CHAPTER 6

CONCLUSION

6.1 Conclusion

In the case of simply connected region, the method of conducting paper analogue is reliable to determine shear stress distribution including magnitude of maximum shear stress occured in normal condition and also torsional stiffness. But it is inaccurateto determine the stress concentration which shear stress is increased rapidly in a small region and the region where the increment of conjugate function is high comparing to other regions of the specimen.

6.2 Suggestion for future work

6.2.1 To determine the stress concentration at the reentrant corner more accurately.

To determine the stress concentration at the reentrant corner accurately, the following attempts may be done.

1. The specimen is enlarged, since the size of conducting paper is limited, so it is impossible for conducting paper analogue to follow this way, but for other type of electrical analogue such as electrolytic tank it is practicable.

2. The fillet region should be enlarged by transforming the cross section to a more suitable form. All points in the region of cross section are mapped on to another region by suitable transformation and then form a new shape of cross section. The boundary potential supplied to the points on the transformed region must correspond to those on the original cross-section. When equipotential lines are obtained in the transformed cross-section, they are re-transformed to the original cross-section, then other results can be determined by the previous method.

Example for the use of transformation:-

$$u + iv = m \coth \frac{x - iy}{2}$$

as D.S. Ross[1] used in case of circular shaft containing a key way, transform the equal angle in z - region on to a new shape in w - region. Real and imaginary parts are obtained as followed:-

$$u = \frac{m \sinh x}{\cosh x - \cos y}, v = \frac{m \sin y}{\cosh x - \cos y}$$

The original and transformed cross - section are shown in fig.23(a) and (b)

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[1] Ross, D.S. and Qureshi, I.H. "Boundary Value Problems in Two Dimension Elasticity by Conducting Paper Analogue." Journal of Scientific Instruments, Vol. 40(1963): P.513 - 517.

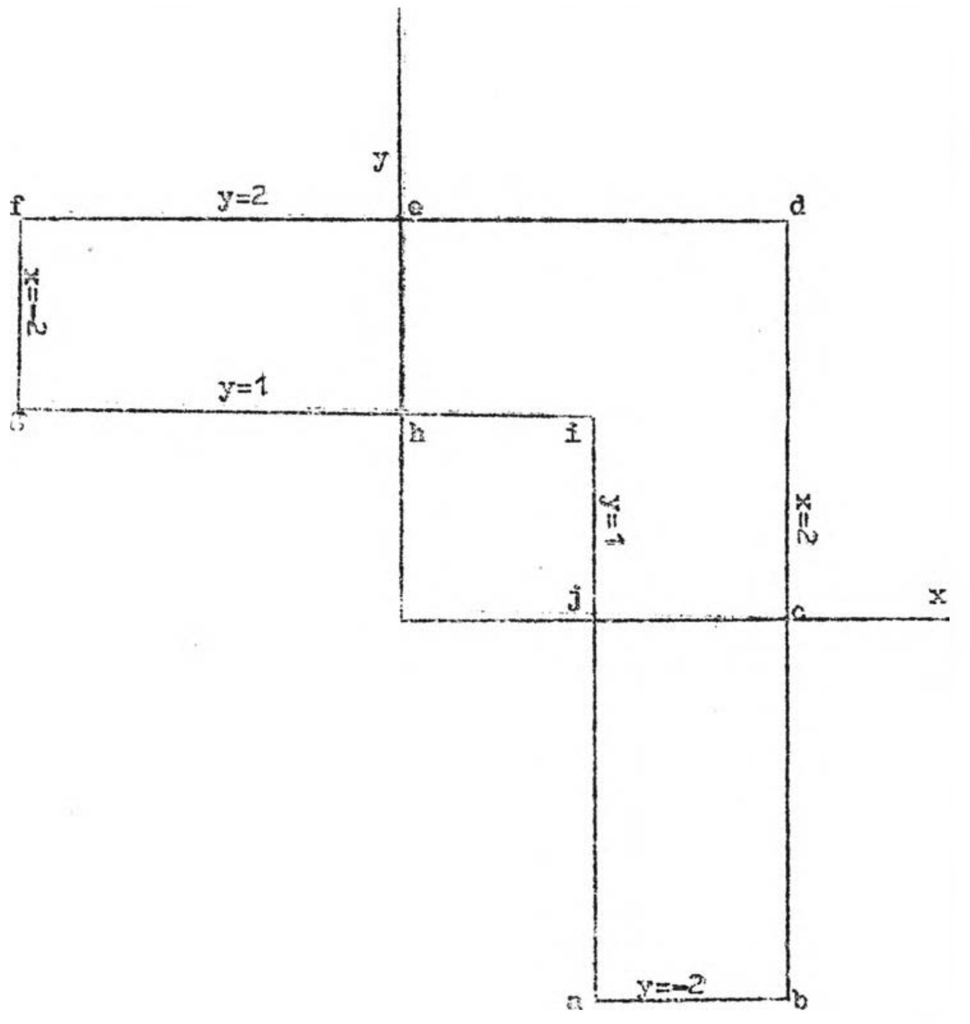

Fig. 23(a)

The cross-section of the equal angle in z-region.

