DIRICHLET SERIES AND POWER SERIES: SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

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In this thesis, we consider two classes of algebraic differential equations, namely, linear differential equations and generalized Riccati differential equations. We are interested in certain properties of the coefficients and exponents of their solutions. For Dirichlet series solutions, we give the pattern of its coefficients and exponents. For Power series solutions, we provide the bound of its coefficients.

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CONTENTS

CHAPTER I
INTRODUCTION

In this chapter, we collect basic concepts and state the results of earlier works that are related to our work in this thesis.

An algebraic differential equation, abbreviated *ADE*, is an equation of the form

$$F(s, f, f', \ldots, f^{(\nu)}) = 0$$

where $\nu \in \mathbb{N} \cup \{0\}$, $F(T_1, \ldots, T_{\nu+2}) \in \mathbb{C}[T_1, \ldots, T_{\nu+2}]$, and f is a function of s. If f satisfies an ADE, then it is said to be *differentially algebraic*.

Example 1.1. The function $f(s) = se^s$ is differentially algebraic, because it satisfies an ADE

$$F(f, f', f'') = f'' - 2f' + f = 0.$$

Throughout, by a *Dirichlet series*, we mean a convergent series of the form

$$\varphi(s) = \sum_{i=0}^{\infty} a_i e^{-\lambda_i s}; \quad a_i \in \mathbb{C},\ a_0 \neq 0,\ \lambda_i \in \mathbb{R},\ \lambda_0 < \lambda_1 < \lambda_2 < \cdots,\ \lim_{i \to \infty} \lambda_i = \infty. \tag{1.1}$$

Here a_i is called a *coefficient* and λ_i is called an *exponent* of the series. If a_i are nonzero for infinitely many i, we call $\varphi(s)$ an *infinite Dirichlet series.* The following theorem gives the domain of convergence of the convergent Dirichlet series.

Theorem 1.2. *[4, Theorem 1.10] If a Dirichlet series converges at the point* $z_0 = x_0 + iy_0$, *then it converges at every point of the half-plane Re(z)=* $x > x_0$. *Moreover, the convergence is uniform in every angle* $|arg(z - z_0)| \leq \frac{\pi}{2} - \delta$, *where* $\delta > 0$.

The next theorem provides the uniqueness for Dirichlet series.

Theorem 1.3. *[6, Lemma 8.13] If a Dirichlet series $\varphi(s)$, convergent in a half-plane $Re(s) > x_0 (x_0 \in \mathbb{R})$, has infinitely many roots $s_1, s_2, \ldots$, lying in an angle $|arg\ s| \leq \frac{\pi}{2} - \epsilon$ (where $\epsilon > 0$) and tending to ∞, then $a_0 = a_1 = \ldots = 0$ and the function $\varphi(s)$ vanishes identically.*

From the latter theorem, it follows that if a convergent Dirichlet series $\sum_{i=0}^{\infty} a_i e^{-\lambda_i s}$ as in (1.1) equals to zero, then we get $a_0 = a_1 = \ldots = 0$.

In 1920, Ostrowski, [5], proved the following results revealing astonishing properties of the exponents and coefficients of a Dirichlet series satisfying an ADE.

I. ([5, Theorem 6]) If a convergent Dirichlet series $\varphi(s)$ satisfies an ADE, then there exists $i \in \mathbb{N}$ such that for each $j > i$, the exponent λ_j can be written as an integral linear combination of $\lambda_0, \lambda_1, \ldots, \lambda_{i-1}$.

II. ([5, Theorem 8]) If a convergent Dirichlet series $\varphi(s)$ satisfies an ADE, then there exists a finite number of coefficients $a_0, a_1, \ldots, a_k$ such that all other coefficients a_h, with $h > k$, can be expressed as rational functions with rational coefficients in $a_0, a_1, \ldots, a_k$.

Our first objective here is to investigate a Dirichlet series satisfying a linear algebraic differential equation,

$$F(f, f', \ldots, f^{(n)}) = c + b_0 f + b_1 f' + \cdots + b_n f^{(n)} = 0,$$

where $n \in \mathbb{N}$ and $c, b_0, \ldots, b_n \in \mathbb{C}$.
Next, we consider a nonlinear algebraic differential equation of the form

$$F(f, f') := f' + Bf^k + Cf^{\ell} + D = 0, \tag{1.2}$$

where $B(\neq 0)$, C, D are polynomials with complex coefficients, and $k > \ell$ are positive integers. Such equation is called a *generalized Riccati differential equation* because the equation (1.2) with $k = 2$ and $\ell = 1$ is the well-known *Riccati differential equation*. Moreover, the Riccati differential equation relates to the second-order linear equation,

$$u'' + pu' + qu = 0 \tag{1.3}$$

where p, q are complex polynomials and u is a function of s. If we assume that u is a solution of (1.3) and replace $v = \frac{u'}{u}$ in (1.3), we get the Riccati differential equation,

$$v' + v^2 + pv + q = 0. \tag{1.4}$$

Hence, if we find a solution of (1.4), saying $v(s)$, we get that $u(s) = Ce^{\int v(s)ds}$, which C is a constant, is a solution of (1.3).

Based on Ostrowski's method, we aim to refine his results on the coefficients and exponents of Dirichlet series solutions. We divide our results, shown in Chapter II, into two cases.

- For the case of $\ell = 1$, we determine the exact Dirichlet series solution of (1.2), and prove that all exponents can be writen as a linear combination of the smallest nonzero exponent.

- For the case of $k = 3$ and $\ell = 2$, we show that convergent Dirichlet series solutions of (1.2) are plentiful.

In the rest of this thesis, we consider a power series,

$$\sum_{k=0}^{\infty} f_k z^k, \quad \text{with } f_k \in \mathbb{C},$$

that satisfies some ADE.

In 1935, K. Popken [3, Theorem 16] proved an interesting theorem yielding bounds on the coefficients of power series solutions. His theorem states that if a power series

$$\sum_{k=0}^{\infty} f_k z^k,$$

where $f_k \in \mathbb{C}$ satisfies an ADE, then there exist two positive constants γ_1 and γ_2 such that

$$|f_k| \leq \gamma_1 (k!)^{\gamma_2}$$

for all $k \in \mathbb{N}$. In Chapter III, we aim to derive bounds for the coefficients of a power series satisfying:

(i) a linear differential equation

$$F(f, f') := f\,' + Bf + C = 0,$$

where $B(\neq 0),\ C$ are polynomials with complex coefficients,

(ii) a Riccati differential equation

$$F(f, f') := f\,' + Bf^2 + Cf + D = 0,$$

where $B(\neq 0),\ C,\ D$ are polynomials with complex coefficients.

CHAPTER II
DIRICHLET SERIES SOLUTIONS

In this chapter, we consider Dirichlet series satisfying Linear differential equations and generalized Riccati equations.

2.1 Linear differential equations

In this section, we investigate properties of the coefficients and exponents of a Dirichlet series satisfying a linear differential equation of the form

$$F(f, f^{'}, \dots, f^{(n)}) = c + b_0 f + b_1 f^{'} + \cdots + b_n f^{(n)} = 0, \tag{2.1}$$

where $n \in \mathbb{N}$, and $c, b_0, \dots, b_n \in \mathbb{C}, b_n \neq 0$.

Theorem 2.1. *Assume that a convergent Dirichlet series* $\varphi(s)$ *as in* (1.1)*, with* $\lambda_0 \geq 0$*, is a solution of the linear differential equation* (2.1)*. Then*

$$\varphi(s) = -\frac{c}{b_0} + \sum_{i=1}^{k} a_i e^{-\lambda_i s} \text{ or } \varphi(s) = \sum_{i=0}^{k-1} a_i e^{-\lambda_i s},$$

where $k \leq n$ *and all* λ_i *are real roots of the polynomial*

$$b_0 - b_1 x + b_2 x^2 - b_3 x^3 + \cdots + (-1)^n b_n x^n = 0.$$

Proof. We split our proof into two cases according to the value of c.
Case $c \neq 0$. Substituting $\varphi(s)$ in (2.1), we have

$$c + b_0 \sum_{i=0}^{\infty} a_i e^{-\lambda_i s} - b_1 \sum_{i=0}^{\infty} \lambda_i a_i e^{-\lambda_i s} + \cdots + (-1)^n b_n \sum_{i=0}^{\infty} \lambda_i^n a_i e^{-\lambda_i s} = 0 \tag{2.2}$$

If $\lambda_0 = 0$, then by comparing the coefficients in (2.2), we get

$$\begin{aligned} c + b_0 a_0 &= 0 \\ (b_0 - b_1\lambda_1 + b_2\lambda_1^2 - b_3\lambda_1^3 + \cdots + (-1)^n b_n \lambda_1^n)a_1 &= 0 \\ &\vdots \end{aligned}$$

From the first equation, since $c \neq 0$, $b_0 \neq 0$, we get $a_0 = -\frac{c}{b_0}$. Since $\lambda_1, \lambda_2, \ldots$ are real numbers, the previous equations imply that $\lambda_1, \lambda_2, \ldots$ are real roots of the polynomial

$$b_0 - b_1 x + b_2 x^2 - b_3 x^3 + \cdots + (-1)^n b_n x^n = 0,$$

which has atmost n real roots. Therefore there are finitely many λ_i, so

$$\varphi(s) = -\frac{c}{b_0} + \sum_{i=1}^{k} a_i e^{-\lambda_i s}.$$

for some $k \in \mathbb{N}$. If $\lambda_0 > 0$, then by comparing the coefficients in (2.2), because all terms except the first term c on the left hand side are exponential functions, we get $c = 0$ a contradiction.

Case $c = 0$. By comparing the coefficients in (2.2), we get

$$\begin{aligned} b_0 - b_1\lambda_0 + b_2\lambda_0^2 - b_3\lambda_0^3 + \cdots + (-1)^n b_n \lambda_0^n &= 0 \\ b_0 - b_1\lambda_1 + b_2\lambda_1^2 - b_3\lambda_1^3 + \cdots + (-1)^n b_n \lambda_1^n &= 0 \\ &\vdots \end{aligned}$$

Since $\lambda_0, \lambda_1, \ldots$ are real numbers, their are all real roots of the polynomial

$$b_0 - b_1 x + b_2 x^2 - b_3 x^3 + \cdots + (-1)^n b_n x^n = 0.$$

Finally, we get $\varphi(s) = \sum_{i=0}^{k-1} a_i e^{-\lambda_i s}$ similar to the previous case. □

2.2 Generalized Riccati equations

We now consider a Dirichlet series solution of a generalized Riccati differential equation of the form

$$F(f,f') := f\,' + Bf^k + Cf^\ell + D = 0,$$

where $B(\neq 0)$, C, D are complex constants, and $k > \ell$ are positive integers. Spliting into two cases, we will first consider such differential equation with $\ell = 1$, and then with $\ell = 2$ and $k = 3$. We show that there are more than one class of convergent Dirichlet series solutions. To prove our result, we recall now the theorem [2, Theorem 3.7] about the expansion of multinomial.

Proposition 2.2. *Let n and r be positive integers and let x_i, $i = 1, 2, \ldots, r$, be real numbers. Then*

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1+k_2+\cdots+k_r=n} \binom{n}{k_1, k_2, \ldots, k_r} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r},$$

where the summation is taken over all $k_i = 0, 1, \ldots, n, (i = 1, 2, \ldots, r)$, such that $k_1 + k_2 + \cdots + k_r = n$ and $\binom{n}{k_1,k_2,\ldots,k_r} = \frac{n!}{k_1!k_2!\cdots k_{r-1}!k_r!}$.

