CHAPTER IV

TRANSFORMATION SEMIGROUPS

The notation P_X, T_X, I_X, M_X, E_X and G_X where X is a set and the notation C_I and D_I where I is any interval on $\boldsymbol{R}$ with $|I|>1$ are recalled from Chapter I, page 6. Again, since P_X, T_X, I_X and G_X are regular for any set X, we deduce from Theorem 1.1 that they are members of $\boldsymbol{BQ}$ for any set X. We know that $M_X = E_X = G_X$ if and only if X is finite. Then for any finite set X, M_X and E_X belong to $\boldsymbol{BQ}$. We show in this chapter that $M_X \notin \boldsymbol{BQ}$ and $E_X \notin \boldsymbol{BQ}$ if X is infinite. This is the first purpose. Consequently, by Theorem 1.1and Theorem 1.2, both M_X and E_X are not regular, not left simple and not right simple if X is infinite. In fact, these can be proved directly. The second purpose is to show that $C_I \notin \boldsymbol{BQ}$ and $D_I \notin \boldsymbol{BQ}$ for any nonempty interval I with $|I| > 1$. Observe that all of these semigroups have an identity.

First, we shall introduce a direct proof that both M_X and E_X are not regular for the case that X is an infinite set. Since X is infinite, X contains a countably infinite subset $\{a_1, a_2, a_3, \ldots\}$ where $a_i \neq a_j$ if $i \neq j$. Define $\alpha, \beta : X \to X$ by

$$x\alpha = \begin{cases} a_{2n} & \text{if } x = a_n \text{ for some } n \in N, \\ x & \text{otherwise,} \end{cases}$$

$$x\beta = \begin{cases} a_{\frac{n}{2}} & \text{if } x = a_n \text{ for some even integer } n \in N, \\ a_1 & \text{if } x = a_n \text{ for some odd integer } n \in N, \\ x & \text{otherwise .} \end{cases}$$

Then $\alpha \in M_X$ and $\beta \in E_X$. Suppose there exists $\lambda \in M_X$ such that $\alpha = \alpha\lambda\alpha$. Then for $n \in \boldsymbol{N}$, $a_n\alpha = a_n\alpha\lambda\alpha = (a_{2n}\lambda)\alpha$. Since α is one-to-one, $a_{2n}\lambda = a_n$ for every $n \in \boldsymbol{N}$. Thus $\{a_2, a_4, a_6, \ldots\}\lambda = \{a_1, a_2, a_3, \ldots\}$. If $x \in X - \{a_1, a_2, a_3, \ldots\}$, then $x\alpha =$

$x\alpha\lambda\alpha = (x\lambda)\alpha$, so $x\lambda = x$. This proves that $(X - \{a_1, a_3, a_5, \ldots\})\lambda = X$. Consequently, λ is not one-to-one, a contradiction. Hence M_X is not regular.

Next, suppose that there exists $\mu \in E_X$ such that $\beta = \beta\mu\beta$. Then for every odd integer $n \in N$, $a_1 = a_n\beta = a_n\beta\mu\beta = (a_1\mu)\beta$ which implies that $a_1\mu \in \{a_2\} \cup \{a_1, a_3, a_5, \ldots\}$. If $n \in N - \{1\}$, $a_n = a_{2n}\beta = a_{2n}\beta\mu\beta = (a_n\mu)\beta$, so by the definition of β, $a_n\mu = a_{2n}$. If $x \in X - \{a_1, a_2, a_3, \ldots\}$, then $x = x\beta = x\beta\mu\beta = (x\mu)\beta$ which implies that $x\mu = x$. This proves that $X\mu = \{a_1\mu\} \cup \{a_4, a_6, a_8, \ldots\} \cup (X - \{a_1, a_2, a_3, \ldots\})$. Since $a_1\mu$ is only one value in $\{a_2\} \cup \{a_1, a_3, a_5, \ldots\}$, it follows that $X\mu \neq X$, a contradiction. Hence E_X is not regular.

If X is an infinite set , it can be easily proved that both M_X and E_X are neither left simple nor right simple because $M_X - G_X$ and $E_X - G_X$ are ideals of M_X and E_X , respectively . To prove these , let $\alpha \in M_X$, $\beta \in E_X$, $\gamma \in M_X - G_X$ and $\lambda \in E_X - G_X$. Then $\alpha\gamma$, $\gamma\alpha \in M_X$ and $\beta\lambda$, $\lambda\beta \in E_X$. Since $\mathrm{Im}\alpha\gamma \subseteq \mathrm{Im}\gamma \neq X$, $\alpha\gamma \in M_X - G_X$. If $\gamma\alpha \in G_X$, then $\mathrm{Im}\alpha \supseteq \mathrm{Im}\gamma\alpha = X$, so $\alpha \in G_X$ which implies that $\gamma = (\gamma\alpha)\alpha^{-1} \in G_X$, a contradiction . Thus $\gamma\alpha \in M_X - G_X$. Since λ is not one-to-one , $\lambda\beta$ is not one-to-one , so $\lambda\beta \in E_X - G_X$. If $\beta\lambda \in G_X$, then β is one-to-one , so $\beta \in G_X$ which implies that $\lambda = \beta^{-1}(\beta\lambda) \in G_X$, a contradiction . Hence $\beta\lambda \in E_X - G_X$.

