>Theorem 4.6</u> Let K be a semifield of type III and $a \in K$, $d \in K\setminus\{a\}$ be such that $(K\setminus\{a\},\cdot)$ is a group and $a\cdot x = d\cdot x$ for all $x \in K$. Let $A = \{\sum_{i<\infty} n_i d^{m_i} \mid m_i, n_i \in \mathbb{Z}^+\}$ and K' the prime semifield of K. Then A is is a multiplicatively cancellative semiring and $K' \cong$ (the quotient ratio semiring of A) $\cup$ $\{a\}$.

<u>Proof</u> Let e be the identity of $(K\setminus\{a\},\cdot)$, $D = K\setminus\{a\}$ and $S = \{x \in D \mid a+x = a\}$.

By Theorem 4.2, we have that $a,e \in K$, so $d = a\cdot e \in K'$. Thus $d \in K'\setminus\{a\}$. Since $K'\setminus\{a\}$ is a ratio semiring (by Theorem 2.41), we get that $A \subseteq K'\setminus\{a\}$. Clearly A is a semiring. Since $A \subseteq K'\setminus\{a\}$ which is a group under multiplication so A is M.C. Let B be the quotient ratio semiring of A. Then we get that $B \subseteq K'\setminus\{a\}$ (since B is the smallest ratio semiring containing A).

Let $S' = B \cap S$. Claim that

(1) $S' = \Phi$ or S' is an additive subsemigroup of $I_B(d)$.

(2) $B\setminus S' = \Phi$ or $B\setminus S'$ is an ideal of $(B,+)$

(3) If $B\setminus S' = \Phi$ then $|B| = 1$.

To show (1), we assume that $S\neq\Phi$. Let $x \in S'$, then $x \in B$ and $x \in S$. Since $S \subseteq I_D(d)$ (by Theorem 2.50), $x \in I_D(d)$. So $x+d = d$. Thus $x \in I_B(d)$. Hence $S' \subseteq I_B(d)$. Let $x, y \in S'$. Then $x, y \in B$ and $x, y \in S$. Since B is a ratio semiring and S is an additive subsemigroup of $I_D(d)$, $x+y \in B$ and $x+y \in S$. Thus $x+y \in S'$. So we have (1).

To show (2), we assume that $B\setminus S' \neq \Phi$. Let $x \in B\setminus S'$ and $y \in B$. Since $B\setminus S' = B\cap S'^{c} = B\cap(B\cap S)^{c} = B\cap(B^{c}\cup S^{c}) = (B\cap B^{c})\cup(B\cap S^{c}) = B\cap S^{c}$ $= B\setminus S \subseteq D\setminus S$, $B\setminus S' = B\setminus S \subseteq D\setminus S$. So we get that $x \in D\setminus S$. Since $D\setminus S$ is an ideal of $(D,+)$ (by Theorem 2.50), $x+y \in D\setminus S$. Since $x,y \in B$, $x+y \in B$. Thus $x+y \in B\setminus S$ and hence $x+y \in B\setminus S'$ (since $B\setminus S' = B\setminus S$). So we have (2).

To show (3), we assume that $B\setminus S' = \Phi$. Then $B = S'$. Since $S' \subseteq I_B(d)$, we get that $B = I_B(d)$. Then $d+x = d$ $\forall x \in B$. Thus d is an additive zero of B, then by Theorem 1.13 and Proposition 1.15, $|B| = 1$. So we have (3)

Since $S' \subseteq S$ and $B\setminus S' \subseteq D\setminus S$, $a+x = a$ $\forall x \in S'$ and $a+x = d+x$ $\forall x \in B\setminus S'$. By Theorem 2.43, $a+a = a$ or $a+a = d+d$. Since $B \cup \{a\} \subseteq K$, $ax = dx$ $\forall x \in B \cup \{a\}$. By Theorem 2.51 we obtain that $B \cup \{a\}$ is a semifield of type III. Hence $K' \subseteq B \cup \{a\}$. Since $B \subseteq K'\setminus \{a\}$, $B \cup \{a\} \subseteq K'$. Therefore $K' \cong B \cup \{a\}$ so we have the Theorem.

From now on we shall compute the prime semifields of semifields of type III.

Remark 4.7 Let $K = \{a,1\}$. Define + and $\cdot$ on K as the following tables :

(1)

$\cdot$	a	1
a	1	1
1	1	1

and

+	a	1
a	a	a
1	a	1

(2)

$\cdot$	a	1
a	1	1
1	1	1

and

+	a	1
a	a	1
1	1	1

(3)

$\cdot$	a	1
a	1	1
1	1	1

and

+	a	1
a	1	a
1	a	1

(4)

$\cdot$	a	1
a	1	1
1	1	1

and

+	a	1
a	1	1
1	1	1

By Theorem 2.51 we get that (1),(2),(3) and (4) are all semifields of type III.

Remark 4.8 Let $\mathbb{Q}^+$ have the usual addition and multiplication. Then $(\mathbb{Q}^+,+,\cdot)$ is a ratio semiring. Let $d \in \mathbb{Q}^+$ and a a symbol not representing any element in $\mathbb{Q}^+$. Extend + and $\cdot$ from $\mathbb{Q}^+$ to $\mathbb{Q}^+ \cup \{a\}$ by $ax = xa = dx$ $\forall x \in \mathbb{Q}^+$, $a^2 = d^2$, $a+x = x+a = d+x$ $\forall x \in \mathbb{Q}^+$ and $a+a = d+d$. Then by Theorem 2.51, we obtain that $\mathbb{Q}^+ \cup \{a\}$ is a semifield of type III.

Remark 4.9 Let <d> be notation for the set of symbols $\{d^n \mid n \in \mathbb{Z}\}$. Define + and $\cdot$ on <d> by $d^m + d^n = d^n$ where $k = \min\{m,n\}$ and $d^m \cdot d^n = d^{m+n}$. We shall show that $(\langle d\rangle,+,\cdot)$ is a ratio semiring. Clearly, $(\langle d\rangle,+,\cdot)$ is a commutative group. For $\ell,m,n \in \mathbb{Z}$.

$$(d^\ell + d^m) + d^n = d^k = d^\ell + (d^m + d^n) \quad \text{where } k = \min\{\ell,m,n\}$$

and we get that

$$(d^\ell + d^m)\cdot d^n = d^k \cdot d^n = d^{k+n} \quad \text{where } k = \min\{\ell,m\} \quad \text{and}$$

$$d^\ell \cdot d^n + d^m \cdot d^n = d^{\ell+n} + d^{m+n} = d^r \quad \text{where } r = \min\{\ell+n,\ m+n\}.$$

Since $k = \min\{\ell,m\}$, we get that $k+n = \min\{\ell+n, m+n\}$. Thus $r = k+n$. Hence $(d^{\ell}+d^{m})\cdot d^{n} = d^{\ell}\cdot d^{n} + d^{m}\cdot d^{n}$.

Therefore $(\langle d\rangle,+,\cdot)$ is a ratio semiring. Let a be a symbol not representing any element in $\langle d\rangle$ and $n_0 \in \mathbb{Z}^+$ be fixed. Let $S_1 = \{d^n \mid n \in \mathbb{Z},\ n \geq n_0\}$, then $\langle d\rangle \setminus S_1 = \{d^n \mid n \in \mathbb{Z},\ n < n_0\}$. Clearly, $I_{\langle d\rangle}(d) = \{d^n \mid n \in \mathbb{Z},\ n \geq 1\}$. It is easy to show that S_1 is an additive subsemigroup of $I_{\langle d\rangle}(d)$ and $\langle d\rangle \setminus S_1$ is an ideal of $(\langle d\rangle,+)$. Extend + and $\cdot$ from $\langle d\rangle$ to $K = \langle d\rangle \cup \{a\}$ as follows ;

(1) $ax = xa = dx$ for all $x \in \langle d\rangle$ and $a^2 = d^2$,

(2) $a+x = x+a = a$ for all $x \in S_1$ and

$a+x = x+a = d+x$ for all $x \in \langle d\rangle \setminus S_1$,

(3) $a+a = a$ or d.

Then by Theorem 2.51, we obtain that $\langle d\rangle \cup \{a\}$ is a semifield of type III.

Remark 4.10 Let $(\langle d\rangle,+,\cdot)$ be the ratio semiring given in Remark 4.9. Let a be a symbol not representing any element in $\langle d\rangle$. Extend + and $\cdot$ from $\langle d\rangle$ to $\langle d\rangle \cup \{a\}$ by $ax = xa = dx$ $\forall x \in \langle d\rangle$, $a^2 = d^2$, $a+x = x+a = d+x$ $\forall x \in \langle d\rangle$ and $a+a = a$ or d. By Theorem 2.51, we obtain that $\langle d\rangle \cup \{a\}$ is a semifield of type III.

Remark 4.11 Let $\langle d\rangle$ be notation for the set of symbol $\{d^n \mid n \in \mathbb{Z}\}$. Define + and $\cdot$ on $\langle d\rangle$ by $d^m + d^n = d^k$ where $k = \max.\{m,n\}$ and $d^m \cdot d^n = d^{m+n}$. Similarly as remark 4.9, we can show that $(\langle d\rangle,+,\cdot)$ is a ratio semiring. Let a be a symbol not representing any element in $\langle d\rangle$ and $n_0 \in \mathbb{Z}$, $n_0 \leq 1$. Let $S_1 = \{a^n \mid n \in \mathbb{Z},\ n \leq n_0\}$, then $\langle d\rangle \setminus S_1 = \{d^n \mid n \in \mathbb{Z},\ n > n_0\}$. Clearly, $I_{\langle d\rangle}(d) = \{d^n \mid n \in \mathbb{Z},\ n \leq 1\}$. It is easy to show that S_1 is an additive subsemigroup of $I_{\langle d\rangle}(d)$ and

$\langle d \rangle \setminus S_1$ is an ideal of $(\langle d \rangle,+)$. Extend + and $\cdot$ from $\langle d \rangle$ to $\langle d \rangle \cup \{a\}$ as follows ;

(1) $ax = xa = dx$ for all $x \in \langle d \rangle$ and $a^2 = d^2$,

(2) $a+x = x+a = a$ for all $x \in S_1$ and

$a+x = x+a = d+x$ for all $x \in \langle d \rangle \setminus S_1$,

(3) $a+a = a$ or d .

Then by Theorem 2.51 we obtain that $\langle d \rangle \cup \{a\}$ is a semifield of type III.

<u>Remark 4.12</u> Let $(\langle d \rangle,+,\cdot)$ be the ratio semiring given in Remark 4.11. Let a be a symbol not representing any element in $\langle d \rangle$. Extend + and $\cdot$ from $\langle d \rangle$ to $\langle d \rangle \cup \{a\}$ by $ax = xa = dx$ $\forall x \in \langle d \rangle$, $a^2 = d^2$, $a+x = x+a = d+x$ $\forall x \in \langle d \rangle$ and $a+a = a$ or d. By Theorem 2.51, we get that $\langle d \rangle \cup \{a\}$ is a semifield of type III.

<u>Remark 4.13</u> Let $\mathbb{Q}^+$ have the usual addition and multiplication. Let $\mathbb{Q}^+ \cdot \langle d \rangle$ be notation for the set of symbols $\{xd^n \mid x \in \mathbb{Q}^+ \text{ and } n \in \mathbb{Z}\}$ Define $\oplus$ and $\odot$ on $\mathbb{Q}^+ \cdot \langle d \rangle$ as follows ;

$$xd^m \oplus yd^n = \begin{cases} xd^m & \text{if } m < n \\ (x+y)d^m & \text{if } m = n \\ yd^n & \text{if } n < m \end{cases}$$

and $xd^m \odot yd^n = (xy)d^{m+n}$.

Claim that $(\mathbb{Q}^+ \cdot \langle d \rangle, \oplus, \odot)$ is a ratio semiring.

Clearly $(\mathbb{Q}^+ \cdot \langle d \rangle, \odot)$ is a commutative group. Let $x,y,z \in \mathbb{Q}^+$ and $\ell,m,n \in \mathbb{Z}$. We shall show that

(1) $(xd^\ell \oplus yd^m) \oplus zd^n = xd^\ell \oplus (yd^m \oplus zd^n)$ and

(2) $(xd^{\ell} \oplus yd^{m}) \odot zd^{n} = xd^{\ell} \odot zd^{n} \oplus yd^{m} \odot zd^{n}$

To show (1), we will consider the following cases :

<u>Case 1</u> $\ell = m = n$.

$(xd^{\ell} \oplus yd^{m}) \oplus zd^{n} = (x+y)d^{\ell} \oplus zd^{n} = ((x+y)+z)d^{\ell} = (x+(y+z))d^{\ell} = xd^{\ell} \oplus (y+z)d^{\ell}$

$= xd^{\ell} \oplus (yd^{\ell} \oplus zd^{\ell}) = xd^{\ell} \oplus (yd^{m} \oplus zd^{n})$.

<u>Case 2</u> $\ell = m \neq n$.

<u>Subcase 2.1</u> $m < n$. Then $\ell < n$.

$(xd^{\ell} \oplus yd^{m}) \oplus zd^{n} = (x+y)d^{\ell} \oplus zd^{n} = (x+y)d^{\ell}$.

$(xd^{\ell} \oplus (yd^{m} \oplus zd^{n}) = xd^{\ell} \oplus yd^{m} = (x+y)d^{\ell}$

<u>Subcase 2.2</u> $m > n$. Then $\ell > n$.

$(xd^{\ell} \oplus yd^{m}) \oplus zd^{n} = (x+y)d^{\ell} \oplus zd^{n} = zd^{n}$

$xd^{\ell} \oplus (yd^{m} \oplus zd^{n}) = xd^{\ell} \oplus zd^{n} = zd^{n}$

<u>Case 3</u> $\ell = n \neq m$.

<u>Subcase 3.1</u> $n < m$. Then $\ell < m$.

$(xd^{\ell} \oplus yd^{m}) \oplus zd^{n} = xd^{\ell} \oplus zd^{n} = (x+z)d^{\ell}$

$xd^{\ell} \oplus (yd^{m} \oplus zd^{n}) = xd^{\ell} \oplus zd^{n} = (x+z)d^{\ell}$

<u>Subcase 3.2</u> $n > m$. Then $\ell > m$.

$(xd^{\ell} \oplus yd^{m}) \oplus zd^{n} = yd^{m} \oplus zd^{n} = yd^{m}$

$xd^{\ell} \oplus (yd^{m} \oplus zd^{n}) = xd^{\ell} \oplus yd^{m} = yd^{m}$

<u>Case 4</u> $n = m \neq \ell$. Then

$(xd^{\ell} \oplus yd^{m}) \oplus zd^{n} = (yd^{m} \oplus xd^{\ell}) \oplus zd^{n} = yd^{m} \oplus (xd^{\ell} \oplus zd^{n})$ (by case 3) $= yd^{m} \oplus (zd^{n} \oplus xd^{\ell}) = (yd^{m} \oplus zd^{n}) \oplus xd^{\ell}$ (by case 2) $= xd^{\ell} \oplus (yd^{m} \oplus zd^{n})$.