Let

$$\chi_{n+1} = (a_0 + a_1 e^{-\lambda_1 s} + a_2 e^{-2\lambda_1 s} + \cdots + a_n e^{-n\lambda_1 s})^k.$$

Applying the multinomail expansion, we obtain

$$\begin{aligned}\chi_{n+1} &= \sum_{k_1+k_2+\cdots+k_{n+1}=k} \binom{k}{k_1, k_2, \ldots, k_{n+1}} a_0^{k_1} \left(a_1 e^{-\lambda_1 s}\right)^{k_2} \cdots \left(a_n e^{-n\lambda_1 s}\right)^{k_{n+1}}, \\ &= \sum_{k_1+k_2+\cdots+k_{n+1}=k} \binom{k}{k_1, k_2, \ldots, k_{n+1}} a_0^{k_1} a_1^{k_2} \cdots a_n^{k_{n+1}} e^{-(k_2+2k_3+\cdots+nk_{n+1})\lambda_1 s}.\end{aligned}$$

Set $t_n = k_{n+1}$. We get

$$\chi_{n+1} = \sum_{t_0+t_1+\cdots+t_n=k} \binom{k}{t_0, t_1, \ldots, t_n} a_0^{t_0} a_1^{t_1} \cdots a_n^{t_n} e^{-(t_1+2t_2+\cdots+nt_n)\lambda_1 s}.$$

As the result, the coefficient of the term with exponent $(n+1)\lambda_1$ in χ_{n+1} is

$$\sum_{\substack{t_0+\cdots+t_n=k \\ t_1+2t_2+\cdots+nt_n=n+1}} \binom{k}{t_0, \ldots, t_n} a_0^{t_0} a_1^{t_1} \cdots a_n^{t_n}.$$

Proposition 2.3. *If a finite Dirichlet series $\varphi(s) = \sum_{i=0}^{n} a_i e^{-\lambda_i s}$ satisfies* (1.2), *then $\varphi(s) = a_0$ is the only solution of* (1.2) *with $Ba_0^k + Ca_0^\ell + D = 0$.*

Proof. If $\lambda_0 = 0$, then $\varphi(s) = a_0 + \sum_{i=1}^{n} a_i e^{-\lambda_i s}$ and (1.2) becomes

$$0 = -\sum_{i=1}^{n} a_i \lambda_i e^{-\lambda_i s} + B\left(a_0 + \sum_{i=1}^{n} a_i e^{-\lambda_i s}\right)^k + C\left(a_0 + \sum_{i=1}^{n} a_i e^{-\lambda_i s}\right)^\ell.$$

For $n \geq 1$, since $k > \ell$, equating the terms with the highest exponent we get $Ba_n^k e^{-k\lambda_n s} = 0$ yielding $B = 0$, which is a contradiction. Hence, the only solution is $\varphi(s) = a_0$ with $Ba_0^k + Ca_0^\ell + D = 0$.

If $\lambda_0 \neq 0$, then for $n \geq 0$, equating the terms with the highest exponent in (1.2), we get $Ba_n^k e^{-k\lambda_n s} = 0$ yielding $B = 0$, which is a contradiction. $\square$

The above result deals with finite Dirichlet series. *Throughout the rest of the thesis, we shall consider only infinite Dirichlet series.*

The following procedure is very important to the proofs of results in this chapter because we often use this method whenever we want to compare the exponents of a Dirichlet series satisfiying (1.2).

For $n \in \mathbb{N}$, rewrite the infinite Dirichlet series (1.1) as

$$\varphi(s) = \sum_{i=0}^{n-1} a_i e^{-\lambda_i s} + \sum_{i=n}^{\infty} a_i e^{-\lambda_i s} =: \varphi_{1,n}(s) + \varphi_{2,n}(s).$$

By Taylor series expansion of $F(\varphi_{1,n}+\varphi_{2,n},\varphi_{1,n}^{'}+\varphi_{2,n}^{'})$ and (1.2), we get

$$\begin{aligned}
0 = F(\varphi,\varphi^{'}) &= F(\varphi_{1,n}+\varphi_{2,n},\varphi_{1,n}^{'}+\varphi_{2,n}^{'}) \\
&= F(\varphi_{1,n},\varphi_{1,n}^{'}) + \left(\varphi_{2,n}\frac{\partial F(\varphi_{1,n},\varphi_{1,n}^{'})}{\partial \varphi_{1,n}} + \varphi_{2,n}^{'}\frac{\partial F(\varphi_{1,n},\varphi_{1,n}^{'})}{\partial \varphi_{1,n}^{'}}\right) \\
&\quad + \left(\frac{\varphi_{2,n}^2}{2!}\frac{\partial^2 F(\varphi_{1,n},\varphi_{1,n}^{'})}{\partial^2 \varphi_{1,n}}\right) + \cdots + \left(\frac{\varphi_{2,n}^k}{k!}\frac{\partial^k F(\varphi_{1,n},\varphi_{1,n}^{'})}{\partial^k \varphi_{1,n}}\right) \\
&= F(\varphi_{1,n},\varphi_{1,n}^{'}) + \varphi_{2,n}(kB\varphi_{1,n}^{k-1} + \ell C\varphi_{1,n}^{l-1}) + \varphi_{2,n}^{'} \\
&\quad + \frac{\varphi_{2,n}^2}{2!}\left\{k(k-1)B\varphi_{1,n}^{k-2} + \ell(\ell-1)C\varphi_{1,n}^{\ell-2}\right\} + \cdots + \varphi_{2,n}^k B.
\end{aligned}$$

For brevity, let

$$\begin{aligned}
A_n :&= \varphi_{2,n}\left(kB\varphi_{1,n}^{k-1} + \ell C\varphi_{1,n}^{\ell-1}\right) + \varphi_{2,n}^{'} \\
&= a_n e^{-\lambda_n s}\left\{kB(a_0 e^{-\lambda_0 s})^{k-1} + \ell C(a_0 e^{-\lambda_0 s})^{\ell-1}\right\} - \lambda_n a_n e^{-\lambda_n s} \\
&\quad + \text{terms with higher exponents} \qquad (2.3)
\end{aligned}$$

and

$$\begin{aligned}
E_n :&= \frac{\varphi_{2,n}^2}{2!}\left\{k(k-1)B\varphi_{1,n}^{k-2} + \ell(\ell-1)C\varphi_{1,n}^{l-2}\right\} + \cdots + \varphi_{2,n}^k B \\
&= \frac{a_n^2 e^{-2\lambda_n s}}{2!}\left\{k(k-1)B(a_0 e^{-\lambda_0 s})^{k-2} + \ell(\ell-1)C(a_0 e^{-\lambda_0 s})^{\ell-2}\right\} \\
&\quad + \text{terms with higher exponents.}
\end{aligned}$$

Thus

$$0 = F(\varphi,\varphi^{'}) = F(\varphi_{1,n},\varphi_{1,n}^{'}) + A_n + E_n. \qquad (2.4)$$

Since our approach involves equating exponents, the term E_n can then be ignored because the exponent in its first term (i.e., the term of least exponent) is

$$\min\left\{2\lambda_n + \lambda_0(k-2), 2\lambda_n + \lambda_0(\ell-2)\right\}$$

while the exponent of the first term of A_n is

$$\min\{\lambda_n + \lambda_0(k-1), \lambda_n + \lambda_0(\ell-1), \lambda_n\}.$$

Comparing these two qualities: if $\lambda_0 \geq 0$, we obtain that

$$\begin{aligned}\min\{2\lambda_n + \lambda_0(k-2), 2\lambda_n + \lambda_0(\ell-2)\} &= 2\lambda_n + \lambda_0(\ell-2) \\ &> \lambda_n + \lambda_0(\ell-1), \qquad \lambda_n - \lambda_0 > 0 \\ &= \min\{\lambda_n + \lambda_0(k-1), \lambda_n + \lambda_0(\ell-1), \lambda_n\},\end{aligned}$$

on the other hand, if $\lambda_0 < 0$, we obtain that

$$\begin{aligned}\min\{2\lambda_n + \lambda_0(k-2), 2\lambda_n + \lambda_0(\ell-2)\} &= 2\lambda_n + \lambda_0(k-2) \\ &> \lambda_n + \lambda_0(k-1), \qquad \lambda_n - \lambda_0 > 0 \\ &= \min\{\lambda_n + \lambda_0(k-1), \lambda_n + \lambda_0(\ell-1), \lambda_n\}.\end{aligned}$$

Furthermore, if the first term of A_n is nonzero, then $F(\varphi_{1,n}, \varphi'_{1,n})$ must contain a term whose exponent is the same as that of A_n. If $\lambda_0 = 0$, then the first term of A_n is

$$(kBa_0^{k-1} + \ell Ca_0^{\ell-1} - \lambda_n)a_n e^{-\lambda_n s}. \tag{2.5}$$

If $\lambda_0 > 0$, then the first term of A_n is

$$(C - \lambda_n)a_n e^{-\lambda_n s} \quad \text{if } \ell = 1, \quad \text{and} \quad -\lambda_n a_n e^{-\lambda_n s} \quad \text{if } \ell > 1. \tag{2.6}$$

If $\lambda_0 < 0$, then the first term of A_n is

$$kBa_n a_0^{k-1} e^{-(\lambda_n + (k-1)\lambda_0)}.$$

The number ℓ appearing in (1.2) is closely related to λ_0, as seen in the next lemma.

Lemma 2.4. *Assume that an infinite Dirichlet series $\varphi(s)$ as in* (1.1) *satisfies*

(1.2). *If* $\ell \geq 2$, *then* $\lambda_0 = 0$, *while* $\lambda_0 \geq 0$ *if* $\ell = 1$.

Proof. Taking $n = 1$ in (2.4), i.e.,

$$\varphi(s) = a_0 e^{-\lambda_0 s} + \sum_{i=1}^{\infty} a_i e^{-\lambda_i s}, \quad F(\varphi_{1,1}, \varphi'_{1,1}) = -\lambda_0 a_0 e^{-\lambda_0 s} + B a_0^k e^{-k\lambda_0 s} + C a_0^{\ell} e^{-\ell\lambda_0 s} + D.$$

For the case $\ell \geq 2$, comparing the exponents in (2.4), we see that either the coefficient of the first term of A_1 is zero, or there is a term in $F(\varphi_{1,1}, \varphi'_{1,1})$ having the same exponent as the first term of A_1.

If $\lambda_0 > 0$, then the expression $F(\varphi_{1,1}, \varphi'_{1,1})$ contains a nonzero term with exponent λ_0; that is $-\lambda_0 a_0 e^{-\lambda_0 s}$, but

$$A_1 = -\lambda_1 a_1 e^{-\lambda_1 s}$$

starts with a term with exponent λ_1. Thus the term $-\lambda_0 a_0 e^{-\lambda_0 s}$ must vanish away, i.e., $\lambda_0 = 0$, which is a contradiction.

If $\lambda_0 < 0$, then

$$k\lambda_0 < \lambda_1 + (k-1)\lambda_0 < \lambda_1 + (\ell - 1)\lambda_0 < \lambda_1,$$

and so the first nonzero term of A_1 is $a_0^{k-1} a_1 k B e^{-(\lambda_1 + (k-1)\lambda_0)s}$, by replacing (2.3) with $n = 1$, while the first nonzero term of $F(\varphi_{1,1}, \varphi'_{1,1})$ is $B a_0^k e^{-k\lambda_0 s}$, so it must vanish and then implies that $B = 0$ which is again a contradiction.
Therefore, $\lambda_0 = 0$.

We next treat the case $\ell < 2$, i.e., $\ell = 1$. If $\lambda_0 < 0$, then the first term of A_1 is $(C - \lambda_1) a_1 e^{-\lambda_1 s}$ and

$$F(\varphi_{1,1}, \varphi'_{1,1}) = (C - \lambda_0) a_0 e^{-\lambda_0 s} + B a_0^k e^{-k\lambda_0 s} + D.$$

Comparing the first term, since $k\lambda_0 < \lambda_0 < \lambda_1$, by (2.4), we get

$$Ba_0^k e^{-k\lambda_0 s} = 0,$$

so it implies that $B = 0$ which is a contradiction. Therefore $\lambda_0 \geq 0$ □

Lemma 2.4 tells us that the least exponent, λ_0, in an infinite Dirichlet series satisfying (1.2) is a nonnegative real number.

We proceed now to obtain information about the coefficients B, C, D in (1.2).