Theorem 4.1. *For a set X, $M_X \in$ **BQ** if and only if X is finite.*

Proof. As mentioned above, $M_X \in$ **BQ** if X is finite.

For the converse, assume that X is an infinte set. Then X contains a countably infinite subset. Let $\{a_i \mid i \in N\} \subseteq X$ where $a_i \neq a_j$ if $i \neq j$.

Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a_{2n} & \text{if } x = a_n \text{ for some } n \in N, \\ x & \text{otherwise} . \end{cases}$$

$$x\beta = \begin{cases} a_{n+2} & \text{if } x = a_n \text{ for some } n \in N, \\ x & \text{otherwise} \end{cases}$$

and

$$x\gamma = \begin{cases} a_{n+1} & \text{if } x = a_n \text{ for some } n \in N, \\ x & \text{otherwise .} \end{cases}$$

Then $\beta, \gamma \in M_X$. For $n \in N$, we have that $a_n\alpha\beta = a_{2n}\beta = a_{2n+2} = a_{2(n+1)} = a_{n+1}\alpha = a_n\gamma\alpha$. If $x \in X - \{a_1, a_2, a_3, \ldots\}$, then $x\alpha\beta = x = x\gamma\alpha$. This proves that $\alpha\beta = \gamma\alpha$. Therefore $\alpha\beta = \gamma\alpha \in \alpha M_X \cap M_X\alpha$. Let $\lambda = \alpha\beta$. Then $\lambda \in \alpha M_X \cap M_X\alpha \subseteq BM_X \cap M_XB$ since $\alpha \in B$. To show that $\lambda \notin B$. Suppose that $\lambda \in B$. Then $\lambda \in \alpha M_X\alpha \cup \{\alpha\}$.Since $a_1\lambda = a_1\alpha\beta = a_2\beta = a_4 \neq a_2 = a_1\alpha$, we have $\lambda \neq \alpha$. Then $\lambda \in \alpha M_X\alpha$, so $\lambda = \alpha\eta\alpha$ for some $\eta \in M_X$. It follows that for $n \in N$, $a_{n+1}\alpha = a_n\gamma\alpha = a_n\alpha\beta = a_n\lambda = a_n\alpha\eta\alpha = (a_{2n}\eta)\alpha$ which implies that $a_{2n}\eta = a_{n+1}$ for all $n \in N$ because α is one-to-one. Consequently,

$$\{a_2, a_4, a_6, \ldots\}\eta = \{a_2, a_3, a_4, \ldots\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

If $x \in X - \{a_1, a_2, a_3, \ldots\}$, then $x\alpha = x = x\alpha\beta = x\alpha\eta\alpha = (x\eta)\alpha$ which implies that $x = x\eta$ since α is one-to-one. Thus

$$(X - \{a_1, a_2, a_3, \ldots\})\eta = X - \{a_1, a_2, a_3, \ldots\}. \quad \ldots\ldots\ldots\ldots(2)$$

From (1) and (2), we have

$$(X - \{a_1, a_3, a_5, \ldots\})\eta = X - \{a_1\}$$

which is impossible since η is one-to-one. Hence $\lambda \notin B$, so $\lambda \in (BM_X \cap M_XB) - B$. Thus $BM_X \cap M_XB \not\subseteq B$. Now we have that B is a bi-ideal but not a quasi-ideal of M_X. Therefore $M_X \notin \boldsymbol{BQ}$.

Hence the theorem is completely proved. □

Theorem 4.2. *For a set X,* $E_X \in \boldsymbol{BQ}$ *if and only if* X *is finite.*

Proof. As mentioned previously, $E_X \in \boldsymbol{BQ}$ if X is a finite set.

Conversely, assume that X is an infinite set. Let $\{a_i \mid i \in N\} \subseteq X$ where $a_i \neq a_j$ if $i \neq j$. Define $\alpha, \beta, \gamma : X \to X$ by

$$x\alpha = \begin{cases} a_{\frac{n}{2}} & \text{if } x = a_n \text{ and } n \text{ is even}, \\ a_1 & \text{if } x = a_n \text{ and } n \text{ is odd}, \\ x & \text{otherwise}, \end{cases}$$

$$x\beta = \begin{cases} a_{n-1} & \text{if } x = a_n \text{ for } n \in N - \{1\}, \\ x & \text{otherwise} \end{cases}$$

and

$$x\gamma = \begin{cases} a_{n-2} & \text{if } x = a_n \text{ for } n \in N - \{1,2\}, \\ a_1 & \text{if } x = a_1 \text{ or } x = a_2, \\ x & \text{otherwise}. \end{cases}$$

Since $\operatorname{Im}\alpha = \operatorname{Im}\beta = \operatorname{Im}\gamma = X$, $\alpha, \beta, \gamma \in E_X$. Set $B = (\alpha)_b$, then $B = \alpha E_X \alpha \cup \{\alpha\}$. Next, we shall show that $\alpha\beta = \gamma\alpha$. Since $x\alpha = x\beta = x\gamma = x$ for all $x \in X - \{a_1, a_2, a_3, \ldots\}$, it follows that $x\alpha\beta = x = x\gamma\alpha$ for all $x \in X - \{a_1, a_2, a_3, \ldots\}$. Next, to show that $a_n\alpha\beta = a_n\gamma\alpha$ for all $n \in N$, let n be given.

