CHAPTER IV

QUANTUM TRANSPORT FOR CLEAN SYSTEM

Introduction

In this chapter we would like to determine the transport property of a two-dimensional system without imperfections in the system. This is a limiting case for the system of our consideration on transportation with imperfections. For this clean system, the Hamiltonian is

$$H = \frac{1}{2m}(\vec{p} + \frac{e}{c}\vec{A})^2 + \vec{f}\cdot\vec{r}. \tag{111}$$

This describes an electron moving in a two dimensional system under an applied constant magnetic field $\vec{B}$ in perpendicular direction and an external electric field $\vec{f} = -e\vec{\mathcal{E}}$. $\vec{A}$ in this expression is the vector potential of the magnetic field satisfying $\vec{B} = \nabla \times \vec{A}$.

In path integrals scheme, we would like to formulate this problem from the Lagrangian, $\mathcal{L}(\vec{r}, \dot{\vec{r}}; \vec{f})$. From eq.(111), it can be written in the form

$$\mathcal{L}(\vec{r}, \dot{\vec{r}}; \vec{f}) = \frac{m}{2}\dot{\vec{r}}^2 + \frac{e}{c}\vec{A}\cdot\dot{\vec{r}} + \vec{f}\cdot\vec{r} \tag{112}$$

We will choose the symmtric gauge for the vector potential, $\vec{A} = (1/2)(yB, -xB)$. To treat this problem in more details, we would like to use the matrix notations;

$$r = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \bar{r} = (x, y) \quad \text{and} \quad \epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Then the Lagrangian in eq.(112) becomes

$$\mathcal{L}(r,\dot{r};f) \;=\; \frac{1}{2}\bar{\dot{r}}\cdot m\cdot \dot{r} + \frac{eB}{2c}\bar{\dot{r}}\epsilon r - \bar{r}f \qquad (113)$$

We shall use this Lagrangian in our clean system transport problem. In next section, we would like to determine the density of states of the system, and to see how an electron has energy under this condition. We shall determine its transport coefficients in the following section.

Density of States

From the path integrals formulation, the density of states, $n(E)$, can be determined from the Fourier transform of the propagator at the origin,

$$n(E) \;=\; \frac{A}{2\pi\hbar}\int_{\infty}^{\infty} dT K(0,0,;T)e^{iET/\hbar} \qquad (114)$$

where $K(0,0;T)$ is the Feynman propagator of this system at the origin, $r_T = 0$ and $r_0 = 0$. The expression of the propagator is

$$K(r_T, r_0; T) \;=\; \int \mathcal{D}[r_t] e^{(i/\hbar)\int_0^T \mathcal{L}(r,\dot{r};f)dt} \qquad (115)$$

where $\mathcal{L}(r,\dot{r};f)$ is the Lagrangian from eq.(113). Since $\mathcal{L}(r,\dot{r};f)$ has a quadratic form, then its corresponding propagator will have exact expression, in the form

$$K(r_t, r_0; T) \;=\; F(T)e^{(i/\hbar)S_c(r;f)} \qquad (116)$$

where $F(T)$ is called the *pre factor* function which is a function of the time interval T only and can be evaluated from the classical action by the expression

$$F(T) \;=\; \left(\frac{1}{2\pi i\hbar}\right)\left(\det\left[\frac{\partial^2 S_{cl}(r_T, r_0; f)}{\partial r_T \partial r_0}\right]\right)^{-1/2} \qquad (117)$$

which is called the *van Vleck-Pauli* determinant. The classical action of this system, $S_{cl}(r_T, r_0; f)$, is well known, it takes the form

$$\begin{aligned} S_{cl}(r_T, r_0; f) &= \frac{m\Omega}{2sin(\Omega T/2)}((r_T^2 + r_0^2)cos(\Omega T/2) - 2r_T e^{\Omega\epsilon T/2} r_0) \\ &+ \frac{r_T}{sin(\Omega T/2)} \int_0^T dt e^{\Omega\epsilon(T-t)/2} sin(\Omega t/2) f \\ &+ \frac{r_0}{sin(\Omega T/2)} \int_0^T dt e^{-\Omega\epsilon t/2} sin(\Omega(T-t)/2) f \\ &+ \frac{1}{m} \int_0^T dt \int_0^T dt' f_t G(t,t') f_{t'} \end{aligned} \quad (118)$$

where $G(t, t')$ is the Green's function of this problem, it takes the form

$$\begin{aligned} G(t,t') &= -\frac{1}{(\Omega/2)sin(\Omega T/2)} e^{\Omega\epsilon(t-t')/2} (sin(\Omega(T-t)/2) sin(\Omega t'/2) H(t-t') \\ &+ sin(\Omega t/2) sin(\Omega(T-t')/2) H(t'-t)) \end{aligned} \quad (119)$$