<u>Case 5</u> ℓ, m, n are all distinct. Let $k = \min\{\ell, m, n\}$, then

$$(xd^{\ell} \oplus yd^{m}) \oplus zd^{n} = xd^{\ell} \oplus (yd^{m} \oplus zd^{n}) = \begin{cases} xd^{\ell} & \text{if } k = \ell , \\ yd^{m} & \text{if } k = m , \\ zd^{n} & \text{if } k = n . \end{cases}$$

To show (2) , we will consider the following cases :

Case 1 $\ell = m$.

$$(xd^{\ell} \oplus yd^{m}) \odot zd^{n} = (x+y)d^{\ell} \odot zd^{n} = ((x+y)z)d^{\ell+n} = (xz+yz)d^{\ell+n}$$

$$= (xz)d^{\ell+n} \oplus (yz)d^{\ell+n} = xd^{\ell} \odot zd^{n} \oplus yd^{\ell} \odot zd^{n}.$$

Case 2 $\ell \neq m$.

Subcase 2.1 $\ell < m$. Then $\ell+n < m+n$.

$$(xd^{\ell} \oplus yd^{m}) \odot zd^{n} = xd^{\ell} \odot zd^{n} = (xz)d^{\ell+n}$$

$$xd^{\ell} \odot zd^{n} \oplus yd^{m} \odot zd^{n} = (xz)d^{\ell+n} \oplus (yz)d^{m+n} = (xz)d^{\ell+n}.$$

Subcase 2.2 $\ell > m$.

$(xd^{\ell} \oplus yd^{m}) \odot zd^{n} = (yd^{m} \oplus xd^{\ell}) \odot zd^{n} = yd^{m} \odot zd^{n} \oplus xd^{\ell} \odot zd^{n}$ (by Subcase 2.1) $= xd^{\ell} \odot zd^{n} \oplus yd^{m} \odot zd^{n}$

Therefore $(\mathbb{Q}^{+}\cdot\langle d\rangle, \oplus, \odot)$ is a ratio semiring.

Let $n_0 \in \mathbb{Z}$, $n_0 \geq 2$ and $S_1 = \{xd^{n} \mid x \in \mathbb{Q}^{+} \text{ and } n \in \mathbb{Z}, n \geq n_0\}$. Then $\mathbb{Q}^{+}\cdot\langle d\rangle \setminus S_1 = \{xd^{n} \mid x \in \mathbb{Q}^{+} \text{ and } n \in \mathbb{Z}, n < n_0\}$. Clearly ,

$I_{\mathbb{Q}^{+}\cdot\langle d\rangle}(1d) = \{xd^{n} \mid x \in \mathbb{Q}^{+} \text{ and } n \in \mathbb{Z}, n \geq 2\}$. It is easy to show that S_1 is an additive subsemigroup of $I_{\mathbb{Q}^{+}\cdot\langle d\rangle}(1d)$ and $\mathbb{Q}^{+}\cdot\langle d\rangle \setminus S_1$ is an ideal of $(\mathbb{Q}^{+}\cdot\langle d\rangle, \oplus)$. Let a be a symbol not representing any element in $\mathbb{Q}^{+}\cdot\langle d\rangle$. Extend + and $\cdot$ from $\mathbb{Q}^{+}\cdot\langle d\rangle$ to $\mathbb{Q}^{+}\cdot\langle d\rangle \cup \{a\}$ as follows ;

(1) $a \odot z = z \odot a = 1d \odot z$ for all $z \in \mathbb{Q}^{+}\cdot\langle d\rangle$ and $a \odot a = 1d \odot 1d$,

(2) $a \oplus z = z \oplus a = a$ for all $z \in S_1$ and

$a \oplus z = z \oplus a = 1d \oplus z$ for all $z \in \mathbb{Q}^{+}\cdot\langle d\rangle \setminus S_1$,

(3) $a \oplus a = 1d \oplus 1d$.

Then by Theorem 2.51 we obtain that $\mathbb{Q}^{+}\cdot\langle d\rangle \cup \{a\}$ is a semifield of type III.

Remark 4.14 Let $(\mathbb{Q}^{+}\cdot\langle d\rangle, \oplus, \odot)$ be the ratio semiring given in remark 4.13. Let a be a symbol not representing any element of $\mathbb{Q}^{+}\cdot\langle d\rangle$. Extend $\oplus$ and $\odot$ from $\mathbb{Q}^{+}\cdot\langle d\rangle$ to $\mathbb{Q}^{+}\cdot\langle d\rangle \cup \{a\}$ by $a \odot z = z \odot a = 1d \odot z$ for all $z \in \mathbb{Q}^{+}\cdot\langle d\rangle$, $a \odot a = 1d \odot 1d$, $a \oplus z = z \oplus a = 1d \oplus z$ for all $z \in \mathbb{Q}^{+}\cdot\langle d\rangle$ and $a \oplus a = 1d \oplus 1d$. By Theorem 2.51, we obtain that $\mathbb{Q}^{+}\cdot\langle d\rangle \cup \{a\}$ is a semifield of type III.

Remark 4.15 Let $\mathbb{Q}^{+}$ have the usual addition and multiplication. Let $\mathbb{Q}^{+}\cdot\langle d\rangle$ be notation for the set of symbols $\{xd^{n} \mid x \in \mathbb{Q}^{+} \text{ and } n \in \mathbb{Z}\}$. Define $\oplus$ and $\odot$ on $\mathbb{Q}^{+}\cdot\langle d\rangle$ as follows ;

$$xd^{m} \oplus yd^{n} = \begin{cases} xd^{m} & \text{if } m > n, \\ (x+y)d^{m} & \text{if } m = n, \\ yd^{n} & \text{if } n > m, \end{cases}$$

$$\text{and } xd^{m} \odot yd^{n} = (xy)d^{m+n}.$$

Similarly as Remark 4.13, we can show that $(\mathbb{Q}^{+}\cdot\langle d\rangle, \oplus, \odot)$ is a ratio semiring. Let $n_0 \in \mathbb{Z}$, $n_0 < 1$ and $S_1 = \{xd^{n} \mid x \in \mathbb{Q}^{+} \text{and } n \in \mathbb{Z}, n \leqslant n_0\}$. Then we get that $\mathbb{Q}^{+}\cdot\langle d\rangle$ $S_1 = \{xd^{n} \mid x \in \mathbb{Q}^{+} \text{ and } n \in \mathbb{Z},\ n \geqslant n_0\}$. Clearly $I_{\mathbb{Q}^{+}\cdot\langle d\rangle}(1d) = \{xd^{n} \mid x \in \mathbb{Q}^{+} \text{ and } n \in \mathbb{Z},\ n < 1\}$. It is easy to show that S_1 is an additive subsemigroup of $I_{\mathbb{Q}^{+}\cdot\langle d\rangle}(1d)$ and $\mathbb{Q}^{+}\cdot\langle d\rangle \setminus S_1$ is an ideal of $(\mathbb{Q}^{+}\cdot\langle d\rangle, \oplus)$. Let a be a symbol not representing any element of $\mathbb{Q}^{+}\cdot\langle d\rangle$. Extend $\oplus$ and $\odot$ from $\mathbb{Q}^{+}\cdot\langle d\rangle$ to $\mathbb{Q}^{+}\cdot\langle d\rangle \cup \{a\}$ as follows ;

(1) $a \odot z = z \odot a = 1d \odot z$ for all $z \in \mathbb{Q}^{+}\cdot\langle d\rangle$ and $a \odot a = 1d \odot 1d$,

(2) $a \oplus z = z \oplus a = a$ for all $z \in S_1$ and

$a \oplus z = z \oplus a = 1d \oplus z$ for all $z \in \mathbb{Q}^{+}\cdot\langle d\rangle \setminus S$,

(3) $a \oplus a = 1d \oplus 1d$.

Then by Theorem 2.51, we obtain that $\mathbb{Q}^{+}\langle d\rangle \cup \{a\}$ is a semifield of type III.

Remark 4.16 Let $(\mathbb{Q}^{+}\langle d\rangle, \oplus, \odot)$ be the ratio semiring given in Remark 4.15. Let a be a symbol not representing any element of $\mathbb{Q}^{+}\langle d\rangle$. Extend $\oplus$ and $\odot$ from $\mathbb{Q}^{+}\langle d\rangle$ to $\mathbb{Q}^{+}\langle d\rangle \cup \{a\}$ by $a \odot z = z \odot a = 1d \odot z$ $\forall z \in \mathbb{Q}^{+}\langle d\rangle$. $a \odot a = 1d \odot 1d$, $a \oplus z = z \oplus a = 1d \oplus z \,\forall z \in \mathbb{Q}^{+}\cdot\langle d\rangle$ and $a \oplus a = 1d \oplus 1d$. By Theorem 2.51, $\mathbb{Q}^{+}\cdot\langle d\rangle \cup \{a\}$ is a semifield of type III.

Remark 4.17 Let $\langle x,y\rangle$ be notation for the set of symbols $\{x^m y^n \mid m, n \in \mathbb{Z}.\}$. Define + and $\cdot$ on $\langle x,y\rangle$ as follows ;

$$x^k y^\ell + x^m y^n = x^r y^s \text{ where } r = \max\{k,m\} \text{ and } s = \min\{\ell, n\}$$

$$\text{and } x^k y^\ell \cdot x^m y^n = x^{k+m} y^{\ell+n}.$$

Claim that $(\langle x,y\rangle, +, \cdot)$ is a ratio semiring.

Clearly $(\langle x,y\rangle, \cdot)$ is a commutative group. To show the claim we need only to show that for $m_1, m_2, m_3, n_1, n_2, n_3 \in \mathbb{Z}$

$$(1)\ (x^{m_1} y^{n_1} + x^{m_2} y^{n_2}) + x^{m_3} y^{n_3} = x^{m_1} y^{n_1} + (x^{m_2} y^{n_2} + x^{m_3} y^{n_3})$$

and
$$(2)\ (X^{m_1} y^{n_1} + x^{m_2} y^{n_2}) \cdot x^{m_3} y^{n_3} = x^{m_1} y^{n_1} \cdot x^{m_3} y^{n_3} + x^{m_2} y^{n_2} \cdot x^{m_3} y^{n_3}.$$

First we shall show (1). Let m_1, m_2, m_3, n_1, n_2 and $n_3 \in \mathbb{Z}$. By definition of +, we get that

$$(x^{m_1} y^{n_1} + x^{m_2} y^{n_2}) + x^{m_3} y^{n_3} = x^r y^s = x^{m_1} y^{n_1} + (x^{m_2} y^{n_2} + x^{m_3} y^{n_3})$$

where $r = \max.\{m_1, m_2, m_3\}$ and $s = \min.\{n_1, n_2, n_3\}$.

To show (2), let m_1, m_2, m_3, n_1, n_2 and $n_3 \in \mathbb{Z}$. Then

$$(x^{m_1} y^{n_1} + x^{m_2} y^{n_2}) \cdot x^{m_3} y^{n_3} = x^r y^s \cdot x^{m_3} y^{n_3} = x^{r+m_3} y^{s+n_3}$$

where $r = \max\{m_1, m_2\}$ and $s = \min\{n_1, n_2\}$ and

$$x^{m_1}y^{n_1} \cdot x^{m_3}y^{n_3} + x^{m_2}y^{n_2} \cdot x^{m_3}y^{n_3} = x^{m_1+m_3}y^{n_1+n_3} + x^{m_2+m_3}y^{n_2+n_3}$$

$= x^p y^q$ where $p = \max\{m_1 + m_3, m_2 + m_3\}$ and $q = \min\{n_1 + n_3, n_2 + n_3\}$.

Since $r = \max\{m_1, m_2\}$, $r+m_3 = \max\{m_1 + m_3, m_2 + m_3\}$ and since $s = \min\{n_1, n_2\}$, $s+n_3 = \min\{n_1 + n_3, n_2 + n_3\}$. Thus $r+m_3 = p$ and $s+n_3 = q$, so we have (2). So we have the claim i.e. $(\langle x,y\rangle, +, \cdot)$ is a ratio semiring.

Clearly $I_{\langle x,y\rangle}(xy) = \{x^m y^n \mid m,n \in \mathbb{Z},\ m \leqslant 1 \leqslant n\}$. Let $m_0, n_0 \in \mathbb{Z}$ be such that $m_0 \leqslant 1 \leqslant n_0$. Define $S_1 = \{x^m y^n \mid m \leqslant m_0 \text{ and } n \geqslant n_0\}$. Then $\langle x, y\rangle \setminus S_1 = \{x^m y^n \mid m > m_0 \text{ or } n < n_0\}$. Clearly S_1 is an additive sub-semigroup of $I_{\langle x,y\rangle}(xy)$. Claim that $\langle x,y\rangle \setminus S_1$ is an ideal of $(\langle x,y\rangle, +)$. Let $z \in \langle x,y\rangle \setminus S_1$ and $w \in \langle x,y\rangle$. Then $z = x^m y^n$ where $m > m_0$ or $n < n_0$ and $w = x^k y^\ell$ for some $k, \ell \in \mathbb{Z}$. Consider $z+w = x^m y^n + x^k y$. We get that

$$z+w = x^m y^n + x^k y^\ell = \begin{cases} x^m y^n & \text{if } k \leqslant m \text{ and } n \leqslant \ell, \\ x^m y^\ell & \text{if } k \leqslant m \text{ and } \ell < n, \\ x^k y^n & \text{if } m < k \text{ and } n \leqslant \ell, \\ x^k y^\ell & \text{if } m < k \text{ and } \ell < n. \end{cases}$$

In all cases we see that $z+w \in \langle x,y\rangle \setminus S_1$. So we have the claim i.e. $\langle x,y\rangle \setminus S_1$ is an ideal of $(\langle x,y\rangle, +)$.

Let a be a symbol not representing any element of $\langle x,y\rangle$. Extend $+$ and $\cdot$ from $\langle x,y\rangle$ to $\langle x,y\rangle \cup \{a\}$ as follows ;

(1) $a \cdot w = w \cdot a = xy \cdot w$ for all $w \in \langle x,y\rangle$ and $a \cdot a = xy \cdot xy$,

(2) $a + w = w + a = a$ for all $w \in S_1$ and

$a + w = w + a = xy + w$ for all $w \in \langle x,y\rangle \setminus S_1$,

(3) $a + a = a$ or xy.

Then by Theorem 2.51 we obtain that $\langle x,y\rangle \cup \{a\}$ is a semifield of type III.

