โครงการ

การเรียนการสอนเพื่อเสริมประสบการณ์

ชื่อโครงการ ผลคูณอนันต์ ตัวอย่าง และการประยุกต์

Infinite Product Example and Application

ชื่อนิสิต นายสิทธิชัย เพิ่มพงศ์โกศล **เลขประจำตัว** 5833547623

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์

ปีการศึกษา 2561

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ผลคูณอนันต์ ตัวอย่าง และการประยุกต์

นายสิทธิชัย เพิ่มพงศ์โกศล

โครงงานนี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรวิทยาศาสตรบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2561

Infinite Product Example and Application

Mr. Sitthichai Pempongkoson

A Project Submitted in Partial Fulfillment of the Requirements
for the Degree of Bachelor of Science Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2018

หัวข้อโครงงาน	ผลคูณอนันต์ ตัวอย่าง และการประยุกต์
โดย	นายสิทธิชัย เพิ่มพงศ์โกศล เลขประจำตัว 5833547623
สาขาวิชา	คณิตศาสตร์
อาจารย์ที่ปรึกษาโครงงาน	ผู้ช่วยศาสตราจารย์ ดร.เก่ง วิบูลย์ธัญญ์

ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์ คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย อนุมัติให้นับโครงงานฉบับนี้เป็นส่วนหนึ่ง ของการศึกษาตามหลักสูตรปริญญาบัณฑิต ในรายวิชา 2301499 โครงงานวิทยาศาสตร์ (Senior Project)

.... หัวหน้าภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์

(ศาสตราจารย์ ดร.กฤษณะ เนียมมณี)

คณะกรรมการสอบโครงงาน

.... อาจารย์ที่ปรึกษาโครงงาน

(ผู้ช่วยศาสตราจารย์ ดร.เก่ง วิบูลย์ธัญญ์)

.... กรรมการ

(รองศาสตราจารย์ ดร.วิชาญ ลิ่วกีรติยุตกุล)

.... กรรมการ

(ศาสตราจารย์ ดร. ไพศาล นาคมหาชลาสินธุ์)

สิทธิชัย เพิ่มพงศ์โกศล: ผลคูณอนันต์ ตัวอย่าง และการประยุกต์ (INFINITE PRODUCT: EXAMPLE AND APPLICATION) อ.ที่ปรึกษาโครงงาน: ผศ.ดร.เก่ง วิบูลย์ธัญญ์ , 28 หน้า

เนื้อหาของผลบวกอนันต์ถูกกล่าวถึงในวิชาแคลคูลัส และการวิเคราะห์เชิงจริง อย่างไรก็ตาม แนวคิด ของผลคูณอนันต์มีความเกี่ยวข้องกับผลบวกอนันต์ แต่ไม่มีในรายวิชาของหลักสูตรปริญญาตรี โครงงานนี้ จะศึกษาเกี่ยวกับบทนิยาม, ทฤษฎีบท และตัวอย่างของผลคูณอนันต์ เช่น ผลคูณอนันต์ของฟังก์ชันไซน์ รวมถึงตัวอย่างของเศษส่วนย่อยของฟังก์ชันโคแทนเจนต์ สุดท้ายนี้จะให้การประยุกต์ที่น่าสนใจของผลคูณอนันต์

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....
สาขาวิชา .. คณิตศาสตร์ .. ลายมือชื่อ อ.ที่ปรึกษาโครงงาน ....
ปีการศึกษา ..... 2561 .....

5833547623 : MAJOR MATHEMATICS.
KEYWORDS: INFINITE PRODUCT

SITTHICHAI PEMPONGKOSON. ADVISOR: ASST.PROF. KENG WIBOONTON, PH.D., 28 PP.

The content of infinite sums is discussed extensively in calculus and real analysis course. However, the concept of infinite products, which is closely related to that of infinite sums, is not mentioned in any regular undergraduate analysis course. In this project, we study the definition of infinite products, examples of infinite products such as the infinite product of the sine function with the associated examples including infinite partial fraction of the cotangent function. Finally, we give some interesting applications of infinite products.

Department	.Mathematics and Computer Science.	Student's Signature	...Sitthichai...
Field of Study	...Mathematics...	Advisor's Signature	...Keng Wiboonton......
Academic Year	.......2018.......		

Acknowledgments

I would like to express my sincere thanks to my project advisor, Assistant Professor Dr. Keng Wiboonton for his invaluable help and constant encouragement throughout the course of this project. I am most graceful to work with him. I receive many things from him for teaching, advice and work experience. I would like to express my special thanks to my project committee: Associate Professor Dr. Wicharn Lewkeeratiyutkul and Professor Dr. Paisan Nakmahachalasint. Their suggestions and comments are my sincere appreciation. Moreover, I feel very thankful to all of my teachers who have taught me abundant knowledge for supporting me a scholarship to do the project comfortably. I lastly wish to express my thankfulness to my friends and my family for their encouragement throughout my study.

