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STRATEGIES FOR PLAYING CLOSING OCTAGONS GAME
IN SOME SITUATIONS

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เกมการปิดรูปแปดเหลี่ยมเป็นเกมเชิงคอมบินาทอริกสำหรับผู้เล่น 2 คน ซึ่งดัดแปลงมาจากเกมจุดและกล่องและเกมจุดและรูปหกเหลี่ยม เกมเริ่มต้นที่อาร์เรย์ของรูปแปดเหลี่ยมขนาด $m \times n$ และผู้เล่นทั้งสองมี 0 แต้ม ผู้เล่นทั้งสองผลัดกันระบายสีลงบนด้านแต่ละด้านของรูปแปดเหลี่ยม ผู้เล่นที่ระบายสีลงบนด้านที่ 8 ของรูปแปดเหลี่ยม k รูป จะได้รับ k แต้ม และได้ระบายสีลงบนด้านเพิ่มอีก 1 ด้าน เกมจบเมื่อทุกด้านของรูปแปดเหลี่ยมทุกรูปถูกระบายสีทั้งหมด ผู้เล่นที่มีแต้มมากที่สุดคือผู้ชนะ เกมนี้ถูกทำการแปลงเป็นเกมใหม่ซึ่งเป็นเกมเกี่ยวกับกราฟ เพื่อความสะดวกในการวิเคราะห์เกม กฎบางข้อถูกเพิ่มเข้าไปในเกม และเกมที่ถูกเพิ่มกฎเข้าไปนี้เรียกว่าเกมปกติ วิทยานิพนธ์ฉบับนี้ได้วิเคราะห์เกี่ยวกับรูปแบบของเกมและสร้างกลยุทธ์สำหรับการเล่นเกมในสถานการณ์ต่างๆ

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ปีการศึกษา .... 2560 ....

THITIPHUT LEELATHANAKIT : STRATEGIES FOR PLAYING CLOSING OCTAGONS GAME IN SOME SITUATIONS

ADVISOR : ASST. PROF. RATINAN BOONKLURB, Ph.D., 46 pp.

Closing Octagons game is a combinatorial game for two players that we modify from Dots and Boxes game and Dots and Hexagons game. The game starts with an $m \times n$ array of octagons such that every two adjacent octagons has one common side and 0 points for both players. Two players alternately turn by the following rules. (i) A player moves by coloring one side of an octagon. (ii) A player who colors the eighth side of k octagons earns k points and takes one more move. The game ends when every side of octagons has been colored and a player having the most points wins. This game is formulated into the new game using graphs. In order to analyze the game, more rules are added and the game with these additional rules is called normal game. In this thesis, we analyze patterns of the game and give strategies on how to play the game in several situations.

Department : Mathematics and Computer Science Student's Signature : ....

Field of Study : ......Mathematics...... Advisor's Signature : ....

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CONTENTS

CHAPTER I
INTRODUCTION

Combinatorial game [3] is a 2-person game with perfect information and no change moves. Such a game is determined by a set of states, including the initial state which is a state at the beginning of the game. Play turns from one state to another, with the players usually alternating turns, until the terminal state is reached. The terminal state is a state that no moves are possible. After that one player is declared the winner and the other the loser. However, two players can draw if no one win.

Dots and Boxes game [1] is one of combinatorial games, the game starts with an $(m+1) \times (n+1)$ array of dots and 0 points for both players. Two players alternately turn by drawing a vertical or horizontal line between two adjacent dots. A player who draws the fourth line of k square boxes of size 1×1 earns k points and draws one more line. The game ends when every two adjacent dots has a line and a player having the most points wins. The game has been studied by several researchers, for examples, Lenhardt [5] analyzed $1 \times n$ Dots and Boxes game and Buzzard and Ciere [2] constructed a highly efficient algorithm for playing Dots and Boxes game optimally. In 2016, Ratiprasit, Simadhamnand and Teeravichayangoon derived Dots and Hexagons game from Dots and Boxes game. They changed square boxes to hexagonal boxes and used the same rules of playing as in the Dots and Boxes game.