Case 1: $n = 1$. Then $a_n\alpha\beta = a_1\alpha\beta = a_1\beta = a_1 = a_1\alpha = a_1\gamma\alpha = a_n\gamma\alpha$.

Case 2: $n = 2$. Then $a_n\alpha\beta = a_2\alpha\beta = a_1\beta = a_1 = a_1\alpha = a_2\gamma\alpha = a_n\gamma\alpha$.

Case 3: $n = 3$. Then $a_n\alpha\beta = a_3\alpha\beta = a_1\beta = a_1 = a_1\alpha = a_3\gamma\alpha = a_n\gamma\alpha$.

Case 4: $n \geq 4$ and n is even. Then $\frac{n}{2} \in N - \{1\}$, $n - 2 \in N$ and n - 2 is even, so

$a_n\alpha\beta = a_{\frac{n}{2}}\beta = a_{\frac{n}{2}-1} = a_{\frac{n-2}{2}} = a_{n-2}\alpha = a_n\gamma\alpha$.

Case 5: $n \geq 4$ and n is odd. Then $n - 2 \in N$ and n - 2 is odd, so $a_n\alpha\beta = a_1\beta = a_1 = a_{n-2}\alpha = a_n\gamma\alpha$.

Now we have $\alpha\beta = \gamma\alpha$. Let $\lambda = \alpha\beta$. Then $\lambda \in \alpha E_X \cap E_X\alpha \subseteq BE_X \cap E_XB$. Since $a_4\alpha\beta = a_2\beta = a_1 \neq a_2 = a_4\alpha$, we have that $\lambda \neq \alpha$. Suppose that $\lambda \in \alpha E_X\alpha$. Then $\lambda = \alpha\eta\alpha$ for some $\eta \in E_X$. If $x \in X - \{a_1, a_2, a_3, \ldots\}$, then $x = x\alpha\beta = x\alpha\eta\alpha = (x\eta)\alpha$, so by the definition of α, we have $x\eta = x$. Thus

$$(X - \{a_1, a_2, a_3, \ldots\})\eta = X - \{a_1, a_2, a_3, \ldots\}. \quad \ldots\ldots\ldots\ldots(1)$$

By the definition of α, we have that for $k \in N - \{1\}$ and $x \in X$, $x\alpha = a_k$ implies that $x = a_{2k}$. But for $n \in N - \{1, 2\}$, $a_{n-1} = a_n\beta = a_{2n}\alpha\beta = a_{2n}\alpha\eta\alpha = (a_n\eta)\alpha$, so $a_n\eta = a_{2(n-1)}$ for all $n \in N - \{1, 2\}$. It follows that

$$(\{a_3, a_4, a_5, \ldots\})\eta = \{a_4, a_6, a_8, \ldots\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (1) and (2), we have

$$(X - \{a_1, a_2\})\eta = X - \{a_1, a_2, a_3, a_5, a_7, \ldots\}$$

which is impossible since $\eta : X \to X$ and $\text{Im}\,\eta = X$. This proves that $\lambda \notin \alpha E_X\alpha$. Then $\lambda \notin \alpha E_X\alpha \cup \{\alpha\}$ since $\lambda \neq \alpha$. But $B = \alpha E_X\alpha \cup \{\alpha\}$, so $\lambda \notin B$. Because $\lambda \in BE_X \cap E_XB$, we get $\lambda \in (BE_X \cap E_XB) - B$. Thus B is not a quasi-ideal of E_X. Consequently, $E_X \notin \boldsymbol{BQ}$.

Hence we prove that $E_X \in \boldsymbol{BQ}$ if and only if X is finite. □

To prove that C_I and $D_I \notin \boldsymbol{BQ}$ for any interval I on $\boldsymbol{R}$ with $|I| > 1$, we first note that for such an interval I, there exist $a, b \in I$ such that $a < b$ which implies that $[a,b] \subseteq I$ since I is an interval.