<u>Remark 4.18</u> Let $(\langle x,y\rangle, +, \cdot)$ be the ratio semiring given in Remark 4.17. Let a be a symbol not representing any element of $\langle x,y\rangle$. Extend + and $\cdot$ from $\langle x,y\rangle$ to $\langle x,y\rangle \cup \{a\}$ by $a \cdot w = w \cdot a = xy \cdot w$ $\forall w \in \langle x,y\rangle$, $a \cdot a = xy \cdot xy$, $a+w = w+a$ $\forall w \in \langle x,y\rangle$ and $a+a = a$ or xy. Then by Theorem 2.51 we obtain that $\langle x,y\rangle \cup \{a\}$ is a semifield of type III.

<u>Definition 4.19</u> Let D be a ratio semiring and let 0 be a symbol not representing any element of D. We have shown in Theorem 3.5 that we can extend the binary operations of D to $D \cup \{0\}$ making $D \cup \{0\}$ into a 0-semifield. Let $(D\cup\{0\})[x]$ be the set of all polynomials coefficient in $D \cup \{0\}$. Define $D[x] = (D\cup\{0\})[x] \setminus \{0\}$. Then $D[x]$ is a semiring. If $D[x]$ is M.C. then define $D(x)$ to be the <u>quotient ratio semiring</u> of $D[x]$.

<u>Remark</u> 4.20 Let $D = \{1\}$ be a ratio semiring. Then D is A.C. but we see that $D \cup \{0\}$ the 0-semifield of definition 4.19 is not A.C. because $1+1 = 1+0$.

The next proposition will show that $D \cup \{0\}$ is also A.C. if D is A.C. and infinite.

<u>Proposition 4.21</u> Let D be an infinite ratio semiring and $D \cup \{0\}$ the 0-semifield of definition 4.19. Then $D \cup \{0\}$ is A.C. iff D is A.C.

<u>Proof</u> Assume that $D \cup \{0\}$ is A.C. Let $a, b, c \in D$ be such that $a+b = a+c$. Since $a, b, c \in D \cup \{0\}$ and $D \cup \{0\}$ is A.C., $b = c$. Thus D is A.C..

Conversely assume that D is A.C. Let $a,b,c \in D \cup \{a\}$ be such that $a+b = a+c$. We must show that $b = c$. If $a = 0$, then $b = c$. Suppose that $a \neq 0$. Consider b and c.

<u>Case 1</u> Both of them are 0. Then $b = c$.

<u>Case 2</u> Both of them are not 0. Then $a,b,c \in D$ are such that $a+b = a+c$. Thus $b = c$ since D is A.C.

<u>Case 3</u> One is 0 the other is not. We may assume that $b = 0$ and $c \neq 0$. Then $a = a+c$, so $1 = 1+a^{-1}c$. Let $x = a^{-1}c$, then $1+x = 1$. By induction, we obtain that $1+x^n = 1 \quad \forall n \in \mathbb{Z}^+$. Let $m,n \in \mathbb{Z}, m < n$. Then $x^m + x^n = x^m(1+x^{n-m}) = x^m\, 1 = x^m$. Thus $x^m + x^n = x^m$ for all $m,n \in \mathbb{Z}$, $m < n$. Then $x+x^2 = x+x^3$. Since D is A.C., $x^2 = x^3$. Hence $x = 1$. So we get that $1+1 = 1$. Let $z \in D \setminus \{1\}$. If $1+z = 1$, then $1+z = 1+1$. Since D is A.C., $z = 1$ which is a contradiction. Thus $1+z \neq 1$. Now $1+z = (1+1)+z = 1+(1+z)$. Then $1+z = 1+(1+z)$, so $1 = 1+(1+z)^{-1}$. Thus $1+(1+z)^{-1} = 1+1$. Since D is A.C., $(1+z)^{-1} = 1$. Hence $1+z = 1$ which is a contradiction. Thus this case cannot occur.

Therefore we get that $D \cup \{0\}$ is A.C.. #

<u>Remark</u> 4.22 Let $D = \{1\}$ be a ratio semiring. Then clearly D is A.C. Consider $D[x]$.

Let $f(x) = 1+x$, $g(x) = 1+x+x^2$ and $h(x) = 1+x^2$. We see that

$$f(x)g(x) = (1+x)(1+x+x^2) = 1+x+x^2+x^3 \quad \text{and}$$

$$f(x)h(x) = (1+x)(1+x^2) = 1+x+x^2+x^3$$

So we have that $f(x)g(x) = f(x)h(x)$ and $g(x) \neq h(x)$. Thus $D[x]$ is not M.C..

The next theorem will show that if D is A.C. and D is infinite then $D[x]$ is M.C..

Theorem 4.23 Let D be an infinite ratio semiring. Then D[x] is M.C. iff D is A.C..

Proof Assume that D[x] is M.C. Let $a,b,c \in D$ be such that $a+b = a+c$. Let $f(x) = 1+x$, $g(x) = a+bx+ax^2$ and $h(x) = a+cx+ax^2$. Consider $f(x)\cdot g(x)$ and $f(x)\cdot h(x)$.

$$f(x)\cdot g(x) = (1+x)(a+bx+ax^2) = a+(a+b)x+(a+b)x^2+ ax^3$$

$$f(x)\cdot h(x) = (1+x)(a+cx+ax^2) = a+(a+c)x+(a+c)x^2+ ax^3$$

Since $a+b = a+c$, we get that $f(x)g(x) = f(x)h(x)$. Since D[x] is M.C., we get that $g(x) = h(x)$. Hence $b = c$. Therefore D is A.C.

Conversely, assume that D is A.C. Let $f(x)$, $g(x)$ and $h(x) \in D[x]$ be such that $f(x)g(x) = f(x)h(x)$.

Suppose that $f(x) = \sum_{i=0}^{k} a_i x^i$, $g(x) = \sum_{i=0}^{\ell} b_i x^i$ and $h(x) = \sum_{i=0}^{m} c_i x^i$ where $a_k, b_\ell, c_m \neq 0$. Then $f(x)g(x) = \sum_{i=0}^{k+\ell} d_i x^i$ where $d_i = \sum_{j=0}^{i} a_{i-j}b_j$, $i = 0,1,\ldots,k+\ell$ and $f(x)h(x) = \sum_{i=0}^{k+m} f_i x^i$ where $f_i = \sum_{j=0}^{i} a_{i-j}c_j$. Since $f(x)g(x) = f(x)h(x)$, $k+\ell = k+m$ and $d_i = f_i$ $\forall\, i = 0,1,\ldots,k+\ell$. Let n be the smallest non negative integer such that $a_n \neq 0$. Since $d_n = f_n$, then

$a_nb_0+ a_{n-1}b_1+\ldots+ a_0b_n = a_nc_0+ a_{n-1}c_1+\ldots+ a_0c_n$. Since $a_0= a_1= \ldots = a_{n-1}= 0$, $a_nb_0= a_nc_0$. Thus $b_0= c_0$.

Now consider d_{n+1} and f_{n+1}. Since $d_{n+1}= f_{n+1}$, we get that

$a_{n+1}b_0+ a_nb_1+ a_{n-1}b_2+ \ldots + a_0b_{n+1}= a_{n+1}c_0+ a_nc_1+ a_{n-1}c_2+\ldots$ $\ldots+ a_0c_{n+1}$. Since $a_0= a_1= \ldots = a_{n-1}= 0$, $a_{n+1}b_0+ a_nb_1= a_{n+1}c_0 + a_nc_1$. Since $b_0= c_0$, $a_{n+1}b_0 = a_{n+1}c_0$. By Proposition 4.21

we have that $D \cup \{0\}$ is A.C. Since $a_{n+1}b_0$, a_nb_1, $a_nc_1 \varepsilon D \cup \{0\}$ and $D \cup \{0\}$ is A.C., $a_nb_1 = a_nc_1$. Hence $b_1 = c_1$.

Consider d_{n+2} and f_{n+2}. Since $d_{n+2} = f_{n+2}$, we get that

$a_{n+2}b_0 + a_{n+1}b_1 + a_nb_2 = a_{n+2}c_0 + a_{n+1}c_1 + a_nc_2$ (since $a_0 = a_1 = \ldots$ $\ldots = a_{n-1} = 0$). Since $b_0 = c_0$ and $b_1 = c_1$, $a_{n+2}b_0 + a_{n+1}b_1 = a_{n+2}c_0 + a_{n+1}c_1$. Since $D \cup \{0\}$ is A.C., $a_nb_2 = a_nc_2$. Hence $b_2 = c_2$. Using the same proof we then get $b_i = c_i$ for all i. Thus $g(x) = h(x)$. Therefore $D[x]$ is M.C.. #

Remark 4.24 Let $D = \{1\}$ be a ratio semiring. Then clearly D is A.C. Consider $D[x]$.

Let $f(x) = 1+x$, $g(x) = 1+x^2$ and $h(x) = x+x^2$. We see that

$$f(x)+g(x) = (1+x)+(1+x^2) = 1+x+x^2 \quad \text{and}$$
$$f(x)+h(x) = (1+x)+(x+x^2) = 1+x+x^2$$

So we have that $f(x)+g(x) = f(x)+h(x)$ and $g(x) \neq h(x)$. Thus $D[x]$ is not A.C.

The next theorem will show that if D is A.C. and D is infinite, then $D[x]$ is A.C.

Theorem 4.25 Let D be an infinite ratio semiring. Then $D[x]$ is A.C. iff D is A.C..

Proof Assume that $D[x]$ is A.C. Let $a,b,c \varepsilon D$ be such that $a+b = a+c$. Since a,b,c are all polynomials in $D[x]$ which is A.C., so $b = c$. Thus D is A.C..

Conversely, assume that D is A.C. Let $f(x)$, $g(x)$ and $h(x) \varepsilon D[x]$ be such that $f(x)+g(x) = f(x)+h(x)$. Let $f(x) = \sum_{i=0}^{n} a_ix^i$,

$g(x) = \sum_{i=0}^{n} b_i x^i$ and $h(x) = \sum_{i=0}^{n} c_i x^i$. Then $\sum_{i=0}^{n} (a_i+b_i)x^i = \sum_{i=0}^{n} (a_i+c_i)x^i$.

Thus $a_i + b_i = a_i + c_i$ $\forall$ $i = 0,1,\ldots,n$. By Proposition 4.21 we have that $D \cup \{0\}$ is A.C. Since $a_i, b_i, c_i \in D \cup \{0\}$ such that $a_i + b_i = a_i + c_i$ for all $i = 0,1,\ldots,n$, we get that $b_i = c_i$ $\forall$ $i = 0,1,\ldots,n$. Thus $g(x) = h(x)$. Therefore $D[x]$ is A.C. #

Remark 4.26 Let $\mathbb{Q}^+$ have the usual addition and multiplication. Then $\mathbb{Q}^+$ is a ratio semiring and since $\mathbb{Q}^+$ is A.C. Then by Proposition 4.21, we obtain that $\mathbb{Q}^+[x]$ is M.C. So $\mathbb{Q}^+(x)$ is the quotient ratio semiring of $\mathbb{Q}^+[x]$. Let a be a symbol not representing any element of $\mathbb{Q}^+(x)$. Extend + and $\cdot$ from $\mathbb{Q}^+(x)$ to $\mathbb{Q}^+(x) \cup \{a\}$ by $a \cdot z = z \cdot a = x \cdot z$ $\forall$ $z \in Q^+(x)$, $a^2 = x^2$, $a+z = z+a = x+z$ $\forall$ $z \in \mathbb{Q}^+(x)$ and $a+a = x+x$. By Theorem 2.51, $\mathbb{Q}^+(x) \cup \{a\}$ is a semifield of type III.

Definition 4.27 Let D be a ratio semiring and $E \subseteq D$. Then E is called a C-set iff

1) $x, y \in E \Longrightarrow xy^{-1} \in E$

2) $x \in E$ and $y \in D \Longrightarrow \frac{x+y}{1+y} \in E$.

Let D be a ratio semiring and E a C-set in D.. Define a relation $\sim$ on D by $x \sim y$ iff $xy^{-1} \in E$. Clearly $\sim$ is reflexive. Let $x, y \in D$ be such that $x \sim y$. Then $xy^{-1} \in E$. By condition 1), $(E,\cdot) \leqslant (D,\cdot)$. Thus $(xy^{-1})^{-1} \in E$. Since $yx^{-1} = (xy^{-1})^{-1}$, $yx^{-1} \in E$. Thus $y \sim x$. Let $x, y, z \in D$ be such that $x \sim y$ and $y \sim z$. Then $xy^{-1} \in E$ and $yz^{-1} \in E$. So $(xy^{-1})(yz^{-1}) \in E$ because $(E,\cdot) \leqslant (D,\cdot)$. Thus $xz^{-1} \in E$. Hence $x \sim z$. So $\sim$ is an equivalence relation.

Let $D/_E$ be the set of all equivalence classes in D. Let α, $\beta \in D/_E$. Define + and $\cdot$ on $D/_E$ in the following way:

Choose $x \in \alpha$ and $y \in \beta$ and let $\alpha\cdot\beta = [xy]$ and $\alpha+\beta = [x+y]$. To show + and $\cdot$ are well-defined, let $x' \in \alpha$ and $y' \in \beta$. Then $x'\cdot x^{-1} \in E$ and $y'\cdot y^{-1} \in E$. Since $(E,\cdot) \leqslant (D,\cdot)$, $(x'\cdot x^{-1})\cdot(y'\cdot y^{-1}) \in E$. Thus $(x'\cdot y')\cdot(xy)^{-1} \in E$, so $x'\cdot y' \sim xy$. Hence $[xy] = [x'y']$. So $\cdot$ is well-defined. Since $\frac{x'+y'}{x+y'} = \frac{x'x^{-1} + y'x^{-1}}{1 + y'x^{-1}}$ and $x'x^{-1} \in E$, so $\frac{x'+y'}{x+y'} \in E$.

Thus $x' + y' \sim x + y'$. Similarly we can show that $x + y' \sim x + y$. So $x' + y' \sim x + y$. Hence $[x' + y'] = [x+y]$, so + is well-defined.

Claim that $D/_E$ is a ratio semiring.

Let $\alpha \in D/_E$. Choose $x \in \alpha$. Then $\alpha\cdot[1] = [x][1] = [x\cdot 1] = [x] = \alpha$, so $[1]$ is the multiplicative identity. Let $\beta = [x^{-1}]$ Then $\alpha\beta = [x][x^{-1}] = [xx^{-1}] = [1]$ so every element has a multiplicative inverse. Clearly $\cdot$ is commutative and associative. Thus $(D/_E,\cdot)$ is a commutative group, and clearly $(D/_E,+)$ is a commutative semigroup.