Lemma 2.5. *If an infinite Dirichlet series $\varphi(s)$ as in* (1.1) *satisfies* (1.2)*, then the three polynomials B, C, D are complex constants.*

Proof. We need only consider the two possibilities: $\lambda_0 = 0$ and $\lambda_0 > 0$.
Case $\lambda_0 = 0$. With $n = 1$ in (2.4), i.e., writing $\varphi(s) = a_0 + \sum_{i\geq 1} a_i e^{-\lambda_i s}$, we see that the first term of A_1 is $(kBa_0^{k-1} + \ell Ca_0^{\ell-1} - \lambda_1)a_1 e^{-\lambda_1 s}$, and

$$F(\varphi_{1,1}, \varphi_{1,1}^{'}) = Ba_0^k + Ca_0^\ell + D.$$

Comparing the coefficients in (2.4), we have

$$Ba_0^k + Ca_0^\ell + D = 0 \quad \text{and} \quad kBa_0^{k-1} + \ell Ca_0^{\ell-1} - \lambda_1 = 0. \tag{2.7}$$

Next, with $n = 2$ in (2.4), i.e., $\varphi(s) = a_0 + a_1 e^{-\lambda_1 s} + \sum_{i\geq 2} a_i e^{-\lambda_i s}$, the first term in A_2 is

$$(kBa_0^{k-1} + \ell Ca_0^{\ell-1} - \lambda_2)a_2 e^{-\lambda_2 s} \neq 0,$$

because by the latter of (2.7), if the first term in A_2 is zero, we get $\lambda_1 = \lambda_2$ which contradicts to the definition of our Dirichlet series, and then

$$F(\varphi_{1,2}, \varphi_{1,2}^{'}) = -\lambda_1 a_1 e^{-\lambda_1 s} + B(a_0 + a_1 e^{-\lambda_1 s})^k + C(a_0 + a_1 e^{-\lambda_1 s})^\ell + D.$$

The coefficient of the term $e^{-2\lambda_1 s}$ in $F(\varphi_{1,2}, \varphi_{1,2}^{'})$ is $\left(B\binom{k}{2}a_0^{k-2}a_1^2 + C\binom{\ell}{2}a_0^{\ell-2}a_1^2\right) e^{-2\lambda_1 s}$.

If $\lambda_2 \neq 2\lambda_1$, then the coefficient of the term $e^{-2\lambda_1 s}$ in $F(\varphi_{1,2}, \varphi_{1,2}^{'})$ is zero because the first term in A_2 has the exponent λ_2 , i.e.,

$$B\binom{k}{2}a_0^{k-2}a_1^2 + C\binom{\ell}{2}a_0^{\ell-2}a_1^2 = 0. \tag{2.8}$$

Using the latter of (2.7) and (2.8), we deduce that B and C are complex constants, and using the former of (2.7) again we have that D is also a complex constant.

If $\lambda_2 = 2\lambda_1$, then the coefficient of the term with exponent $2\lambda_1$ in $F(\varphi_{1,2}, \varphi_{1,2}^{'})$ and the term with exponent λ_2 in A_2 add up to zero, i.e.,

$$B\binom{k}{2}a_0^{k-2}a_1^2 + C\binom{\ell}{2}a_0^{\ell-2}a_1^2 + (kBa_0^{k-1} + \ell ca_0^{\ell-1} - \lambda_2)a_2 = 0. \tag{2.9}$$

Again using (2.7) and (2.9), we deduce that $B, C, D \in \mathbb{C}$.

Case $\lambda_0 > 0$. With $n = 1$ in (2.4), i.e., $\varphi(s) = a_0 e^{-\lambda_0 s} + \sum_{i\geq 1} a_i e^{-\lambda_i s}$, Lemma 2.4 yields $\ell = 1$. In this case the first term of A_1 is $(C - \lambda_1)a_1 e^{-\lambda_1 s}$, and

$$F(\varphi_{1,1}, \varphi_{1,1}^{'}) = (C - \lambda_0)a_0 e^{-\lambda_0 s} + Ba_0^k e^{-k\lambda_0 s} + D.$$

Comparing the coefficients in (2.4), we have $D = 0$ and $C = \lambda_0 \in \mathbb{C}$. Thus, $(C - \lambda_1)a_1 e^{-\lambda_1 s} \neq 0$, and so $\lambda_1 = k\lambda_0$ and $B = (k-1)\lambda_0 a_1 a_0^{-k} \in \mathbb{C}$. □

From Lemma 2.5, throughout the rest of the thesis, we assume that $B(\neq 0), C, D$ are complex numbers.

2.2.1 The case $\ell = 1$

In this section we treat the case $\ell = 1$ and $k \geq 2$ in (1.2), i.e., the differential equation takes the form

$$F(f, f') := f' + Bf^k + Cf + D = 0; \quad B(\neq 0), C, D \in \mathbb{C}. \tag{2.10}$$

Theorem 2.6. *Assume that an infinite Dirichlet series $\varphi(s)$ as in* (1.1) *satisfies* (2.10).

(i) For $\lambda_0 = 0$, let $c_1 = 1$, $c_2 = \binom{k}{2}/\lambda_1$, and for $n \geq 3$, let $(n-1)\lambda_1 c_n$ be

$$\sum_{\substack{q_0+q_1+\cdots+q_{n-1}=k \\ q_1+2q_2+\cdots+(n-1)q_{n-1}=n}} \binom{k}{q_0, \ldots, q_{n-1}} \left(\frac{c_2(\lambda_1 - C)}{k}\right)^{q_2} \cdots \left(\frac{c_{n-1}(\lambda_1 - C)}{k}\right)^{q_{n-1}}.$$

If $c_n \neq 0 \quad (n \geq 3)$, then $Ba_0^k + Ca_0 + D = 0$, $\lambda_1 = kBa_0^{k-1} + C$, and for $n \geq 2$, we have $\lambda_n = n\lambda_1$ and $a_n = Bc_n a_0^{k-n} a_1^n$, where $\binom{k}{q_0,\ldots,q_{n-1}} = \frac{k}{q_0!\cdots q_{n-1}!}$.

(ii) For $\lambda_0 > 0$, let $d_1 = 1/(k-1)C$, and for $n \geq 2$, let

$$d_n = \frac{d_1}{n} \sum_{\substack{t_0+t_1+\cdots+t_{n-1}=k \\ t_0+kt_1+\cdots+((n-1)k-(n-2))t_{n-1}=nk-(n-1)}} \binom{k}{t_0, \ldots, t_{n-1}} (Bd_1)^{t_1} \cdots (Bd_{n-1})^{t_{n-1}}. \tag{2.11}$$

Then $D = 0$, $\lambda_0 = C$, and for $n \geq 1$, we have $\lambda_n = (nk - (n-1))C$ and $a_n = Bd_n a_0^{nk-(n-1)}$.

Proof. (i) Since $\lambda_0 = 0$, using (2.5), the first term of A_n is

$$(kBa_0^{k-1} + C - \lambda_n)a_n e^{-\lambda_n s}.$$

Taking $n = 1$ in (2.4), i.e., $\varphi(s) = a_0 + \sum_{i\geq 1} a_i e^{-\lambda_i s}$, the first term of A_1 is $(kBa_0^{k-1} + C - \lambda_1)a_1 e^{-\lambda_1 s}$, and $F(\varphi_{1,1}, \varphi_{1,1}') = Ba_0^k + Ca_0 + D$. Comparing the coefficients in (2.4), we get $Ba_0^k + Ca_0 + D = 0$ and $\lambda_1 = kBa_0^{k-1} + C$, i.e.,

$$0 \neq B = \frac{\lambda_1 - C}{ka_0^{k-1}}.$$

Taking $n = 2$ in (2.4), i.e., $\varphi(s) = a_0 + a_1 e^{-\lambda_1 s} + \sum_{i\geq 2} a_i e^{-\lambda_i s} = \varphi_{1,2} + \varphi_{2,2}$, the first term of A_2 is $(\lambda_1 - \lambda_2)a_2 e^{-\lambda_2 s} \neq 0$. The term with exponent $2\lambda_1$ in

$$F(\varphi_{1,2}, \varphi_{1,2}') = -\lambda_1 a_1 e^{-\lambda_1 s} + B(a_0 + a_1 e^{-\lambda_1 s})^k + C(a_0 + a_1 e^{-\lambda_1 s}) + D,$$

is $\binom{k}{2}Ba_0^{k-2}a_1^2e^{-2\lambda_1 s} \neq 0$. Comparing the coefficients in (2.4), we get $\lambda_2 = 2\lambda_1$ and

$$(\lambda_1 - \lambda_2)a_2 + \binom{k}{2}Ba_0^{k-2}a_1^2 = 0,$$

and so $a_2 = Bc_2a_0^{k-2}a_1^2$, where $c_2 = \binom{k}{2}/\lambda_1$.
Taking $n = 3$ in (2.4), i.e.,

$$\varphi(s) = a_0 + a_1e^{-\lambda_1 s} + a_2e^{-\lambda_2 s} + \sum_{i\geq 3} a_ie^{-\lambda_i s} = \varphi_{1,3} + \varphi_{2,3},$$

the first term of A_3 is $(\lambda_1 - \lambda_3)a_3e^{-\lambda_3 s} \neq 0$. Replacing $\lambda_2 = 2\lambda_1$ and using the multinomial expansion, the term with exponent $3\lambda_1$ in

$$F(\varphi_{1,3}, \varphi_{1,3}^{'}) = -\lambda_1a_1e^{-\lambda_1 s} - \lambda_2a_2e^{-\lambda_2 s} + B(a_0 + a_1e^{-\lambda_1 s} + a_2e^{-\lambda_2 s})^k$$
$$+ C(a_0 + a_1e^{-\lambda_1 s} + a_2e^{-\lambda_2 s}) + D,$$

has coefficient equal to

$$B \sum_{\substack{t_0+t_1+t_2=k \\ t_1+2t_2=3}} \binom{k}{t_0, t_1, t_2} a_0^{t_0}a_1^{t_1}a_2^{t_2} = B \sum_{\substack{t_0+t_1+t_2=k \\ t_1+2t_2=3}} \binom{k}{t_0, t_1, t_2} a_0^{t_0}a_1^{t_1} \left(\frac{\lambda_1 - C}{ka_0^{k-1}}c_2a_0^{k-2}a_1^2\right)^{t_2}$$
$$= B \sum_{\substack{t_0+t_1+t_2=k \\ t_1+2t_2=3}} \binom{k}{t_0, t_1, t_2} \left(\frac{c_2(\lambda_1 - C)}{k}\right)^{t_2} a_0^{k-3}a_1^3 = B2\lambda_1c_3a_0^{k-3}a_1^3 \neq 0.$$

Equating these terms, we get $\lambda_3 = 3\lambda_1$ and $(\lambda_1 - \lambda_3)a_3 + 2B\lambda_1c_3a_0^{k-3}a_1^3 = 0$, i.e., $a_3 = Bc_3a_0^{k-3}a_1^3$. Continuing inductively, assume $\lambda_n = n\lambda_1 \quad (n > 1)$ and $a_n = Bc_na_0^{k-n}a_1^n$. Writing

$$\varphi(s) = \sum_{i=0}^{n} a_ie^{-\lambda_i s} + \sum_{i\geq n+1} a_ie^{-\lambda_i s} = \varphi_{1,n+1}(s) + \varphi_{2,n+1}(s),$$

we see that the first term of A_{n+1} is $(\lambda_1 - \lambda_{n+1})a_{n+1}e^{-\lambda_{n+1}s} \neq 0$. Replacing $\lambda_n = n\lambda_1$ and using the multinomial expansion, the coefficient of the term with exponent $(n+1)\lambda_1$ in $F(\varphi_{1,n+1}, \varphi'_{1,n+1})$ is

$$B \sum_{\substack{t_0+\cdots+t_n=k \\ t_1+\cdots+nt_n=n+1}} \binom{k}{t_0,\ldots,t_n} a_0^{t_0} a_1^{t_1} \cdots a_n^{t_n}$$
$$= B \sum_{\substack{t_0+\cdots+t_n=k \\ t_1+\cdots+nt_n=n+1}} \binom{k}{t_0,\ldots,t_n} \left(\frac{c_2(\lambda_1 - C)}{ka_0^{k-1}} a_0^{k-2} a_1^2\right)^{t_2} \cdots \left(\frac{c_n(\lambda_1 - C)}{ka_0^{k-1}} a_0^{k-n} a_1^n\right)^{t_n}$$
$$= B \sum_{\substack{t_0+\cdots+t_n=k \\ t_1+\cdots+nt_n=n+1}} \binom{k}{t_0,\ldots,t_n} \left(\frac{c_2(\lambda_1 - C)}{k}\right)^{t_2} \cdots \left(\frac{c_n(\lambda_1 - C)}{k}\right)^{t_n} a_1^{n+1} a_0^{k-(n+1)}$$
$$= nB\lambda_1 c_{n+1} a_1^{n+1} a_0^{k-(n+1)} \neq 0.$$

Comparing the coefficients in (2.4), we get $\lambda_{n+1} = (n+1)\lambda_1$ and

$$(\lambda_1 - \lambda_{n+1})a_{n+1} + nB\lambda_1 c_{n+1} a_1^{n+1} a_0^{k-(n+1)} = 0,$$

i.e., $a_{n+1} = Bc_{n+1} a_0^{k-(n+1)} a_1^{n+1}$.

