# CHAPTER II

# INTERVAL SEMIGROUPS OF REAL NUMBERS

From Theorem 1.7 and the cancellation property of $\boldsymbol{R} - \{0\}$ under usual multiplication, we have that all the regular multiplicative interval semigroups on $\boldsymbol{R}$ are $\boldsymbol{R}$, $\{0\}$, $\{1\}$, $(0, \infty)$ and $[0, \infty)$. Then by Theorem 1.1, all of them belong to $\boldsymbol{BQ}$. The first purpose of this chapter is to show that these five intervals are the only multiplicative interval semigroups on $\boldsymbol{R}$ which belong to $\boldsymbol{BQ}$.

We also have from Theorem 1.8 and the cancellation property of $\boldsymbol{R}$ under usual addition that all the regular additive interval semigroups on $\boldsymbol{R}$ are only $\boldsymbol{R}$ and $\{0\}$. $\boldsymbol{R}$ and $\{0\}$ are shown to be the only additive interval semigroups on $\boldsymbol{R}$ which belong to $\boldsymbol{BQ}$. This is the second purpose of this chapter.

**Theorem 2.1.** *For a multiplicative interval semigroup $S$ on $\boldsymbol{R}$, $S \in \boldsymbol{BQ}$ if and only if $S$ is one of the following intervals: $\boldsymbol{R}$, $\{0\}$, $\{1\}$, $(0, \infty)$ and $[0, \infty)$.*

**Proof.** As mentioned above, if $S$ is $\boldsymbol{R}$, $\{0\}$, $\{1\}$, $(0, \infty)$ or $[0, \infty)$, then $S \in \boldsymbol{BQ}$.

Next, we recall from Theorem 1.7 that these are all types of multiplicative interval semigroups on $\boldsymbol{R}$ .

(1) $\boldsymbol{R}$, (2) $\{0\}$, (3) $\{1\}$, (4) $(0, \infty)$, (5) $[0, \infty)$,

(6) $(a, \infty)$ where $a \geq 1$,

(7) $[a, \infty)$ where $a \geq 1$,

(8) $(0, b)$ where $0 < b \leq 1$,

(9) $(0, b]$ where $0 < b \leq 1$,

(10) $[0, b)$ where $0 < b \leq 1$,

(11) $[0, b]$ where $0 < b \leq 1$,

(12) $(a, b)$ where $-1 \leq a < 0 < a^2 \leq b \leq 1$,

(13) $(a, b]$ where $-1 \leq a < 0 < a^2 \leq b \leq 1$,

(14) $[a, b)$ where $-1 < a < 0 < a^2 < b \leq 1$,

(15) $[a, b]$ where $-1 \leq a < 0 < a^2 \leq b \leq 1$.

To prove the converse, it is equivalent to show that none of the multiplicative interval semigroups of types (6) - (15) belongs to $\boldsymbol{BQ}$.

**Case 1:** $S$ is of type (6) or (7). Then there exists $a \in \boldsymbol{R}$ such that $a \geq 1$ and $S = (a, \infty)$ or $[a, \infty)$. Let

$$B = (a + 1, a + 2) \cup ((a + 1)^2, \infty).$$

Then $B \subseteq S$ and $B^2 \subseteq ((a + 1)^2, \infty)$. Thus $S^1B^2 \subseteq (a(a + 1)^2, \infty)$. Since $a \geq 1$, $a(a + 1)^2 \geq (a + 1)^2$, so $(a(a + 1)^2, \infty) \subseteq ((a + 1)^2, \infty)$ which implies that $S^1B^2 \subseteq B$. Hence $B$ is a bi-ideal of $S$. Since $a + \frac{1}{2} \in S$ and $a + \frac{3}{2} \in B$, $(a + \frac{1}{2})(a + \frac{3}{2}) \in SB$. But

$$(a + \frac{1}{2})(a + \frac{3}{2}) = a^2 + 2a + \frac{3}{4} < a^2 + 2a + 1 = (a + 1)^2$$

and

$$a + 2 \leq a(a + 2) = a^2 + 2a < a^2 + 2a + \frac{3}{4} = (a + \frac{1}{2})(a + \frac{3}{2})$$

since $a \geq 1$, so $(a + \frac{1}{2})(a + \frac{3}{2}) \notin B$. Thus $SB \not\subseteq B$. Hence $B$ is a bi-ideal of $S$ which is not a quasi-ideal. Therefore $S \notin \boldsymbol{BQ}$.

**Case 2:** $S$ is of type (8), (9), (10) or (11). Then there exists $b \in \boldsymbol{R}$ such that $0 < b \leq 1$ and $S = (0, b)$, $(0, b]$, $[0, b)$ or $[0, b]$. Let

$$B = (0, \frac{b^2}{4}) \cup (\frac{b}{4}, \frac{b}{2}).$$

Then $B \subseteq S$ and $B^2 \subseteq (0, \frac{b^2}{4}) \subseteq B$. Then $SB^2 \subseteq (0, \frac{b^3}{4})$. Since $0 < b \leq 1$, $\frac{b^3}{4} \leq \frac{b^2}{4}$, so $SB^2 \subseteq (0, \frac{b^2}{4}) \subseteq$ B. Then $S^1B^2 = SB^2 \cup B^2 \subseteq B$. Therefore $B$ is a bi-ideal of $S$. Since $\frac{b}{4} < \frac{3b}{8} < \frac{b}{2}$, $\frac{3b}{8} \in B$. But $\frac{2b}{3} \in S$, so $\frac{b^2}{4} = (\frac{2b}{3})(\frac{3b}{8}) \in SB$. This implies that $SB \not\subseteq B$ since $\frac{b^2}{4} \notin B$. Hence $S \notin \boldsymbol{BQ}$.