Let $\alpha, \beta, \gamma \in D/_E$. Choose $x \in \alpha$, $y \in \beta$, $z \in \gamma$. Then $(\alpha+\beta)\cdot\gamma = ([x]+[y])[z] = [x+y][z] = [(x+y)z] = [xz+yz] = [xz]+[yz] = [x][z]+[y][z] = \alpha\cdot\gamma + \beta\cdot\gamma$.

Hence $(D/_E,+,\cdot)$ is a ratio semiring. So we have the claim.

<u>Remark 4.28</u> Let $\mathbb{Q}^+(x)$ be the quotient ratio semiring of $\mathbb{Q}^+[x]$ given in remark 4.23. Let L be a C-set in $\mathbb{Q}^+(x)$ such that $1+x, \frac{1+x}{x} \in \mathbb{Q}^+(x) \setminus L$ and $\mathbb{Q}^+ \cap L = \{1\}$ and $\gamma x \notin L\ \forall \gamma \in \mathbb{Q}^+$. Then $\mathbb{Q}^+(x)/_L$ is a ratio semiring. Let $W = \mathbb{Q}^+(x)/_L$ and $w = [x]$. Claim that $[1]$, $w^2 \notin I_W(w)$. Suppose that $[1] \in I_W(w)$. Then $[1] + w = w$, so $[1]+[x] = [x]$. Then $[1+x] = [x]$, so $\frac{1+x}{x} \in L$ which is a contradiction. Thus $[1] \notin I_W(w)$. Similarly we can show that $w^2 \notin I_W(w)$.

Choose $S_1 \subseteq I_W(w)$ such that either ($S_1 = \phi$) or (S_1 is an additive subsemigroup of $I_W(w)$ and $W \setminus S_1$ is an additive ideal of W)

Let a be a symbol not representing any element of W. Extend + and $\cdot$ from W to $W \cup \{a\}$ by

(1) $a \cdot y = y \cdot a = w \cdot y \quad \forall y \in W$ and $a^2 = w^2$,

(2) $a+y = y+a = a \quad \forall y \in S_1$ and

$a+y = y+a = w+y \quad \forall y \in W \setminus S_1$,

(3) $a+a = w+w$.

Then by Theorem 2.51, $W \cup \{a\}$ is a semifield of type III. Hence $\mathbb{Q}^+(x)/_L \cup \{a\}$ is a semifield of type III.

<u>Theorem 4.29</u> Let K be a semifield of type III and $a \in K$, $d \in K \setminus \{a\}$ be such that $(K \setminus \{a\}, \cdot)$ is a group and $ax = dx \quad \forall x \in K$. Let K' the prime semifield of K. Then $K' \cong \mathbb{Q}^+ \cup \{a\}$ as in Remark 4.8 or $K' \cong \{a,1\}$ as in Remark 4.7 (1) or (2) or (3) or (4) or $K' \cong \langle d\rangle \cup \{a\}$ as in Remark 4.9 or Remark 4.10 or Remark 4.11 or Remark 4.12 or $K' \cong \mathbb{Q}^+\langle d\rangle \cup \{a\}$ as in Remark 4.13 or Remark 4.14 or Remark 4.15 or Remark 4.16 or $K' \cong \langle x\rangle\langle y\rangle \cup \{a\}$ as in Remark 4.17 or Remark 4.18 or $K' \cong \mathbb{Q}^+(x) \cup \{a\}$ as in Remark 4.26 or $K' \cong \mathbb{Q}^+(x)/_L \cup \{a\}$ as in Remark 4.28.

<u>Proof</u> Let $D = K \setminus \{a\}$, $S = \{x \in D \mid a+x = a\}$ and 1 the identity of $(D, \cdot)$.

<u>Case 1</u> $d = 1$.

By Theorem 2.41, D is a ratio semiring. Let D_1 be the smallest ratio subsemiring of D. By Proposition 1.18, $D_1 \cong \{1\}$ with $1 \cdot 1 = 1$, $1+1 = 1$ or $D_1 \cong \mathbb{Q}^+$ with the usual addition and multiplication.

<u>Subcase 1.1</u> $D_1 \cong \{1\}$ with $1 \cdot 1 = 1$ and $1+1 = 1$. Thus $1+1 = 1$. By Theorem 2.43, $a+a = a$ or $a+a = d+d = 1+1 = 1$ and $a+1 = 1+a$

$d+1 = 1+1 = 1$. So we have 4 cases to consider. They are (1),(2),(3) and (4) as in Remark 4.7. Thus $K' \cong \{a,1\}$ as in Remark 4.7 (1) or (2) or (3) or (4).

Subcase 1.2 $D_1 \cong \mathbb{Q}^+$ with the usual addition and multiplication. Then up to isomorphism we can consider $\mathbb{Q}^+ \subseteq D$. Claim that $S \cap \mathbb{Q}^+ = \Phi$. Suppose not, then $\exists\, x \in \mathbb{Q}^+$ and $x \in S$. By Theorem 2.50, $S \subseteq I_D(d) = I_D(1)$. So $x \in I_D(1)$. Then $x+1 = 1$, which is a contradiction since $x, 1 \in \mathbb{Q}^+$. Thus $S \cap \mathbb{Q}^+ = \Phi$. So we have the claim. Thus $\mathbb{Q}^+ \subseteq D \setminus S$, so $a+x = d+x = 1+x\ \forall x \in \mathbb{Q}^+$ and by Theorem 2.43 (1) and (2), $a+a = d+d = 1+1$. Since $\mathbb{Q}^+ \cup \{a\} \subseteq K$, $ax = xa = dx\ \ \forall x \in \mathbb{Q}^+ \cup \{a\}$. By Theorem 2.51, we obtain that $\mathbb{Q}^+ \cup \{a\}$ is a semifield as in Remark 4.8. Thus $\mathbb{Q}^+ \cup \{a\}$ is a subsemifield of K, so $K' \subseteq \mathbb{Q}^+ \cup \{a\}$. By Theorem 2.41, $K' \setminus \{a\}$ is a ratio semiring. Thus $K' \setminus \{a\}$ is a ratio subsemiring of $K \setminus \{a\}$, so $\mathbb{Q}^+ \subseteq K' \setminus \{a\}$. Thus $\mathbb{Q}^+ \cup \{a\} \subseteq K' \setminus \{a\}$. Therefore $K' \cong \mathbb{Q}^+ \cup \{a\}$ as in Remark 4.8.

Case 2 $d \neq 1$.

Subcase 2.1 $d^2 \in I_D(d)$ and $1 \in I_D(d)$.

Then $d = d+d^2 = 1\cdot d + d\cdot d = (1+d)d = d\cdot d = d^2$. Thus $d = 1$, a contradiction. Thus this case cannot occur.

Subcase 2.2 $d^2 \in I_D(d)$ and $1 \in D \setminus I_D(d)$.

Then $d = d+d^2 = 1\cdot d + d\cdot d = (1+d)d$. Since $1, d \in D = K \setminus \{a\}$ which is a ratio semiring, $1+d \neq a$. Then $1 = d\cdot d^{-1} = ((1+d)d)d^{-1} = (1+d)(d\cdot d^{-1}) = (1+d)\cdot 1 = 1+d$. Thus $1+d = 1$. Claim that $1+d^n = 1$ $\forall\, n \in \mathbb{Z}^+$. We shall prove this by induction. Clearly it is true for $n = 1$. Assume it is true for $n-1$. That is $1+d^{n-1} = 1$. Thus $d = d+d^n$. Then $1 = 1+d = 1+(d+d^n) = (1+d)+d^n = 1+d^n$. Thus $1+d^n = 1$. So we have the claim. Let $m, n \in \mathbb{Z}$ be such that $m < n$. Then $d^m+d^n =$

$1\cdot d^m + d^{n-m}\cdot d^m = (1+d^{n-m})\cdot d^m = 1\cdot d^m = d^m$ (since $n-m \in \mathbb{Z}^+$ and by the claim) Thus $d^m + d^n = d^m$ for all $m, n \in \mathbb{Z}$, $m < n$ ......(1). For $m \in \mathbb{Z}$, $m \leqslant 0$. Claim that $d^m \in D \setminus S$. If $m = 0$, then $d^0 = 1 \in D\setminus S$ since $1 \in D \setminus I_D(d)$ and $S \subseteq I_D(d)$. If $m < 0$, then $d^m = d^m + d^0 = d^m + 1 \in D\setminus S$ since $D\setminus S$ is an ideal of $(D,+)$.

Thus $d^m \in D\setminus S$ for all $m \leqslant 0$. ....(2)

Subcase 2.2.1 $1+1 = 1$.

Then $d^n + d^n = d^n$ $\forall n \in \mathbb{Z}$. So we have that

$d^m + d^n = d^m$ for all $m, n \in \mathbb{Z}$, $m \leqslant n$. ....(3)

Let $\langle d\rangle = \{d^n \mid n \in \mathbb{Z}\}$. Clearly $(\langle d\rangle,\cdot) \leqslant (K\setminus\{a\},\cdot)$ and by (3), we can show that $\langle d\rangle$ is an additive subsemigroup of $K\setminus\{a\}$. Thus $(\langle d\rangle,+,\cdot)$ is a ratio semiring.

Subcase 2.2.1.1 There exists $m \in \mathbb{Z}^+$ such that $d^m \in S$. Choose the smallest $n_0 \in \mathbb{Z}^+$ such that $d^{n_0} \in S$. Then $n_0 \geqslant 1$ (since $d^m \in D\setminus S$ $\forall m \in \mathbb{Z}$, $m \leqslant 0$). So we get that $d^k \in D\setminus S$ $\forall k < n_0$. Claim that $d^n \in S$ $\forall n \geqslant n_0$. Let $n \geqslant n_0$, then $d^{n_0} = d^{n_0} + d^n$. Now $a = a+d^{n_0} = a + (d^{n_0} + d^n) = (a+d^{n_0})+d^n = a+d^n$. Thus $a+d^n = a$. Therefore $d^n \in S$. So we have the claim. Let $S_1 = \{d^n \mid n \geqslant n_0\}$. Then $\langle d\rangle\setminus S_1 = \{d^n \mid n < n_0\}$. Thus $S_1 \subseteq S$ and $\langle d\rangle\setminus S_1 \subseteq D\setminus S$. Clearly $(S_1,+) \leqslant (I_{\langle d\rangle}(d),+)$ and $\langle d\rangle\setminus S_1$ is an ideal of $(\langle d\rangle,+)$. Since $S_1 \subseteq S$ and $\langle d\rangle\setminus S_1 \subseteq D\setminus S$, $a+x = a$ $\forall x \in S_1$ and $a+x = d+x$ $\forall x \in \langle d\rangle\setminus S_1$. By Theorem 2.43, $a+a = a$ or $a+a = d+d = d$ since $1+1 = 1$. Since $\langle d\rangle \cup \{a\} \subseteq K$, $ax = xa = dx$ $\forall x \in \langle d\rangle \cup \{a\}$. By Theorem 2.51, $\langle d\rangle \cup \{a\}$ is the semifield given in Remark 4.9. Thus $K' \subseteq \langle d\rangle \cup \{a\}$. Claim that $K' \cong \langle d\rangle \cup \{a\}$. By Theorem 4.2, $a, 1 \in K'$ and $(K'\setminus\{a\},\cdot)$ is a group. Then $d = d\cdot 1 = a\cdot 1 \in K'\setminus\{a\}$. Thus $d \in K'\setminus\{a\}$. Hence $\langle d\rangle \subseteq K'\setminus\{a\}$. Therefore $\langle d\rangle \cup \{a\} \subseteq K'$. Hence $K' \cong \langle d\rangle \cup \{a\}$ as in Remark 4.9

Subcase 2.2.1.2 There does not exist an $m \in \mathbb{Z}^+$ such that $d^m \in S$. Thus $d^n \in D \setminus S$ $\forall n \in \mathbb{Z}^+$ and from (2) we then get that $d^n \in D \setminus S$ $\forall n \in \mathbb{Z}$, so $a+d^n = d+d^n$ $\forall n \in \mathbb{Z}$. By Theorem 2.43, $a+a = a$ or $a+a = d+d = d$ since $1+1 = 1$. Since $\langle d \rangle \cup \{a\} \subseteq K$, $ax = xa = dx$ $\forall x \in \langle d \rangle \cup \{a\}$. By Theorem 2.51, $\langle d \rangle \cup \{a\}$ is the semifield given in Remark 4.10. The same as before, we can show that $K' \cong \langle d \rangle \cup \{a\}$ as in Remark 4.10

Subcase 2.2.2 $1+1 \neq 1$.

By Proposition 1.18, $\mathbb{Q}^+$ with the usual addition and multiplication is the smallest ratio subsemiring of $K' \setminus \{a\}$. Then up to isomorphism we can consider $\mathbb{Q}^+ \subseteq K' \setminus \{a\}$.

Subcase 2.2.2.1 $d \in \mathbb{Q}^+$. Claim that $\mathbb{Q}^+ \cap S = \Phi$. If $x \in \mathbb{Q}^+$ and $x \in S$, then $x \in I_D(d)$ (since $S \subseteq I_D(d)$). Thus $x+d = d$, a contradiction since $x, d \in \mathbb{Q}^+$. So we have the claim. Thus $\mathbb{Q}^+ \subseteq D \setminus S$, so $a+x = x+a = d+x$ $\forall x \in \mathbb{Q}^+$ and $ax = xa = dx$ $\forall x \in \mathbb{Q}^+ \cup \{a\}$. By Theorem 2.43 (1) and (2) we get that $a+a = d+d$. By Theorem 2.51, we obtain that $\mathbb{Q}^+ \cup \{a\}$ is the semifield given in Remark 4.8. Since $\mathbb{Q}^+ \subseteq K' \setminus \{a\}$, $\mathbb{Q}^+ \cup \{a\} \subseteq K'$. Since K' is the smallest subsemifield of K, $K' \subseteq \mathbb{Q}^+ \cup \{a\}$. Therefore $K' \cong \mathbb{Q}^+ \cup \{a\}$ as in Remark 4.8.

Subcase 2.2.2.2 $d \notin \mathbb{Q}^+$.

Consider $\mathbb{Q}^+ \cdot \langle d \rangle = \{xd^n \mid x \in \mathbb{Q}^+, n \in \mathbb{Z}\}$. Clearly $(\mathbb{Q}^+ \cdot \langle d \rangle, \cdot)$ is a subgroup of $(K \setminus \{a\}, \cdot)$. Claim that $\mathbb{Q}^+ \cdot \langle d \rangle$ is a ratio subsemiring of $K' \setminus \{a\}$. Since a and $1 \in K'$, $d = a \cdot 1 \in K'$. Since $\mathbb{Q}^+ \subseteq K' \setminus \{a\}$, which is a group under multiplication, $\mathbb{Q}^+ \cdot \langle d \rangle \subseteq K' \setminus \{a\}$. To show the claim, we need only show that $\mathbb{Q}^+ \cdot \langle d \rangle$ is a subsemigroup of $K' \setminus \{a\}$ under addition. Let $x, y \in \mathbb{Q}^+, m, n \in \mathbb{Z}$. Consider $xd^m + yd^n$. If $m = n$

then $xd^m + yd^n = xd^m + yd^m = (x+y)d^m \in \mathbb{Q}^+ \cdot \langle d \rangle$. Now suppose that $m \neq n$. We may assume that $m < n$.

