Chapter 2

Introduction of the Tunneling Time

2.1 Order of Magnitude of the Tunneling Time

In this thesis, we study some other possible ways of formulating the tunneling time. Let us first consider a typical example and order of magnitude of the tunneling time in condensed matter physics. An electron with an effective mass of $m^* = 0.07\ m_e$, where m_e is the bare mass of the electron. tunneling through $GaAs/AlGaAs/GaAs$ structure [12]. If the height of the barrier is $V_0 = 0.23\ eV$ and width of the barrier is $d = 60 \times 10^{-10} m$, then for the incident energy $E = 0.115\ eV$, the tunneling time is about $6 \times 10^{-15} s$. The question loosely formulated as "*How much time does tunneling take* ?" is not new. Early answers were given in the 1940s and 1950s [27] with alternatives proposed in the 1960s [28,29]. The prospect of high-speed devices based on tunneling structures in semiconductors (see [30] for example) has brought new urgency to the problem. An understanding of the time-dependent aspects of tunneling is clearly required for the construction of a kinetic theory for such a system. The simple question of the tunneling time seems a natural one from which to start. But simplicity can be deceptive: tunneling times have continued to be controversial throughout the 1980s.

2.2 Tunneling and the Uncertainty Principle

At first sight, the tunneling of a particle looks like a paradoxical problem. since if the height of the barrier is greater than the total energy of the particle

$$E = \frac{1}{2m}p^2 + V(x) \tag{2.1}$$

then in the region II (see Fig. 2.1) where $V(x) > E$, the kinetic energy $\frac{1}{2m}p^2$ (p is the momentum of the particle) is negative and p is imaginary. but this is not correct. At the root of this paradox is our assumption that at each instant we know both the kinetic and the potential energies separately. or in other words we can assign values to the coordinate x and the momentum p simultaneously and this is in the violation of the uncertainty principle. Here we want to know whether it is possible to determine the position of the particle when it is moving under the barrier or not. For this we observe that the particle can be at the point x where $V(x) > E$ but then according to the uncertainty principle its momentum is uncertain by an amount Δp. Thus if we know the position of the particle to be x. then its total energy cannot be E.

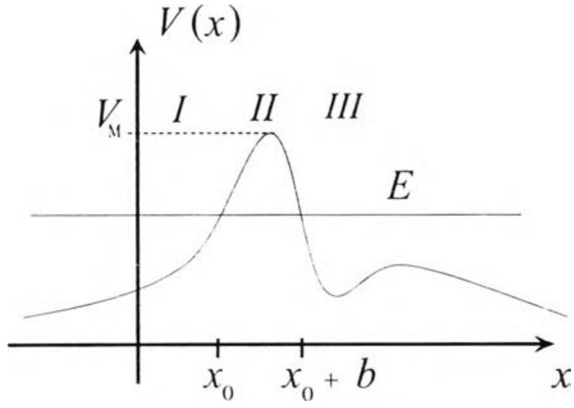

Figure 2.1: A particle moves in the potential $V(x)$ which has the height of the barrier greater than the total energy of the particle.

Since the transmission amplitude during tunneling is proportional to

$$\exp\left\{-\frac{i}{\hbar}\int_{x_0}^{x}\sqrt{2m(V(x)-E)}dx\right\} \tag{2.2}$$

where x_0 is the classical turning point, then the probability of finding the particle which is coming from the left to be in the right of the barrier, i.e. $x_0 + b$ is proportional to the square of this amplitude or to the factor

$$\exp\left\{-\frac{2}{\hbar}\int_{x_0}^{x_0+b}\sqrt{2m(V(x)-E)}dx\right\}. \tag{2.3}$$