**Case 3:** $S$ is of type (12), (13), (14) or (15). Then there exist $a,b \in \boldsymbol{R}$ such that

(i) $S = (a, b)$, $(a, b]$ or $[a, b]$ and $-1 \leq a < 0 < a^2 \leq b \leq 1$, or

(ii) $S = [a, b)$ and $-1 < a < 0 < a^2 < b \leq 1$.

Set

$$B = (\frac{a}{2}, \frac{a^2}{4}) \cup (\frac{a^2}{3}, \frac{a^2}{2}).$$

Then $B \subseteq S$ and $B^2 \subseteq (\frac{a^3}{4}, \frac{a^2}{4})$. Since $b \leq 1$ and $a < 0$, we have $\frac{a^3}{4} \leq \frac{a^3 b}{4}$ $< 0$. Also, $0 < \frac{a^4}{4} \leq \frac{a^2 b}{4}$ since $a^2 \leq b \leq 1$. These implies that $[a, b](\frac{a^3}{4}, \frac{a^2}{4}) \subseteq (\frac{a^3}{4}, \frac{a^2 b}{4}) \subseteq (\frac{a^3}{4}, \frac{a^2}{4})$. Now we have $B^2 \subseteq (\frac{a^3}{4}, \frac{a^2}{4})$ and $[a, b](\frac{a^3}{4}, \frac{a^2}{4}) \subseteq (\frac{a^3}{4}, \frac{a^2}{4})$. It follows that $S^1B^2 = SB^2 \cup B^2 \subseteq (\frac{a^3}{4}, \frac{a^2}{4})$. But $\frac{a}{2} < \frac{a^3}{4} < 0$, so $(\frac{a^3}{4}, \frac{a^2}{4}) \subseteq B$. Hence $S^1B^2 \subseteq B$. Therefore $B$ is a bi-ideal of $S$. Since $\frac{a}{2} < \frac{a}{3}$ $< 0$ and $a < \frac{3a}{4} < 0$, we have that $\frac{a}{3} \in B$ and $\frac{3a}{4} \in S$, so $\frac{a^2}{4} = (\frac{3a}{4})(\frac{a}{3})$ $\in SB - B$. Therefore $B$ is not a quasi-ideal of $S$. Hence $S \notin \boldsymbol{BQ}$. □

**Theorem 2.2** *For an additive interval semigroup $S$ on $\boldsymbol{R}$, $S \in \boldsymbol{BQ}$ if and only if $S = \boldsymbol{R}$ or $S = \{0\}$.*

**Proof.** We recall from Theorem 1.8 that these are all types of additive interval semigroups on $\boldsymbol{R}$.

(1) $\{0\}$, (2) $\boldsymbol{R}$,

(3) $(a, \infty)$ where $a \geq 0$, (4) $[a, \infty)$ where $a \geq 0$,

(5) $(-\infty, b)$ where $b \leq 0$, (6) $(-\infty, b]$ where $b \leq 0$.

As mentioned above, to prove the theorem, it suffices to show that none of the additive interval semigroups on $\boldsymbol{R}$ of types (3) - (6) belongs to $\boldsymbol{BQ}$.

**Case 1:** $S$ is of type (3) or (4). Then $S = (a, \infty)$ or $[a, \infty)$ for some $a \in \boldsymbol{R}$ such that $a \geq 0$. Set

$$B = (a + 2, a + 3] \cup (2a + 4, \infty).$$

Then $B + B \subseteq (2a + 4, \infty) \subseteq B$ and $S + B + B \subseteq (3a + 4, \infty) \subseteq B$. Thus $(S + B + B) \cup (B + B) \subseteq B$, so $B$ is a bi-ideal of $S$. Since $a + 1 \in S$ and $a + 3 \in B$, $2a + 4 = (a + 1) + (a + 3) \in S + B$. But $2a + 4 \notin B$, so $B$ is not a quasi-ideal of $S$. Therefore $S \notin \boldsymbol{BQ}$.

**Case 2:** $S$ is of type (5) or (6). Then $S = (-\infty, b)$ or $(-\infty, b]$ for some $b \in \boldsymbol{R}$ such that $b \leq 0$. Let

$$B = (-\infty, 2b - 4) \cup [b - 3, b - 2).$$

Then $B + B \subseteq (-\infty, 2b - 4) \subseteq B$ and $S + B + B \subseteq (-\infty, 3b - 4) \subseteq B$ since $b \leq 0$, so $B$ is a bi-ideal of $S$. Since $2b - 4 = (b - 1) + (b - 3) \in S + B$ and $2b - 4 \notin B$, this shows that $B$ is not a quasi-ideal of $S$. Hence $S \notin \boldsymbol{BQ}$. □