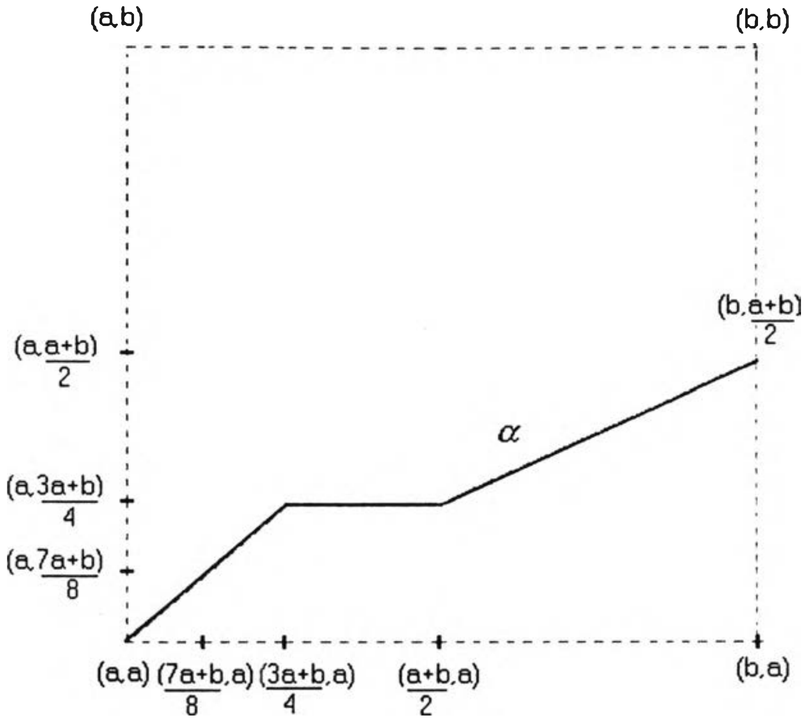
For any map α and $A \subseteq \operatorname{Dom}\alpha$, let $\alpha|_A$ denote the restriction of α to A.

Theorem 4.3. *For an interval I on $\boldsymbol{R}$ with $|I| > 1$, $C_I \notin \boldsymbol{BQ}$.*

Proof. Assume that I is an interval on $\boldsymbol{R}$ and $|I| > 1$. To show that $C_I \notin \boldsymbol{BQ}$, let $a, b \in I$ be such that $a < b$. Then $[a, b] \subseteq I$. Since $\frac{a+b}{2}$ is the middle point of $[a, b]$, $\frac{a+b}{2} \in I$. Also $\frac{3a+b}{4} \in I$ since $\frac{3a+b}{4} = \frac{1}{2}(a + \frac{a+b}{2})$ which is the middle point of $[a, \frac{a+b}{2}]$ and $[a, \frac{a+b}{2}] \subseteq [a, b] \subseteq I$. If $x \in \boldsymbol{R}$ is such that $\frac{a+b}{2} < x \leq b$, then $\frac{3a+b}{2} = a + \frac{a+b}{2} < a+x \leq a+b$ and so $\frac{3a+b}{4} < \frac{a+x}{2} \leq \frac{a+b}{2}$. Therefore we have $\frac{a+x}{2} \in I$ for every $x \in \boldsymbol{R}$ such that $\frac{a+b}{2} < x \leq b$. Next, we define $\alpha : I \to I$ by

$$x\alpha = \begin{cases} x & \text{if } x \leq \frac{3a+b}{4}, \\ \frac{3a+b}{4} & \text{if } \frac{3a+b}{4} < x \leq \frac{a+b}{2}, \\ \frac{a+x}{2} & \text{if } \frac{a+b}{2} < x \leq b, \\ \frac{a+b}{2} & \text{if } x > b. \end{cases}$$

Then α is continuous on I, so $\alpha \in C_I$. The graph of α on $[a, b]$ can be given as follows:



Since $\frac{7a+b}{8} = \frac{1}{2}(a + \frac{3a+b}{4})$, $\frac{7a+b}{8}$ is the middle point of $[a, \frac{3a+b}{4}]$.

From the graph of α, we have

$$x\alpha^{-1} = \{x\} \text{ for all } x \in (\frac{7a+b}{8}, \frac{3a+b}{4}) \text{ . } \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

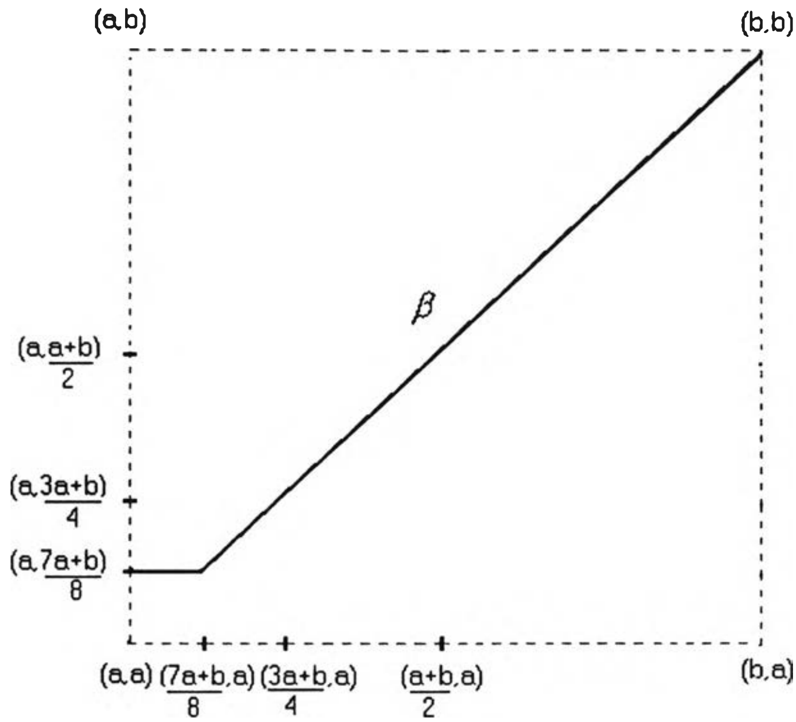
and

$$x\alpha^{-1} = \{2x-a\} \text{ for all } x \in (\frac{3a+b}{4}, \frac{a+b}{2}). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Set $B = (\alpha)_b$. Then $B = \alpha C_I \alpha \cup \{\alpha\}$. To show that B is not a quasi-ideal of C_I, define $\beta : I \to I$ by

$$x\beta = \begin{cases} \frac{7a+b}{8} & \text{if } x \leq \frac{7a+b}{8}, \\ x & \text{if } x > \frac{7a+b}{8}. \end{cases}$$