Contents

Chapter 1

Introduction

The content of infinite sums is discussed extensively in calculus and real analysis course. However, the concept of infinite products, which is closely related to that of infinite sums, is not mentioned in any regular undergraduate analysis course. In this project, we study about infinite products.

In Chapter 2, we give a formal definition of infinite products, together with basic examples including important infinite products such as Viète's formula and Wallis product. We also give theorems concerning the convergence of infinite products.

In Chapter 3, we give an infinite product of sine using fourier series. We also devise the partial fraction of cotangent using the infinite product of sine.

In the last chapter, we give an application of infinite products to a geometric problem.

Chapter 2

Infinite products

Definition 1. *Let a_n be a sequence of real numbers. An infinite product*

$$\prod_{n=1}^{\infty} a_n = a_1 a_2 a_3 \cdots$$

is said to converge if there exists an $m \in \mathbb{N}$ such that the $a_n{}'s$ are nonzero for all $n \geq m$ and the limit of partial products $\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdots a_n$

$$\lim_{n\to\infty} \prod_{k=m}^{n} a_k = \lim_{n\to\infty} (a_m \cdot a_{m+1} \cdots a_n) \tag{1}$$

converges to a nonzero value, say p. In this case we define

$$\prod_{n=1}^{\infty} a_n = a_1 \cdot a_2 \cdot a_3 \cdots a_{m-1} \cdot p.$$

The definition is of course independent of the m chosen such that the $a_n{}'s$ are nonzero for all $n \geq m$. The infinite product $\prod_{n=1}^{\infty} a_n$ diverges if it doesn't converge; That is either there are infinitely many zero $a_n{}'s$ or the limit (1) diverges or the limit (1) converges to zero. In this latter case, we say that the infinite product diverges to zero.

See the proof in [1].

Example 2. *1. The product $1 \cdot 2 \cdot 3 \cdots$ diverges.*

2. *The product $\prod_{n=1}^{\infty}(1 + \frac{1}{n})$ diverges.*
3. *The product $\prod_{n=2}^{\infty}\left(1 - \frac{2}{n(n+1)}\right)$ converges to $\frac{1}{3}$.*
4. *The product $\prod_{n=2}^{\infty}(1 - \frac{1}{n})$ diverges to zero.*

Proof. 1. Consider

$$\begin{aligned}
1 \cdot 2 \cdot 3 \cdots &= \prod_{n=1}^{\infty} n \\
&= \lim_{k \to \infty} \prod_{n=1}^{k} n \\
&= \lim_{k \to \infty} k!.
\end{aligned}$$

Hence the product $1 \cdot 2 \cdot 3 \cdots$ diverges.

2. Consider

$$\begin{aligned}
\prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) &= \lim_{k \to \infty} \prod_{n=1}^{k} \left(1 + \frac{1}{n}\right) \\
&= \lim_{k \to \infty} \left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \cdots \left(1 + \frac{1}{k}\right) \\
&= \lim_{k \to \infty} \left(\frac{2}{1}\right) \left(\frac{3}{2}\right) \left(\frac{4}{3}\right) \cdots \left(\frac{k+1}{k}\right) \\
&= \lim_{k \to \infty} (k+1).
\end{aligned}$$

Hence the product $\prod_{n=1}^{\infty}(1 + \frac{1}{n})$ diverges.

3. Consider

$$\begin{aligned}
\lim_{k \to \infty} \prod_{n=2}^{k} \left(1 - \frac{2}{n(n+1)}\right) &= \lim_{k \to \infty} \prod_{n=2}^{k} \frac{n^2 + n - 2}{n(n+1)} \\
&= \lim_{k \to \infty} \prod_{n=2}^{k} \frac{(n-1)(n+2)}{n(n+1)} \\
&= \lim_{k \to \infty} \left(\frac{1 \cdot 4}{2 \cdot 3} \; \frac{2 \cdot 5}{3 \cdot 4} \; \frac{3 \cdot 6}{4 \cdot 5} \; \frac{4 \cdot 7}{5 \cdot 6} \cdots \frac{(k-1)(k+2)}{k(k+1)}\right) \\
&= \lim_{k \to \infty} \left(\frac{1}{3} \cdot \frac{k+2}{k}\right) \\
&= \frac{1}{3}.
\end{aligned}$$

Hence the product $\prod_{n=2}^{\infty} \left(1 - \frac{2}{n(n+1)}\right)$ converges to $\frac{1}{3}$.