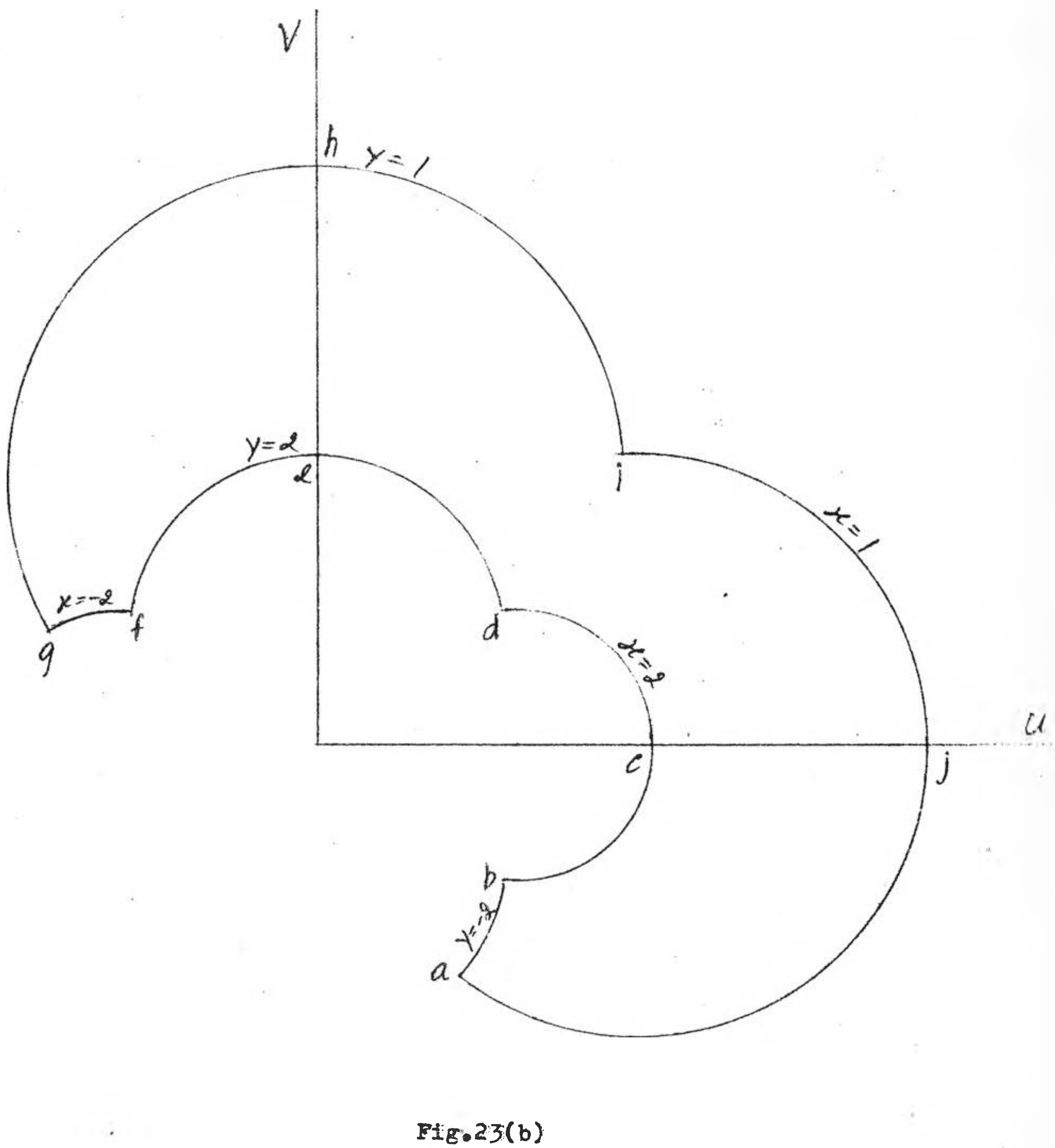

Fig.23(b)

The cross-section of the equal angle in w-region

(The transformation of z-region)