Then β is continuous on I , so $\beta \in C_I$ .We can give the graph of β on $[a, b]$ as follows :



We claim that $\alpha\beta = \beta\alpha$. To prove the claim, we first note that

$$x\alpha = x \text{ for all } x \in I \cap (-\infty, \frac{3a+b}{4}]. \qquad \text{......(3)}$$

$$(I \cap [\frac{7a+b}{8}, \infty))\alpha \subseteq I \cap [\frac{7a+b}{8}, \infty). \qquad \text{......(4)}$$

$$(I \cap (-\infty, \frac{7a+b}{8}])\beta = \{\frac{7a+b}{8}\}. \qquad \text{......(5)}$$

and

$$x\beta = x \text{ for all } x \in I \cap [\frac{7a+b}{8}, \infty). \qquad \text{......(6)}$$

Now we are ready to show that $\alpha\beta = \beta\alpha$. Let $x \in I$.

Case 1: $x \in I \cap (-\infty, \frac{7a+b}{8})$. From (3) and (5), we have $x\alpha\beta = x\beta = \frac{7a+b}{8}$ and $x\beta\alpha = (\frac{7a+b}{8})\alpha = \frac{7a+b}{8}$, so $x\alpha\beta = \frac{7a+b}{8} = x\beta\alpha$.

Case 2: $x \in I \cap [\frac{7a+b}{8}, \infty)$. From (4) and (6), $x\alpha\beta = x\alpha$. From (6), $x\beta\alpha = x\alpha$. Then $x\alpha\beta = x\alpha = x\beta\alpha$.

Now we have $\alpha\beta = \beta\alpha$. Let $\lambda = \alpha\beta$. Then $\lambda = \alpha\beta = \beta\alpha \in \alpha C_I \cap C_I\alpha \subseteq BC_I \cap C_IB$. From (3) and (5), we have $a\lambda = a\alpha\beta = a\beta = \frac{7a+b}{8} \neq a = a\alpha$, so $\lambda \neq \alpha$. To show that $\lambda \notin \alpha C_I\alpha$, suppose that $\lambda \in \alpha C_I\alpha$. Then $\lambda = \alpha\eta\alpha$ for some $\eta \in C_I$, so $\alpha\beta = \alpha\eta\alpha$. From (3) and (6), we have that for every $x \in (\frac{7a+b}{8}, \frac{3a+b}{4})$, $x\alpha\beta = x\beta = x$ and $x\alpha\eta\alpha = (x\eta)\alpha$. It follows from (1) that

$$x\eta = x \text{ for all } x \in (\frac{7a+b}{8}, \frac{3a+b}{4}). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(*)$$

Next, let $y \in (\frac{3a+b}{4}, \frac{a+b}{2})$. From (2), $(2y-a)\alpha = y$ which implies that $(2y-a)\alpha\beta = y\beta = y$ by (6). Then $(2y-a)\lambda = y$ and $(2y-a)\alpha\eta\alpha = (y\eta)\alpha$. It follows that $(y\eta)\alpha = y$. From (2), $y\eta = 2y - a$.

This proves that

$$x\eta = 2x - a \text{ for all } x \in (\frac{3a+b}{4}, \frac{a+b}{2}). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(**)$$

From (*) and (**), we have that

$$\lim_{x \to (\frac{3a+b}{4})^-} x\eta = \frac{3a+b}{4}$$

and

$$\lim_{x \to (\frac{3a+b}{4})^+} x\eta = 2(\frac{3a+b}{4}) - a$$
$$= \frac{3a+b}{2} - a$$
$$= \frac{a+b}{2}.$$

Since $a \neq b$, $\frac{3a+b}{4} \neq \frac{a+b}{2}$ which implies that $\lim_{x \to (\frac{3a+b}{4})^+} x\eta \neq \lim_{x \to (\frac{3a+b}{4})^-} x\eta$.

Hence η is not continuous at $\frac{3a+b}{4}$ which is a contradiction. Therefore $\lambda \notin \alpha C_I\alpha$, so $\lambda \notin \alpha C_I\alpha \cup \{\alpha\} = B$. Now we have $\lambda \in (BC_I \cap C_IB) - B$. Hence B is not a quasi-ideal of C_I.