With this classical action, we can get the corresponding prefactor $F(T)$, eq.(117), in the form

$$F(T) = \frac{1}{2\pi i\hbar} \frac{m\Omega/2}{sin(\Omega T/2)} \quad (120)$$

Then from the resulting classical action S_{cl}, eq.(118), and its corresponding prefactor $F(T)$, eq.(120), the propagator, eq.(116), at the origin will take the form

$$\begin{aligned} K(0,0;T) &= \frac{1}{2\pi i\hbar} \frac{m\Omega/2}{sin(\Omega T/2)} \\ &\cdot \exp\{-\frac{i}{\hbar} \int_0^T dt \int_0^T dt' f_t e^{\Omega\epsilon(t-t')/2} f_{t'} \\ &\cdot (sin(\Omega(T-t)/2) sin(\Omega t'/2) H(t-t') \\ &+ sin(\Omega t/2) sin(\Omega(T-t')/2) H(t'-t))\} \end{aligned} \quad (121)$$

Since

$$f = -e\mathcal{E} = -e\begin{pmatrix} \mathcal{E}_x \\ \mathcal{E}_y \end{pmatrix}$$

then we can evaluate the integrals on the exponents of eq.(121) to be

$$\begin{aligned} & \frac{i}{\hbar}\frac{e^2\mathcal{E}^2}{m(\Omega/2)sin(\Omega T/2)}\int_0^T dt\int_0^t dt' cos(\Omega(t-t')/2)sin(\Omega(T-t)/2)sin(\Omega t'/2) \\ = & \frac{i}{\hbar}\frac{(e\mathcal{E})^2}{m\Omega}\int_0^T \frac{t sin(\Omega(T-t)/2)sin(\Omega t/2)}{sin(\Omega T/2)}dt \end{aligned} \qquad (122)$$

In the limit of large magnetic field, $\Omega \to \infty$, we can approximate this integration to be

$$\frac{1}{4\hbar}\frac{(e\mathcal{E})^2}{m\Omega}T^2,$$

then eq.(121) becomes

$$K(0,0;T) = \frac{1}{2\pi i\hbar}\frac{m\Omega/2}{sin(\Omega T/2)}\exp\left\{-\frac{1}{4\hbar}\frac{(e\mathcal{E})^2}{m\Omega}T^2\right\}. \qquad (123)$$

Then its corresponding density of states becomes

$$\begin{aligned} n(E) = & \left(\frac{A}{2\pi\hbar}\right)\frac{m\Omega/2}{\pi\hbar} \\ & \cdot\int_{-\infty}^{\infty} dT \exp\left\{i(E-(n+\frac{1}{2})\hbar\Omega)\frac{T}{\hbar} - \frac{1}{4\hbar}\frac{(e\mathcal{E})^2}{m\Omega}T^2\right\} \end{aligned} \qquad (124)$$

where we have used

$$\frac{1}{sin(\Omega T/2)} = 2i\sum_{n=0}^{\infty} e^{i(n+1/2)\Omega T}$$

Let us define the notations of the magnetic length and energy as

$$l_B^2 = \frac{\hbar}{m\Omega}, \quad E_\Omega = \hbar\Omega \quad \text{and} \quad E_\mathcal{E} = e\mathcal{E}l_B$$

and dimensionless parameters

$$\tau = \Omega T, \quad E'_{\mathcal{E}} = \frac{E_{\mathcal{E}}}{E_{\Omega}} \quad \text{and} \quad E' = \frac{E}{E_{\Omega}}$$