Now if we want a non-negligible probability then we must have

$$2b\sqrt{2m(V_M-E)} \approx \hbar \tag{2.4}$$

where V_M is the maximum height of the potential. To find the position of the particle inside the barrier. we have to measure its coordinate with an accuracy $\Delta x < b$, therefore the uncertainty in momentum is

$$(\Delta p)^2 = \hbar^2/4(\Delta x)^2 = \hbar^2/4b^2. \tag{2.5}$$

By substituting b from Eq. (2.4) to Eq.(2.5) we find

$$(\Delta p)^2/2m = V_M - E. \tag{2.6}$$

Thus the kinetic energy of the particle must be greater then the difference between the height of the barrier V_M and the total energy E [31]. A result similar to Eq.(2.5) and Eq.(2.6) can be obtained from the time-energy uncertainty relation [32] i.e.,

$$\Delta E\Delta t \approx \hbar/2. \tag{2.7}$$

Again let us denote the energy of the incident particle by E. For a very short time Δt. the uncertainty in the energy is ΔE. and for sufficiently small Δt. the energy of the particle $E + \Delta E$ is greater than the hight of the barrier V_M. Tunneling takes place if in the time interval Δt the particle can traverse the barrier. For a rectangular barrier of width b this Δt is given by

$$\Delta t = b/\sqrt{(\frac{2}{m})(E + \Delta E - V_M)}. \tag{2.8}$$

From Eq.(2.7) and Eq.(2.8) we find ΔE to be the solution of the quadratic equation

$$(\Delta E)^2 - \frac{\hbar^2}{2mb^2}\Delta E + \frac{\hbar^2}{2mb^2}(V_M - E) = 0 \tag{2.9}$$

and the condition for ΔE to be real is given by

$$\frac{\hbar^2}{8mb^2} > V_M - E \tag{2.10}$$

which is the same as Eq.(2.5).

2.3 Time Delay in Tunneling

Suppose that we have two particles both starting at the point $-a$ (see Fig. 2.3) and both reaching the point a on the other side of the barrier $V(x)$, but one travelling the distance of $-a$ to a in free space (see Fig. 2.2)and other reaching a from $-a$ by tunneling.

We assume that the barrier exists only for $-a < x < a$. Denoting the mass and the energy of each particle by m and E respectively, the travel time of the first (free) particle between the point $-a$ and a according to classical mechanics is equal to

$$t_1 = \frac{2a}{v_0} = \frac{2ma}{\sqrt{2mE}}, \tag{2.11}$$

where v_0 is the velocity of the particle. For the second particle if $E > V(x)$ the classical travel time is

$$t_2 = \int_{-a}^{a} dx m/\sqrt{2m(E - V(x))}. \tag{2.12}$$

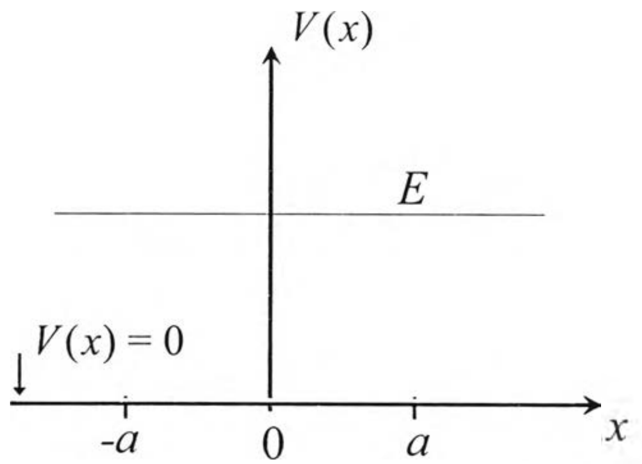

Figure 2.2: The free particle with total energy E moves in one-dimensional space.

Therefore the time delay due to the presence of the barrier is given by

$$\tau_c = t_2 - t_1 = \int_{-a}^{a} dxm \left\{ 1/\sqrt{2m(E - V(x))} - 1/\sqrt{2mE} \right\}. \tag{2.13}$$

Now we will study this problem in the semi-classical approximation. To this end we start with the Schroedinger equation with the potential $V(x)$.