4. Consider

$$\begin{aligned}\lim_{k\to\infty}\prod_{n=2}^{k}\left(1-\frac{1}{n}\right) &= \lim_{k\to\infty}\prod_{n=2}^{k}\frac{n-1}{n}\\ &= \lim_{k\to\infty}\left(\frac{1}{2}\frac{2}{3}\frac{3}{4}\cdots\frac{k-1}{k}\right)\\ &= \lim_{k\to\infty}\frac{1}{k}\\ &= 0.\end{aligned}$$

Hence the product $\prod_{n=2}^{\infty}(1-\frac{1}{n})$ diverges to zero. □

Theorem 3. *If $a_k \geq 0$ for all k, then the product $\prod_{k=1}^{\infty}(1+a_k)$ converges if and only if the series $\sum_{k=1}^{\infty} a_k$ converges. The same is true if $a_k \leq 0$ for all k.*

See the proof in [1].

Example 4. *The product $\prod_{n=2}^{\infty}\left(1-\frac{2}{n(n+1)}\right)$ converges.*

Proof. Assume $a_k = \frac{-2}{k(k+1)}$ and $b_k = \frac{1}{k^2}$.

So

$$\begin{aligned}\lim_{k\to\infty}\frac{a_k}{b_k} &= \lim_{k\to\infty}\frac{\frac{-2}{k(k+1)}}{\frac{1}{k^2}}\\ &= \lim_{k\to\infty}\frac{-2k^2}{k^2+k+1}\\ &= -2.\end{aligned}$$

Since $\sum_{n=2}^{\infty} b_k = \frac{1}{k^2}$ is a p-serie, $\sum_{n=2}^{\infty} b_k$ converges.

By comparison test, $\sum_{n=2}^{\infty}\frac{-2}{n(n+1)}$ converges. Since $a_k \leq 0$ for all k, the product $\prod_{n=2}^{\infty}\left(1-\frac{2}{n(n+1)}\right)$ converges. □

2.1 Vieta's product formula

The infinite product formula

$$\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots=\frac{2}{\pi}$$

was discovered in 1593 by the French lawyer and mathematician Francois Vieta (1540-1603), whose name was rendered into Latin as Franciscus Vieta. The formula derives special interest from the unexpected appearance of the number π. It can be written more explicitly as

$$\lim_{n\to\infty}(b_1 b_2 \cdots b_n) = \frac{2}{\pi}$$

where $b_1 = \sqrt{\frac{1}{2}}$ *and* $b_{n+1} = \sqrt{\frac{1}{2} + \frac{1}{2}b_n}$, $n = 1, 2, \ldots$. See the proof in [1].

2.2 Wallis product formula

The remarkable Wallis product formula is

$$\lim_{n\to\infty} \frac{2\cdot 2}{1\cdot 3}\,\frac{4\cdot 4}{3\cdot 5}\,\frac{6\cdot 6}{5\cdot 7}\cdots\frac{(2n)(2n)}{(2n-1)(2n+1)} = \frac{\pi}{2}. \tag{1}$$

It was first recorded by John Wallis (1616-1703) as early as 1656. Equivalent formulation are

$$\lim_{n\to\infty}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(2n)^2}\right) = \frac{2}{\pi} \tag{2}$$

and

$$\lim_{n\to\infty}\frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}} = \sqrt{\pi}. \tag{3}$$

The factors in (2) are simply the reciprocals of those in (1). To see that (1) and (3) are equivalent, observe first that because $\frac{2n+1}{2n} \to 1$, the formula (1) implies

$$\lim_{n\to\infty}\frac{2^2 4^2 6^2 \cdots (2n-2)^2}{3^2 5^2 7^2 \cdots (2n-1)^2}(2n) = \frac{\pi}{2}$$

Taking square roots, we deduce that

$$\begin{aligned}\sqrt{\frac{\pi}{2}} &= \lim_{n\to\infty}\frac{2\cdot 4\cdot 6\cdots(2n-2)}{3\cdot 5\cdot 7\cdots(2n-1)}\sqrt{2n}\\ &= \lim_{n\to\infty}\frac{2^2 4^2 6^2\cdots(2n-2)^2(2n)^2}{(2n)!\sqrt{2n}}\\ &= \frac{1}{\sqrt{2}}\lim_{n\to\infty}\frac{2^{2n}(n!)^2}{(2n)!\sqrt{n}},\end{aligned}$$

which shows that (1) implies (3). Since all steps are reversible, a similar argument shows that (3) implies (1). See the proof in [1].

Chapter 3

Examples

3.1 Infinite product of sine

In this chapter, we show some results about uniform convergence, the Weierstrass M-test and some properties of uniform convergence.

Theorem 5 (Weierstrass M-test)**.** *Assume* $|f_n(x)| \leq M_n$ *for all* $x \in D$ *and the series of real number* $\sum_{n=1}^{\infty} M_n$ *converges, so series* $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly on D.*
See the proof in [2].

Example 6. $\displaystyle\sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2}$ *converges uniformly in each interval* $[-a, a]$ *with* $0 < a < 1$.