(ii) We assume for the time being that $d_n \neq 0$ for all $n \in \mathbb{N}$. Using Lemma 2.4, we get $\lambda_0 > 0$. Since $\lambda_0 > 0$, from (2.6), the first term of A_n is $(C - \lambda_n)a_n e^{-\lambda_n s}$. Taking $n = 1$ in (2.4), i.e.,

$$\varphi(s) = a_0 e^{-\lambda_0 s} + \sum_{i \geq 1} a_i e^{-\lambda_i s},$$

the first term of A_1 is $(C - \lambda_1)a_1 e^{-\lambda_1 s}$, and

$$F(\varphi_{1,1}, \varphi'_{1,1}) = (C - \lambda_0)a_0 e^{-\lambda_0 s} + Ba_0^k e^{-k\lambda_0 s} + D.$$

Comparing the coefficients in (2.4), we get $D = 0,\ C = \lambda_0 > 0$. Since $(C - \lambda_1)a_1 \neq 0$, we have

$$\lambda_1 = k\lambda_0 = kC, \text{ and } Ba_0^k + (C - \lambda_1)a_1 = 0,$$

i.e., $a_1 = Bd_1 a_0^k$, where $d_1 = 1/(k-1)C$.

Taking $n = 2$ in (2.4), i.e.,

$$\varphi(s) = a_0 e^{-\lambda_0 s} + a_1 e^{-\lambda_1 s} + \sum_{i \geq 2} a_i e^{-\lambda_i s} = \varphi_{1,2} + \varphi_{2,2},$$

the first term of A_2 is $(C - \lambda_2) a_2 e^{-\lambda_2 s} \neq 0$. The term with exponent $(2k-1)\lambda_0$ in

$$\begin{aligned} F(\varphi_{1,2}, \varphi_{1,2}') = & -\lambda_0 a_0 e^{-\lambda_0 s} - \lambda_1 a_1 e^{-\lambda_1 s} + B(a_0 e^{-\lambda_0 s} + a_1 e^{-\lambda_1 s})^k \\ & + C(a_0 e^{-\lambda_0 s} + a_1 e^{-\lambda_1 s}) + D, \end{aligned}$$

which is the right term after the one with exponent $k\lambda_0$, has the coefficient equal to

$$\begin{aligned} B \sum_{\substack{t_0+t_1=k \\ t_0+kt_1=2k-1}} \binom{k}{t_0, t_1} a_0^{t_0} a_1^{t_1} &= B \sum_{\substack{t_0+t_1=k \\ t_0+kt_1=2k-1}} \binom{k}{t_0, t_1} a_0^{t_0} (B d_1 a_0^k)^{t_1} \\ &= B \sum_{\substack{t_0+t_1=k \\ t_0+kt_1=2k-1}} \binom{k}{t_0, t_1} (B d_1)^{t_1} a_0^{2k-1} \\ &= B \frac{2d_2}{d_1} a_0^{2k-1} \\ &= B 2(k-1) C d_2 a_0^{2k-1} \neq 0. \end{aligned}$$

Comparing these terms, we get $\lambda_2 = (2k-1)\lambda_0$ and

$$(C - \lambda_2) a_2 + 2(k-1) d_2 C B a_0^{2k-1} = 0,$$

i.e., $a_2 = B d_2 a_0^{2k-1}$. To carry out the induction step, assume that

$$\lambda_n = (nk - (n-1))C \text{ and } a_n = B d_n a_0^{nk-(n-1)}.$$

Writing $\varphi(s) = \varphi_{1,n+1}(s) + \varphi_{2,n+1}(s)$, we see that the first term of A_{n+1} is

$$(C - \lambda_{n+1}) a_{n+1} e^{-\lambda_{n+1} s} \neq 0,$$

and

$$F(\varphi_{1,n+1},\varphi'_{1,n+1}) = -\sum_{i=0}^{n}\lambda_i a_i e^{-\lambda_i s} + \left(\sum_{i=0}^{n} a_i e^{-\lambda_i s}\right)^k + \sum_{i=0}^{n} a_i e^{-\lambda_i s} + D$$
$$= -(\lambda_0 a_0 e^{-Cs} + \cdots + \lambda_n a_n e^{-(nk-(n-1))Cs})$$
$$+ \left(a_0 e^{-Cs} + a_1 e^{-kCs} + a_2 e^{-(2k-1)Cs} + \cdots + a_n e^{-(nk-(n-1))Cs}\right)^k$$
$$+ a_0 e^{-Cs} + a_1 e^{-kCs} + a_2 e^{-(2k-1)Cs} + \cdots + a_n e^{-(nk-(n-1))Cs} + D.$$

Then the term with exponent $((n+1)k-n)C$ in $F(\varphi_{1,n+1},\varphi'_{1,n+1})$ following the term with exponent $(nk-(n-1))C$ has coefficient equal to

$$B \sum_{\substack{t_0+t_1+\cdots+t_n=k \\ t_0+kt_1\cdots+(nk-(n-1))t_n=(n+1)k-n}} \binom{k}{t_0,\ldots,t_n} a_0^{t_0}\cdots a_n^{t_n}$$
$$= B \sum_{\substack{t_0+\cdots+t_n=k \\ t_0+\cdots+(nk-(n-1))t_n=(n+1)k-n}} \binom{k}{t_0,\ldots,t_n} (Bd_1)^{t_1}\cdots(Bd_n)^{t_n} a_0^{(n+1)k-n}$$
$$= (n+1)(k-1)Cd_{n+1}Ba_0^{(n+1)k-n} \neq 0.$$

Comparing these terms, we get $\lambda_{n+1} = ((n+1)k-n)C$ and

$$(C-\lambda_{n+1})a_{n+1} + (n+1)(k-1)Cd_{n+1}Ba_0^{(n+1)k-n} = 0,$$

i.e., $a_{n+1} = Bd_{n+1}a_0^{(n+1)k-n}$. There remains to verify that $d_n \neq 0$ for all $n \in \mathbb{N}$. This follows immediately from $d_1 = 1/(k-1)C$ and the following claim.

Claim. We have

$$d_n = \frac{1}{n}B^{n-1}d_1^n D_n \qquad (n \geq 2),$$

where all D_n's are positive real numbers.

Proof of Claim. Note first that $d_2 = \frac{d_1}{2}\binom{k}{k-1,1}Bd_1 = \frac{1}{2}Bd_1^2 D_2$ with $D_2 = k$, i.e., the claim holds for $n = 2$.

Note that the second condition in the summation of (2.11) can be transfered to

$$t_1 + 2t_2 + \cdots + (n-1)t_{n-1} = n - 1, \tag{2.12}$$

because

$$\begin{aligned}
nk - (n-1) &= t_0 + kt_1 + \cdots + ((n-1)k - (n-2))t_{n-1}, \\
nk - (n-1) - k &= t_0 + kt_1 + (2k-1)t_2 + \cdots + ((n-1)k - (n-2))t_{n-1} \\
&\quad - (t_0 + t_1 + \cdots + t_{n-1}), \\
(n-1)(k-1) &= k(t_1 + 2t_2 + \cdots + (n-1)t_{n-1}) - (t_2 + 2t_3 + \cdots + (n-2)t_{n-1}) \\
&\quad - (t_1 + t_2 + \cdots + t_{n-1}) \\
&= k(t_1 + 2t_2 + \cdots + (n-1)t_{n-1}) - (t_1 + 2t_2 + \cdots + (n-1)t_{n-1}) \\
&= (k-1)(t_1 + 2t_2 + \cdots + (n-1)t_{n-1}).
\end{aligned}$$

Suppose that $d_n = \frac{1}{n}B^{n-1}d_1^n D_n$ for all natural numbers $n \geq 2$, where all D_n are positive real numbers. Hence, by (2.12) and the induction hypothesis, we have

$$\begin{aligned}
d_{n+1} &= \frac{d_1}{n+1} \sum_{\substack{t_0+t_1+\cdots+t_n=k \\ t_1+2t_2+\cdots+nt_n=n}} \binom{k}{t_0, \ldots, t_n} (Bd_1)^{t_1} \cdots (Bd_n)^{t_n} \\
&= \frac{d_1}{n+1} \sum_{\substack{t_0+t_1+\cdots+t_n=k \\ t_1+2t_2+\cdots+nt_n=n}} \binom{k}{t_0, \ldots, t_n} B^{t_1+\cdots+t_n} d_1^{t_1} \left(\frac{Bd_1^2 D_2}{2}\right)^{t_2} \cdots \left(\frac{B^{n-1}d_1^n D_n}{n}\right)^{t_n} \\
&= \frac{d_1}{n+1} \sum_{\substack{t_0+t_1+\cdots+t_n=k \\ t_1+2t_2+\cdots+nt_n=n}} \binom{k}{t_0, \ldots, t_n} (Bd_1)^{t_1+2t_2+\cdots+nt_n} \left(\frac{D_2}{2}\right)^{t_2} \cdots \left(\frac{D_n}{n}\right)^{t_n} \\
&= \frac{B^n d_1^{n+1}}{n+1} \sum_{\substack{t_0+t_1+\cdots+t_n=k \\ t_1+2t_2+\cdots+nt_n=n}} \binom{k}{t_0, \ldots, t_n} \left(\frac{D_2}{2}\right)^{t_2} \cdots \left(\frac{D_n}{n}\right)^{t_n} \\
&:= \frac{B^n d_1^{n+1} D_{n+1}}{n+1} \neq 0.
\end{aligned}$$

□

Applying Lemma 2.5 and the case $k = 2$ in (2.10) of Theorem 2.6, which corresponds to the classical Riccati equation, lead to:

Corollary 2.7. *Assume that an infinite Dirichlet series $\varphi(s)$, as in (1.1), satisfies the Riccati equation:*

$$f' + Bf^2 + Cf + D = 0.$$

(i) If $\lambda_0 = 0$, then $\varphi(s) = a_0 + \frac{a_1\lambda_1}{\lambda_1 e^{\lambda_1 s} - Ba_1}$ with $a_0 = (\lambda_1 - C)/2B$, $C^2 - \lambda_1^2 = 4BD$ and $|Ba_1 e^{-\lambda_1 s}/\lambda_1| < 1$.

(ii) If $\lambda_0 > 0$, then $D = 0$, $\lambda_0 = C$, and $\varphi(s) = \frac{a_0 C}{Ce^{Cs} - a_0 B}$ with $|Ba_0 e^{-Cs}/C| < 1$.