then eq.(124) becomes

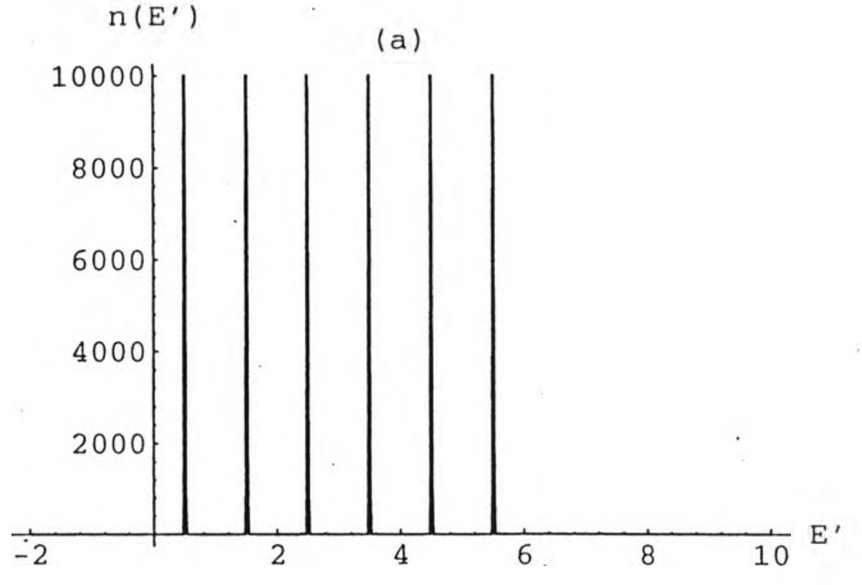
$$\begin{aligned} n(E') &= \frac{1}{4\pi}\left(\frac{A}{\pi l_B^2}\right)\sum_{n=0}^{\infty}\int_{-\infty}^{\infty} d\tau \exp\left\{i(E'-(n+1/2))\tau - \frac{1}{4}{E'_{\mathcal{E}}}^2\tau^2\right\} \\ &= \left(\frac{A}{\pi l_B^2}\right)\sum_{n=0}^{\infty}\frac{1}{{E'_{\mathcal{E}}}^2} e^{-\frac{(E'-(n+1/2))^2}{{E'_{\mathcal{E}}}^2}}. \end{aligned} \tag{125}$$

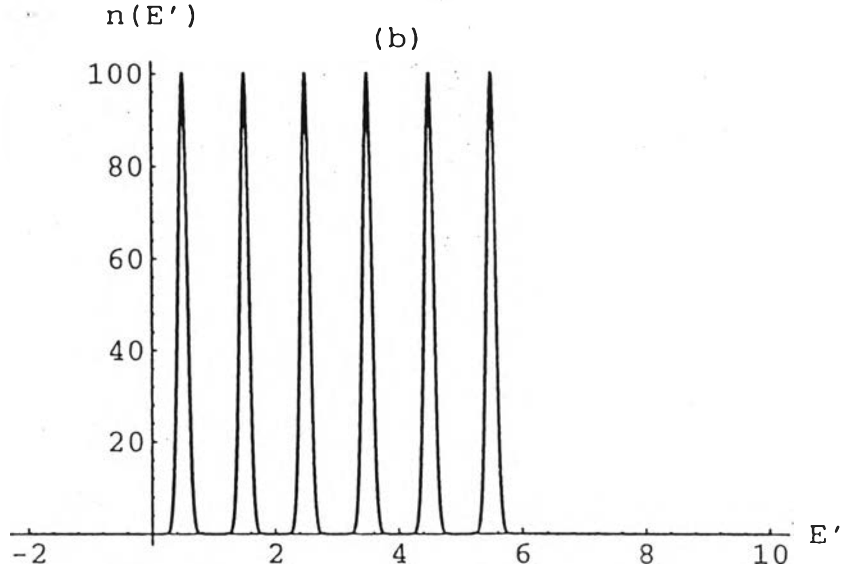
The resulting density of states in eq.(125) shows that the electric field $\mathcal{E}$ lift out the degeneracy of Landau levels from the delta function of $n_0(E)$ in the absence of the electric field,