6.2.2 Torsion problem of an anisotropic shaft solved by conducting paper analogue.

Let the anisotropic shaft has three mutual orthogonal planes of elastic symmetry. The axis of the shaft is perpendicular to one of the plane of elastic symmetry. For this special case, the analytical equations have been solved in rather complicated forms.[1] These equations can be reduced to the simple forms of isotropic shaft by choosing the co-ordinate axes of the cross-section to coincide with the longitudinal plane of elastic symmetry and introducing new variables.

$$u = \sqrt{\frac{\mu_2}{\mu_1}}\, x\,, \quad v = y$$

$$\psi_1(x,y) = \sqrt{\frac{\mu_2}{\mu_1}}\, \varphi(u, v)$$

It means the boundary equation changed from f (x,y) = 0 to F(x,y) = 0. Then the equations of isotropic shaft become

$$\frac{\partial^2 \psi_1}{\partial u^2} + \frac{\partial^2 \psi_1}{\partial v^2} = 0$$

and boundary condition

$$\frac{\partial \psi_1}{\partial \nu} = v \cos (u, \nu) - u \cos (v, \nu) \quad \text{on } C'$$

where ν is the normal to the boundary C'

$$\tau_{xz} = \alpha\mu_1 \left[\frac{\partial \psi_1}{\partial u} - v\right]$$

$$\tau_{yz} = \alpha\sqrt{\mu_1\mu_2} \left[\frac{\partial \psi_1}{\partial v} + u\right]$$

$$M = \frac{\alpha\mu_1^2}{\sqrt{\mu_1\mu_2}} \iint_R \left(u\frac{\partial \psi_1}{\partial v} - v\frac{\partial \psi_1}{\partial u} + u^2 + v^2\right) du\,dv$$

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[1] Sokolnikoff, I.S. Mathematical Theory of Elasticity. P.193-197.

Since the function $\varphi_1(x,y)$ is harmonic, it is possible to introduce a conjugate function $\psi(u,y)$ related to $\varphi_1(u,v)$ by Cauchy - Riemann equations

$$\frac{\partial \varphi_1}{\partial u} = \frac{\partial \psi}{\partial v} \;, \quad \frac{\partial \varphi_1}{\partial v} = -\frac{\partial \psi}{\partial u}$$

and also introduced in the same way as isotropic shaft

$$\phi(u,v) = \psi(u,v) - \tfrac{1}{2}(u^2 + v^2)$$

The following equations are proved by the same method as isotropic shaft.

$$\tau_{xz} = \alpha \mu_1 \left(\frac{\partial \psi}{\partial v} - v \right)$$

$$\tau_{yz} = \alpha \sqrt{\mu_1 \mu_2} \left(-\frac{\partial \psi}{\partial u} + u \right)$$

$$\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} = 0$$

B.C. $\quad \psi = \frac{1}{2}(u^2 + v^2) + c$

$$\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} = -2$$

B.C. $\quad \phi = c$

$$M = \frac{2 \alpha \mu_1^2}{\sqrt{\mu_1 \mu_2}} \iint_R \phi \, du\,dv$$

These equations are different from the equations of isotropic shaft only by the constants of the shear stress components and torsional stiffness. So the conducting paper analogue can be extended to solve these problems by transforming the cross - section required which is governed by equation

$f(x,y) = 0$ to be a new cross - section which is governed by the equation $F(u,v) = 0$. Boundary potential is set by mean of boundary condition $\psi = \frac{1}{2}(u^2 + v^2) + c$. Then shear stress components and torsional stiffness can be determined from equipotential line by the above equation.

According to Sokolnikoff[1], the co - ordinate axes are not chosen to coincide with the longitudinal planes of elastic symmetry. It is considered that conducting paper analogue is also applicable, but transformed cross-section may be awkward and the equation of shear stress components are in more complicated forms.

6.2.3 Torsion problem of multiply connected region shaft solved by conducting paper analogue.

In case of shaft whose cross - section is multiply connected region, the conjugate function ψ must satisfy

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad \text{in } R$$

$$\text{and B.C. } \psi_i = \tfrac{1}{2}(x^2+y^2) + k_i \qquad \text{on } C_i \ (i = 0,1,2\ldots n)$$

C_0 is the exterior contour of shaft $C_1, C_2, C_3, \ldots, C_n$ are the contours of holes. R is the region inside C_0 and outside $C_1, C_2, \ldots C_n$.

The value of k_i should be determined so that the displacements are single value. The value of k_0 can be assigned arbitrarily, but the remaining n constant k_i must be determined so that they satisfy the set of n conditions.