Proof. (i) When $\lambda_0 = 0$, from the proof of Theorem 2.6(i), we get $B = (\lambda_1 - C)/2a_0$ and $Ba_0^2 + Ca_0 + D = 0$, i.e.,

$$a_0 = (\lambda_1 - C)/2B \text{ and } C^2 - \lambda_1^2 = 4BD.$$

Since $c_2 = 1/\lambda_1$, $c_3 = Ba_0/\lambda_1^2$, assuming $c_{n-1} = (Ba_0)^{n-3}/\lambda_1^{n-2}$, we see that

$$\begin{aligned}
c_n &= \frac{1}{(n-1)\lambda_1} \sum_{\substack{q_0+\cdots+q_{n-1}=2 \\ q_1+\cdots+(n-1)q_{n-1}=n}} \binom{2}{q_0,\ldots,q_{n-1}} \left(\frac{c_2(\lambda_1 - C)}{2}\right)^{q_2} \cdots \left(\frac{c_{n-1}(\lambda_1 - C)}{2}\right)^{q_{n-1}} \\
&= \frac{1}{(n-1)\lambda_1} \sum_{\substack{q_0+\cdots+q_{n-1}=2 \\ q_1+\cdots+(n-1)q_{n-1}=n}} \binom{2}{q_0,\ldots,q_{n-1}} \left(\frac{2Ba_0}{\lambda_1 2}\right)^{q_2} \cdots \left(\frac{(Ba_0)^{n-3} 2Ba_0}{\lambda_1^{n-2} 2}\right) q_{n-1} \\
&= \frac{1}{(n-1)\lambda_1} \sum_{\substack{q_0+\cdots+q_{n-1}=2 \\ q_1+\cdots+(n-1)q_{n-1}=n}} \binom{2}{q_0,\ldots,q_{n-1}} \left(\frac{Ba_0}{\lambda_1}\right)^{q_2} \cdots \left(\frac{(Ba_0)^{n-2}}{\lambda_1^{n-2}}\right)^{q_{n-1}} \\
&= \frac{1}{(n-1)\lambda_1} \sum_{\substack{q_1+\cdots+q_{n-1}=2 \\ q_1+\cdots+(n-1)q_{n-1}=n}} \binom{2}{q_1,\ldots,q_{n-1}} \frac{(Ba_0)^{n-2}}{\lambda_1^{n-2}}
\end{aligned}$$

Since $\sum_{\substack{q_1+\cdots+q_{n-1}=2 \\ q_1+\cdots+(n-1)q_{n-1}=n}} \binom{2}{q_1,\ldots,q_{n-1}} = n - 1,$

$$c_n = \frac{(Ba_0)^{n-2}}{\lambda_1^{n-1}}.$$

It is easy to check that $c_n \neq 0$ for all $n \in \mathbb{N}$. Then, from the Theorem 2.6 (i), we have $\lambda_n = n\lambda_1$ and

$$a_n = Bc_n a_0^{2-n} a_1^n = \frac{B(Ba_0)^{n-2} a_0^{2-n} a_1^n}{\lambda_1^{n-1}} = \left(\frac{Ba_1}{\lambda_1}\right)^{n-1} a_1$$

for all $n \in \mathbb{N}$. Substituting $a_n = (Ba_1/\lambda_1)^{n-1} a_1$ into $\varphi(s)$, we get

$$\varphi(s) = \sum_{i=0}^{\infty} a_i e^{-\lambda_i s} = a_0 + \sum_{i=1}^{\infty} \left(\frac{Ba_1}{\lambda_1}\right)^{i-1} a_1 e^{-i\lambda_1 s} = a_0 + \frac{a_1\lambda_1}{\lambda_1 e^{\lambda_1 s} - Ba_1}.$$

(ii) When $\lambda_0 > 0$, since $d_1 = 1/C,\ d_2 = B/C^2$, assuming that $d_{n-1} = B^{n-2}/C^{n-1}$, by Theorem 2.6 (ii), we get

$$\begin{aligned} d_n &= \frac{1}{nC} \sum_{\substack{t_0+\cdots+t_{n-1}=2 \\ t_1+\cdots+(n-1)t_{n-1}=n-1}} \binom{2}{t_0, \ldots, t_{n-1}} (Bd_1)^{t_1} \cdots (Bd_{n-1})^{t_{n-1}} \\ &= \frac{1}{nC} \sum_{\substack{t_0+\cdots+t_{n-1}=2 \\ t_1+\cdots+(n-1)t_{n-1}=n-1}} \binom{2}{t_0, \ldots, t_{n-1}} \left(\frac{B}{C}\right)^{t_1} \cdots \left(\frac{B^{n-1}}{C^{n-1}}\right)^{t_{n-1}} \\ &= \frac{B^{n-1}}{C^n}. \end{aligned}$$

Using Theorem 2.6 again, we have

$$a_n = Bd_n a_0^{n+1} = (B/C)^n a_0^{n+1} \text{ and } \lambda_n = (n+1)C$$

for all $n \in \mathbb{N} \cup \{0\}$, and so

$$\varphi(s) = \sum_{i=0}^{\infty} a_i e^{-\lambda_i s} = \sum_{i=0}^{\infty} \left(\frac{B}{C}\right)^i a_0^{i+1} e^{-(i+1)Cs} = \frac{a_0 C}{Ce^{Cs} - a_0 B}.$$

□

2.2.2 The case $\ell = 2,\ k = 3$

In this section we treat the case $k = 3,\ \ell = 2$ of (1.2), i.e., the differential equation takes the form

$$F(f, f') := f' + Bf^3 + Cf^2 + D = 0; \quad B(\neq 0), C, D \in \mathbb{C}. \tag{2.13}$$

We show that for $D = 0$, the equation (2.13) has a unique infinite Dirichlet series solution, while for $D \neq 0$, there generally are more than one infinite Dirichlet series solutions.

Theorem 2.8. *If an infinite Dirichlet series $\varphi(s)$ as in (1.1) satisfies*

$$f' + Bf^3 + Cf^2 = 0; \qquad B(\neq 0), C \in \mathbb{C},$$

then $Ba_0 + C = 0,\ \lambda_0 = 0,\ \lambda_1 = Ba_0^2$ and for $n \geq 2$, we have $\lambda_n = n\lambda_1,\ a_n = c_n a_1^n / a_0^{n-1}$, where $c_1 = 1$ and $(n-1)c_n$ for all $n \geq 2$ are the coefficient of x^n in the expansion of

$$(1 + c_1 x + c_2 x^2 + \cdots + c_{n-1}x^{n-1})^2 (c_1 x + c_2 x^2 + \cdots + c_{n-1}x^{n-1}).$$

Proof. From (2.3), we have

$$A_n = \varphi_{2,n}(3B\varphi_{1,n}^2 + 2C\varphi_{1,n}) + \varphi_{2,n}^{'}.$$

By Lemma 2.4, we know that $\lambda_0 = 0$. From (2.5), the first term of A_n is

$$(3Ba_0^2 + 2Ca_0 - \lambda_n)a_n e^{-\lambda_n s}.$$

Taking $n = 1$ in (2.4), i.e., $\varphi(s) = a_0 + \sum_{i \geq 1} a_i e^{-\lambda_i s}$, the first term of A_1 is

$$(3Ba_0^2 + 2Ca_0 - \lambda_1)a_1 e^{-\lambda_1 s},$$

and

$$F(\varphi_{1,1}, \varphi_{1,1}^{'}) = Ba_0^3 + Ca_0^2.$$

Comparing terms, we get $\lambda_1 = 3Ba_0^2 + 2Ca_0$ and $Ba_0^3 + Ca_0^2 = 0$. Thus,

$$Ba_0 + C = 0 \text{ and } \lambda_1 = Ba_0^2.$$

Next, taking $n = 2$ in (2.4), i.e., $\varphi(s) = a_0 + a_1 e^{-\lambda_1 s} + \sum_{i\geq 2} a_i e^{-\lambda_i s} = \varphi_{1,2}(s) + \varphi_{2,2}(s)$. In this case the first term of A_2 is $(\lambda_1 - \lambda_2)a_2 e^{-\lambda_2 s} \neq 0$. The term with exponent $2\lambda_1$ in $F(\varphi_{1,2}, \varphi_{1,2}^{'}) = -\lambda_1 a_1 e^{-\lambda_1 s} + B(a_0 + a_1 e^{-\lambda_1 s})^3 + C(a_0 + a_1 e^{-\lambda_1 s})^2$ is $3Ba_0a_1^2 + Ca_1^2 = 2Ba_0a_1^2 \neq 0$. Comparing these terms, we get $\lambda_2 = 2\lambda_1$ and $(\lambda_1 - \lambda_2)a_2 + 2Ba_0a_1^2 = 0$. Thus,

$$a_2 = \frac{2Ba_0a_1^2}{\lambda_1} = \frac{2a_1^2}{a_0} = \frac{c_2a_1^2}{a_0},$$

where c_2 is the coefficient of x^2 in the expansion of $(1 + c_1x)^2(c_1x)$.

For $n \in \mathbb{N}$, assume that

$$\lambda_n = n\lambda_1 \text{ and } a_n = c_n a_1^n / a_0^{n-1},$$

where $(n-1)c_n$ is the coefficient of x^n of

$$(1 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1})^2(c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1}).$$

Writing $\varphi(s) = \varphi_{1,n+1}(s) + \varphi_{2,n+1}(s)$, the first term of A_{n+1} is

$$(\lambda_1 - \lambda_{n+1})a_{n+1}e^{-\lambda_{n+1}s} \neq 0.$$

The term with exponent $(n+1)\lambda_1$ in $F(\varphi_{1,n+1}, \varphi_{1,n+1}^{'})$ is

$$B \sum_{\substack{t_0+\cdots+t_n=3 \\ t_1+\cdots+nt_n=n+1}} \binom{3}{t_0, \ldots, t_n} a_0^{t_0} a_1^{t_1} \cdots a_n^{t_n} + C \sum_{\substack{q_0+\cdots+q_n=2 \\ q_1+\cdots+nq_n=n+1}} \binom{2}{q_0, \ldots, q_n} a_0^{q_0} a_1^{q_1} \cdots a_n^{q_n}$$

$$= \left(\sum_{\substack{t_0+\cdots+t_n=3 \\ t_1+\cdots+nt_n=n+1}} \binom{3}{t_0,\dots,t_n} c_2^{t_2}\cdots c_n^{t_n} - \sum_{\substack{q_0+\cdots+q_n=2 \\ q_1+\cdots+nq_n=n+1}} \binom{2}{q_0,\dots,q_n} c_2^{q_2}\cdots c_n^{q_n} \right) \frac{Ba_1^{n+1}}{a_0^{n-2}}$$

$$=: Q\frac{Ba_1^{n+1}}{a_0^{n-2}}.$$

In the second sum, we must have $q_0 = 0$, which shows that the second sum is less than the case $t_0 = 1$ in the first sum, and this implies that $Q > 0$. Consequently,

$$\lambda_{n+1} = (n+1)\lambda_1 \text{ and } (\lambda_1 - \lambda_{n+1})a_{n+1} + QBa_1^{n+1}/a_0^{n-2} = 0,$$

i.e.,

$$a_{n+1} = QBa_1^{n+1}/n\lambda_1 a_0^{n-2} = Qa_1^{n+1}/na_0^n,$$

because of $\lambda_1 = Ba_0^2$. Since Q is the coefficient of x^{n+1} in the expansion $(1+c_1x+\cdots+c_nx^n)^3 - (1+c_1x+\cdots+c_nx^n)^2 = (1+c_1x+\cdots+c_nx^n)^2(c_1x+\cdots+c_nx^n)$, putting $nc_{n+1} = Q$, we get $a_{n+1} = c_{n+1}a_1^{n+1}/a_0^n$, as desired. □

We next solve (2.13) with $D \neq 0$. We show that it generally has more than one infinite Dirichlet series solutions; one such solution is given explicitly in the following theorem.

Theorem 2.9. *The infinite Dirichlet series* $\varphi(s) = a_0 + \sum_{i=1}^{\infty} a_i e^{-\lambda_i s}$ *with*

$$\lambda_1 = 3Ba_0^2 + 2C,\ a_0 = -C/3B,\ Ba_0^3 + Ca_0^2 + D = 0$$

and

$$\lambda_n = (2n-1)\lambda_1,\ a_n = \left(\frac{Ba_1^2}{4\lambda_1}\right)^{n-1}\binom{2(n-1)}{n-1}a_1 \quad (n \in \mathbb{N})$$

satisfies the differrential equation

$$f\,' + Bf^3 + Cf^2 + D = 0 \qquad (B(\neq 0), C, D(\neq 0) \in \mathbb{C}) \tag{2.14}$$

provided that $\left|Ba_1^2 e^{-2\lambda_1 s}/4\lambda_1\right| < 1/4.$

Proof. Setting $Y = Ba_1^2 e^{-2\lambda_1 s}/4\lambda_1$ and assuming $|Y| < 1/4$, we have

$$\varphi(s) = -\frac{C}{3B} + \sum_{i=1}^{\infty} \left(\frac{Ba_1^2}{4\lambda_1}\right)^{i-1} \binom{2(i-1)}{i-1} a_1 e^{-(2i-1)\lambda_1 s} = -\frac{C}{3B} + \frac{a_1 e^{-\lambda_1 s}}{\sqrt{1-4Y}}.$$

It is easy to check that this infinite Dirichlet series satisfies (2.13). □

To find another possible infinite Dirichlet series solution of (2.13), we begin with:

Theorem 2.10. *The infinite Dirichlet series* $\varphi(s) = a_0 + \sum_{i\geq 1} a_i e^{-\lambda_i s}$, *as in* (1.1), *with*

$$\lambda_0 = 0 < \lambda_1 = 3Ba_0^2 + 2Ca_0, \ \ \lambda_n = n\lambda_1 \quad (n \geq 2) \tag{2.15}$$

$$Ba_0^3 + Ca_0^2 + D = 0, \ \ a_2 = \frac{(3Ba_0 + C)a_1^2}{\lambda_1} \tag{2.16}$$

and the remaining coefficients recursively determined by

$$\begin{aligned} a_{n+1} = & \frac{1}{n\lambda_1} \sum_{\substack{q_1+\cdots+q_n=2 \\ q_1+\cdots+nq_n=n+1}} (3Ba_0 + C) \binom{2}{q_1, \ldots, q_n} a_1^{q_1} a_2^{q_2} \cdots a_n^{q_n} \\ & + \sum_{\substack{t_1+\cdots+t_n=3 \\ t_1+\cdots+nt_n=n+1}} \binom{3}{t_1, \ldots, t_n} a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n} \qquad (n \geq 3), \end{aligned} \tag{2.17}$$

satisfies the differential equation (2.14).