This proves that $C_I \notin \boldsymbol{BQ}$, as required. □

Theorem 4.4. *For any interval I on $\boldsymbol{R}$ with $|I| > 1$, $D_I \notin \boldsymbol{BQ}$.*

Proof. Let I be an interval on $\boldsymbol{R}$ and $|I| > 1$. Then all the possible types of I are as follows :

1. $I = \boldsymbol{R}$,
2. $I = [c, \infty)$ or (c, ∞) for some $c \in \boldsymbol{R}$,
3. $I = (-\infty, d]$ or $(-\infty, d)$ for some $d \in \boldsymbol{R}$

and

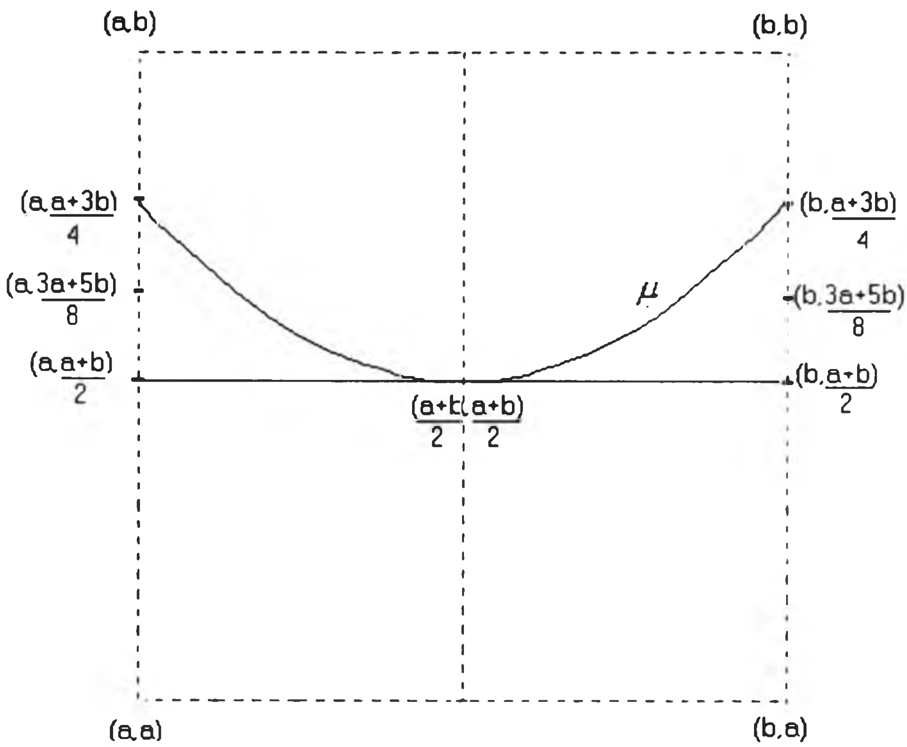
4. $I = [c, d], [c, d), (c, d]$ or (c, d) for some $c, d \in \boldsymbol{R}$ such that $c < d$.

Choose $a, b \in \boldsymbol{R}$ such that $a < b$ and a, b have the following special properties. If I is of type 1, a, b can be any points in I. If I is of type 2, choose $a = c$. If I is of type 3, choose $b = d$. If I is of type 4, choose $a = c$ and $b = d$. Then for every type of I, $(a, b) \subseteq I$.

Define $\mu : \boldsymbol{R} \to \boldsymbol{R}$ by

$$x\mu = \frac{1}{b-a}(x - \frac{a+b}{2})^2 + \frac{a+b}{2}$$

for all $x \in \boldsymbol{R}$. Then μ is differentiable on $\boldsymbol{R}$ and the graph of μ on $[a, b]$ is as follows :



Observe that μ is decreasing on $(-\infty, \frac{a+b}{2}]$ and increasing on $[\frac{a+b}{2}, \infty)$.

From the definition of μ, we have

$$\begin{aligned} b\mu = a\mu &= \frac{1}{b-a}(a-\frac{a+b}{2})^2+\frac{a+b}{2} \\ &= \frac{b-a}{4}+\frac{a+b}{2} \\ &= \frac{a+3b}{4}. \end{aligned}$$

Since $\frac{a+3b}{4}$ is the middle point of $[\frac{a+b}{2}, b]$, $a\mu = b\mu \in (a, b)$. It follows from what we choose a and b that $I\mu \subseteq I$ for every type of I. Let $\alpha = \mu|_I$. Then

$\alpha : I \to I$ and

$$x\alpha = \frac{1}{b-a}(x - \frac{a+b}{2})^2 + \frac{a+b}{2} \quad . \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots(1)$$

for all $x \in I$. Since μ is differentiable on $\boldsymbol{R}$, we have $\alpha \in D_I$. Set $B = (\alpha)_b$. Then $B = \alpha D_I \alpha \cup \{\alpha\}$. Let $\zeta_1, \zeta_2 : \boldsymbol{R} \to \boldsymbol{R}$ be the straight lines define by

$$x\zeta_1 = \frac{1}{2}(x - \frac{a+b}{2}) + \frac{a+b}{2}$$

and

$$x\zeta_2 = \frac{1}{4}(x - \frac{a+b}{2}) + \frac{a+b}{2}$$

for all $x \in \boldsymbol{R}$. Then ζ_1 and ζ_2 are increasing functions on $\boldsymbol{R}$ and