Proof. Let t be in each interval $[-a, a]$ with $0 < a < 1$. Assume $f_n(x) = \dfrac{2t}{t^2 - n^2}$ and $M_n = \dfrac{2a}{n^2 - a^2}$. Computing the maximum of each component, the numerator must be maximum and denominator must be minimum, so

$$f_n(t) = \left| \frac{2t}{t^2 - n^2} \right| \leq \frac{2a}{n^2 - a^2}.$$

Since $\sum_{n=1}^{\infty} M_n$ converge by use comparison test with p-series (n=2). By the Weierstrass M-test, $\displaystyle\sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2}$ converges uniformly in each interval $[-a, a]$ with $0 < a < 1$. □

Theorem 7. *If* $\sum_{n=1}^{\infty} f_n(x) = S(x)$ *converges uniformly on* $[a,b]$ *and* f_n *is on* $[a,b]$ *for all* $n \in \mathbb{N}$,

$$\int_a^b S(x)dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)dx.$$

See the proof in [2].

Theorem 8. *If* $\sum_{n=1}^{\infty} f_n'(x)$ *converges uniformly on* (a,b), f_n' *is continuous on* (a,b) *for all* $n \in \mathbb{N}$ *and* $S(x) = \sum_{n=1}^{\infty} f_n(x)$ *then* $S'(x) = \sum_{n=1}^{\infty} f_n'(x)$ *on* (a,b).
See the proof in [2].

Next, we present basic definitions and theorems about Fourier series.

Definition 9. *Trigonometric series are series of sine functions and cosine functions, of the form,*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)) \tag{1}$$

or

$$\frac{a_0}{2} + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(x) + b_2 \sin(x) + \cdots$$

where a_0,a_1,b_1,a_2,b_2,$\ldots$ *are real numbers.*
See the proof in [2].

Definition 10. *Let* f *be an integrable function having a period of* 2π. *The Fourier series of* f *are trigonometric series (1) having coefficients* a_n *and* b_n *given by*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \quad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \quad n = 1, 2, 3, \ldots.$$

See the proof in [2].

Theorem 11. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* $[0, 2\pi]$. *If* f *have a period of* 2π, *then the Fourier series of* f *converges to the mean of right-hand limits and left-hand limits. That is*

$$\frac{f(x^-) + f(x^+)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$$

where $f(x^-) = \lim_{t\to x^-} f(t)$ *and* $f(x^+) = \lim_{t\to x^+} f(t)$.
See the proof in [2].

Theorem 12. *Let $x \in \mathbb{R}$. Then*

$$\sin \pi x = \pi x \left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right)\cdots.$$

Proof. Let $c \in \mathbb{R} \setminus \mathbb{Z}$ be not an integer. Assume $f(x) = \cos(cx)$ for $-\pi \leq x \leq \pi$. Then the Fourier series of f is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)).$$

Since f is an integrable function having a period of 2π, by Definition 10, the coefficient a_0 is

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(cx)dx \\
&= \frac{1}{\pi}\left(\frac{\sin(cx)}{c}\right)\Big|_{-\pi}^{\pi} \\
&= \frac{1}{\pi}\left(\frac{\sin(c\pi)}{c} - \frac{\sin(-c\pi)}{c}\right) \\
&= \frac{2}{c\pi}\sin(c\pi).
\end{aligned}$$

Then coefficient a_n is

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(cx)\cos(nx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos(cx+nx)\cos(cx-nx)}{2}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(cx+nx)+\cos(cx-nx)dx \\
&= \frac{1}{2\pi}\left(\frac{\sin(cx+nx)}{c+n}+\frac{\sin(cx-nx)}{c-n}\Big|_{-\pi}^{\pi}\right) \\
&= \frac{1}{2\pi}\left(\frac{\sin(c\pi+n\pi)}{c+n}+\frac{\sin(c\pi-n\pi)}{c-n}-\frac{\sin(-c\pi-n\pi)}{c+n}-\frac{\sin(-c\pi+n\pi)}{c-n}\right) \\
&= \frac{1}{2\pi}\left(\frac{2\sin(c\pi+n\pi)}{c+n}+\frac{2\sin(c\pi-n\pi)}{c-n}\right) \\
&= \frac{1}{\pi}\left(\frac{\sin(c\pi+n\pi)}{c+n}+\frac{\sin(c\pi-n\pi)}{c-n}\right) \\
&= \frac{1}{\pi}\left(\frac{(c-n)\sin(c\pi+n\pi)+(c+n)\sin(c\pi-n\pi)}{c^2-n^2}\right) \\
&= \frac{1}{\pi}\left(\frac{(c-n)(\sin c\pi\cos n\pi+\cos c\pi\sin n\pi)}{c^2-n^2}+\frac{(c+n)(\sin c\pi\cos n\pi-\cos c\pi\sin n\pi)}{c^2-n^2}\right) \\
&= \frac{1}{\pi}\left(\frac{(c-n)(\sin c\pi\cos n\pi)+(c+n)(\sin c\pi\cos n\pi)}{c^2-n^2}\right) \\
&= \frac{1}{\pi}\left(\frac{2c\sin c\pi\cos n\pi}{c^2-n^2}\right) \\
&= \frac{2c}{\pi}\sin c\pi\frac{(-1)^n}{c^2-n^2}.
\end{aligned}$$