Proof. By Lemma 2.4, we have $\lambda_0 = 0$. From (2.5), the first term of A_n is

$$(3Ba_0^2 + 2Ca_0 - \lambda_n)a_n e^{-n\lambda_1 s}.$$

For $n = 1$, using similar arguments as in the proof of Theorem 2.8, we get

$$0 < \lambda_1 = 3Ba_0^2 + 2Ca_0, \ \ Ba_0^3 + Ca_0^2 + D = 0.$$

For $n = 2$, using (2.15), the first term of A_2 is $-\lambda_1 a_2 e^{-\lambda_2 s}$, while the term with exponent $2\lambda_1$ in $F(\varphi_{1,2}, \varphi_{1,2}^{'})$ is $3Ba_0a_1^2 + Ca_1^2$. Comparing these terms, we get a_2 as given in (2.16). Proceeding by induction from $\varphi(s) = \varphi_{1,n+1}(s) + \varphi_{2,n+1}(s)$, using (2.15), we see that the first term of A_{n+1} is $-n\lambda_1 a_{n+1}e^{-(n+1)\lambda_1 s}$, while the coefficient of the term with exponent $(n+1)\lambda_1$ in $F(\varphi_{1,n+1}, \varphi_{1,n+1}^{'})$ is

$$B \sum_{\substack{t_0+\cdots+t_n=3 \\ t_1+\cdots+nt_n=n+1}} \binom{3}{t_0,\ldots,t_n} a_0^{t_0}a_1^{t_1}\cdots a_n^{t_n} + C \sum_{\substack{q_0+\cdots+q_n=2 \\ q_1+\cdots+nq_n=n+1}} \binom{2}{q_0,\ldots,q_n} a_0^{q_0}a_1^{q_1}\cdots a_n^{q_n}$$

$$= \sum_{\substack{q_1+\cdots+q_n=2 \\ q_1+\cdots+nq_n=n+1}} \left\{ B\binom{3}{1,q_1,\ldots,q_n} a_0 + C\binom{2}{q_1,\ldots,q_n} \right\} a_1^{q_1}\cdots a_n^{q_n}$$

$$+ \sum_{\substack{t_1+\cdots+t_n=3 \\ t_1+\cdots+nt_n=n+1}} \binom{3}{t_1,\ldots,t_n} a_1^{t_1}\cdots a_n^{t_n}, \qquad (q_0 = 0 \text{ and } t_0 = 0 \text{ or } 1)$$

$$= \sum_{\substack{q_1+\cdots+q_n=2 \\ q_1+\cdots+nq_n=n+1}} (3Ba_0 + C)\tbinom{2}{q_1,\ldots,q_n} a_1^{q_1}\cdots a_n^{q_n} + \sum_{\substack{t_1+\cdots+t_n=3 \\ t_1+\cdots+nt_n=n+1}} \tbinom{3}{t_1,\ldots,t_n} a_1^{t_1}\cdots a_n^{t_n}$$

Comparing the terms, (2.17) follows. □

Using Theorem 2.10, another solution of (2.13) is now easily constructed.

Corollary 2.11. *Adopting the notation of Theorem 2.10, if* $|3Ba_0 + C| \leq 1$, *and* $|a_1| = \frac{4}{|B|} \leq \lambda_1$, *then for* $|e^{-\lambda_1 s}| < 1/4$ *the corresponding Dirichlet series* (1.1) *converges and satisfies* (2.14) .

Proof. We first show that $|a_n| \leq \binom{2(n-1)}{n-1}\frac{4}{|B|} = \binom{2(n-1)}{n-1}|a_1|$. This is trivial for $n = 1$. Using Theorem 2.10, (2.17), and the induction hypothesis, we get

$$|a_{n+1}| \leq \frac{1}{n\lambda_1} \sum_{\substack{q_1+\cdots+q_n=2 \\ q_1+\cdots+nq_n=n+1}} \binom{2}{q_1,\ldots,q_n} |a_1|^{q_1}\cdots|a_n|^{q_n}$$

$$+ \frac{1}{n\lambda_1} \sum_{\substack{t_1+\cdots+t_n=3 \\ t_1+\cdots+nt_n=n+1}} |B|\binom{3}{t_1,\ldots,t_n} |a_1|^{t_1}\cdots|a_n|^{t_n}$$

$$\leq \frac{1}{n\lambda_1} \sum_{\substack{q_1+\cdots+q_n=2 \\ q_1+\cdots+nq_n=n+1}} \binom{2}{q_1,\ldots,q_n} |a_1|^{q_1} (2|a_1|)^{q_2} \cdots \left(\binom{2(n-1)}{n-1}|a_1|\right)^{q_n}$$

$$+ \frac{1}{n\lambda_1} \sum_{\substack{t_1+\cdots+t_n=3 \\ t_1+\cdots+nt_n=n+1}} |B| \binom{3}{t_1,\ldots,t_n} |a_1|^{t_1} (2|a_1|)^{t_2} \cdots \left(\binom{2(n-1)}{n-1}|a_1|\right)^{t_n}$$

$$= \frac{1}{n\lambda_1} \sum_{\substack{q_1+\cdots+q_n=2 \\ q_1+\cdots+nq_n=n+1}} \binom{2}{q_1,\ldots,q_n} \binom{2(1)}{1}^{q_2} \cdots \binom{2(n-1)}{n-1}^{q_n} |a_1|^2$$

$$+ \frac{1}{n\lambda_1} \sum_{\substack{t_1+\cdots+t_n=3 \\ t_1+\cdots+nt_n=n+1}} |B| \binom{3}{t_1,\ldots,t_n} |a_1|^{t_1} \binom{2(1)}{1}^{t_2} \cdots \binom{2(n-1)}{n-1}^{t_n} |a_1|^3$$

$$\leq \frac{|a_1|}{n} \sum_{\substack{q_1+\cdots+q_n=2 \\ q_1+\cdots+nq_n=n+1}} \binom{2}{q_1,\ldots,q_n} \binom{2(1)}{1}^{q_2} \cdots \binom{2(n-1)}{n-1}^{q_n}$$

$$+ \frac{4|a_1|}{n} \sum_{\substack{t_1+\cdots+t_n=3 \\ t_1+\cdots+nt_n=n+1}} \binom{3}{t_1,\ldots,t_n} \binom{2(1)}{1}^{t_2} \cdots \binom{2(n-1)}{n-1}^{t_n},$$

using $|a_1|^2 \leq a_1\lambda_1$ and $|a_1|^3 \leq |a_1| \frac{4}{|B|}\lambda_1$,

$$\leq \frac{|a_1|}{n} \left(\sum_{\substack{q_1+\cdots+q_n=2 \\ q_1+\cdots+nq_n=n+1}} \tbinom{2}{q_1,\ldots,q_n} 2^{q_2} \cdots \tbinom{2(n-1)}{n-1}^{q_n} + \sum_{\substack{t_1+\cdots+t_n=3 \\ t_1+\cdots+nt_n=n+1}} 4\tbinom{3}{t_1,\ldots,t_n} 2^{t_2} \cdots \tbinom{2(n-1)}{n-1}^{t_n} \right).$$

Observe that the term $\sum_{\substack{q_1+\cdots+q_n=2 \\ q_1+\cdots+nq_n=n+1}} \binom{2}{q_1,\ldots,q_n} \binom{2(1)}{1}^{q_2} \cdots \binom{2(n-1)}{n-1}^{q_n}$ is the coefficient of x^{n-1} in the expansion $\left(1 + 2x + \cdots + \binom{2(n-1)}{n-1} x^{n-1}\right)^2$ which is also the coefficient of x^{n-1} in the expansion

$$\left(\sum_{j=0}^{\infty} \binom{2(j-1)}{j-1} x^{j-1} \right)^2 = \frac{1}{1-4x} \quad \text{(convergent when } |x| < 1/4)$$

Similarly, the term $\sum_{\substack{t_1+\cdots+t_n=3 \\ t_1+\cdots+nt_n=n+1}} \binom{3}{t_1,\ldots,t_n} \binom{2(1)}{1}^{t_2} \cdots \binom{2(n-1)}{n-1}^{t_n}$ is the coefficient of x^{n-2} in the expansion $(1-4x)^{-2/3}$.

Note that $\binom{2n}{n} - \frac{2^n}{n}\left(2^{n-2} + \frac{3(5)\cdots(2n-3)}{(n-2)!}\right)$

$$
\begin{aligned}
&= 2^n \left(\frac{3\cdot 5\cdots(2n-1)}{n!} - \left(\frac{2^{n-2}}{n} + \frac{3\cdot 5\cdots(2n-3)}{n(n-2)!} \right) \right) \\
&= 2^n \left(\frac{3\cdot 5\cdots(2n-1)}{n!} - \frac{(n-1)!2^{n-2}}{n!} - \frac{3\cdot 5\cdots(2n-3)(n-1)}{n!} \right) \\
&= \frac{2^n}{n!} \left(3\cdot 5\cdots(2n-3)(2n-1-(n-1)) - \frac{(n-1)!2^{n-1}}{2} \right) \\
&= \frac{2^n}{n!} \left(3\cdot 5\cdots(2n-3)n - 2\cdot 4\cdots(2n-4)(2n-2)\cdot\frac{1}{2} \right) \\
&= \frac{2^n}{n!} \{3\cdot 5\cdots(2n-3)n - 2\cdot 4\cdots(2n-4)(n-1)\} \geq 0,
\end{aligned}
$$

so $\frac{2^n}{n}\left(2^{n-2} + \frac{3(5)\cdots(2n-3)}{(n-2)!}\right) \leq \binom{2n}{n}$ It follows that

$$
|a_{n+1}| \leq |a_1| \frac{2^n}{n} \left(2^{n-2} + \frac{3(5)\cdots(2n-3)}{(n-2)!} \right) \leq \frac{4}{|B|}\binom{2n}{n}.
$$

The corresponding Dirichlet series solution thus converges as the following majorization shows

$$
\begin{aligned}
|\varphi(s)| = \left| \sum_{j=0}^{\infty} a_i e^{-\lambda_j s} \right| &\leq \sum_{j=0}^{\infty} \binom{2(j-1)}{j-1} \frac{4}{|B|} |e^{-(j-1)\lambda_1 s}||e^{-\lambda_1 s}| \\
&= \frac{4|e^{-\lambda_1 s}|}{|B|\sqrt{1-4|e^{-\lambda_1 s}|}} \quad \text{(provided that } |e^{-\lambda_1 s}| < 1/4).
\end{aligned}
$$

□

CHAPTER III
POWER SERIES SOLUTIONS

In this chapter, we provide an upper bound for the coefficients of a power series,

$$f = \sum_{k=0}^{\infty} f_k z^k \quad \text{where } f_k \in \mathbb{C} \tag{3.1}$$

which satisfies:

(i) a linear differential equation; $F(f, f') = f\,' + Bf + C = 0$, where $B(\neq 0),\ C$ are polynomials with complex coefficients,

(ii) a Riccati differential equation; $F(f, f') = f\,' + Bf^2 + Cf + D = 0$, where $B(\neq 0),\ C,\ D$ are polynomials with complex coefficients.

The proofs of our results in both cases, depend directly on the following Lemma, quoted from [3].

Lemma 3.1. *[3] Let $h \geq 1$ and $N \geq 0$ be any two integers, and let $\omega_0, \omega_1, \ldots, \omega_N$ be $N+1$ power series with real or complex coefficients. Then*

$$\left(\frac{d}{dz}\right)^h (\omega_0 \omega_1 \cdots \omega_N) = h! \sum_{\lambda_0, \lambda_1, \ldots, \lambda_N} \frac{\omega_0^{(\lambda_0)}}{\lambda_0!} \frac{\omega_1^{(\lambda_1)}}{\lambda_1!} \cdots \frac{\omega_N^{(\lambda_N)}}{\lambda_N!}$$

where the summation taken over all ordered systems of integers $\lambda_0, \lambda_1, \ldots, \lambda_N$ such that

$$\lambda_0 \geq 0, \lambda_1 \geq 0, \ldots, \lambda_N \geq 0 \quad \textit{and} \quad \lambda_0 + \lambda_1 + \cdots + \lambda_N = h$$

It is clear that this lemma also holds for $h = 0$.

