CHAPTER II

MINIMUM SEMILATTICE CONGRUENCES

The purpose of this chapter is to study the minimum semilattice congruences on factorizable semigroups. It is shown that in any factorizable semigroup S with identity, the group of units forms a class of the minimum semilattice congruence on S. However, this property is not a property of any regular semigroup with identity. A counter example is given.

Let S be a semigroup with a left identity e. Suppose ρ is a semilattice congruence on S. Then for any $x \in S$, $(x\rho)(e\rho) = (e\rho)(x\rho) = ex\rho = x\rho$. Then $e\rho$ is the identity of S/ρ. Hence, from Lemma 1.1, the following proposition follows :

2.1 Proposition. Let a semigroup S be factorizable as GE and e is the identity of G. If ρ is a semilattice congruence on S, then $e\rho$ is the identity of S/ρ.

Let S be a semigroup. A congruence ρ on S is a semilattice congruence if and only if for all $a, b \in S$, $a\rho a^2$ and $ab\rho ba$. Then, arbitrary intersection of semilattice congruences on S is a semilattice congruence on S. Hence, the intersection of all semilattice congruences on S is the minimum semilattice congruence on S.

Throughout this chapter, for any semigroup S, let η denote the minimum semilattice congruence on S.

Let T be a subsemigroup of a semigroup S. The semigroup T is called a filter of S if for any a, b $\in$ S, ab $\in$ T implies a $\in$ T and b $\in$ T. If a $\in$ S, let N(a) denote the smallest filter of S containing a; that is, N(a) is the intersection of all filters of S containing a.

Let S be a semigroup. It has been proved in [4] that the minimum semilattice congruence on S, η, defined as follows: aηb if and only if N(a) = N(b), and hence for a $\in$ S, aη = {x $\in$ S / N(x) = N(a)}. Therefore a$\eta \subseteq$ N(a) for all a $\in$ S. In general, for a $\in$ S, aη and N(a) are not necessarily equal. An example is given as follows :

Example. Let S = {1, 2, 3, ...} and define an operation * on S as follows : x * y = maximum {x, y}. Then (S, *) is a semilattice having 1 as its identity. Because S is a semilattice, η on S is the identity congruence. Then 2η = {2}. It is clearly seen that the smallest filter on S containing 2 is {1, 2}; that is, N(2) = {1, 2}. Therefore 2$\eta \neq$ N(2). #

Let ρ be a congruence on a semigroup S. Let a $\in$ S. Then, aρ is an idempotent of S/ρ if and only if aρ forms a subsemigroup of S. Hence, if ρ is a semilattice congruence on S, then every ρ-class forms a subsemigroup of S.

2.2 Lemma. Let S be a semigroup with a left identity e. Then eη = N(e); that is, eη is the smallest filter containing e.

Proof : Because $e \in e\eta \subseteq N(e)$, it suffices to show that $e\eta$ is a filter of S. Since η is a semilattice congruence on S, $e\eta$ is a subsemigroup of S. Let $a, b \in S$ such that $ab \in e\eta$. Thus $ab\eta = e\eta$. Because e is a left identity of S and η is a semilattice congruence on S, it follows that $a\eta = (ea)\eta = (e\eta)(a\eta) = (ab)\eta a\eta = (a^2\eta)(b\eta) = ab\eta = e\eta$ and $b\eta = (eb)\eta = (e\eta)(b\eta) = (ab)\eta(b\eta) = (a\eta)(b^2\eta) = ab\eta = e\eta$. Therefore $a, b \in e\eta$. This proves $e\eta$ is a filter of S, as desired. #

Let ρ be a semilattice congruence on a semigroup S. Let $a \in S$. If a' is an inverse of a in S, then $a\rho = a'\rho$. To prove this, let $a' \in S$ such that $a = aa'a$ and $a' = a'aa'$. Since ρ is a semilattice congruence on S, $a\rho = (aa'a)\rho = (a\rho)^2(a'\rho) = a\rho a'\rho = a\rho(a'\rho)^2 = a'\rho a\rho a'\rho = (a'aa')\rho = a'\rho$.