Case 1 (of claim) $x = y$. Then $xd^m + yd^n = xd^m + xd^n = x(d^m + d^n) = xd^m \in \mathbb{Q}^+ \cdot \langle d \rangle$.

Case 2 (of claim) $x > y$. Then $\exists\, \ell \in \mathbb{Q}^+$ such that $x = \ell + y$. Then

$$xd^m + yd^n = (\ell+y)d^m + yd^n = \ell d^m + yd^m + yd^n = \ell d^m + y(d^m + d^n)$$
$$= \ell d^m + yd^m = (\ell+y)d^m = xd^m \in \mathbb{Q}^+ \cdot \langle d \rangle .$$

Case 3 (of claim) $x < y$. Let $z \in K$ and $n \in \mathbb{Z}^+$. Define $nz = nz = \underbrace{z+z+\ldots+z}_{n \text{ times}}$ Since $y = \left[\frac{y}{x}\right]x + \gamma$ where $0 \leqslant \gamma < x$, then $xd^m + yd^n =$

$$xd^m + \left(\left[\frac{y}{x}\right]x + \gamma\right)d^n = xd^m + \left[\frac{y}{x}\right]xd^n + \gamma d^n = (xd^m + \gamma d^n) + \left[\frac{y}{x}\right]xd^n$$
$$= xd^m + \left[\frac{y}{x}\right]xd^n \text{ (by case 2) } = xd^m + (\underbrace{xd^n + xd^n + \ldots + xd^n}_{\left[\frac{y}{x}\right] \text{ times}})$$
$$= x(d^m + \underbrace{d^n + d^n + \ldots + d^n}_{\left[\frac{y}{x}\right] \text{ times}}) = xd^m \in \mathbb{Q}^+ \cdot \langle d \rangle .$$

So we get that

$$xd^m + yd^n = \begin{cases} (x+y)d^m & \text{if } m = n, \\ xd^m & \text{if } m < n, \\ yd^n & \text{if } m > n. \end{cases} \quad \ldots\ldots.(4)$$

So we have the claim. Therefore $\mathbb{Q}^+ \cdot \langle d \rangle$ is a ratio subsemiring of $K' \setminus \{a\}$.

Let $x \in \mathbb{Q}^+$. If $xd \in S$, then $xd \in I_D(d)$ since $S \subseteq I_D(d)$. Thus $xd+d = d$, so $x+1 = 1$, a contradiction since $x, 1 \in \mathbb{Q}^+$. Hence $xd \in D \setminus S$ $\forall x \in \mathbb{Q}^+$. For $n \in \mathbb{Z}$, $n < 1$ we have that $xd^n = xd^n + xd$. Since $xd \in D \setminus S$ which is an ideal of $(D,+)$, $xd^n + xd \in D \setminus S$.

Thus $xd^n \in D\setminus S \quad \forall x \in \mathbb{Q}^+, \forall n \in \mathbb{Z}, n \leqslant 1$ ....(5)

Subcase 2.2.2.2.1 There exists $n \in \mathbb{Z}^+$ such that $d^n \in S$. Let n_0 be the smallest positive integer such that $d^{n_0} \in S$. By (5) we get that $n_0 \geqslant 2$. Let $S_1 = \{xd^n \mid x \in \mathbb{Q}^+, n \geqslant n_0\}$. Then $\mathbb{Q}^+\cdot\langle d\rangle\setminus S_1 = \{xd^n \mid x \in \mathbb{Q}^+, n < n_0\}$. Claim that $S_1 \subseteq S$ and $\mathbb{Q}^+\cdot\langle d\rangle\setminus S_1 \subseteq D\setminus S$.

To show $S_1 \subseteq S$, let $x \in \mathbb{Q}^+$ and $n \in \mathbb{Z}^+$, $n \geqslant n_0$.

Case 1 (of claim) $n = n_0$.

Subcase 1.1 $x = 1$. Then $xd^{n_0} = 1\cdot d^{n_0} = d^{n_0} \in S$.

Subcase 1.2 $x < 1$. Then $\exists\, \ell \in \mathbb{Q}^+$ such that $1 = x+\ell$. Thus $d^{n_0} = xd^{n_0} + \ell d^{n_0}$. Then $a = a+d^{n_0} = a+(xd^{n_0} + \ell d^{n_0}) = (a+xd^{n_0}) + \ell d^{n_0}$.

If $a+xd^{n_0} \neq a$, then $a+xd^{n_0} = d+xd^{n_0}$ (by Theorem 2.43). Thus $a = (d+xd^{n_0}) + \ell d^{n_0} = d+\ell d^{n_0} = d$ (since $n_0 \geqslant 2$ and by (4)). Hence $a = d$, a contradiction. Thus $a+xd^{n_0} = a$, so $xd^{n_0} \in S$.

Subcase 1.3 $x > 1$. Then $x = [x]+\ell$ where $0 \leqslant \ell < 1$. Then $xd^{n_0} = [x]d^{n_0} + \ell d^{n_0} = (\underbrace{d^{n_0}+\ldots+d^{n_0}}_{[x] \text{ times}}) + \ell d^{n_0}$. By subcase 1.1 and Subcase 1.2, $d^{n_0}, \ell d^{n_0} \in S$. Since $(S,+) \leqslant (I_D(d),+)$, we get that $(\underbrace{d^{n_0}+\ldots+d^{n_0}}_{[x] \text{ times}}) + \ell d^{n_0} \in S$. Hence $xd^{n_0} \in S$.

Case 2 (of claim) $n > n_0$. Then $xd^{n_0} = xd^{n_0} + xd^n$ (by (4)). By case 1, $xd^{n_0} \in S$. Then $a = a+xd^{n_0} = a+(xd^{n_0} + xd^n) = (a+xd^{n_0}) + xd^n = a+xd^n$. Thus $a+xd^n = a$. Hence $xd^n \in S$.

Therefore we get that $S_1 \subseteq S$.

To show $\mathbb{Q}^+\cdot\langle d\rangle\setminus S_1 \subseteq D\setminus S$, let $x \in \mathbb{Q}^+$, $n < n_0$. If $n \leqslant 1$, then by (5) we get that $xd^n \varepsilon D\setminus S$. Suppose that $1 < n < n_0$.

Case 1 (of claim) $x = 1$. Then $xd^n = 1\cdot d^n = d^n$. Since n_0 is the smallest positive integer such that $d^{n_0} \varepsilon S$, $d^n \varepsilon D\setminus S$.

Case 2 (of claim) $x > 1$. Then $\exists\, \ell \,\varepsilon\, \mathbb{Q}^+$ such that $x = 1+\ell$. Then $xd^n = (1+\ell)d^n = 1\cdot d^n + \ell d^n = d^n + \ell d^n$. By case 1, $d^n \varepsilon D\setminus S$. Thus $d^n + \ell d^n \,\varepsilon\, D\setminus S$ since $D\setminus S$ is an ideal of $(D,+)$. Hence $xd^n \varepsilon D\setminus S$

Case 3 (of claim) $x < 1$. Then $\exists\, k \,\varepsilon\, \mathbb{Z}^+$ such that $kx > 1$. By case 2, $(kx)d^n \varepsilon D\setminus S$. Since $(kx)d^n = \underbrace{(x+\ldots+x)}_{k \text{ times}}d^n = \underbrace{xd^n+\ldots+xd^n}_{k \text{ times}}$, we get that if $xd^n \varepsilon S$, then $\underbrace{(xd^n+\ldots+xd^n)}_{k \text{ times}} \varepsilon S$. Thus $(kx)d^n \varepsilon S$, which is a contradiction. Hence $xd^n \varepsilon D\setminus S$. So we have the claim i.e. $S_1 \subseteq S$ and $\mathbb{Q}^+ \cdot \langle d\rangle \setminus S_1 \subseteq D\setminus S$. Thus $a+x = x+a = a \quad \forall x \,\varepsilon\, S_1$ and $a+x = x+a = d+x \quad \forall\, x \,\varepsilon\, \mathbb{Q}^+ \cdot \langle d\rangle \setminus S_1$. By Theorem 2.43 (1) and (2), we get that $a+a = d+d$. Since $\mathbb{Q}^+ \cdot \langle d\rangle \cup \{a\} \subseteq K$, $ax = xa = dx \quad \forall\, x \,\varepsilon\, \mathbb{Q}^+ \langle d\rangle \cup \{a\}$. Clearly $(S_1,+) \leqslant (I_{\mathbb{Q}^+ \cdot \langle d\rangle}(d), +)$ and $\mathbb{Q}^+ \cdot \langle d\rangle \setminus S_1$ is an ideal of $(\mathbb{Q}^+ \cdot \langle d\rangle,+)$.
By Theorem 2.51, $\mathbb{Q}^+ \cdot \langle d\rangle \cup \{a\}$ is the semifield given in Remark 4.13.
Thus $K' \subseteq \mathbb{Q}^+ \cdot \langle d\rangle \cup \{a\}$. Since $a, 1 \,\varepsilon\, K'$, we get that $d = a\cdot 1 \,\varepsilon\, K'$. Since $\mathbb{Q}^+ \subseteq K' \setminus \{a\}$, $\mathbb{Q}^+ \cdot \langle d\rangle \subseteq K\setminus\{a\}$ because $(K'\setminus\{a\},\cdot)$ is a group. Hence $\mathbb{Q}^+ \cdot \langle d\rangle \cup \{a\} \subseteq K'$. Thus $K' \cong \mathbb{Q}^+ \cdot \langle d\rangle \cup \{a\}$ as in Remark 4.13

Subcase 2.2.2.2.2 There does not exist an $n \,\varepsilon\, \mathbb{Z}^+$ such that $d^n \varepsilon S$. Then $d^n \varepsilon D\setminus S \quad \forall\, n \,\varepsilon\, \mathbb{Z}^+$. By (2), we have that $d^m \varepsilon D\setminus S \quad \forall\, m \,\varepsilon\, \mathbb{Z}$, $m \leqslant 0$. Hence $d^n \varepsilon D\setminus S \quad \forall\, n \,\varepsilon\, \mathbb{Z}$. Claim that $\mathbb{Q}^+ \cdot \langle d\rangle \subseteq D\setminus S$. Let $x \,\varepsilon\, \mathbb{Q}^+$ and $n \,\varepsilon\, \mathbb{Z}$.

Case 1 (of claim) $n \leqslant 1$. Then by (5), $xd^n \varepsilon D\setminus S$.

Case 2 (of claim) $n > 1$.

<u>Subcase 2.1</u> (of claim) $x = 1$. Then $xd^n = 1d^n = d^n \in D \setminus S$.

<u>Subcase 2.2</u> (of claim) $x > 1$. Then $\exists\, \ell \in Q^+$ such that $x = 1 + \ell$. Thus $xd^n = d^n + \ell d^n$. Since $d^n \in D \setminus S$ which is an ideal of $(D,+)$, $d^n + \ell d^n \in D \setminus S$. Hence $xd^n \in D \setminus S$.

<u>Subcase 2.3</u> (of claim) $x < 1$. Then $\exists\, k \in \mathbb{Z}^+$ such that $kx > 1$. By Subcase 2.2, $(kx)d^n \in D \setminus S$. Since $(kx)d^n = (\underbrace{x+\ldots+x}_{k \text{ times}})d^n = \underbrace{xd^n+\ldots+xd^n}_{k \text{ times}}$, we get that if $xd^n \in S$, then $(kx)d^n \in S$ (since $(S,+) \leqslant (I_D(d),+)$). Hence $(kx)d^n \in S$, a contradiction. Thus $xd^n \in D \setminus S$.

So we have the claim, i.e. $\mathbb{Q}^+\langle d\rangle \subseteq D \setminus S$. Thus $a+x = x+a = d+x$ $\forall x \in \mathbb{Q}^+ \langle d\rangle$ and, by Theorem 2.43 (1) and (2), we get that $a+a = d+d$. Since $\mathbb{Q}^+ \langle d\rangle \subseteq K$, $ax = xa = dx$ $\forall x \in \mathbb{Q}^+ \langle d\rangle \cup \{a\}$. By Theorem 2.51, $\mathbb{Q}^+ \langle d\rangle \cup \{a\}$ is the semifield given in Remark 4.14. Using the same proof as before, we obtain that $K' \cong \mathbb{Q}^+ \cdot \langle d\rangle \cup \{a\}$.

<u>Subcase 2.3</u> $d^2 \in D \setminus I_D(d)$ and $1 \in I_D(d)$. Then $1+d = d$. By induction, we can show that $1+d^k = d^k$ $\forall k \in \mathbb{Z}^+$. Let $m, n \in \mathbb{Z}$ be such that $m < n$. Then $d^m + d^n = 1 \cdot d^m + d^{n-m} \cdot d^m = (1+d^{n-m})d^m = d^{n-m} \cdot d^m = d^n$. Thus

$$d^m + d^n = d^n \quad \forall\, m, n \in \mathbb{Z},\ m < n. \qquad \ldots\ldots\ldots\ldots(6)$$

For $m \in \mathbb{Z}^+$, $m \geqslant 2$, claim that $d^m \in D \setminus S$. Since $d^2 \notin I_D(d)$ and $S \subseteq I_D(d)$, $d^2 \in D \setminus S$. Assume that $m > 2$, then $d^m = d^2 + d^m$. Thus $a+d^m = a+(d^2+d^m) = (a+d^2) + d^m = (d+d^2) + d^m = d^2 + d^m = d^m$ since $d^2 \in D \setminus S$ and by (6). Hence $a+d^m = d^m$. Thus $a+d^m \neq a$. Therefore $d^m \in D \setminus S$. So we have the claim,

$$d^m \in D \setminus S \quad \forall\, m \in \mathbb{Z}^+,\ m \geqslant 2. \qquad \ldots\ldots\ldots\ldots\ldots(7)$$

<u>Subcase 2.3.1</u> $1+1 = 1$.

Thus $d^n + d^n = d^n \; \forall n \in \mathbb{Z}$. So we have that

$d^m + d^n = d^n \; \forall m, n \in \mathbb{Z}, \; m \leqslant n$. ....(8)

Let $\langle d\rangle = \{d^m \mid n \in \mathbb{Z}\}$. Clearly $(\langle d\rangle, \cdot) \leqslant (K \setminus \{a\}, \cdot)$ and by (8), we can easily show that $\langle d\rangle$ is an additive subsemigroup of $K \setminus \{a\}$. Hence $\langle d\rangle$ is a ratio subsemiring of $K \setminus \{a\}$.

