CHAPTER V

CONCLUSION

5.1 Conclusion

We define group divisible designs with two groups and three associate classes, $\mathsf{GDD}(m,n;\lambda_1,\lambda_1',\lambda_2)$, and investigate the existence problem of such GDDs. By considering a GDD graphically, the following necessary conditions are obviously obtained:

(i) every vertex in the graph $\lambda_1 K_m \vee_{\lambda_2} \lambda_1' K_n$ has even degree,

(ii) the number of edges in the graph $\lambda_1 K_m \vee_{\lambda_2} \lambda_1' K_n$ is divisible by three and

(iii) if $\lambda_2 = 0$, then the number of edges in each of the two graphs $\lambda_1 K_m$ and $\lambda_1' K_n$ must be divisible by three.

We prove that these conditions are sufficient for the existence of GDDs when $m \neq 2$, $n \neq 2$, $\lambda_1 \geq \lambda_2$ and $\lambda_1' \geq \lambda_2$. To summarize, the construction of each case is provided in Table 5.1.

Eventually, we conclude all the constructions in the following main theorem.

Theorem 5.1. (Main Theorem) Let m and n be positive integers such that $m \neq 2$, $n \neq 2$ and $mn \neq 1$. Let λ_1, λ_1' and λ_2 be nonnegative integers such that $\lambda_1 \geq \lambda_2$ and $\lambda_1' \geq \lambda_2$. There exists a $\mathsf{GDD}(m,n;\lambda_1,\lambda_1',\lambda_2)$ if and only if

(i) $2|(\lambda_1(m-1)+\lambda_2 n)$,

(ii) $2|(\lambda_1'(n-1)+\lambda_2 m)$,

(iii) $6|(\lambda_1 m(m-1)+\lambda_1' n(n-1)+2\lambda_2 mn)$ and

(iv) if $\lambda_2 = 0$, there exists a $\mathsf{TS}(m;\lambda_1)$ and a $\mathsf{TS}(n;\lambda_1')$.

$m \setminus n$	0	1	2	3	4	5
0	$2\|\lambda_1$, $2\|\lambda_1'$ Theorem 3.3	$2\|(\lambda_1+\lambda_2)$ Theorem 3.2	$2\|\lambda_1$, $6\|\lambda_1'$ Theorem 3.7	$2\|(\lambda_1+\lambda_2)$ Theorem 3.2	$2\|\lambda_1$, $2\|\lambda_1'$ Theorem 3.3	$3\|\lambda_1'$ $2\|(\lambda_1+\lambda_2)$ Theorem 3.18
1		$6\|\lambda_2$ Theorem 3.2	$2\|(\lambda_1'+\lambda_2)$ $3\|(\lambda_1'+2\lambda_2)$ Theorem 3.2	$2\|\lambda_2$ Theorem 3.2	$2\|(\lambda_1'+\lambda_2)$ $3\|\lambda_2$ Theorem 3.2	$3\|(\lambda_1'+2\lambda_2)$ $2\|\lambda_2$ Theorem 3.2
2			$2\|\lambda_1$, $2\|\lambda_1'$ $3\|(\lambda_1+\lambda_1'+\lambda_2)$ Theorem 4.3	$3\|\lambda_1$ $2\|(\lambda_1+\lambda_2)$ Theorem 3.9	$2\|\lambda_1$. $2\|\lambda_1'$ $3\|(\lambda_1+2\lambda_2)$ Theorem 3.12	$2\|(\lambda_1+\lambda_2)$ $3\|(\lambda_1+\lambda_1'+\lambda_2)$ Theorem 4.6
3				$2\|\lambda_2$ Theorem 3.2	$2\|(\lambda_1'+\lambda_2)$ Theorem 3.2	$3\|\lambda_1'$, $2\|\lambda_2$ Theorem 3.18
4					$2\|\lambda_1$, $2\|\lambda_1'$ $3\|\lambda_2$ Theorem 3.3	$3\|(\lambda_1'+2\lambda_2)$ $2\|(\lambda_1+\lambda_2)$ Theorem 3.2
5						$2\|\lambda_2$ $3\|(\lambda_1+\lambda_1'+\lambda_2)$ Theorem 4.9

Table 5.1: All possible GDDs

5.2 Open Problem

It is natural to follow through on the case when $m = 2$ or $n = 2$ as an open problem. This open problem has more necessary conditions. When either $m = 2$ or $n = 2$, each edge in the group of size two (simply called *pure edge*) must belong to a triangle which contains two edges between the groups (simply called *cross edges*). Thus, the number of edges in the group of size two must be at most half of the number of edges between two groups. In particular,

(1) if $m = 2$, then $\lambda_1 \leq \lambda_2 n$ and

(2) if $n = 2$, then $\lambda_1' \leq \lambda_2 m$.

When both $m = 2$ and $n = 2$, each pure edge requires two cross edges to form a triangle, and these edges are in exactly one triangle. Thus, the number of pure edges must be at most half of the number of cross edges. That is $\lambda_1 + \lambda_1' \le \frac{4\lambda_2}{2} = 2\lambda_2$.

On the other hand, we note that any triangle in this case contains one pure edge and two cross edges, and these triangles are edge-disjoint. Hence, half of the number of cross edges in the graph must be at most the number of pure edges, or $2\lambda_2 = \frac{4\lambda_2}{2} \le \lambda_1 + \lambda_1'$. Therefore, when $m = 2$ and $n = 2$, we have that $\lambda_1 + \lambda_1' = 2\lambda_2$. We conclude these necessary conditions in the following theorem.

Theorem 5.2. (Necessary Conditions) *Let m and n be positive integers. Let λ_1, λ_1' and λ_2 be nonnegative integers such that $\lambda_1 \ge \lambda_2$ and $\lambda_1' \ge \lambda_2$. If there exists a* $\mathsf{GDD}(m, n; \lambda_1, \lambda_1', \lambda_2)$, *then*

(i) $2|(\lambda_1(m-1) + \lambda_2 n)$,

(ii) $2|(\lambda_1'(n-1) + \lambda_2 m)$,

(iii) $6|(\lambda_1 m(m-1) + \lambda_1' n(n-1) + 2\lambda_2 mn)$,

(iv) *if $\lambda_2 = 0$, then there exist a* $\mathsf{TS}(m; \lambda_1)$ *and a* $\mathsf{TS}(n; \lambda_1')$.

(v) *if $m = 2$, then $\lambda_1 \le \lambda_2 n$,*

(vi) *if $n = 2$, then $\lambda_1' \le \lambda_2 m$ and*

(vii) *if $m = 2$ and $n = 2$, then $\lambda_1 + \lambda_1' = 2\lambda_2$.*