$$\begin{aligned} a\zeta_1 &= \frac{1}{2}(a - \frac{a+b}{2}) + \frac{a+b}{2} \\ &= \frac{1}{2}(\frac{a-b}{2}) + \frac{a+b}{2} \\ &= \frac{3a+b}{4} \quad , \end{aligned}$$

$$\begin{aligned} b\zeta_1 &= \frac{1}{2}(b - \frac{a+b}{2}) + \frac{a+b}{2} \\ &= \frac{1}{2}(\frac{b-a}{2}) + \frac{a+b}{2} \\ &= \frac{a+3b}{4} \quad , \end{aligned}$$

$$\begin{aligned} a\zeta_2 &= \frac{1}{4}(a - \frac{a+b}{2}) + \frac{a+b}{2} \\ &= \frac{1}{4}(\frac{a-b}{2}) + \frac{a+b}{2} \end{aligned}$$

$$= \frac{5a+3b}{8}$$

and

$$b\zeta_2 = \frac{1}{4}(b - \frac{a+b}{2}) + \frac{a+b}{2}$$

$$= \frac{1}{4}(\frac{b-a}{2}) + \frac{a+b}{2}$$

$$= \frac{3a+5b}{8} \, .$$

Since $\frac{3a+b}{4}$ is the middle point of $[a, \frac{a+b}{2}]$ and $\frac{a+3b}{4}$ is the middle point of $[\frac{a+b}{2}, b]$, it follows that $a\zeta_1, b\zeta_1 \in (a, b)$. We have that $\frac{5a+3b}{8}$ is the middle point of $[\frac{3a+b}{4}, \frac{a+b}{2}]$ and $\frac{3a+5b}{8}$ is the middle point of $[\frac{a+b}{2}, \frac{a+3b}{4}]$. It follows that $a\zeta_2$, $b\zeta_2 \in (a, b)$. Consequently, $I\zeta_1 \subseteq I$ and $I\zeta_2 \subseteq I$. Let $\beta = \zeta_1|_I$ and $\gamma = \zeta_2|_I$. Then $\beta, \gamma \in D_I$ and

$$x\beta = \frac{1}{2}(x - \frac{a+b}{2}) + \frac{a+b}{2}$$

and

$$x\gamma = \frac{1}{4}(x - \frac{a+b}{2}) + \frac{a+b}{2}$$

for all $x \in I$.

Next, to show that $\alpha\gamma = \beta\alpha$, let $x \in I$. Then

$$x\alpha\gamma = (\frac{1}{b-a}(x - \frac{a+b}{2})^2 + \frac{a+b}{2})\gamma$$

$$= \frac{1}{4}(\frac{1}{b-a}(x - \frac{a+b}{2})^2 + \frac{a+b}{2} - \frac{a+b}{2}) + \frac{a+b}{2}$$

$$= \frac{1}{4(b-a)}(x - \frac{a+b}{2})^2 + \frac{a+b}{2}$$

and

$$x\beta\alpha = (\frac{1}{2}(x - \frac{a+b}{2}) + \frac{a+b}{2})\alpha$$

$$= \frac{1}{b-a}(\frac{1}{2}(x - \frac{a+b}{2}) + \frac{a+b}{2} - \frac{a+b}{2})^2 + \frac{a+b}{2}$$

$$= \frac{1}{4(b-a)}(x - \frac{a+b}{2})^2 + \frac{a+b}{2} \ .$$

Hence $\alpha\gamma = \beta\alpha$. Let $\lambda = \alpha\gamma$. Then

$$x\lambda = \frac{1}{4(b-a)}(x - \frac{a+b}{2})^2 + \frac{a+b}{2} \qquad \text{..........................(2)}$$

for all $x \in I$ and $\lambda = \alpha\gamma = \beta\alpha \in \alpha D_I \cap D_I\alpha \subseteq BD_I \cap D_IB$. We have that $\lambda \neq \alpha$ since $a \neq b$,

$$(\frac{3a+b}{4})\lambda = (\frac{3a+b}{4})\alpha\gamma$$

$$= (\frac{1}{b-a}(\frac{3a+b}{4} - \frac{a+b}{2})^2 + \frac{a+b}{2})\gamma$$

$$= (\frac{1}{b-a}(\frac{a-b}{4})^2 + \frac{a+b}{2})\gamma$$

$$= (\frac{7a+9b}{16})\gamma$$

$$= \frac{1}{4}(\frac{7a+9b}{16} - \frac{a+b}{2}) + \frac{a+b}{2}$$

$$= \frac{31a+33b}{64}$$

and

$$(\frac{3a+b}{4})\alpha = \frac{1}{b-a}(\frac{3a+b}{4} - \frac{a+b}{2})^2 + \frac{a+b}{2}$$

$$= \frac{1}{b-a}(\frac{a-b}{4})^2 + \frac{a+b}{2}$$

$$= \frac{b-a}{16} + \frac{a+b}{2}$$

$$= \frac{7a+9b}{16}.$$

Next, suppose that $\lambda \in \alpha D_I\alpha$. Then there exists $\eta \in D_I$ such that $\lambda = \alpha\eta\alpha$. Then $x\lambda = ((x\alpha)\eta)\alpha$ for all $x \in I$. By (1) and (2), we have that for every $x \in I$,