<u>Subcase 2.3.1.1</u> There exists $n \in \mathbb{Z}$ such that $d^n \in S$. By (7), we get that $n \leqslant 1$. Let n_0 be the largest integer such that $d^{n_0} \in S$. Thus $n_0 \leqslant 1$. Let $S_1 = \{d^n \mid n \leqslant n_0\}$. Then $\langle d\rangle \setminus S_1 = \{d^n \mid n > n_0\}$. Claim that $S_1 \subseteq S$ and $\langle d\rangle \setminus S_1 \subseteq D \setminus S$. To show $S_1 \subseteq S$, let $n \in \mathbb{Z}$, $n \leqslant n_0$. By (8), we get that $d^{n_0} = d^{n_0} + d^n$. Thus $a = a + d^{n_0} = a + (d^{n_0} + d^n) = (a + d^{n_0}) + d^n = a + d^n$ (since $d^{n_0} \in S$). Thus $a + d^n = a$. Hence $d^n \in S$. To show $\langle d\rangle \setminus S_1 \subseteq D \setminus S$, let $n \in \mathbb{Z}$, $n > n_0$. If $n \geqslant 2$, then by (7), $d^n \in D \setminus S$. If $n_0 < n < 2$, then by the choice of n_0 we get that $d^n \in D \setminus S$. So we have the claim. Clearly $(S_1, +) \leqslant (I_{\langle d\rangle}(d), +)$ and $\langle d\rangle \setminus S_1$ is an ideal of $(\langle d\rangle, +)$. By the claim, we get that $a + x = x + a = a \;\; \forall x \in S_1$, $a + x = x + a = d + x \;\; \forall x \in \langle d\rangle \setminus S_1$ and by Theorem 2.43, $a + a = a$ or $a + a = d + d = d$ (since $1+1 = 1$). Since $\langle d\rangle \cup \{a\} \subseteq K$, $ax = xa = dx \;\; \forall x \in \langle d\rangle \cup \{a\}$. By Theorem 2.51, $\langle d\rangle \cup \{a\}$ is the semifield given in Remark 4.11. Using the same proof as before, we obtain that $K' \cong \langle d\rangle \cup \{a\}$.

<u>Subcase 2.3.1.2</u> There does not exist an $n \in \mathbb{Z}$ such that $d^n \in S$. Thus $d^n \in D \setminus S \;\; \forall n \in \mathbb{Z}$, so $a + d^n = d + d^n \;\; \forall n \in \mathbb{Z}$. By Theorem 2.43, $a + a = a$ or $a + a = d + d = d$ (since $1+1 = 1$). Since $\langle d\rangle \cup \{a\} \subseteq K$, $ax = dx \;\; \forall x \in \langle d\rangle \cup \{a\}$. By Theorem 2.51, $\langle d\rangle \cup \{a\}$ is the semifield given in Remark 4.12 and using the same proof as before, we obtain that $K' \cong \langle d\rangle \cup \{a\}$.

Subcase 2.3.2 $1+1 \neq 1$.

By Theorem 2.41, $K' \setminus \{a\}$ is a ratio semiring. Since $1+1 \neq 1$, we get that $\mathbb{Q}^+$ with the usual + and • is the smallest ratio subsemiring of $K' \setminus \{a\}$ (Proposition 1.18). Then, up to isomorphism, we can consider $\mathbb{Q}^+ \subseteq K' \setminus \{a\}$.

Subcase 2.3.2.1 $d \in \mathbb{Q}^+$.

Claim that $\mathbb{Q}^+ \cap S = \Phi$. Suppose that $\exists\, x \in \mathbb{Q}^+$ and $x \in S$. Since $S \subseteq I_D(d)$, $x \in I_D(d)$. Thus $x+d = d$ which is a contradiction since $x, d \in \mathbb{Q}^+$. So we have the claim. Thus $\mathbb{Q}^+ \subseteq D \setminus S$. Hence $a+x = d+x$ $\forall x \in \mathbb{Q}^+$. By Theorem 2.43 (1) and (2), we get that $a+a = d+d$. Since $\mathbb{Q}^+ \cup \{a\} \subseteq K$, $ax = dx$ $\forall x \in \mathbb{Q}^+ \cup \{a\}$. By Theorem 2.51, $\mathbb{Q}^+ \cup \{a\}$ is the semifield given in Remark 4.8 and using the same proof as before we can show that $K' \cong \mathbb{Q}^+ \cup \{a\}$.

Subcase 2.3.2.2 $d \notin \mathbb{Q}^+$.

Consider $\mathbb{Q}^+ \langle d \rangle = \{xd^n \mid x \in \mathbb{Q}^+, n \in \mathbb{Z}\}$. Clearly $(\mathbb{Q}^+ \langle d \rangle, \cdot) \leqslant (K \setminus \{a\}, \cdot)$. Claim that $\mathbb{Q}^+ \langle d \rangle$ is a ratio subsemiring of $K' \setminus \{a\}$. Since $a, 1 \in K'$, $d = a \cdot 1 \in K'$. So $d \in K' \setminus \{a\}$. Since $\mathbb{Q}^+ \subseteq K' \setminus \{a\}$ which is a group under multiplication, $\mathbb{Q}^+ \langle d \rangle \subseteq K' \setminus \{a\}$. To show the claim we need only show that $\mathbb{Q}^+ \langle d \rangle$ is an additive subsemigroup of $K' \setminus \{a\}$. Let $x, y \in \mathbb{Q}^+$ $y \in \mathbb{Q}^+$, $m, n \in \mathbb{Z}$. Consider $xd^m + yd^n$. If $m = n$, then $xd^m + yd^n = xd^m + yd^m = (x+y)d^m \in \mathbb{Q}^+ \langle d \rangle$. Suppose that $m \neq n$. We may assume that $m < n$.

Case 1 (of claim) $x = y$. Then $xd^m + yd^n = xd^m + xd^n = x(d^m + d^n) = xd^n \in \mathbb{Q}^+ \langle d \rangle$ by (6).

Case 2 (of claim) $x < y$. Then $\exists\, \ell \in \mathbb{Q}^+$ such that $y = x+\ell$. Then $xd^m + yd^n = xd^m + (x+\ell)d^n = xd^m + xd^n + \ell d^n = x(d^m + d^n) + \ell d^n = xd^n + \ell d^n = (x+\ell)d^n = yd^n$.

Case 3 (of claim) $x > y$. Let $z \in K$ and $n \in \mathbb{Z}^+$. Define $nz = z+\ldots$ $\ldots+z$ (n times). Since $x = \left[\frac{x}{y}\right] y+\gamma$ where $0 \leqslant \gamma < y$, $xd^m + yd^n$

$$= (\left[\tfrac{x}{y}\right]y+\gamma)d^m + yd^n = \left[\tfrac{x}{y}\right]y\, d^m + \gamma d^m + yd^n = \left[\tfrac{x}{y}\right]y\, d^m + (\gamma d^m + yd^n)$$

$$= \left[\tfrac{x}{y}\right]y\, d^m + yd^n \text{ (by case 2)} = \underbrace{(y+y+\ldots+y)}_{\left[\frac{x}{y}\right] \text{ times}}d^m + yd^n = \underbrace{yd^m+yd^m+\ldots+yd^m}_{\left[\frac{x}{y}\right] \text{ times}}$$

$$+ yd^n = y(\underbrace{d^m+\ldots+d^m}_{\left[\frac{x}{y}\right] \text{ times}}+d^n) = yd^n \text{ (by (6))}.$$

So we get that

$$xd^m + yd^n = \begin{cases} (x+y)d^m & \text{if } m = n, \\ xd^m & \text{if } m > n, \\ yd^n & \text{if } n > m. \end{cases} \quad \ldots\ldots\ldots\ldots(9)$$

Thus $\mathbb{Q}^+ \langle d \rangle$ is an additive subsemigroup of $K' \setminus \{a\}$. Therefore $\mathbb{Q}^+ \langle d \rangle$ is a ratio subsemiring of $K' \setminus \{a\}$.

Claim that $xd^m \in D \setminus S$ $\forall x \in \mathbb{Q}^+$, $\forall m \in \mathbb{Z}^+$, $m \geqslant 2$. Let $x \in \mathbb{Q}^+$, $m \in \mathbb{Z}^+$, $m \geqslant 2$.

Case 1 (of claim) $x = 1$. Then $xd^m = 1 \cdot d^m = d^m \in D \setminus S$ (by (7)).

Case 2 (of Claim) $x > 1$. Then $\exists \ell \in \mathbb{Q}^+$ such that $x = 1+\ell$. Then $xd^m = d^m + \ell d^m$.

Since $d^m \in D \setminus S$ which is an ideal of $(D,+)$, $d^m + \ell d^m \in D \setminus S$. Thus $xd^m \in D \setminus S$.

Case 3 (of claim) $x < 1$. Let $x \in \mathbb{Z}^+$ be such that $nx > 1$. By case 2, $(nx)d^m \in D \setminus S$. Since $(nx)d^m = \underbrace{(x+x+\ldots+x)}_{n \text{ times}}d^m = \underbrace{xd^m + xd^m + \ldots + xd^m}_{n \text{ times}}$,

we get that if $xd^m \in S$, then $(nx)d^m \in S$ because $(S,+) \leqslant (I_D(d),+)$, a contradiction. Thus $xd^m \in D \setminus S$. So we have the claim, i.e.,

$$xd^m \in D \setminus S \quad \forall x \in \mathbb{Q}^+ \forall m \in \mathbb{Z}^+, m \geqslant 2 \,.\ldots\ldots\ldots\ldots(10)$$

Subcase 2.3.2.2.1 There exists $n \in \mathbb{Z}$ such that $d^n \in S$. By (7), $n \leqslant 1$. Let n_0 be the largest integer such that $d^{n_0} \in S$. Let $S_1 = \{xd^n \mid x \in \mathbb{Q}^+, n \in \mathbb{Z}, n \leqslant n_0\}$. Then $\mathbb{Q}^+ \langle d\rangle \setminus S_1 = \{xd^n \mid x \in \mathbb{Q}^+, n \in \mathbb{Z}, n > n_0\}$. Claim that $S_1 \subseteq S$ and $\mathbb{Q}^+ \langle d\rangle \setminus S_1 \subseteq D \setminus S$.

To show $S_1 \subseteq S$, let $x \in \mathbb{Q}^+$ and $n \in \mathbb{Z}$, $n \leqslant n_0$.

Case 1 (of claim) $n = n_0$.

Subcase 1.1 $x = 1$. Then $xd^n = 1 \cdot d^{n_0} = d^{n_0} \in S$.

Subcase 1.2 $x < 1$. Then $\exists\, \ell \in \mathbb{Q}^+$ such that $1 = x+\ell$. Then $d^{n_0} = 1\, d^{n_0} = (x+\ell)d^{n_0} = xd^{n_0} + \ell d^{n_0}$. Thus $d^{n_0} = xd^{n_0} + \ell d^{n_0}$. Since $d^{n_0} \in S$, $a = a+d^{n_0} = a+(xd^{n_0}+\ell d^{n_0}) = (a+xd^{n_0}) + \ell d^{n_0}$. If $a+xd^{n_0} \neq a$, then $a+xd^{n_0} = d+xd^{n_0}$. Thus $a = (d+xd^{n_0}) + \ell d^{n_0}$, which is a contradiction because $K \setminus \{a\}$ is a ratio semiring. Hence $a+xd^{n_0} = a$, so $xd^{n_0} \in S$.

Subcase 1.3 $x > 1$. Then $x = [x]+\ell$ where $0 \leqslant \ell < 1$. If $\ell = 0$, then $x = [x]$. $xd^n = [x]d^{n_0} = \underbrace{d^{n_0}+\ldots+d^{n_0}}_{[x] \text{ times}}$. By Subcase 1.1, $d^{n_0} \in S$. Thus $[x]d^{n_0} \in S$ since $(S,+) \leqslant (I_D(d),+)$. Hence $xd^{n_0} \in S$. If $0 < \ell < 1$, then $x = [x]+\ell$. Thus $xd^n = xd^{n_0} = [x]d^{n_0} + \ell d^{n_0}$. By Subcase 1.2, $\ell d^{n_0} \in S$. It $\ell = 0$, then by Subcase 1.3, $[x]d^{n_0} \in S$. Hence $[x]d^{n_0} + \ell d^{n_0} \in S$ since $(S,+) \leqslant (I_D(d),+)$. Therefore $xd^n \in S$.

Case 2 (of claim) $n < n_0$. By (9), $xd^{n_0} = xd^{n_0} + xd^n$. By case 1, $xd^{n_0} \in S$. Then $a = a+xd^{n_0} = a+(xd^{n_0}+xd^n) = (a+xd^{n_0})+xd^n = a+xd^n$. Thus $a+xd^n = a$. Hence $xd^n \in S$. Therefore $S_1 \subseteq S$.

To show $\mathbb{Q}^+ \langle d\rangle \setminus S_1 \subseteq D \setminus S$, let $x \in \mathbb{Q}^+$, $n \in \mathbb{Z}$, $n > n_0$. If $n \geqslant 2$ then by (10), $xd^n \in D \setminus S$. Suppose that $n_0 < n < 2$. We want to show that $xd^n \varepsilon D \setminus S$

<u>Case 1</u> (of claim) $x = 1$. Then $xd^n = 1 \cdot d^n = d^n$. By the choice of n_0 we get that $d^n \varepsilon D \setminus S$. Thus $xd^n \varepsilon D \setminus S$.

<u>Case 2</u> (of claim) $x > 1$. Then $\exists\, \ell \,\varepsilon\, \mathbb{Q}^+$ such that $x = 1+\ell$. Then $xd^n = (1+\ell)d^n = 1\, d^n + \ell d^n = d^n + \ell d^n$. By case 1, $d^n \varepsilon D \setminus S$. Since $D \setminus S$ is an ideal of $(D,+)$, $d^n + \ell d^n \,\varepsilon\, D \setminus S$. Thus $xd^n \varepsilon D \setminus S$.