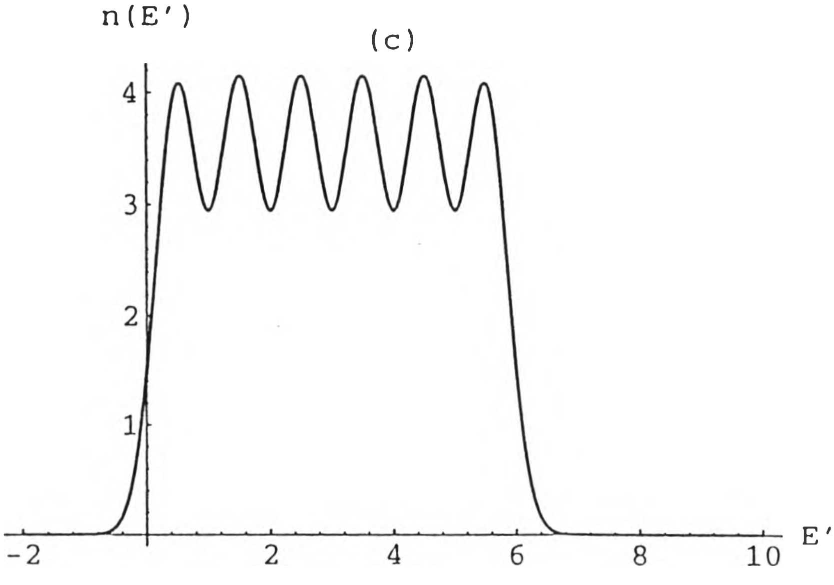
$$n_0(E') = \left(\frac{A}{\pi l_B^2}\right)\sum_{n=0}^{\infty}\delta(E'-(n+1/2)). \tag{126}$$

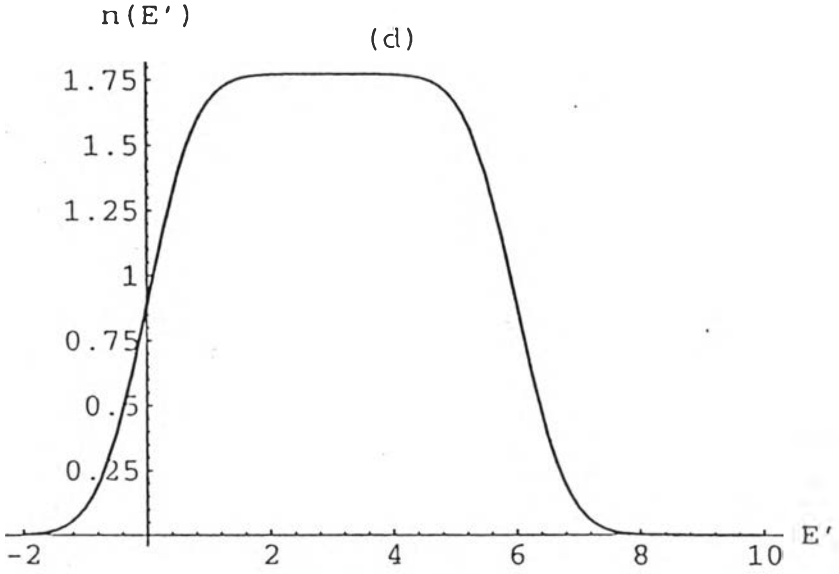
We also see that the density of states in eq.(125) will become the delta function as in eq.(126) when the electric field is small. This shows that the degeneracy comes back when the electric field is weak and it is lifted out when the electric field is strong. Figure(4.1) shows the evolution of the density of states, $n(E')$, at any electric field strength.

Figure(4.1). Plots of $n(E')$ for the first six Landau levels as a function of an electron energy E' at different values of the electric energy $E_{\mathcal{E}}$, (a) 0.01, (b) 0.05, (c) 0.1 and (d) 1.0, in the unit of magnetic energy.


n(E')
(a)
10000
8000
6000
4000
2000
-2
2
4
6
8
10
E'


n(E')
(b)
100
80
60
40
20
-2
2
4
6
8
10
E'


n(E')
(c)
4
3
2
1
-2
2
4
6
8
10
E'


n(E')
(d)
1.75
1.5
1.25
1
0.75
0.5
0.25
-2
2
4
6
8
10
E'

Transport

With the Lagrangian in eq.(112), we can write down the corresponding density matrix in the form

$$\begin{aligned}\rho(r_T, r'_T) &= \int dr_0 \int dr'_0 \rho(r_0, r'_0) \\ &\int \mathcal{D}[r_t] \int \mathcal{D}[r'_t] \exp\left\{\frac{i}{\hbar}\int_0^T dt(\mathcal{L}(r,\dot{r};f) - \mathcal{L}(r',\dot{r}';f'))\right\} \quad (127)\end{aligned}$$