$$\frac{1}{4(b-a)}(x-\frac{a+b}{2})^2+\frac{a+b}{2}$$

$$= ((\frac{1}{b-a}(x-\frac{a+b}{2})^2+\frac{a+b}{2})\eta)\alpha$$

$$= \frac{1}{b-a}((\frac{1}{b-a}(x-\frac{a+b}{2})^2+\frac{a+b}{2})\eta-\frac{a+b}{2})^2+\frac{a+b}{2}$$

which implies that for every $x\in I$,

$$\frac{1}{4}(x-\frac{a+b}{2})^2 = ((\frac{1}{b-a}(x-\frac{a+b}{2})^2+\frac{a+b}{2})\eta-\frac{a+b}{2})^2.$$

Then for $x \in I$,

$$(\frac{1}{b-a}(x-\frac{a+b}{2})^2+\frac{a+b}{2})\eta = \pm\frac{1}{2}(x-\frac{a+b}{2})+\frac{a+b}{2}. \quad \text{............(3)}$$

Fix $c \in (\frac{a+b}{2}, b)$. Then $[\frac{a+b}{2}, c) \subseteq I$ and for $t \in [\frac{a+b}{2}, c\alpha)$, there exists $x \in [\frac{a+b}{2}, c)$ such that $\frac{1}{b-a}(x-\frac{a+b}{2})^2+\frac{a+b}{2} = t$ which implies $x = \sqrt{(b-a)(t-\frac{a+b}{2})}+\frac{a+b}{2}$. Therefore by (3), for $t\in [\frac{a+b}{2}, c)$,

$$t\eta = \pm\frac{1}{2}\sqrt{(b-a)(t-\frac{a+b}{2})}+\frac{a+b}{2}.$$

Then $(\frac{a+b}{2})\eta = \frac{a+b}{2}$ and for $t \in (\frac{a+b}{2}, c\alpha)$, $t\eta \neq \frac{a+b}{2}$. Since η is differentiable on I, η is continuous on $(\frac{a+b}{2}, c\alpha)$, so $(\frac{a+b}{2}, c\alpha)\eta$ is an interval on $\boldsymbol{R}$. Then if there exist $t_1, t_2 \in (\frac{a+b}{2}, c\alpha)$ such that

$$t_1\eta = \frac{1}{2}\sqrt{(b-a)(t_1 - \frac{a+b}{2})} + \frac{a+b}{2}$$

and

$$t_2\eta = -\frac{1}{2}\sqrt{(b-a)(t_2 - \frac{a+b}{2})} + \frac{a+b}{2},$$

then $\frac{a+b}{2} \in (t_2\eta, t_1\eta) \subseteq (\frac{a+b}{2}, c\alpha)\eta$ which is a contradiction since $t\eta \neq \frac{a+b}{2}$ for every $t \in (\frac{a+b}{2}, c\alpha)$. Hence we have

$$t\eta = \frac{1}{2}\sqrt{(b-a)(t - \frac{a+b}{2})} + \frac{a+b}{2} \quad \text{for all } t \in (\frac{a+b}{2}, c\alpha)$$

or

$$t\eta = -\frac{1}{2}\sqrt{(b-a)(t - \frac{a+b}{2})} + \frac{a+b}{2} \quad \text{for all } t \in (\frac{a+b}{2}, c\alpha).$$

Assume that $t\eta = \frac{1}{2}\sqrt{(b-a)(t - \frac{a+b}{2})} + \frac{a+b}{2}$ for all $t \in (\frac{a+b}{2}, c\alpha)$. Then for $h > 0$ with $\frac{a+b}{2} + h \in (\frac{a+b}{2}, c\alpha)$,

$$\begin{aligned} \frac{1}{h}((\frac{a+b}{2} + h)\eta - (\frac{a+b}{2})\eta) &= \frac{1}{h}(\frac{1}{2}\sqrt{(b-a)h} + \frac{a+b}{2} - \frac{a+b}{2}) \\ &= \frac{1}{2}(\frac{\sqrt{b-a}}{\sqrt{h}}) \end{aligned}$$

which implies that $\lim_{h \to 0^+} \frac{1}{h}((\frac{a+b}{2} + h)\eta - (\frac{a+b}{2})\eta)$ does not exist. Then η is not differentiable at $\frac{a+b}{2}$. Similarly, if $t\eta = -\frac{1}{2}\sqrt{(b-a)(t - \frac{a+b}{2})} + \frac{a+b}{2}$ for all $t \in (\frac{a+b}{2}, c\alpha)$, we also have that η is not differentiable at $\frac{a+b}{2}$.

This proves that $\lambda \notin \alpha D_I \alpha \cup \{\alpha\}$, so $\lambda \notin B$. Hence $\lambda \in BD_IB - B$, so B is not a quasi-ideal of D_I. Consequently, $D_I \notin \boldsymbol{BQ}$.

Therefore the theorem is completely proved. □