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi + [E - V(x)]\Psi = 0, \tag{2.14}$$

and use the WKB approximation to find the solution of Eq.(2.14)

$$\Psi(x) \approx N_C \exp\left\{ \frac{i}{\hbar} \int_{a}^{x} \sqrt{2m(E - V(x))}dx \right\}, -a \le x \le a. \tag{2.15}$$

where N_C is the normalization constant and again we have assumed that $E \ge V(x)$ in this interval. For $x > a$, we can write the wave function as

$$\Psi(x) = N_C \exp\left\{ \frac{i}{\hbar}\sqrt{2mE}x + i\eta(E) \right\}, x > a \tag{2.16}$$

which is the wave function for a free particle. The phase shift $\eta(E)$, which is dimensionless, is caused by the presence of the potential $V(x)$ and is given by [33]

$$\eta(E) = \frac{1}{\hbar} \int_{-a}^{a} dx \left\{ \sqrt{2m(E - V(x))} - \sqrt{2mE} \right\} \tag{2.17}$$

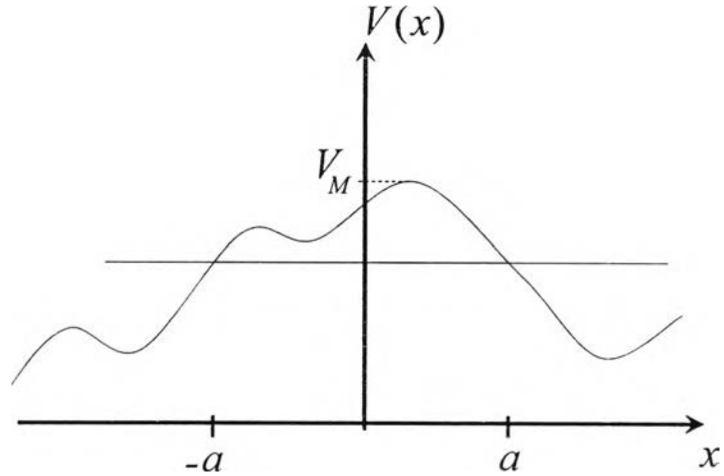

Figure 2.3: The particle is in the barrier V(x). In the region $[-a, a]$, the potential $V(x)$ greater than the total energy of the particle.

This relation is obtained by matching the inside and the outside solutions, Eq.(2.15) and Eq.(2.16) respectively. If we differentiate $\eta(E)$ with respect to E we have

$$\hbar\frac{\partial\eta(E)}{\partial E} = \int_{-a}^{a}\left\{\frac{1}{\sqrt{2m(E-V(x))}} - \frac{1}{\sqrt{2mE}}\right\} m dx. \tag{2.18}$$

By comparing Eq.(2.13) and Eq.(2.18) we find that $\hbar\partial\eta(E)/\partial E$ corresponds to the time delay in classical mechanics.

Next we want to derive a full quantum mechanical expression for the time delay. We first observe that $mdx/\sqrt{2m(E-V(x))}$ and $mdx/\sqrt{2mE}$ are the classical probabilities of finding a free particle and a particle moving under the influence of the potential $V(x)$ in the range dx. In quantum mechanics these probabilities are given by $|\Psi_0(x, E)|^2\, dx$ and $|\Psi(x, E)|^2\, dx$ respectively, where $\Psi_0(x, E)$ is the wave function for a free particle. An approach by Smith [34] and others [35] yields a collision time for a scattering event. If τ_q is defined as the ratio of the number of particles within the barrier to the incident flux $j = \hbar k/m$, where $p = \hbar k$ is the momentum of the particle, we find for the number of particles in the barrier n,

$$n = N_0\int_{-a}^{a}|\Psi(x, E)|^2\, dx. \tag{2.19}$$

where N_0 is the total number of particles. We can write the quantum mechanical time delay τ_q as

$$\tau_q = \frac{1}{j(k)} \int_{-a}^{a} \left(|\Psi(x, E)|^2 - |\Psi_0(x, E)|^2 \right) dx \tag{2.20}$$

with the understanding that the incoming flux is $j(k)$.