Then

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(cx)\sin(nx)dx \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} \cos(cx)\sin(nx)dx+\int_{0}^{\pi} \cos(cx)\sin(nx)dx\right).
\end{aligned}$$

Since $\cos(cx)\sin(nx)$ is odd function,

$$\int_{-\pi}^{0} \cos(cx)\sin(nx)dx = -\int_{0}^{\pi} \cos(cx)\sin(nx)dx.$$

Then

$$\begin{aligned} b_n &= \frac{1}{\pi}\left(-\int_0^{\pi} \cos(cx)\sin(nx)dx + \int_0^{\pi}\cos(cx)\sin(nx)dx\right) \\ &= 0. \end{aligned}$$

Hence the fourier series of f is

$$\frac{\frac{2}{c\pi}\sin c\pi}{2} + \sum_{n=1}^{\infty} \frac{2c}{\pi}\sin c\pi \frac{(-1)^n}{c^2-n^2}\cos nx.$$

Since f is a differentiable function on $[-\pi,\pi]$ and f have period 2π, by Theorem 9,

$$\frac{\cos cx + \cos cx}{2} = \cos cx$$

and

$$\frac{\frac{2}{c\pi}\sin c\pi}{2} + \sum_{n=1}^{\infty} \frac{2c}{\pi}\sin c\pi \frac{(-1)^n}{c^2-n^2}\cos nx = \frac{2c}{\pi}\sin c\pi \left(\frac{1}{2c^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{c^2-n^2}\cos nx\right).$$

Then

$$\cos cx = \frac{2c}{\pi}\sin c\pi\left(\frac{1}{2c^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{c^2-n^2}\cos nx\right).$$

Let $t \in \mathbb{R}\backslash\mathbb{Z}$. Fix $x = \pi$ and change c to t :

$$\begin{aligned} \cos \pi t &= \frac{2t}{\pi}\sin \pi t\left(\frac{1}{2t^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{t^2-n^2}\cos n\pi\right) \\ &= \frac{2t}{\pi}\sin\pi t\left(\frac{1}{2t^2} + \sum_{n=1}^{\infty}\frac{1}{t^2-n^2}\right). \end{aligned}$$

Multiply $\frac{1}{\sin \pi t}\cdot\frac{\pi}{2t}$ to both sides,

$$\frac{\pi}{2t}\cot \pi t = \frac{1}{2t^2} + \sum_{n=1}^{\infty}\frac{1}{t^2-n^2}.$$

We give patial fraction of cotangent,

$$\frac{\pi}{2t}\cot\pi t = \frac{1}{2t^2} + \sum_{n=1}^{\infty}\frac{1}{t^2-n^2} \text{ for x} \in \mathbb{R}\backslash\mathbb{Z}.$$

Subtract $\frac{1}{2t^2}$ from both sides,

$$\frac{\pi}{2t}\cot\pi t - \frac{1}{2t^2} = \sum_{n=1}^{\infty}\frac{1}{t^2-n^2}.$$

Multiply $2t$ to both sides,

$$\pi \cot \pi t - \frac{1}{t} = \sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2}.$$

Integrate with respect to t from ϵ to x for $0 < \epsilon < x < 1$,

$$\int_{\epsilon}^{x} \pi \cot \pi t - \frac{1}{t} dt = \int_{\epsilon}^{x} \sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2} dt.$$

Then

$$\begin{aligned}\int_{\epsilon}^{x} \pi \cot \pi t - \frac{1}{t} dt &= \ln \sin \pi t - \ln t - \ln \pi \Big|_{\epsilon}^{x} \\ &= \ln \frac{\sin \pi t}{\pi t} \Big|_{\epsilon}^{x} \\ &= \ln \frac{\sin \pi x}{\pi x} - \ln \frac{\sin \pi \epsilon}{\pi \epsilon}.\end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2}$ converges uniformly in each interval $[-a, a]$ with $0 < a < 1$ and $\frac{2t}{t^2 - n^2}$ is continuous in each interval $[-a, a]$ with $0 < a < 1$, by Theorem 7,

$$\begin{aligned}\int_{\epsilon}^{x} \sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2} dt &= \sum_{n=1}^{\infty} \int_{\epsilon}^{x} \frac{2t}{t^2 - n^2} dt \\ &= \sum_{n=1}^{\infty} \ln t^2 - n^2 \Big|_{\epsilon}^{x} \\ &= \sum_{n=1}^{\infty} \ln \frac{x^2 - n^2}{\epsilon^2 - n^2}.\end{aligned}$$