3.1 Linear differential equations

A linear differential equation that we are interested to study in this part is an ADE of the form

$$F(z, f, f^{'}) = f^{'} + Bf + C = 0, \tag{3.2}$$

where $B = B_0 + B_1 z + \ldots + B_s z^s$, and $C = C_0 + C_1 z + \cdots + C_t z^t$ are polynomials with complex coefficients.

Theorem 3.2. *Assume that a power series $f = \sum_{k=0}^{\infty} f_k z^k$ satisfies (3.2). Let $m = \max\{s, t\}$. Then for each $k \geq m + 3$, there exist constants C_1, C_2 such that*

$$|f_k| \leq \frac{C_1}{k} C_2^{k-m-2}.$$

Proof. Fixed $k \geq m + 1$, by Lemma 3.1,

$$\begin{aligned} 0 &= \left(\frac{d}{dz}\right)^k F(z, f, f^{'}) = \left(\frac{d}{dz}\right)^k (f^{'} + Bf + C) \\ &= f^{(k+1)} + k! \sum_{\lambda_0 + \lambda_1 = k} \frac{B^{(\lambda_0)} f^{(\lambda_1)}}{\lambda_0! \lambda_1!}, \end{aligned}$$

$$|f^{(k+1)}| = \left| k! \sum_{\lambda_0 + \lambda_1 = k} \frac{B^{(\lambda_0)}(z) f^{(\lambda_1)}}{\lambda_0! \lambda_1!} \right|. \tag{3.3}$$

Replacing $z = 0$ again in (3.3), we get

$$\begin{aligned} (k+1)! |f_{k+1}| &= k! \left| \sum_{\lambda_0 + \lambda_1 = k} B_{\lambda_0} f_{\lambda_1} \right| \\ &= k! |B_0 f_k + B_1 f_{k-1} + \cdots + B_s f_{k-s}| \\ &\leq k! A \sum_{k-s}^{k} |f_i| \\ &\leq k! A \sum_{k-s}^{k} 1 \max_{k-s \leq i \leq k} |f_i|, \end{aligned}$$

where $A = \max\{|B_0|, \ldots, |B_s|\}$.

Thus we get $|f_{k+1}| \leq A\left(\frac{s+1}{k+1}\right)\max\{|f_{k-s}|, |f_{k-s+1}|, \ldots, |f_k|\}$, so

$$|f_k| \leq A\left(\frac{s+1}{k}\right)\max\{|f_{k-s-1}|, |f_{k-s}|, \ldots, |f_{k-1}|\} \text{ for each } k \geq m+2.$$

If we define $\mu_{m-s+1}, \ldots, \mu_{m+1}$ to be the real numbers such that

$$|f_{m-s+1}| \leq e^{\mu_{m-s+1}}, \ldots, |f_{m+1}| \leq e^{\mu_{m+1}}$$

and for each $i \geq m+2$, $\mu_i = \ln(A(s+1)) - \ln(i) + \max_{i-s-1 \leq j \leq i-1} \mu_j$, then we claim that $|f_k| \leq e^{\mu_k}$, $k \geq m+2$.

For $k = m+2$,

$$\begin{aligned} |f_{m+2}| &\leq e^{\ln\left(\frac{A(s+1)}{m+2}\max\{|f_{m-s+1}|,\ldots,|f_{m+1}|\}\right)} \\ &\leq e^{\ln\left(\frac{A(s+1)}{m+2}\max\{e^{\mu_{m-s+1}},\ldots,e^{\mu_{m+1}}\}\right)} = e^{\mu_{m+2}}. \end{aligned}$$

Next, we assume that $|f_k| \leq e^{\mu_k}$, for all $k \geq m+2$. Then

$$|f_{k+1}| \leq e^{\ln\left(\frac{A(s+1)}{m+2}\max\{|f_{k-s}|,\ldots,|f_k|\}\right)} \leq e^{\ln\left(\frac{A(s+1)}{k}\max\{e^{\mu_{k-s}},\ldots,e^{\mu_k}\}\right)} = e^{\mu_{k+1}},$$

so we get the claim.

Let $f_{\lambda_1}, \ldots, f_{\lambda_p}$ be all nonzero elements in $\{f_{m-s+1}, \ldots, f_{m+1}\}$. Setting $|f_{\lambda_1}| = e^{\mu_{\lambda_1}}, \ldots, |f_{\lambda_p}| = e^{\mu_{\lambda_p}}$, $\mu_i = 0$ for all $i \in \{m-s+1, \ldots, m+1\} \backslash \{\lambda_1, \ldots, \lambda_p\}$, and

$$\mu_i = \ln(A(s+1)) - \ln(i) + \max_{i-s-1 \leq j \leq i-1} \mu_j \text{ for each } i \geq m+2 \tag{3.4}$$

then we also get $|f_k| \leq e^{\mu_k}, k \geq m+2$.

Since $\mu_k = \ln(A(s+1)) - \ln(k) + \max_{k-s-1 \leq j \leq k-1} \mu_j$ and $\mu_{k+1} = \ln(A(s+1)) -$

$\ln(k+1) + \max_{k-s \le j \le k} \mu_j$,

$$\mu_{k+1} - \mu_k = \ln\left(\frac{k}{k+1}\right) + \max_{k-s \le j \le k} \mu_j - \max_{k-s-1 \le j \le k-1} \mu_j, \text{ for each } k \ge m+2. \tag{3.5}$$

Then there exists γ such that $k - s \le \gamma \le k$ and $\max_{k-s \le j \le k} \mu_j = \mu_\gamma$. We observe that

$$\begin{aligned}\max_{k-s-1 \le j \le k-1} \mu_j &\ge \mu_{\gamma-1}, k-s-1 \le \gamma - 1 \le k-1 \\ &= \mu_{\gamma-1} + \mu_\gamma - \mu_\gamma \\ &= \max_{k-s \le j \le k} \mu_j - (\mu_\gamma - \mu_{\gamma-1}).\end{aligned}$$

Thus $\max_{k-s \le j \le k} \mu_j - \max_{k-s-1 \le j \le k-1} \mu_j \le \mu_\gamma - \mu_{\gamma-1} \le \max\{\mu_{k-s} - \mu_{k-s-1}, \ldots, \mu_k - \mu_{k-1}\}$. Hence, (3.5) becomes

$$\mu_{k+1} - \mu_k = \ln\left(\frac{k}{k+1}\right) + \max\{\mu_{k-s} - \mu_{k-s-1}, \ldots, \mu_k - \mu_{k-1}\}, \text{ for all } k \ge m+2. \tag{3.6}$$

Let $q = \max\{\mu_{m-s+2} - \mu_{m-s+1}, \ldots, \mu_{m+2} - \mu_{m+1}\}$. Next, we claim that $\mu_k - \mu_{k-1} \le \ln\left(\frac{m+2}{m+3}\right) + \max\{\mu_{m-s+2} - \mu_{m-s+1}, \ldots, \mu_{m+2} - \mu_{m+1}\}$ for each $k \ge m+3$. For $k = m+3$, by (3.6),

$$\begin{aligned}\mu_{m+3} - \mu_{m+2} &\le \ln\left(\frac{m+2}{m+3}\right) + \max\{\mu_{m-s+2} - \mu_{m-s+1}, \ldots, \mu_{m+2} - \mu_{m+1}\} \\ &= \ln\left(\frac{m+2}{m+3}\right) + q.\end{aligned}$$

Assume that $\mu_k - \mu_{k-1} \le \ln\left(\frac{k-1}{k}\right) + q$, $k \ge m+3$.
Then $\mu_{k+1} - \mu_k \le \ln\left(\frac{k}{k+1}\right) + \max\{\mu_{k-s} - \mu_{k-s-1}, \ldots, \mu_k - \mu_{k-1}\} \le \ln\left(\frac{k}{k+1}\right) + q$.
Hence,

$$\mu_k - \mu_{k-1} \le \ln\left(\frac{k-1}{k}\right) + q, \text{ for each } k \ge m+3.$$

Then

$$\begin{aligned}\mu_k - \mu_{m+2} &= \mu_k - \mu_{k-1} + \mu_{k-1} - \mu_{k-2} + \cdots + \mu_{m+3} - \mu_{m+2}\\ &\le \ln\left(\frac{k-1}{k}\right) + q + \ln\left(\frac{k-2}{k-1}\right) + q + \cdots + \ln\left(\frac{m+2}{m+3}\right) + q\\ &= \ln\left(\frac{m+2}{k}\right) + (k-m-2)q,\end{aligned}$$

so $\mu_k \le \ln\left(\frac{m+2}{k}\right) + (k-m-2)q + \mu_{m+2}$. From (3.4),

$$\begin{aligned}\mu_{m+2} &= \ln(A(s+1)) - \ln(m+2) + \max\{\mu_{m-s+1}, \ldots, \mu_{m+1}\}\\ &= \ln(A(s+1)) - \ln(m+2) + \max\{0, \ln|f_{\lambda_1}|, \ldots, \ln|f_{\lambda_p}|\}.\end{aligned}$$

Hence,

$$|f_k| \le e^{\mu_k} \le \frac{A(s+1)}{k} \max\{1, |f_{\lambda_1}|, \ldots, |f_{\lambda_p}|\} \left(e^q\right)^{k-m-2},$$

for each $k \ge m+3$. Setting $C_1 = A(s+1)\max\{1, |f_{\lambda_1}|, \ldots, |f_{\lambda_p}|\}$ and $C_2 = e^q$, we get

$$|f_k| \le \frac{C_1}{k} C_2^{k-m-2}.$$

for each $k \ge m+3$.

□

3.2 Riccati differential equations

A Riccati differential equation, an ADE of the form

$$F(z, f, f') = f' + Bf^2 + Cf + D = 0, \tag{3.7}$$

where $B = B_0 + B_1 z + \ldots + B_s z^s$, $C = C_0 + C_1 z + \cdots + C_t z^t$, and $D = D_0 + D_1 z + \cdots + D_w z^w$ are polynomials with complex coefficients.

Theorem 3.3. *Assume that a power series* $f = 1 + \sum_{k=1}^{\infty} f_k z^k$ *satisfies* (3.7).

Then for each $k \in \mathbb{N}$,

$$|f_k| \leq \left(\frac{A}{2}\right)^k \frac{(k+2)!(k+1)}{2},$$

where $A = \max\{|B_0|, \dots, |B_s|, |C_0|, \dots, |C_t|, |D_0|, \dots, |D_w|\}$.