2.4 Dwell Time

The dwell time, τ_D in one dimension was first introduced by Büttiker in 1983 [36] and is the time spent by the particle in any finite region of space, averaged over all incoming particles. τ_D is defined as the ratio of the number of particles within the barrier to the incident flux $j = \hbar k/m$.

$$\tau_D = \frac{1}{j(k)} \int_{-a}^{a} |\Psi(x, k)|^2 \, dx. \tag{2.21}$$

This approach does not distinguish whether, at the end of their stay, particles are reflected or transmitted. This time is the average dwell time of the particles in the region $[-a, a]$. It is also possible to define the dwell time by using Feynman's path integrals. We recall that this approach deals with continuous paths that are not necessarily differentiable in time. Sokolovski and Baskin in 1987 [37] applied this formulation to calculate a traversal time by starting from the construction of quantum traversal time from the generalization of a well-known classical expression. Consider a particle moving along a classical path $x(t)$ connecting the points x_1, t_1 and x_2, t_2 in some potential $V(x)$. The time the particle spends in an arbitrary region Ω is given by the integral

$$t_\Omega^{cl} = \int_{t_1}^{t_2} \Theta_\Omega[x(t)] dt \tag{2.22}$$

where $\Theta_\Omega[x(t)]$ equals 1 if $x \in \Omega$ and 0 otherwise. To find the mean value of t_Ω^{cl} for the transition between arbitrary initial and final state Ψ_i and Ψ_f one has to integrate over initial and final coordinates

$$\tau_D = \langle \Psi_f | \, t^{cl}_{\Omega} \, | \Psi_i \rangle \tag{2.23}$$

or

$$\tau_D = \frac{\int dx_2 \int dx_1 \Psi_f^*(x_2) \int D[x(t)] \; t^{cl}_{\Omega} \exp\left[\frac{i}{\hbar} S[x(t)]\right] \Psi_i(x_1)}{\int dx_2 \int dx_1 \Psi_f^*(x_2) \int D[x(t)] \; \exp\left[\frac{i}{\hbar} S[x(t)]\right] \Psi_i(x_1)}. \tag{2.24}$$

In the particular case when the final state Ψ_f is obtained from Ψ_i by the evolution,

$$\Psi_f(x_2, t_2) = \int dx_1 \int D[x(t)] \; \exp\left\{\frac{i}{\hbar} S[x(t)]\right\} \Psi_i(x_1, t_1).$$

Eq.(2.24) is considerably simplified, so that

$$\tau_D = \int_{t_1}^{t_2} dt \int_{\Omega} dx \, |\Psi(x, t)|^2 \, . \tag{2.25}$$

The details of the calculation are discussed in Appendix A. For the initial wave packet, we can write as

$$\Psi(x, t) = \frac{1}{2\pi} \int dk \Phi(k) \Psi(x, k) \exp\left\{-\frac{i\hbar k^2 t}{2m}\right\}. \tag{2.26}$$

where $|\Phi(k)|^2$ is the probability distribution over wave number k and $\Psi(x, k)$

When $\Psi(x, t < t_1)$ is assumed to be (essentially) zero for $x > x_1$, the lower limit of integration in Eq.(2.25) can be replaced by $-\infty$. We consider t_2 to be ∞ then the upper limit of integration in Eq.(2.25) can be replaced by ∞. Thus replacing the integrals over k and k' by integrating over $D = (k + k')/2$ and $v = (k - k')/2$, one has

$$\tau_D = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dt \int_{\Omega} dx \int dD \int dv \Phi(\frac{D-v}{2}) \Phi(\frac{D+v}{2})$$
$$\times \Psi(x, \frac{D-v}{2}) \Psi^*(x, \frac{D+v}{2}) e^{-\frac{i}{\hbar m} Dvt}. \tag{2.27}$$