Thus

$$\ln \frac{\sin \pi x}{\pi x} - \ln \frac{\sin \pi \epsilon}{\pi \epsilon} = \sum_{n=1}^{\infty} \ln \frac{x^2 - n^2}{\epsilon^2 - n^2}.$$

Taking $\epsilon \to 0$,we have

$$\ln \frac{\sin \pi x}{\pi x} = \sum_{n=1}^{\infty} \ln \frac{x^2 - n^2}{-n^2}.$$

Then

$$\begin{aligned}\sum_{n=1}^{\infty} \ln \frac{x^2 - n^2}{-n^2} &= \sum_{n=1}^{\infty} \ln \left(1 - \frac{x^2}{n^2}\right) \\ &= \ln \left(1 - \frac{x^2}{1^2}\right) + \ln \left(1 - \frac{x^2}{2^2}\right) + \dots \\ &= \ln \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right).\end{aligned}$$

So

$$\ln \frac{\sin \pi x}{\pi x} = \ln \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right).$$

Then

$$\frac{\sin \pi x}{\pi x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right), 0 < x < 1.$$

Since $\dfrac{\sin \pi x}{\pi x}$ is an even function,

$$\frac{\sin \pi x}{\pi x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right), -1 < x < 1.$$

Hence

$$\sin \pi x = \pi x \left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right) \cdots, -1 < x < 1.$$

We show that the right-hand side is periodic with period 2. If $p(x)$ denotes the infinite product (which converges for every $x \in \mathbb{R}$), we want to show that $p(x+2) = p(x)$. Let the partial product be

$$p_n(x) = \pi x \left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right) \cdots \left(1 - \frac{x^2}{n^2}\right).$$

Then

$$\begin{aligned}
p_n(x) &= (-1)^n \pi x \left(\frac{x^2}{1^2} - 1\right)\left(\frac{x^2}{2^2} - 1\right)\left(\frac{x^2}{3^2} - 1\right) \cdots \left(\frac{x^2}{n^2} - 1\right) \\
&= (-1)^n \pi x \left(\frac{x^2 - 1}{1^2}\right)\left(\frac{x^2 - 2^2}{2^2}\right)\left(\frac{x^2 - 3^2}{3^2}\right) \cdots \left(\frac{x^2 - n^2}{n^2}\right) \\
&= \frac{(-1)^n}{(n!)^2} \pi x (x^2 - 1^2)(x^2 - 2^2)(x^2 - 3^2) \cdots (x^2 - n^2) \\
&= \frac{(-1)^n}{(n!)^2} \pi x (x+1)(x-1)(x+2)(x-2)(x+3)(x-3) \cdots (x+n)(x-n) \\
&= \frac{(-1)^n}{(n!)^2} \pi (x-n)(x-n+1) \cdots (x-1)x(x+1) \cdots (x+n-1)(x+n).
\end{aligned}$$

So

$$\begin{aligned}
p_n(x+2) = \frac{(-1)^n}{(n!)^2} \pi (x-n+2)(x-n+3) \cdots (x+1)(x+2)(x+3) \\
\cdots (x+n+1)(x+n+2) \\
= \frac{(-1)^n}{(n!)^2} \pi \frac{(x-n)(x-n+1)}{(x-n)(x-n+1)} (x-n+2)(x-n+3) \cdots (x-1) \\
x(x+1) \cdots (x+n+1)(x+n+2).
\end{aligned}$$

Then

$$p_n(x+2) = \frac{(-1)^n}{(n!)^2}\pi\frac{(x-n)(x-n+1)}{(x-n)(x-n+1)}p_n(x), n = 1, 2, 3, \ldots.$$