Let ρ be a semilattice congruence on a semigroup S and G be a subgroup of S. If e is the identity of G, then for $g \in G$, $g\rho = g\rho g\rho = g\rho g^{-1}\rho = e\rho$. Hence G is contained in a single ρ-class.

From the above fact, Lemma 1.1 and Lemma 2.2, we then have

2.3 Proposition. Let S be a semigroup which is factorizable as GE. Then $G \subseteq e\eta = N(e)$, the smallest filter containing e, where e is the identity of G.

We have a following question : "If a semigroup S is factorizable as GE, does G form a η-class? or, is G equal to $e\eta$?, where e is the identity of G". The answer is "No". It is shown by the following example : Let S be a nontrivial right zero semigroup. Then $E(S) = S$ and for $a \in S$, $\{a\}$ is a subgroup of S and $S = \{a\}E(S)$.

Because for all x, $y \in S$, $xy = y$ and η is a semilattice congruence on S, η is a universal congruence on S. Thus $x\eta = S$ for all $x \in S$. Hence $\{a\} \subsetneq a\eta$ for all $a \in S$.

Let S be a semigroup with identity 1. By Lemma 2.2, $1\eta = N(1)$. Let G be the group of units of S. Then $G \subseteq 1\eta = N(1)$. We show in the next theorem that in any factorizable semigroup S with identity 1, the group of units coincides with 1η. We first give an example to show that in a regular semigroup S with identity 1, the group of units of S and 1η are not necessarily equal.

Example. Let $X = \{x_1, x_2, x_3, \ldots\}$ where $x_i \neq x_j$ if $i \neq j$, and I_X be the symmetric inverse semigroup on the set X. Then G_X, the permutation group on X, is the group of units of I_X. Therefore $G_X \subseteq 1\eta = N(1)$ where 1 is the identity map on X. To show that $G_X \neq 1\eta$, it suffices to show G_X is not a filter of I_X. Let α be the map on X defined by $x_i\alpha = x_{2i}$ for all $i \in \{1, 2, 3, \ldots\}$. Then $\alpha \in I_X$. Moreover, $\alpha\alpha^{-1} = 1 \in G_X$. But α and α^{-1} are not elements of G_X. Hence G_X is not a filter of I_X. Therefore G_X is a proper subset of 1η. #

2.4 Lemma. Let S be a semigroup with identity 1. If S is factorizable, then the group of units of S is a filter of S.

Proof : Assume S is factorizable as GE. Then, by Theorem 1.7, G is the group of units of S. To show G is a filter of S, let $x, y \in S$ such that $xy \in G$. Since $S = GE$, $x = gf$, and $y = hf'$ for some $g, h \in G$, $f, f' \in E$. Thus $gfhf' \in G$ which implies that $fhf' \in G$.

Therefore $(fhf')^{-1}(fhf') = 1$. Hence $1 = (fhf')^{-1}(fhf') = (fhf')^{-1}(fhf')f' = 1f' = f'$. Thus $y = hf' = h \in G$. But $fhf' \in G$ and $f' = 1$. Then $fh \in G$ and so $fhh^{-1} = f \in G$. Therefore $f = 1$, so $x = g \in G$. This proves G is a filter of S as required. #

2.5 Theorem. If S is a factorizable semigroup with identity 1, then the group of units is the η-class containing 1.

Proof : It follows directly from Proposition 2.3 and Lemma 2.4. #

We know that any factorizable inverse semigroup has an identity. Then, from Theorem 2.5, we have that the group of units of a factorizable inverse semigroup S is a η-class of S.

A semigroup S is called η-simple if S has only one η-class.

Hence, the following corollaries follow :

2.6 Corollary. Any η-simple factorizable semigroup with identity is a group.

2.7 Corollary. Any η-simple factorizable inverse semigroup is a group.