Integration over t gives $2\pi(m/\hbar D)\delta(v)$. Finally, integration over d and relabelling $D \to k$ leads to

$$\tau_D = \frac{1}{2\pi} \int dk \, |\Phi(k)|^2 \, \frac{1}{j(k)} \int_{\Omega} dx \, |\Psi(x, k)|^2 \, . \tag{2.28}$$

where $j(k)$ is $\hbar k/m$. Since $\overline{\tau}_D$ is an average over an arbitrary initial wave packet with probability distribution $\frac{1}{2\pi}|\Phi(k)|^2$ over wave numbers k, the average time spent on the interval $[-a, a]$ for particles in a state $\Psi(x, k)$ is

$$\tau_D = \frac{1}{j(k)} \int_{\Omega} dx\, |\Psi(x,k)|^2 \tag{2.29}$$

which is Büttiker's Eq.(2.21) for the dwell time.

2.5 Larmor Precession and the Traversal Time

In 1966 [13] Baz' suggested an appealing thought experiment (see Fig.2.4) to measure separately the time spent in different scattering channels: take advantage of the constant Larmor precession of a spin in a homogeneous magnetic field by covering the region of interest with an infinitesimal field $B = B_0\hat{z}$ (so that only first-order effects need to be retained). With the incident spin $\frac{1}{2}$ particles polarized in the x direction, the time spent in the field region should be proportional to the averaged spin component $\langle \sigma_y \rangle$ of the particles emerging in the given scattering channel.

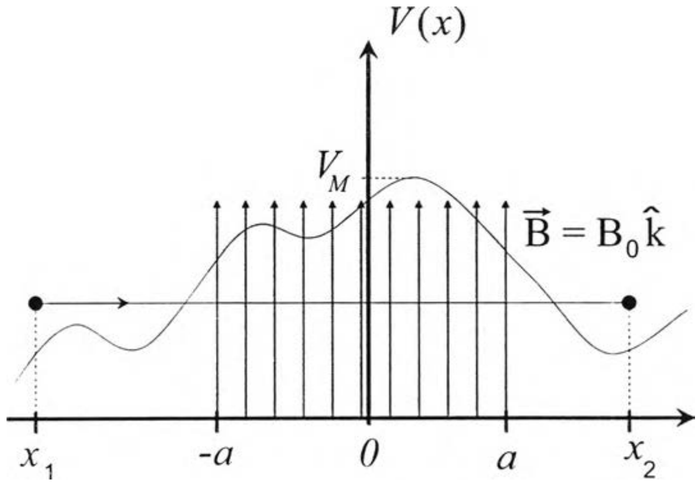

Figure 2.4: The homogeneous magnetic field is at the interval (-a,a).

Let the particle move along the path $\vec{r}(t)$. If in the initial point $\vec{r}_1$ its spin(or angular momentum) is directed normal to the field $\vec{B}$, then in the final point $\vec{r}_2$, it is rotated in the plane normal to $\vec{B}$ by the angle

$$\phi = \omega_l t_{\Omega}^{cl}, \tag{2.30}$$

where t_{Ω}^{cl} is classical traversal time as given by Eq.(2.22) and ω_l is $2\mu B_0/\hbar$ (μ is the Bohr magnetron). From now on we consider potential V to be time independent, hence all the quantities depend on $T = t_2 - t_1$ only. $\vec{B}$ is directed along z, and the axis and the spin of particle ($\sigma = \frac{1}{2}$) is polarized in x direction at $t = 0$. Thus we define the initial state to be

$$\Psi_i(\vec{r}, \sigma_x) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \delta(\vec{r} - \vec{r}_1), \tag{2.31}$$

and the spin-dependent transition amplitude $K(\vec{r}_2, \vec{r}_1; T)$ take the form