Its corresponding equation of motion of the path r_t at the time $t = \tau$ is

$$m\langle\ddot{r}\rangle_\tau + \frac{eB}{c}\epsilon\langle\dot{r}\rangle_\tau + e\mathcal{E} = 0 \quad (128)$$

where the quantum average $< ... >$ is made with respect to the density matrix in eq.(127). At steady state, $<\ddot{r}>_\tau = 0$ and on the moving frame R, $r = R + u$ with no interesting was applied to $\dot{u}$ and $\ddot{u}$, eq.(128) then becomes

$$m\Omega\epsilon v_D + e\mathcal{E} = 0. \quad (129)$$

where we have used $\dot{R} = v_D$ the *drift velocity* of an electron at a steady state. From the matrix notations we have used, we can write the electric field $\mathcal{E}$ as a function of v_D as

$$e\begin{pmatrix}\mathcal{E}_x \\ \mathcal{E}_y\end{pmatrix} = -m\begin{pmatrix}0 & -\Omega \\ \Omega & 0\end{pmatrix}\begin{pmatrix}v_{Dx} \\ v_{Dy}\end{pmatrix}$$

or

$$\begin{pmatrix}v_{Dx} \\ v_{Dy}\end{pmatrix} = \frac{-e}{m}\begin{pmatrix}0 & -\Omega \\ \Omega & 0\end{pmatrix}^{-1}\begin{pmatrix}\mathcal{E}_x \\ \mathcal{E}_y\end{pmatrix} \quad (130)$$

The coefficients of the electric field are the mobility. Eq.(130) shows only the transverse, Hall, component and absence of the longitudinal one.

The mobility μ is defined from the linear coefficient of the electric field from the relation

$$v_D = \mu\mathcal{E} \tag{131}$$

then from eq.(130), we get

$$\mu_0 = -\frac{e}{m}\begin{pmatrix} 0 & -\Omega \\ \Omega & 0 \end{pmatrix}^{-1}. \tag{132}$$

This shows that $\mu_{xy} = -\mu_{yx}$ and

$$\mu_{yx} = \frac{-e}{m\Omega} \text{ and } \mu_{xx} = 0. \tag{133}$$

The conductivity σ is defined from the expression

$$J = \sigma_0\mathcal{E}$$

where the current endsity J can be written as $J = n(-e)v_D$, then we can get the relation between μ_0 and σ_0 as

$$\sigma_0 = n(-e)\mu_0 \tag{134}$$

then we find that the conductivity components take the forms

$$\sigma_{xx} = 0 \quad \text{and} \quad \sigma_{yx} = \frac{ne^2}{m\Omega} \tag{135}$$

At finite temperatures, we can use the *Kubo-Greenwood* formula for the longitudinal component of conductivity;

$$\sigma_{xx} = (-e)\int\left(-\frac{\partial f(E')}{\partial E'}\right)\mu_{xx}(E')n(E')dE' \tag{136}$$

and for the transverse component;

$$\sigma_{yx} = \frac{(-e)}{kT} \int dE' f(E') \mu_{yx}(E') n(E') \quad (137)$$

where $f(E')$ is the Fermi function. We have put *ad hoc* into our problem to keep the many particle effect;

$$f(E') = \frac{1}{\left(1 + e^{(E'-E'_F)/E'_{kT}}\right)} \quad (138)$$

with the dimensionless energies, $E'_F = E_F/E_\Omega$ is Fermi energy, and $E'_{kT} = kT/E_\Omega$ is thermal energy. From eq.(135), then eq.(136) becomes