<u>Case 3</u> (of claim) $x < 1$. Then $\exists\, m \,\varepsilon \mathbb{Z}^+$ such that $mx > 1$. By case 2, $(mx)d^n \varepsilon D \setminus S$. Since $(mx)d^n = (\underbrace{x+x+\ldots+x}_{m \text{ times}})d^n = \underbrace{xd^n+\ldots+xd^n}_{m \text{ times}}$, we get that if $xd^n \varepsilon S$, then $(mx)d^n \varepsilon S$ because $(S,+) \leqslant (I_D\langle d\rangle,+)$, a contradiction. Thus $xd^n \varepsilon D \setminus S$. So we have the claim, i.e. $S_1 \subseteq S$ and $\mathbb{Q}^+ \langle d\rangle \setminus S_1 \subseteq D \setminus S$. Thus $a+x = a \;\forall x \,\varepsilon\, S_1$ and $a+x = d+x \;\; \forall x \,\varepsilon\, \mathbb{Q}^+ \langle d\rangle \setminus S_1$. By Theorem 2.43 (1) and (2) we get that $a+a = d+d$. Since $\mathbb{Q}^+ \langle d\rangle \cup \{a\} \subseteq K$, $ax = dx \;\; \forall x \,\varepsilon\, \mathbb{Q}^+ \langle d\rangle \cup \{a\}$. By Theorem 2.51, $\mathbb{Q}^+ \langle d\rangle \cup \{a\}$ is the semifield given in Remark 4.15. Using the same proof as before we can show that $K' \cong \mathbb{Q}^+ \langle d\rangle \cup \{a\}$.

<u>Subcase 2.3.2.2.2</u> There does not exist an $n \varepsilon \mathbb{Z}$ such that $d^n \varepsilon S$. Thus $d^n \varepsilon D \setminus S \;\; \forall\, n \,\varepsilon \mathbb{Z}$. Claim that $\mathbb{Q}^+ \langle d\rangle \subseteq D \setminus S$. Let $x \,\varepsilon\, \mathbb{Q}^+$ and $n \,\varepsilon \mathbb{Z}$.

<u>Case 1</u> (of claim) $n \geqslant 2$. Then by (10), $xd^n \varepsilon D \setminus S$.

<u>Case 2</u> (of Claim) $n < 2$.

<u>Subcase 2.1</u> $x = 1$. Then $xd^n = 1 \cdot d^n = d^n \varepsilon D \setminus S$.

<u>Subcase 2.2</u> $x > 1$. Then $\exists\, \ell \,\varepsilon\, \mathbb{Q}^+$ such that $x = 1+\ell$. Then $xd^n = d^n + \ell d^n \varepsilon D \setminus S$ since $d^n \varepsilon D \setminus S$ which is an ideal of $(D,+)$. Thus $xd^n \varepsilon D \setminus S$.

<u>Subcase 2.3</u> $x < 1$. Then $\exists\, m \,\varepsilon \mathbb{Z}^+$ such that $mx > 1$. By

subcase 2.2, $(mx)d^n \varepsilon D \setminus S$. Since $(mx)d^n = (\underbrace{x+\ldots+x}_{m\ times})d^n = \underbrace{xd^n+\ldots+xd^n}_{m\ times}$,

we get that if $xd^n \varepsilon S$ then $(mx)d^n \varepsilon S$, which is a contradiction. Hence $xd^n \varepsilon D \setminus S$. So we have the claim, i.e. $\mathbb{Q}^+\langle d\rangle \subseteq D \setminus S$. Thus $a+x = d+x$ $\forall x \varepsilon \mathbb{Q}^+ \langle d\rangle$. By Theorem 2.43 (1) and (2), $a+a = d+d$. Since $\mathbb{Q}^+ \langle c\rangle \cup \{a\} \subseteq K$, $ax = dx$ $\forall x \varepsilon \mathbb{Q}^+ \langle d\rangle \cup \{a\}$. By Theorem 2.51, $\mathbb{Q}^+.\langle d\rangle \cup \{a\}$ is the semifield given in Remark 4.16. Using the same proof as before, we obtain that $K' \cong \mathbb{Q}^+.\langle d\rangle \cup \{a\}$.

<u>Subcase 2.4</u> $d^2 \varepsilon D \setminus I_D(d)$ and $1 \varepsilon D \setminus I_D(d)$.

<u>Subcase 2.4.1</u> $1+1 = 1$.

Consider $1+d$. Since $1 \varepsilon D \setminus I_D(d)$, $1+d \neq d$. Since $d^2 \varepsilon D \setminus I_D(d)$, $1+d \neq 1$. Now $1+d = (1+1)+d = 1+(1+d)$. Let $x = 1+d$, then $1+x = x$. By induction we can show that $1+x^n = x^n$ $\forall n \varepsilon \mathbb{Z}^+$. For $m,n \varepsilon \mathbb{Z}$, $m < n$ we get that $x^m + x^n = x^m(1+x^{n-m}) = x^m x^{n-m} = x^n$, Thus

for $m, n \varepsilon \mathbb{Z}$, $x^m + x^n = x^k$ where $k = \max\{m,n\}$. ....(11)

Next consider $1+d^{-1}$. Since $1+d^{-1} = (1+1)+d^{-1} = 1+(1+d^{-1})$, $1+(1+d^{-1})^{-1} = 1$. Let $y = (1+d^{-1})^{-1}$, then $y = (1+d^{-1})^{-1} = (d^{-1}(1+d))^{-1} = d(1+d)^{-1} = dx^{-1}$. Thus $y = dx^{-1}$ and $1+y = 1$. By induction, we can show that $1+y^n = 1$ $\forall n \varepsilon \mathbb{Z}^+$. For $m, n \varepsilon \mathbb{Z}$, $m < n$, $y^m + y^n = y^m(1+y^{n-m}) = y^m 1 = y^m$. Thus

for $m, n \varepsilon \mathbb{Z}$, $y^m + y^n = y^r$ where $r = \min\{m,n\}$.........(12)

<u>Claim that</u> $1+x^r y^s = x^r$ for all $r, s \varepsilon \mathbb{Z}^+$. ....(13)

We shall prove this by induction on s. Let $r \varepsilon \mathbb{Z}^+$. First, we shall show that (13) holds for $s = 1$.

If $r = 1$, then $1+x^r y = 1+xy = 1+d = x = x^r$ since $d = xy$.

If $r > 1$, then $1+x^r y = 1+x^r(dx^{-1}) = 1+dx^{r-1} = 1+d(1+d)^{r-1} =$

$1+d(1+d+\ldots+d^{r-1}) = 1+d+d^2+\ldots+d^r = (1+d)^r = x^r$ since $(1+d)^n = 1+d+d^2+\ldots+d^n$ for all $n \in \mathbb{Z}^+$ (because $z+z = z$ $\forall z \in D$). Hence $1+x^r y = x^r$. So (13) holds for $s = 1$.

Now assume (13) is true for $s = n$. Thus $1+x^r y^n = x^r$. Since $1+y = 1$, $1+x^r y^{n+1} = (1+y)+ x^r y^{n+1} = 1+(y+x^r y^{n+1}) = 1+(1+x^r y^n)y = 1+x^r y = x^r$ (since (13) holds for $s = 1$ and $s = n$). Hence $1+x^r y^{n+1} = x^r$. So we have the claim.

Let $\langle x,y\rangle = \{x^m y^n \mid m,n \in \mathbb{Z}\}$. Claim that $\langle x,y\rangle$ is a ratio semiring containing d. Since $xy = x(dx^{-1}) = d$, $d \in \langle x,y\rangle$. Clearly $(\langle x,y\rangle,\cdot)$ is a commutative group. To show the claim we need only show that $\langle x,y\rangle$ is closed under addition.

Let $m,n,k,\ell \in \mathbb{Z}$. Consider $x^m y^n + x^k y^\ell$.

<u>Case 1</u> (of claim) $m = k$. Then $x^m y^n + x^k y^\ell = x^m(y^n+y^\ell) = \begin{cases} x^m y^n & \text{if } n \leqslant \ell, \\ x^m y^\ell & \text{if } n > \ell. \end{cases}$

<u>Case 2</u> (of claim) $m < k$.

<u>Subcase (2.1)</u> $n \geqslant \ell$. Then $x^k = x^m + x^k$ and $y^n + y^\ell = y^\ell$ (by (11) and (12)). So $x^m y^n + x^k y^\ell = x^m y^n + (x^m+x^k)y^\ell = x^m y^n + x^m y^\ell + x^k y^\ell$.
$= x^m(y^n+y^\ell) + x^k y^\ell = x^m y^\ell + x^k y^\ell = (x^m+x^k)y^\ell = x^k y^\ell$. Thus $x^m y^n + x^k y^\ell = x^k y^\ell$.

<u>Subcase (2.2)</u> $n < \ell$. Then $x^m y^n + x^k y^\ell = x^m y^n(1+x^{k-m}y^{\ell-n}) = x^m y^n x^{k-m} = x^k y^n$ (by (13)). Thus $x^m y^n + x^k y^\ell = x^k y^n$.

<u>Case 3</u> (of claim) $m > k$.

<u>Subcase (3.1)</u> $n \leqslant \ell$. Then $y^n = y^n + y^\ell$ and $x^m = x^m + x^k$.
Now $x^m y^n + x^k y^\ell = (x^m+x^k)y^n + x^k y^\ell = x^m y^n + x^k y^n + x^k y^\ell$

$= x^m y^n + x^k(y^n+y^\ell) = x^m y^n + x^k y^n = (x^m+x^k)y^n = x^m y^n$. Thus $x^m y^n + x^k y^\ell$ $= x^m y^n$.

Subcase (3.2) $n > \ell$. Then $x^m y^n + x^k y^\ell = x^k y^\ell (1+x^{m-k}y^{n-\ell})$ $= x^k y^\ell x^{m-k} = x^m y^\ell$ (by (13)). Thus $x^m y^n + x^k y^\ell = x^m y^\ell$.

We see that $x^m y^n + x^k y^\ell = x^r y^s$ where $r = \max\{m,k\}$ and $s = \min\{n,\ell\}$. ....(14)

Therefore $\langle x,y\rangle$ is a ratio semiring. So we have the claim.

By (14), we get that $I_{\langle x,y\rangle}(d) = \{x^m y^n \mid m,n \in \mathbb{Z}, m \leqslant 1 \leqslant n\}$. Then $\langle x,y\rangle \setminus I_{\langle x,y\rangle}(d) = \{x^m y^n \mid m,n \in \mathbb{Z}, m > 1 \text{ or } n < 1\}$. Claim that

$\langle x,y\rangle \setminus I_{\langle x,y\rangle}(d) \subseteq D \setminus S$. ....(15)

Let $z \in \langle x,y\rangle \setminus I_{\langle x,y\rangle}(d)$. If $z \in S$, then $z \in I_D(d)$ since $S \subseteq I_D(d)$. Thus $z+d = d$, so $z \in I_{\langle x,y\rangle}(d)$, which is a contradiction. Thus $z \in D\setminus S$. So we have (15).

Now consider $I_{\langle x,y\rangle}(d) \cap S$.

Subcase 2.4.1.1 $I_{\langle x,y\rangle}(d) \cap S = \Phi$. Then

$I_{\langle x,y\rangle}(d) \subseteq D \setminus S$. ....(16)

By (15) and (16), $\langle x,y\rangle \subseteq D \setminus S$. Thus $a+z = d+z$ $\forall z \in \langle x,y\rangle$ and by Theorem 2.43, $a+a = a$ or $a+a = a+d = d$. Since $\langle x,y\rangle \cup \{a\} \subseteq K$, $a \cdot z = dz$ for all $z \in \langle x,y\rangle \cup \{a\}$. By Theorem 2.51, we obtain that $\langle x,y\rangle \cup \{a\}$ is the semifield of type III given in Remark 4.18. Let $A = \{\sum_{i<\infty} n_i d^{m_i} \mid n_i, m_i \in \mathbb{Z}^+\}$, by Theorem 4.6, $B \cup \{a\} \cong K'$ where B is the quotient ratio semiring of A. Since $d \in \langle x,y\rangle$, $A \subseteq \langle x,y\rangle$ Thus $B \subseteq \langle x,y\rangle$. And since $x = 1+d$, $y = dx^{-1}$ so $x,y \in B$. Hence $\langle x,y\rangle \subseteq B$. Therefore $\langle x,y\rangle \cong B$. So we get that $K' \cong \langle x,y\rangle \cup \{a\}$ as in Remark 4.18.

Subcase 2.4.1.2 $I_{<x,y>}(d) \cap S \neq \Phi$.

Let $x^m y^n \varepsilon I_{<x,y>}(d) \cap S$, then $x^m y^n \varepsilon S$ where $m \leqslant 1$ and $n \geqslant 1$. Choose m_1 to be the largest integer such that $x^{m_1} y^n \varepsilon S$ for some $n \geqslant 1$ and choose n_1 to be the smallest integer such that $x^{m_1} y^{n_1} \varepsilon S$. Then $m_1 \leqslant 1 \leqslant n_1$.

For $m \leqslant m_1$ and $n \geqslant n_1$, $x^{m_1} y^{n_1} = x^{m_1} y^{n_1} + x^m y^n$ (by (14)). Then $a = a+x^{m_1} y^{n_1} = a+(x^{m_1} y^{n_1} + x^m y^n) = (a+x^{m_1} y^{n_1}) + x^m y^n = a+x^m y^n$.

Thus $\{x^m y^n \mid m \leqslant m_1$ and $n_1 \leqslant n\} \subseteq S$. ....(17)

By the choice of m_1, we get that

$\{x^m y^n \mid m_1 < m \leqslant 1$ and $1 \leqslant n\} \subseteq D \setminus S$. ....(18)

Choose n_2 to be the smallest integer such that $x^m y^{n_2} \varepsilon S$ for some $m \leqslant 1$ and choose m_2 to be the largest integer such that $x^{m_2} y^{n_2} \varepsilon S$. Claim that $m_1 = m_2$ and $n_1 = n_2$. By the choice of m_1, n_1, n_2, m_2, we obtain that $m_2 \leqslant m_1 \leqslant 1$ and $1 \leqslant n_2 \leqslant n_1$. Then $x^{m_1} y^{n_2} = x^{m_2} y^{n_2} + x^{m_1} y^{n_1}$. Since $(S,+) \leqslant (I_D(d),+)$, $x^{m_2} y^{n_2} + x^{m_1} y^{n_1} \varepsilon S$. Hence $x^{m_1} y^{n_2} \varepsilon S$. By the choice of n_1, we get that $n_1 \leqslant n_2$. Thus $n_1 = n_2$. And by the choice of m_2, we get that $m_1 \leqslant m_2$. Thus $m_1 = m_2$. So we have the claim. Thus we get that n_1 is the smallest integer such that $x^m x^{n_1} \varepsilon S$ for some $m \leqslant 1$. So we get that

$\{x^m y^n \mid m \leqslant 1$ and $1 \leqslant n < n_1\} \subseteq D \setminus S$. ....(19)

From (17), (18) and (19), we get that

$I_{<x,y>}(d) \cap S = \{x^m y^n \mid m \leqslant m_1$ and $n_1 \leqslant n\}$.