$$K(\vec{r}_2, \vec{r}_1; T) = \int D[\vec{r}(t)] \exp \left\{ \frac{i}{\hbar} S[\vec{r}(t)] - \frac{i}{\hbar} \mu B_0 \int_0^T \Theta_\Omega[\vec{r}(t)] dt \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}. \tag{2.32}$$

or in the limit of small B_0,

$$K(\vec{r}_2, \vec{r}_1; T) = K_0(\vec{r}_2, \vec{r}_1; T) \left[1 - \frac{i}{\hbar} \mu B_0 t_{\Omega}^{cl} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + O(B_0^2) \right]. \tag{2.33}$$

Note that Eq.(2.32) and Eq.(2.33) $S[\vec{r}(t)]$ and $K_0(\vec{r}_2, \vec{r}_1; T)$ are the spatial parts of action and transition amplitude.

With the help of Green's function in Eq.(2.33) we obtain the spin orientation in $\vec{r}_2$ as

$$\Psi_f = \frac{1}{\sqrt{2}} \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{i}{\hbar} \mu B_0 t_{\Omega}^{cl} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right] \delta(\vec{r} - \vec{r}_1). \tag{2.34}$$

If follows from Eq.(2.34) that the spin of the particle has nonzero y and z components $\langle \sigma_y \rangle$ and $\langle \sigma_z \rangle$ in $\vec{r}_2$:

$$\langle \sigma_y \rangle = -\frac{2\mu B_0}{\hbar} Re \left[t_{\Omega}^{cl} \right] \tag{2.35}$$

and

$$\langle\sigma_z\rangle = -\frac{2\mu B_0}{\hbar} Im\left[t_\Omega^{cl}\right]. \qquad (2.36)$$

The spin is therefore rotated in xy plane as well as in the plane parallel to the field by the $\phi_\perp$ and $\phi_{//}$, respectively (see Fig 2.5)

Finally, we consider the angles $\phi_\perp$ and $\phi_{//}$ that can be obtained

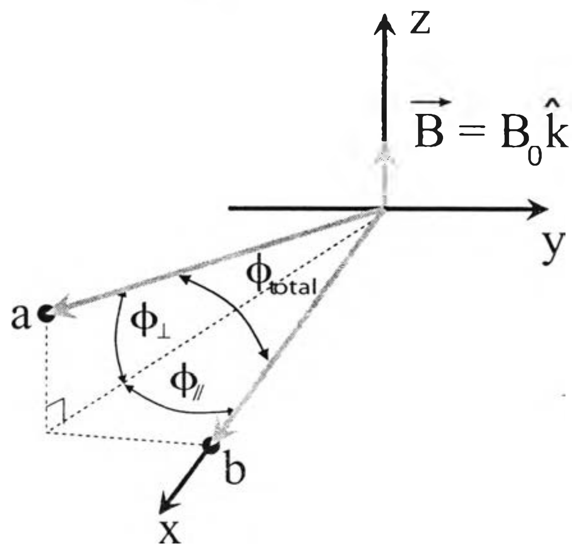

Figure 2.5: The spin orientation of the particle is at the final point a.

$$\phi_\perp = \omega_l Re[t_\Omega^{cl}] \qquad (2.37)$$

and

$$\phi_{//} = \omega_l Im[t_\Omega^{cl}] \qquad (2.38)$$

where ω_l is $2\mu B_0/\hbar$. For the total angle of rotation ϕ_{total}, we have

$$\phi_{total} = \sqrt{\phi_\perp^2 + \phi_{//}^2} = \omega_\perp\left|t_\Omega^{cl}\right|. \qquad (2.39)$$

By measuring the expectation values of σ_y and that of σ_z we obtain two real parameters

$$t_\perp = \frac{\phi_\perp}{\omega_l} = \frac{\langle\sigma_y\rangle}{\omega_l} \qquad (2.40)$$

and

$$t_{//} = \frac{\phi_{//}}{\omega_l} = \frac{\langle\sigma_z\rangle}{\omega_l}. \qquad (2.41)$$