Taking $n \to \infty$, we have

$$p(x+2) = p(x) \text{ for } x \in \mathbb{R}.$$

Hence

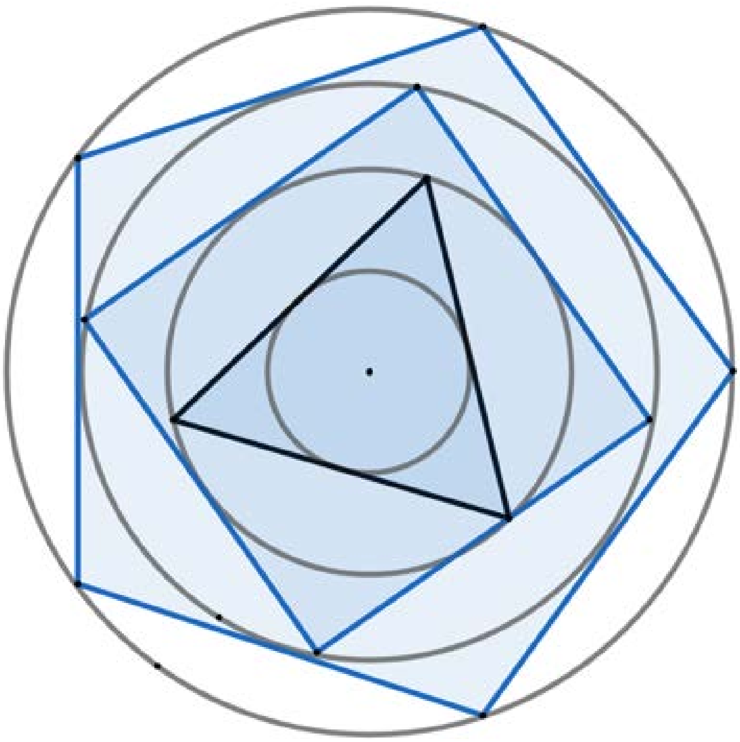
$$\sin \pi x = \pi x\left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right)\cdots \text{ for } x \in \mathbb{R}.$$

□

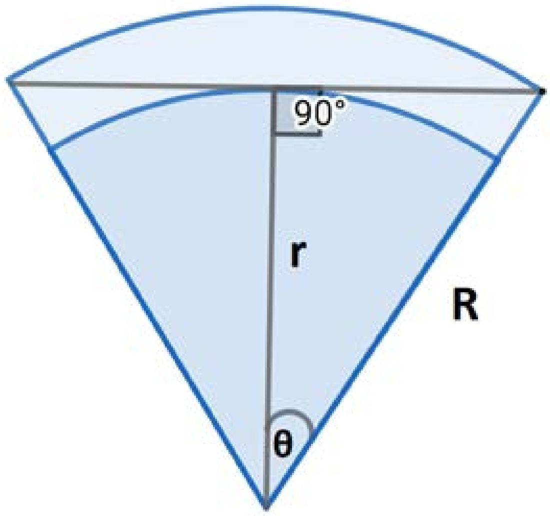
Chapter 4

An application

Problem 1. Suppose that an equilateral triangle is inscribed in a circle of radius 1, then a circle C_3 is circumscribed about the triangle, then a square is circumscribed about C_3, another circle C_4 is circumscribed about the square ,a regular pentagon is circumscribed about C_4, and so on. Is the sequence of the circles C_n bounded?



Proof. Consider the n-gon is circumscribed about a circle of radius r and another circle is circumscribed about the n-gon. The second circle has radius R.



Since θ is $\frac{\pi}{n}$,

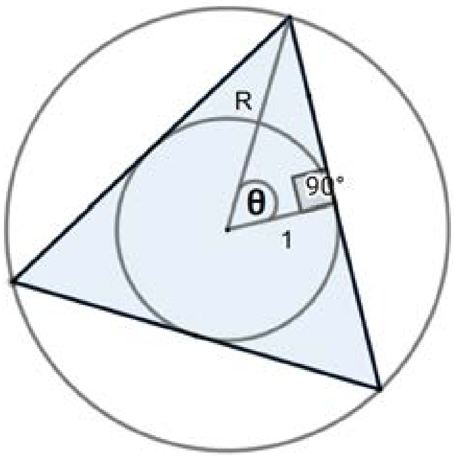
$$\cos\frac{\pi}{n} = \frac{r}{R}.$$

Then

$$R = r\sec\frac{\pi}{n}$$

If C_n has radius R_n ,we will show that $R_n = \prod_{k=3}^{n}\sec\frac{\pi}{k}$.

Basis step ; $n = 3$



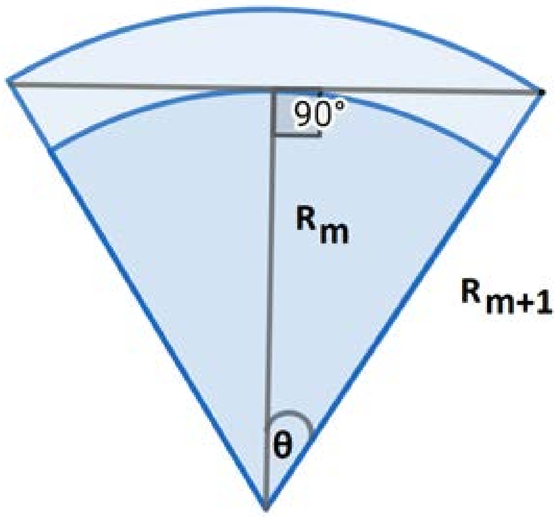
Consider the triangle in the above picture; since the triangle is circumscribed about a circle, θ is $\frac{\pi}{3}$.
By trigonometry,

$$\cos\frac{\pi}{3} = \frac{1}{R}.$$

Then

$$R = \sec\frac{\pi}{3}.$$

Induction step:



Consider triangle in the above picture; since $(m+1)$-gon is circumscribed about a circle, θ is $\frac{\pi}{m+1}$.
By trigonometry,

$$\cos\frac{\pi}{m+1} = \frac{R_m}{R_{m+1}}.$$

Then

$$R_{m+1} = R_m \cdot \sec\frac{\pi}{m+1}.$$

Since $R_m = \prod_{k=3}^{m} \sec\frac{\pi}{k}$ for $m \geq 3$,

$$R_{m+1} = \prod_{k=3}^{m+1} \sec\frac{\pi}{k}.$$

We will prove the circles C_n for $n = 3, 4, 5, \ldots$ are bounded. We show $\lim\limits_{n\to\infty} R_n = \lim\limits_{n\to\infty} \prod_{k=3}^{n} \sec \frac{\pi}{k} = \prod_{k=3}^{\infty} \sec \frac{\pi}{k}$ exists.

Since

$$\prod_{k=3}^{\infty} \sec \frac{\pi}{k} = \prod_{n=1}^{\infty} \sec \frac{\pi}{n+2}.$$

Let

$$a_k = \sec\left(\frac{\pi}{k+2}\right) - 1 = \frac{1}{\cos \frac{\pi}{k+2}} - 1 = \frac{1 - \cos \frac{\pi}{k+2}}{\cos \frac{\pi}{k+2}}$$

and

$$b_k = \left(\frac{\pi}{k+2}\right)^2.$$

Then

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_k}{b_k} &= \lim_{k\to\infty} \frac{\dfrac{1 - \cos \frac{\pi}{k+2}}{\cos \frac{\pi}{k+2}}}{\left(\dfrac{\pi}{k+2}\right)^2} \\
&= \lim_{k\to\infty} \frac{1 - \cos \dfrac{\pi}{k+2}}{\cos \dfrac{\pi}{k+2} \cdot \left(\dfrac{\pi}{k+2}\right)^2} = \frac{1}{2}.
\end{aligned}$$

Since $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \left(\frac{\pi}{k+2}\right)^2 = \sum_{k=1}^{\infty} \frac{\pi^2}{(k+2)^2}$ is a p-series, which is convergent, $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=3}^{\infty} \left(\sec\left(\frac{\pi}{k}\right) - 1\right)$ converges.

By Theorem 3, $\prod_{k=3}^{\infty} \sec \frac{\pi}{k}$ converges. Hence the sequence of the circles C_n is bounded.

$\square$

Bibliography

[1] P. Duren, Invitation to Classical Analysis, *Pure and Applied Undergraduate Texts 17*, *American Mathematical Society* ,131–142 (2012).

[2] ทรงเกียรติ สุเมธกิจการ, **แคลคูลัส : ผลบวกอนันต์**, พิมพ์ครั้งที่ 1, ภาควิชาคณิตศาสตร์ คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย, 33–127 (2556)

The Project Proposal of Course 2301399 Project Proposal
First Semester, Academic Year 2018

Title (Thai) ผลคูณอนันต์ ตัวอย่าง และการประยุกต์

Title (English) Infinite Product Example and Application

Advisor Assistant Professor Dr.Keng Wiboonton

By Mr.Sitthichai Pempongkoson ID 5833547623

Mathematics, Department of Mathematics and Computer Science

Background and Rationale and Scope

The content of infinite sums is discussed extensively in calculus and real analysis course. However, the concept of infinite products, which is closely related to that of infinite sums, is not mentioned in any regular undergraduate analysis course. In this project, we study the definition of infinite products, examples of infinite products such as the infinite product of the sine function with the associated examples including infinite partial fraction of the cotangent function. Finally, we give some interesting applications of infinite products.

Give a decent exploratory discussion of infinite products and their related topics, provide some interesting applications of infinite products.

Objectives

Study definition of infinite products, state and prove important theorems for infinite products and give some interesting applications of infinite products.

Project Activities

1. Study the definition of infinite products.
2. State and prove important theorems for infinite products.
3. Find some interesting applications of infinite products.
4. Write a report.

Activities Table

Project Activities	August 2017 - April 2018								
	Aug	Sep	Oct	Nov	Dec	Jan	Feb	Mar	Apr
1.Study definition of infinite products.									
2.State and prove important theorems for infinite products.									
3.Find some interesting applications of infinite products.									
4.Write a report.									

Benefits

Understanding the topics of infinite products and learning some interesting applications of infinite products.

Equipment

1. Computer
2. Paper
3. Printer
4. Stationery
5. Word processing program

Budget

1. Paper A4

Author's profile

Mr.Sitthichai Pempongkoson

ID 5833547623

Department of Mathematics and Computer Science

Faculty of Science Chulalongkorn University