Let $S_1 = I_{<x,y>}(d) \cap S$. Claim that $<x,y> \cap S = S_1$. Let $M = <x,y>$. Since $M = (M \setminus I_M(d)) \cup I_M(d)$, $<x,y> \cap S = M \cap S =$

$= ((M\setminus I_M(d))\cup I_M(d))\cap S = ((M\setminus I_M(d))\cap S)\cup(I_M(d)\cap S) =$

$\Phi \cup (I_M(d)\cap S)$ (by (15)) $= I_{\langle x,y\rangle}(d)\cap S = S_1$. So we have the claim i.e. $\langle x,y\rangle \cap S = \{x^m y^n \mid m \leqslant m_1$ and $n_1 \leqslant n\} = S_1$. Then $\langle x,y\rangle\setminus S_1$ $= \{x^m y^n \mid m > m_1$ or $n < n_1\}$. We have already shown in Remark 4.17 that $\langle x,y\rangle\setminus S_1$ is an ideal of $(\langle x,y\rangle,+)$. Since $\langle x,y\rangle\setminus S_1 = \langle x,y\rangle\cap S_1^c$ $= \langle x,y\rangle \cap (\langle x,y\rangle\cap S)^c = \langle x,y\rangle\cap(\langle x,y\rangle^c\cup S^c) = (\langle x,y\rangle\cap\langle x,y\rangle^c)\cup$ $(\langle x,y\rangle\cap S^c) = \langle x,y\rangle\setminus S \subseteq D\setminus S$, $\langle x,y\rangle\setminus S_1 \subseteq D\setminus S$. So we have that

$$S_1\subseteq S \text{ and } \langle x,y\rangle\setminus S_1\subseteq D\setminus S.$$

Then $a+z = a$ $\forall z \in S_1$ and $a+z = d+z$ $\forall z \in \langle x,y\rangle\setminus S_1$ and by Theorem 2.43, $a+a = a$ or $a+a = d+d = d$. Since $\langle x,y\rangle \cup \{a\}\subseteq K$, $a\cdot z = d\cdot z$ $z \in \langle x,y\rangle\cup\{a\}$. Then by Theorem 2.51 we obtain that $\langle x,y\rangle\cup\{a\}$ is the semifield given in Remark 4.17 and using the same proof as before, we can show that $K'\cong\langle x,y\rangle\cup\{a\}$.

Subcase 2.4.2 $1+1 \neq 1$.

By Theorem 2.41, $K'\setminus\{a\}$ is a ratio semiring. Since $1+1\neq 1$, by Proposition 1.18, we get that $\mathbb{Q}^+$ with the usual addition and multiplication is the smallest ratio subsemiring of $K'\setminus\{a\}$. Then, up to isomorphism, we can consider $\mathbb{Q}^+\subseteq K'\setminus\{a\}$.

Subcase 2.4.2.1 $d \in \mathbb{Q}^+$.

Claim that $\mathbb{Q}^+\subseteq D\setminus S$. Suppose that $\exists\, x\in\mathbb{Q}^+$ and $x \in S$. Then $x\in I_D(d)$ (since $S\subseteq I_D(d)$), so $x+d = d$ which is a contradiction since $x, d\in\mathbb{Q}^+$. So we have the claim. Then $a+x = d+x$ $\forall x\in\mathbb{Q}^+$ and by Theorem 2.43 (1) and (2), $a+a = d+d$. Since $\mathbb{Q}^+\cup\{a\}\subseteq K$, so $ax = dx$ $\forall x\in\mathbb{Q}^+\cup\{a\}$. By Theorem 2.51, we obtain that $\mathbb{Q}^+\cup\{a\}$ is the semifield given in Remark 4.8. Thus $\mathbb{Q}^+\cup\{a\}$ is a subsemifield of K, so $K'\subseteq\mathbb{Q}^+\cup\{a\}$. Since $\mathbb{Q}^+\subseteq K'\setminus\{a\}$, $\mathbb{Q}^+\cup\{a\}\subseteq K'$. Thus

$K' \cong \mathbb{Q}^+ \cup \{a\}$ as in Remark 4.8.

Subcase 2.4.2.2 $\quad d \notin \mathbb{Q}^+$.

Define $\phi : \mathbb{Q}^+(x) \to K \setminus \{a\}$ as follows :

Let $\frac{F(x)}{G(x)} \varepsilon \mathbb{Q}^+(x)$, define $\phi\left(\frac{F(x)}{G(x)}\right) = \frac{F(d)}{G(d)}$. We must show that ϕ is well-defined. Suppose that $\frac{F'(x)}{G'(x)} = \frac{F(x)}{G(x)}$. We must show that $\frac{F'(d)}{G'(d)} = \frac{F(d)}{G(d)}$. Then $F'(x)\, G(x) = G'(x)\, F(x)$, so $F'(d)\, G(d) = G'(d)\, F(d)$ and hence $\frac{F'(d)}{G'(d)} = \frac{F(d)}{G(d)}$. So ϕ is well-defined. Now we shall show that ϕ is a homomorphism.

Let $\frac{F(x)}{G(x)}, \frac{F'(x)}{G'(x)} \varepsilon \mathbb{Q}^+(x)$. Then $\phi\left(\frac{F(x)}{G(x)} \cdot \frac{F'(x)}{G'(x)}\right) =$

$$\phi\left(\frac{F(x)F'(x)}{G(x)G'(x)}\right) = \frac{F(d)F'(d)}{G(d)G'(d)} = \frac{F(d)}{G(d)} \cdot \frac{F'(d)}{G'(d)} = \phi\left(\frac{F(x)}{G(x)}\right) \cdot \phi\left(\frac{F'(x)}{G'(x)}\right)$$

and $\phi\left(\frac{F(x)}{G(x)} + \frac{F'(x)}{G'(x)}\right) = \phi\left(\frac{F(x)G'(x)+G(x)F'(x)}{G(x)G'(x)}\right) = \frac{F(d)G'(d)+G(d)F'(d)}{G(d)G'(d)}$

$= \frac{F(d)G'(d)}{G(d)G'(d)} + \frac{G(d)F'(d)}{G(d)G'(d)} = \frac{F(d)}{G(d)} + \frac{F'(d)}{G'(d)} = \phi\left(\frac{F(x)}{G(x)}\right) + \phi\left(\frac{F'(x)}{G'(x)}\right)$. Thus ϕ is a homomorphism.

Subcase 2.4.2.2.1 $\quad \phi$ is 1-1.

Then ϕ is 1-1 homomorphism. Thus $\mathbb{Q}^+(x) \cong \operatorname{im}\phi$. So $\operatorname{im}\phi$ is a ratio semiring. Claim that $I_{\operatorname{im}\phi}(d) = \Phi$. To prove this, suppose not. Then $\exists\, y \varepsilon \operatorname{im}\phi$ such that $y \varepsilon I_{\operatorname{im}\phi}(d)$. So $y+d = d$. Since $y \varepsilon \operatorname{im}\phi$, $\exists \frac{F(x)}{G(x)} \varepsilon \mathbb{Q}^+(x)$ such that $y = \phi\left(\frac{F(x)}{G(x)}\right)$ and $d = \phi(x)$. Thus $\phi\left(\frac{F(x)}{G(x)}\right) + \phi(x) = \phi(x)$ and $\phi\left(\frac{F(x)}{G(x)} + x\right) = \phi(x)$. Since ϕ is 1-1, $\frac{F(x)}{G(x)} + x = x$. Thus $F(x) + x\,G(x) = xG(x)$ which is a contradiction since $\mathbb{Q}^+[x]$ is A.C. (Proposition 4.25). So we have the claim i.e. $I_{\operatorname{Im}\phi}(d) = \Phi$. Let $y \varepsilon \operatorname{im}\phi$. If $y \varepsilon S$, then $y \varepsilon I_D(d)$. Thus $y+d = d$, so $y \varepsilon I_{\operatorname{im}\phi}(d)$ which is a contradiction. Hence $\operatorname{im}\phi \subseteq D \setminus S$.

Then $a+y = d+y$ $\forall y \in \text{im}\,\phi$ and by Theorem 2.43 (1) and (2), $a+a = d+d$. Since $\text{im}\,\phi \cup \{a\} \subseteq K$, $ay = dy$ $\forall y \in \text{im}\,\phi \cup \{a\}$. By Theorem 2.51, we obtain that $\text{im}\,\phi \cup \{a\}$ is a semifield. Hence $K' \subseteq \text{im}\,\phi \cup \{a\}$. Since $\mathbb{Q}^+ \subseteq K' \setminus \{a\}$ and $d \in K' \setminus \{a\}$, $\text{im}\phi \subseteq K' \setminus \{a\}$. Thus $\text{im}\phi \cup \{a\} \subseteq K'$. Therefore $K' \cong \text{im}\phi \cup \{a\}$. Since $\text{im}\phi \cong \mathbb{Q}^+(x)$, $\text{im}\phi \cup \{a\} \cong \mathbb{Q}^+(x) \cup \{a\}$ where $\mathbb{Q}^+(x) \cup \{a\}$ is the semifield given Remark 4.26 Hence $K' \cong \mathbb{Q}^+(x) \cup \{a\}$ as in Remark 4.26.

<u>Subcase 2.4.2.2.2</u> ϕ is not 1-1.

Then $\exists\ \frac{F(x)}{G(x)}, \frac{F'(x)}{G'(x)} \in \mathbb{Q}^+(x)$ such that $\frac{F(d)}{G(d)} = \frac{F'(d)}{G'(d)}$. So $\frac{F(d)\ G'(d)}{G(d)\ F'(d)} = 1$. Thus $\phi(\frac{F(x)\ G'(x)}{G(x)\ F'(x)}) = 1$.

Define $\ker\phi = \left\{\frac{F(x)}{G(x)} \in \mathbb{Q}^+(x) \mid \phi(\frac{F(x)}{G(x)}) = 1\right\}$. Claim that $\ker\phi$ is a C-set. Let $y, z \in \ker\phi$. Then $\phi(y) = \phi(z) = 1$. Now $\phi(y \cdot z^{-1}) = \phi(y) \cdot \phi(z^{-1}) = 1 \cdot (\phi(z))^{-1} = 1 \cdot 1^{-1} = 1 \cdot 1 = 1$. Thus $yz^{-1} \in \ker\phi$. Let $y \in \ker\phi$, $z \in \mathbb{Q}^+(x)$. Since $\phi(\frac{y+z}{1+z}) = \frac{\phi(y)+\phi(z)}{\phi(1)+\phi(z)} = \frac{1+\phi(z)}{1+\phi(z)} = 1$, we get that $\frac{y+z}{1+z} \in \ker\phi$. Hence $\ker\phi$ is a C-set. So we have the claim. Thus $\mathbb{Q}^+(x)/_{\ker\phi}$ is a ratio semiring and we obtain that $\mathbb{Q}^+(x)/_{\ker\phi} \cong \text{im}\phi$. Since $d^2, 1 \in D \setminus I_D(d)$, $d^2, 1 \in \text{im}\phi \setminus I_{\text{im}\phi}(d)$ because $I_{\text{im}\phi}(d) \subseteq I_D(d)$. Let $W = \mathbb{Q}^+(x)/_{\ker\phi}$ and $w = [x]$, so $W \cong \text{im}\phi$. Thus w^2 and $[1] \in W \setminus I_W(w)$.

Let $D_1 = \text{im}\phi$ and $S_1 = D_1 \cap S$. Claim that

(1) $S_1 = \Phi$ or $(S_1,+) \leqslant (I_{D_1}(d),+)$,

(2) $D_1 \setminus S_1$ is an ideal of $(D_1,+)$.

To show (1), we assume that $S_1 \neq \Phi$. Let $x, y \in S_1$. Then $x, y \in D$ and

$x, y \in S$. Thus $x+y \in D_1$ and $x+y \in S$ because S is an additive sub-semigroup of $I_D(d)$. Hence $x+y \in S_1$. So we have (1). Now we shall show (2). Since $1 \in D \setminus I_D(d)$ and $S \subseteq I_D(d)$, we get that $1 \in D_1 \setminus S$. Since $D_1 \setminus S_1 = D_1 \cap S_1^c = D_1 \cap (D_1 \cap S)^c = D_1 \cap (D_1^c \cup S^c) = (D_1 \cap D_1^c) \cup (D_1 \cap S^c)$ $= \Phi \cup (D_1 \cap S^c) = D_1 \cap S^c = D_1 \setminus S$, we get that $D_1 \setminus S_1 = D_1 \setminus S$. Thus $1 \in D_1 \setminus S_1$, so $D_1 \setminus S_1 \neq \Phi$. Let $x \in D_1 \setminus S_1$ and $y \in D_1$. Thus $x \in D_1 \setminus S$, so $x \in D \setminus S$. By Theorem 2.50, we have that $D \setminus S$ is an ideal of $(D,+)$, so $x+y \in D \setminus S$. Since $x, y \in D_1$, $x+y \in D_1$. Thus $x+y \in D_1 \setminus S$, so $x+y \in D_1 \setminus S_1$. So we have (2).

Since $S_1 = D_1 \cap S$ and $D_1 \setminus S_1 = D_1 \setminus S$, we get that $S_1 \subseteq S$ and $D_1 \setminus S_1 \subseteq D \setminus S$. Then $a+x = a$ for all $x \in S_1$ and $a+x = d+x$ for all $x \in D_1 \setminus S_1$. By Theorem 2.43 (1) and (2), $a+a = d+d$. Then by Theorem 2.51, we obtain that $D_1 \cup \{a\}$ is a semifield. Thus $\text{im } \phi \cup \{a\}$ is a semifield. Using the same proof as in Subcase 2.4.2.2.1, we get that $K' \cong \text{im } \phi \cup \{a\}$. Since $\mathbb{Q}^+(x)/_{\ker \phi} \cong \text{im } \phi$, we get that $K' \cong \mathbb{Q}^+(x)/_{\ker\phi} \cup \{a\}$.

<u>Claim</u> <u>that</u> $1+x, \frac{1+x}{x} \in \mathbb{Q}^+(x) \setminus \ker \phi$.

Suppose that $1+x \in \ker \phi$. Then $\phi(1+x) = 1$. Thus $1+d = 1$, so $d+d^2 = d$. Hence $d^2 \in I_D(d)$, a contradiction. Thus $1+x \in \mathbb{Q}^+(x)$ $\ker \phi$. Now suppose that $\frac{1+x}{x} \in \ker \phi$. Then $\phi\frac{(1+x)}{x} = 1$. Thus $\frac{1+d}{d} = 1$, so $1+d = d$. Hence $1 \in I_D(d)$. a contradiction. Thus $\frac{1+x}{x} \in \mathbb{Q}^+(x) \setminus \ker \phi$.

Therefore we get that $\mathbb{Q}^+(x)/_{\ker \phi} \cup \{a\}$ is the semifield given in Remark 4.28 and $K' \cong \mathbb{Q}^+(x)/_{\ker \phi} \cup \{a\}$. #