We have the total parameter t_{total} as

$$t_{total} = \frac{\phi_{total}}{\omega_l} \tag{2.42}$$

and we have shown that the natural construction uniting both $t_{\perp}$ and $t_{//}$ is the matrix element of t_{Ω}^{cl}

$$t_{\Omega}^{cl} = t_{\perp} + it_{//}. \tag{2.43}$$

2.6 The Clocked Schroedinger Equation

The controversy over the problem of determining the duration time t_{Ω}^{cl} (also known as the traversal time τ) a quantum particle spends in a specified region of space Ω continues despite numerous attempts to obtain solution(for review. see [12]). So far. various methods including analysis of the time evolution of the wave function and coupling particle' s motion to additional degrees of freedom have failed to provide a definitive answer. Other authors have invoked alternative interpretations of the quantum theory. such as Bohm's theory [27], implying that the required insight into particle's behavior cannot be gained within the conventional quantum mechanics. In the classical Brownian motion. a similar problem can be solved by introducing the generalized ("*clocked*") diffusion equation in both the particle's position x and the traversal time variable τ for the restricted Wiener integral [38]. In the quantum case. a formal analogy lead to "***the** clocked Schroedinger equation* (the clocked SE)" [19]. Sokolovski [19] has claimed that the clocked SE can be derived only from the Feynman path integral. The physical interpretation of the clocked SE is an equation for determining $\Psi(x.t\,|\tau)$. that is the probability amplitude for the particle in x to have spent in the region $\Omega \equiv [a.b]$ (for one-dimension problem) prior to time t. a net duration τ :

$$i\hbar\frac{\partial}{\partial t}\Psi(x.t\,|\tau) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x.t\,|\tau) + V(x)\Psi(x.t\,|\tau) - i\hbar\Theta_{ab}[x]\frac{\partial}{\partial\tau}\Psi(x.t\,|\tau)\,. \tag{2.44}$$

where $\Theta_{ab}[x] = 1$ for $a \leq x \leq b$ and 0 otherwise. Eq.(2.44) conserves the total probability

$$N = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\tau \, |\Psi(x, t\,|\tau)|^2 \,. \tag{2.45}$$

In general, this method was analyzed by Sokolovski in 1998. He analyzed a quantum observable f given by the time average of a dynamical quantity $F[x, \dot{x}]$ along a Feynman path. The time average of a quantity $F[x, \dot{x}]$ (which $\dot{x}$ is the velocity) along a path $x(t)$, $\langle F \rangle_T$. is given by the functional

$$\langle F \rangle_T \equiv f[x(t)] = T^{-1} \int_0^t dt F[x(t), \ \dot{x}(t)] \text{ with } T = t - 0. \tag{2.46}$$

The amplitude at which a particle starting at x' and $t = 0$ and ending at x and t has the value of $f[x(t)]$ exactly to f given by the restricted path integral,

$$K(x, x' \colon t\,|f) = \sum_{over\ all\ paths} \delta(f[x(t)] - f) e^{\frac{i}{\hbar} S[x(t)]}. \tag{2.47}$$

It is the sum over the paths in the class of paths for which $f[x(t)] = f$, with $S[x(t)]$ being the action. Writing the Dirac delta function δ as a Fourier transform of $\delta(f[x(t)] - f)\}$ yields a propagator for the modified action S_λ,

$$S_\lambda[x(t)] = S[x(t)] - \lambda f[x(t)]. \tag{2.48}$$

We have

$$K(x, x' \colon t\,|f) = \int_{-\infty}^{\infty} d\lambda e^{\frac{i}{\hbar}\lambda f} K(x, x' \colon t\,|\lambda) \tag{2.49}$$

where

$$K(x, x' \colon t\,|\lambda) = \sum_{over\ all\ paths} e^{\frac{i}{\hbar} S_\lambda[x(t)]}. \tag{2.50}$$