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[1] Sokolnikoff, I.S. Mathematical Theory of Elasticity. P.193-197

$$\int_{C_i} \left(\frac{\partial \psi}{\partial y} dx - \frac{\partial \psi}{\partial x} dy \right) = 0 \qquad (i = 1,2\ldots,n)$$

When the conducting paper analogue is applied to this case, the problem is what the values of n constants k_i are.

Consider the condition of potential distributed in a thin conducting sheet which is the analogous system, since ψ and V may have different scale.

$$V = S_2 \psi + \sigma_2$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad \text{in R}$$

B.C. $\quad V = S_1 (x^2+y^2) + (c_1+k_i)$ on $C_i \qquad (i = 0,1,2,\ldots,n)$

and set of n condition will be

$$\int_{C_i} \left(\frac{\partial V}{\partial y} dx - \frac{\partial V}{\partial x} dy \right) = 0$$

Since $\quad i_x = -\frac{1}{\rho}\frac{\partial V}{\partial x}, \; i_y = -\frac{1}{\rho}\frac{\partial V}{\partial y}$

then $\quad \int_{C_i} (- i_y dx + i_x dy) = 0$

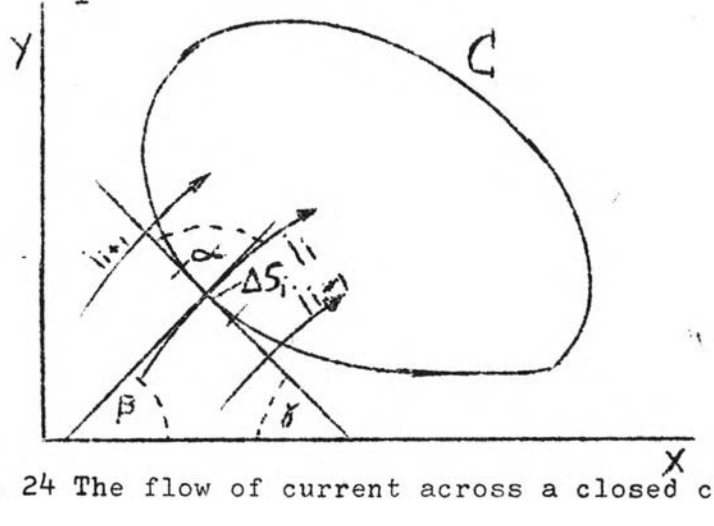

Fig. 24 The flow of current across a closed curve.

Consider the current flow across a closed curve C, the current flow across ΔS which is a portion of curve C is $i \cdot \Delta S \cdot \sin\alpha$.

So the total current flows across boundary C is

$$I = \sum_{i=1}^{n} i_i \cdot \Delta S_i \cdot \sin \alpha_i$$

Since $\alpha_i = 180 - (\beta_i + \gamma_i)$

$$\begin{aligned}
i_i \sin\alpha_i \cdot \Delta S_i &= i_i \sin(\beta_i + \gamma_i) \cdot \Delta S_i \\
&= i_i \sin\beta_i \cos\gamma_i \cdot \Delta S_i + i_i \cos\beta_i \sin\gamma_i \, \Delta S_i \\
&= i_{iy} \cos\gamma_i \cdot \Delta S_i + i_{ix} \sin\gamma_i \cdot \Delta S_i \\
&= i_{iy}\left[\frac{-\Delta x_i}{\Delta S_i}\right] \cdot \Delta S_i + i_{ix} \cdot \frac{\Delta y_i}{\Delta S_i} \cdot \Delta S_i \\
&= -i_{iy}\Delta x_i + i_{xi}\Delta y_i
\end{aligned}$$

$$I = \sum_{i=1}^{n} (-i_{iy}\Delta x_i + i_{ix}\Delta y_i).$$

limit $n \to \infty$

$$I = \int_C (-i_y dx + i_x dy)$$

It is proved that the expression $\int_C (-i_y dx + i_x dy)$ is the total current flow across the boundary C. The set of n conditions $\int_{C_i} (-i_y dx + i_x dy) = 0$ mean that the current flow into any interior boundary C_i and the current flow out of that boundary are equal.

The conducting paper analogue may be extended to solve the problem of multiply connected region. The constant k_o is selected equal to zero, then, the exterior boundary potentials are set so that

$$V_o = S_1 (x^2 + y^2) + c_1 \qquad \text{on } C_o$$

The interior boundary potentials are

$$V_i = S_1 (x^2+y^2) + (c_1 + k_i) \qquad \text{on } C_i$$

The following ways may be done.

1. The value of k_i may be set equal to zero, then gradually increases until the current flow into and out of the boundary are equal.

2. The supply instruments of each boundary are used seperately so the current flow into and out of the instrument must be equal.