Proof. Replacing $z = 0$ in (3.7), we get $f_1 + B_0 f_0^2 + c_0 f_0 + D_0 = 0$, and then

$$|f_1| = |B_0 f_0^2 + C_0 f_0 + D_0| = |B_0 + C_0 + D_0| \leq 3A.$$

Fixed $k \geq 0$, by Lemma 3.1,

$$\begin{aligned} 0 &= \left(\frac{d}{dz}\right)^k F(z, f, f^{'}) = \left(\frac{d}{dz}\right)^k (f^{'} + Bf^2 + Cf + D) \\ &= f^{(k+1)} + k! \sum_{\lambda_0+\lambda_1+\lambda_2=k} \frac{B^{(\lambda_0)}(z) f^{(\lambda_1)} f^{(\lambda_2)}}{\lambda_0!\lambda_1!\lambda_2!} + k! \sum_{\gamma_0+\gamma_1=k} \frac{C^{(\gamma_0)}(z) f^{(\gamma_1)}}{\gamma_0!\gamma_1!} + D^{(k)}(z), \end{aligned}$$

$$|f^{(k+1)}| = \left| k! \sum_{\lambda_0+\lambda_1+\lambda_2=k} \frac{B^{(\lambda_0)}(z) f^{(\lambda_1)} f^{(\lambda_2)}}{\lambda_0!\lambda_1!\lambda_2!} + k! \sum_{\gamma_0+\gamma_1=k} \frac{C^{(\gamma_0)}(z) f^{(\gamma_1)}}{\gamma_0!\gamma_1!} + D^{(k)}(z) \right|. \tag{3.8}$$

Replacing $z = 0$ again in (3.8), we get

$$\begin{aligned} (k+1)!|f_{k+1}| &= k! \left| \sum_{\lambda_0+\lambda_1+\lambda_2=k} B_{\lambda_0} f_{\lambda_1} f_{\lambda_2} + \sum_{\gamma_0+\gamma_1=k} C_{\gamma_0} f_{\gamma_1} + D_k \right|, \\ |f_{k+1}| &\leq \frac{1}{k+1} \left(\sum_{\lambda_0+\lambda_1+\lambda_2=k} |B_{\lambda_0}||f_{\lambda_1}||f_{\lambda_2}| + \sum_{\gamma_0+\gamma_1=k} |C_{\gamma_0}||f_{\gamma_1}| + |D_k| \right). \end{aligned}$$

Let $A = \max\{|B_0|, \dots, |B_s|, |C_0|, \dots, |C_t|, |D_0|, \dots, |D_w|\}$. Then

$$|f_{k+1}| \leq \tfrac{A}{k+1} \left(\sum_{\lambda_0+\lambda_1+\lambda_2=k} |f_{\lambda_1}||f_{\lambda_2}| + \sum_{\gamma_0+\gamma_1=k} |f_{\gamma_1}| + 1 \right)$$

$$\leq \tfrac{A}{k+1} \left(\sum_{\lambda_0+\lambda_1+\lambda_2=k} 1 + \sum_{\gamma_0+\gamma_1=k} 1 + 1 \right) \max\{\max_{\lambda_0+\lambda_1+\lambda_2=k} |f_{\lambda_1}||f_{\lambda_2}|, \max_{\gamma_0+\gamma_1=k} |f_{\gamma_1}|, 1\}.$$

Next, we claim that

$$\max\{\max_{\lambda_0+\lambda_1+\lambda_2=k}|f_{\lambda_1}||f_{\lambda_2}|, \max_{\gamma_0+\gamma_1=k}|f_{\gamma_1}|, 1\} = \max_{\lambda_0+\lambda_1+\lambda_2=k}|f_{\lambda_1}||f_{\lambda_2}|.$$

Since $\max_{\gamma_0+\gamma_1=k}|f_{\gamma_1}| = |f_j|$ for some $0 \leq j \leq k$,

$$\begin{aligned}\max_{\lambda_0+\lambda_1+\lambda_2=k}|f_{\lambda_1}||f_{\lambda_2}| &\geq |f_0||f_j| \quad (\lambda_0 = k-j)\\ &= \max_{\gamma_0+\gamma_1=k}|f_{\gamma_1}| \geq |f_0| = 1,\end{aligned}$$

and our claim is proved. Then we get

$$\begin{aligned}|f_{k+1}| &\leq \frac{A}{k+1}\left(\sum_{\lambda_0+\lambda_1+\lambda_2=k}1 + \sum_{\gamma_0+\gamma_1=k}1+1\right)\max_{\lambda_0+\lambda_1+\lambda_2=k}|f_{\lambda_1}||f_{\lambda_2}|,\\ &= \frac{A}{k+1}\left(\binom{k+2}{2}+k+1+1\right)\max_{\lambda_0+\lambda_1+\lambda_2=k}|f_{\lambda_1}||f_{\lambda_2}|,\\ &= \frac{A}{k+1}\left(\frac{(k+1)(k+4)+2}{2}\right)\max_{\lambda_0+\lambda_1+\lambda_2=k}|f_{\lambda_1}||f_{\lambda_2}|.\end{aligned}$$

Hence, we can conclude that for each $k \geq 1$,

$$\begin{aligned}|f_k| &\leq A\left(\frac{(k+1)(k+2)}{2k}\right)\max_{\lambda_0+\lambda_1+\lambda_2=k-1}|f_{\lambda_1}||f_{\lambda_2}|\\ &= A\alpha_k \max_{\lambda_0+\lambda_1+\lambda_2=k-1}|f_{\lambda_1}||f_{\lambda_2}|, \qquad (3.9)\end{aligned}$$

where $\alpha_k = \frac{(k+1)(k+2)}{2k}$. Next, we define $\mu_0 = \ln|f_0|$, and for $j \geq 1$,

$$\mu_j = \ln A + \ln\alpha_j + \max_{\lambda_0+\lambda_1+\lambda_2=j-1}\{\mu_{\lambda_1}+\mu_{\lambda_2}\}. \qquad (3.10)$$

We claim now that $|f_k| \leq e^{\mu_k}$ for all $k \geq 1$. For $k = 1$, (3.9) and (3.10) imply

$$\begin{aligned}|f_1| &\leq e^{\ln A+\ln\alpha_1+\max_{\lambda_0+\lambda_1+\lambda_2=0}\{\ln|f_{\lambda_1}|+\ln|f_{\lambda_2}|\}}\\ &= e^{\ln A+\ln\alpha_1+\max_{\lambda_0+\lambda_1+\lambda_2=0}\{\mu_{\lambda_1}+\mu_{\lambda_2}\}} = e^{\mu_1}.\end{aligned}$$

Suppose for induction that $f_{k-1} \leq e^{\mu_{k-1}}$. From (3.9), the induction hypothesis, and (3.10)

$$|f_j| \leq e^{\ln A + \ln \alpha_j + \max\limits_{\lambda_0+\lambda_1+\lambda_2=j-1}\{\ln|f_{\lambda_1}|+\ln|f_{\lambda_2}|\}} \leq e^{\ln A + \ln \alpha_j + \max\limits_{\lambda_0+\lambda_1+\lambda_2=j-1}\{\mu_{\lambda_1}+\mu_{\lambda_2}\}} = e^{\mu_j}.$$

Now we have

$$\mu_k = \ln A + \ln \alpha_k + \max_{\lambda_0+\lambda_1+\lambda_2=k-1}\{\mu_{\lambda_1} + \mu_{\lambda_2}\}$$

and

$$\mu_{k+1} = \ln A + \ln \alpha_{k+1} + \max_{\sigma_0+\sigma_1+\sigma_2=k}\{\mu_{\sigma_1} + \mu_{\sigma_2}\}.$$

Thus,

$$\mu_{k+1} - \mu_k = \ln\left(\frac{\alpha_{k+1}}{\alpha_k}\right) + \max_{\sigma_0+\sigma_1+\sigma_2=k}\{\mu_{\sigma_1} + \mu_{\sigma_2}\} - \max_{\lambda_0+\lambda_1+\lambda_2=k-1}\{\mu_{\lambda_1} + \mu_{\lambda_2}\}. \quad (3.11)$$

Let $\pi_0, \pi_1 \in \mathbb{N} \cup \{0\}$ and $\pi_2 \in \mathbb{N}$ be such that $\pi_0 + \pi_1 + \pi_2 = k$, and

$$\max_{\sigma_0+\sigma_1+\sigma_2=k}\{\mu_{\sigma_1} + \mu_{\sigma_2}\} = \mu_{\pi_1} + \mu_{\pi_2}.$$

Then

$$\begin{aligned} \max_{\lambda_0+\lambda_1+\lambda_2=k-1}\{\mu_{\lambda_1} + \mu_{\lambda_2}\} &\geq \mu_{\pi_1} + \mu_{\pi_2-1}, \qquad \pi_0 + \pi_1 + (\pi_2 - 1) = k - 1 \\ &= \mu_{\pi_1} + \mu_{\pi_2-1} + \mu_{\pi_2} - \mu_{\pi_2} \\ &= \max_{\sigma_0+\sigma_1+\sigma_2=k}\{\mu_{\sigma_1} + \mu_{\sigma_2}\} - \{\mu_{\pi_2} - \mu_{\pi_2-1}\}, \end{aligned}$$

so

$$\begin{aligned} \max_{\sigma_0+\sigma_1+\sigma_2=k}\{\mu_{\sigma_1} + \mu_{\sigma_2}\} - \max_{\lambda_0+\lambda_1+\lambda_2=k-1}\{\mu_{\lambda_1} + \mu_{\lambda_2}\} &\leq \{\mu_{\pi_2} - \mu_{\pi_2-1}\} \\ &\leq \max\{\mu_1 - \mu_0, \ldots, \mu_k - \mu_{k-1}\}. \end{aligned}$$

The last inequality is true bacause of $1 \leq \pi_2 \leq k$. Hence, (3.11) becomes

$$\mu_{k+1} - \mu_k \leq \ln\left(\frac{\alpha_{k+1}}{\alpha_k}\right) + \max\{\mu_1 - \mu_0, \ldots, \mu_k - \mu_{k-1}\}. \tag{3.12}$$

Next, we claim that $\mu_{k+1} - \mu_k \leq \ln \alpha_{k+1} - \ln \alpha_1 + \mu_1 - \mu_0$.

We prove this claim by induction. For $k = 1$, using (3.12), we get

$$\mu_2 - \mu_1 \leq \ln \alpha_2 - \ln \alpha_1 + \mu_1 - \mu_0.$$

Suppose that $\mu_k - \mu_{k-1} \leq \ln \alpha_k - \ln \alpha_1 + \mu_1 - \mu_0$. We need to show that $\mu_{k+1} - \mu_k \leq \ln \alpha_{k+1} - \ln \alpha_1 + \mu_1 - \mu_0$. Using (3.12), we obtain

$$\mu_{k+1} - \mu_k \leq \ln \alpha_{k+1} - \ln \alpha_k + \max\{\mu_1 - \mu_0, \ldots, \mu_k - \mu_{k-1}\}.$$

Applying the induction hypothesis yields

$$\begin{aligned}\mu_{k+1} - \mu_k &\leq \ln \alpha_{k+1} - \ln \alpha_k \\ &\quad + \max\{\mu_1 - \mu_0, \ln \alpha_2 - \ln \alpha_1 + \mu_1 - \mu_0, \ldots, \ln \alpha_k - \ln \alpha_1 + \mu_1 - \mu_0\}.\end{aligned}$$

It is easy to see that α_k is an increasing function of variable k, then

$$\mu_{k+1} - \mu_k \leq \ln \alpha_{k+1} - \ln \alpha_1 + \mu_1 - \mu_0,$$

as desired. Then

$$\begin{aligned}\mu_k - \mu_1 &= \mu_k - \mu_{k-1} + \mu_{k-1} - \mu_{k-2} + \cdots + \mu_2 - \mu_1 \\ &\leq (\ln \alpha_k - \ln \alpha_1 + \mu_1 - \mu_0) + (\ln \alpha_{k-1} - \ln \alpha_1 + \mu_1 - \mu_0) \\ &\quad + \cdots + (\ln \alpha_2 - \ln \alpha_1 + \mu_1 - \mu_0) \\ &= \ln(\alpha_k \cdots \alpha_2) + (k-1)(\mu_1 - \mu_0) - (k-1)\ln \alpha_1.\end{aligned}$$

Because of $\mu_0 = \ln|f_0| = 0$,

$$\begin{aligned} \mu_k &\le \ln(\alpha_k \cdots \alpha_2) + k\mu_1 - k\ln\alpha_1 + \ln\alpha_1 \\ &= \ln(\alpha_k \cdots \alpha_1) + k\mu_1 - k\ln\alpha_1. \end{aligned} \tag{3.13}$$

Using (3.10), we get

$$\mu_1 = \ln A + \ln\alpha_1 + \max_{\lambda_0+\lambda_1+\lambda_2=0}\{\mu_{\lambda_1} + \mu_{\lambda_2}\} = \ln A + \ln\alpha_1.$$

Thus, (3.13) becomes

$$\mu_k \le \ln(\alpha_k \cdots \alpha_1) + k\ln A.$$

Finally, we get

$$\begin{aligned} e^{\mu_k} = (\alpha_k \cdots \alpha_1)A^k &= A^k \prod_{n=1}^{k} \alpha_n \\ &= A^k \prod_{n=1}^{k} \frac{(n+2)(n+1)}{2n} \\ &= A^k \frac{(k+2)\cdots 3(k+1)\cdots 2}{2^k k!} \\ &= \left(\frac{A}{2}\right)^k \frac{(k+2)!(k+1)}{2}. \end{aligned}$$

Therefore,

$$|f_k| \le \left(\frac{A}{2}\right)^k \frac{(k+2)!(k+1)}{2}, \quad \text{for all } k \in \mathbb{N}.$$

□

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