Eq.(2.50) is the Feynman propagator for the composite potential $V(x) + \lambda F[x]$, so that, the propagator in Eq.(2.50) satisfies

$$i\hbar \frac{\partial}{\partial t} K(x, x' \colon t\,|\lambda) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} K(x, x' \colon t\,|\lambda) + V(x) K(x, x' \colon t\,|\lambda) + T^{-1}\lambda F[x] K(x, x' \colon t\,|\lambda) \,. \tag{2.51}$$

Suppose that $\Psi(x,t\,|f)$ is the probability amplitude of the particle located in x of having a *history* such that the value of $F[x(t)]$ is equal to f. Then we have

$$\Psi(x,t\,|f) = \int_{-\infty}^{\infty} dx' K(x,x';t\,|f)\,\Psi_i(x,0) \tag{2.52}$$

where $\Psi_i(x,0)$ is an initial state at time $t=0$. Combining Eqs.(2.52), (2.49) and (2.51) we arrive at the generalized SE for $\Psi(x,t\,|f)$,

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t\,|f) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t\,|f) + V(x)\Psi(x,t\,|f) - i\hbar T^{-1}F[x]\frac{\partial}{\partial f}\Psi(x,t\,|f)\,. \tag{2.53}$$

with the usual initial condition

$$\Psi(x,t=0\,|f) = \Psi(x,0)\delta(f) \tag{2.54}$$

where $\delta(x)$ is the Dirac delta function. For the traversal time, the functional $F[x(t)]$ becomes $\Theta_{ab}[x(t)]$ and Eq.(2.53) leads to

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t\,|\tau) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t\,|\tau) + V(x)\Psi(x,t\,|\tau) - i\hbar\Theta_{ab}[x]\frac{\partial}{\partial \tau}\Psi(x,t\,|\tau)\,. \tag{2.55}$$

Sokolovski stated that "in nonrelativistic quantum mechanics, determination of the value f (or τ) of a physical quantity requires expansion of the wave function $\Psi(x,t\,|\tau)$ in terms of the eigenfunctions of the corresponding operator $\mathbf{F}$. This recipe fails, however, if the quantity is defined, not at a given instant, but rather over a period of time. One such example is the much studied tunneling time." He believed that Eq.(2.55) is beyond the SE, so that he called it the generalized Schroedinger equation (the GSE). The GSE can be reduced to the SE by integrating over f (or τ) when $\Psi(x,t\,|\tau)$ is considered as vanishing for $|f|\to\infty$ (or $|\tau|\to\infty$):

$$i\hbar\int_{-\infty}^{\infty} df\,\frac{\partial}{\partial t}\Psi(x,t\,|f) = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} df\,\frac{\partial^2}{\partial x^2}\Psi(x,t\,|f) + V(x)\int_{-\infty}^{\infty} df\,\Psi(x,t\,|f)$$
$$-i\hbar T^{-1}F[x]\int_{-\infty}^{\infty} df\,\frac{\partial}{\partial f}\Psi(x,t\,|f)$$

$$i\hbar \int_{-\infty}^{\infty} df\ \frac{\partial}{\partial t}\Psi(x,t\,|f) = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} df \frac{\partial^2}{\partial x^2}\Psi(x,t\,|f) + V(x)\int_{-\infty}^{\infty} df\,\Psi(x,t\,|f)$$
$$-i\hbar T^{-1}F[x]\left[\Psi(x,t\,|\infty) - \Psi(x,t\,|-\infty)\right] \tag{2.56}$$

and summation of all values of f (which for definiteness we now take to be a single real variable) must erase all information about the particle's past and restore $\Psi(x,t)$.

$$\Psi(x,t) = \int_{-\infty}^{\infty} df\,\Psi(x,t\,|f)\,. \tag{2.57}$$

So we get the SE

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t). \tag{2.58}$$