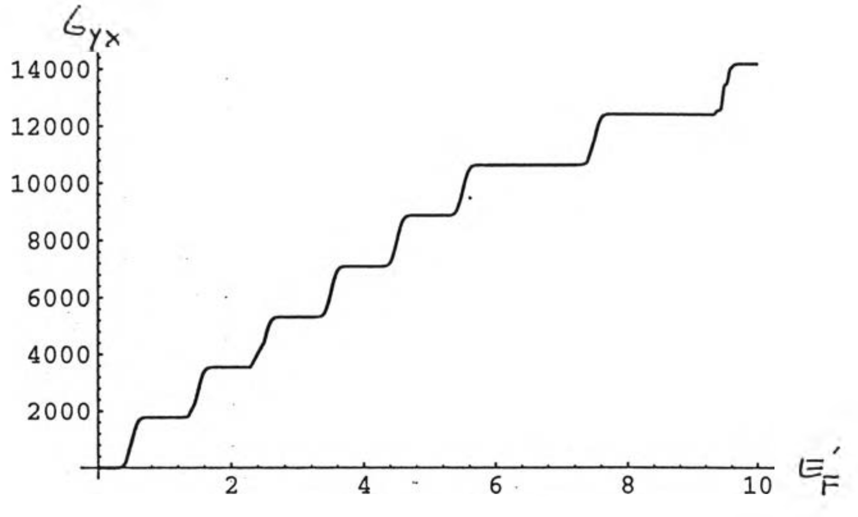
$$\sigma_{xx} = 0 \quad (139)$$

$$\frac{\sigma_{yx}}{A} = \frac{1}{E'_{kT}} \frac{1}{E'^{2}_{\mathcal{E}}} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dE' \frac{e^{-\frac{(E'-(n+1/2))^2}{E'^{2}_{\mathcal{E}}} + \frac{(E'-E'_F)}{E'_{kT}}}}{\left[1 + e^{(E'-E'_F)/E'_{kT}}\right]^2} \quad (140)$$

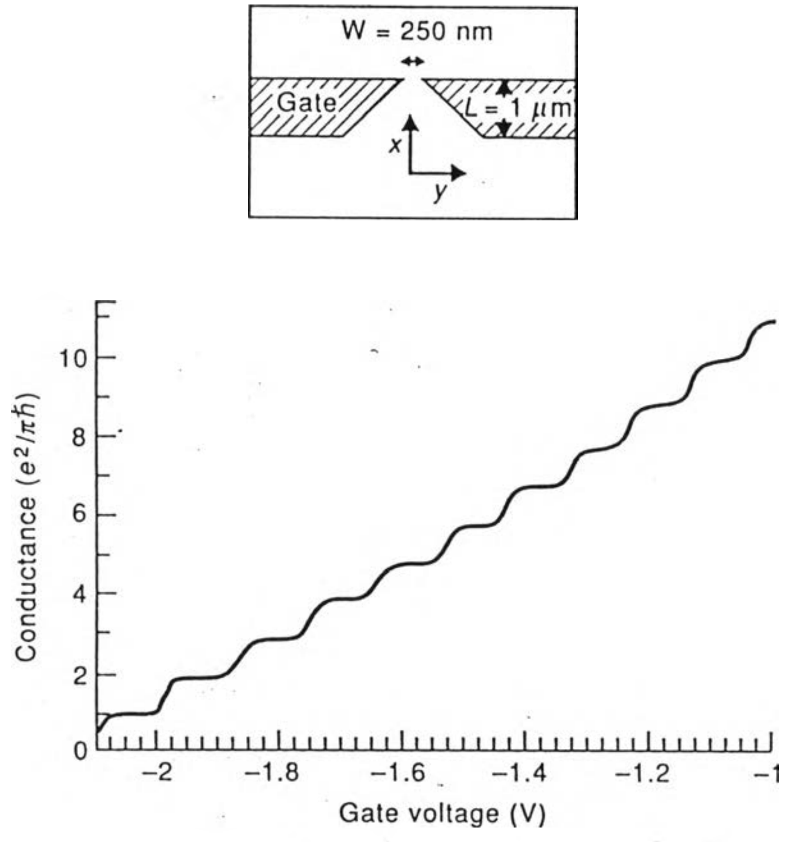
in the unit of (e^2/h). Plots of σ_{yx} as a function of E'_F are shown in figure(4.2).

Discussions

For the density of states, see fig.(4.1), it is shown that the electric field has removed the degeneracy of the system causing the broadening of the Landau levels. For the conductivity curve, fig.(4.2), shows a step-like curve for the transverse component. The longitudinal one does not appear. This result is similar to the experimental result of van Wees which is the point contact measurement, see fig.(4.3).



Figure(4.2). The transverse component of the conductivity, σ_{yx}, is plotted versus the Fermi energy E'_F. This curve is evaluated for the thermal energy $E'_{kT} = 0.1$.



Figure(4.3). Point contact measurement made by van Wees, *et al*,[29] at temperature $T = 0.6K$.

The experimental geometry is shown in the inset.