In this thesis, we modify their games into Closing Octagons game and formulate the game into the new game using graph. In Chapter II, we give some definitions about graphs and combinatorial games. In Chapter III, we introduce Closing Octagons game and the game that is formulated using graph, give some terminologies involving the game, and analyze patterns of graphs, states, moves and turns that

appear in the game. In Chapter IV, we give strategies of playing the game for each player to win or draw or get the most possible points in some situations and some games of size $1 \times n$, $2 \times n$ and 3×3.

CHAPTER II

PRELIMINARIES

In this chapter, we give some definitions about graphs and combinatorial games that are used in this thesis.

2.1 Graph Theory

We use several graph terminologies in this thesis. Some of them are commonly used in several graph theory textbooks, for example, [4] and [7]. However, we slightly modify several terms by using multiset.

Definition 2.1. A *graph* is a pair (V, E) where V is a finite set called a *vertex set* or a set of *vertices*, and E is a finite multiset called an *edge set* or a set of *edges* such that each edge is a multiset of 2 vertices (not necessarily distinct).

We usually use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of a graph G, respectively.

Definition 2.2. Let G be a graph. An edge e of G is a *loop* if there is a vertex v such that $e = \{v, v\}$, and e is a *simple edge* if e is not a loop and occurs once in $E(G)$.

Definition 2.3. Let G be a graph. A vertex v and an edge e of G are *incident* if $v \in e$, and two vertices u and v of G are *adjacent* if there is an edge incident to both u and v.

Graph are represented by drawing a dot for each vertex, and an arc joining two dots u and v for each edge incident to both u and v.

Example 2.4. Let G be a graph that is shown in Figure 2.1. Then, $V(G) = \{u, v, w, x, y\}$ and $E(G) = \{\{v, v\}, \{v, w\}, \{w, y\}, \{x, y\}, \{x, y\}\}$.
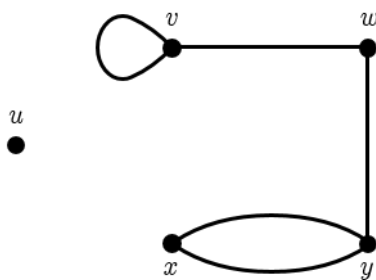
Figure 2.1: A graph G

Definition 2.5. A graph G is a *simple graph* if all edges of G are simple edges.

Definition 2.6. Let v be a vertex of a graph G. The *degree* of v, denoted $d_G(v)$, is the number of times such that v occurs in edges of $E(G)$, and the *weight* of v, denoted $w_G(v)$, is the number of edges incident to v.

Definition 2.7. Let G be a graph. A vertex v of G is an *isolated vertex* if $d_G(v) = 0$, and v is a *leaf* if $d_G(v) = 1$.

Lemma 2.8. *[7] For any graph G, the sum of degree of all vertices of G equals twice of the number of edges of G.*

Definition 2.9. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Definition 2.10. Let G and H be graphs. An *isomorphism* between G and H is a bijection f from $V(G)$ to $V(H)$ such that for each two vertices $u, v \in V(G)$, the number of edges incident to both u and v of G equals the number of edges incident to both $f(u)$ and $f(v)$ of H. Two graphs G and H are *isomorphic* if there is an isomorphism between G and H.

Definition 2.11. A graph G is a *connected graph* if for each two vertices u and v of G, there is a list $u = v_0, v_1, v_2, ..., v_k = v$ of vertices of G such that for all $i \in \{1, 2, 3, ..., k\}$, v_{i-1} and v_i are adjacent.

Definition 2.12. A graph K is a *component* of a graph G if K is a maximal connected subgraph of G, i.e., there is no connected subgraph H of G such that $H \neq K$, $H \neq G$, and H contains K.

Definition 2.13. [4] A *trivial graph* is a graph having exactly 1 vertex and no edges.

Definition 2.14. [4] Let $k \in \mathbb{N}$. A *bouquet graph* or *k-bouquet graph* is a graph having exactly 1 vertex and k loops.

Figure 2.2 shows examples of bouquet graphs.

A 1-bouquet graph

A 2-bouquet graph

A 3-bouquet graph
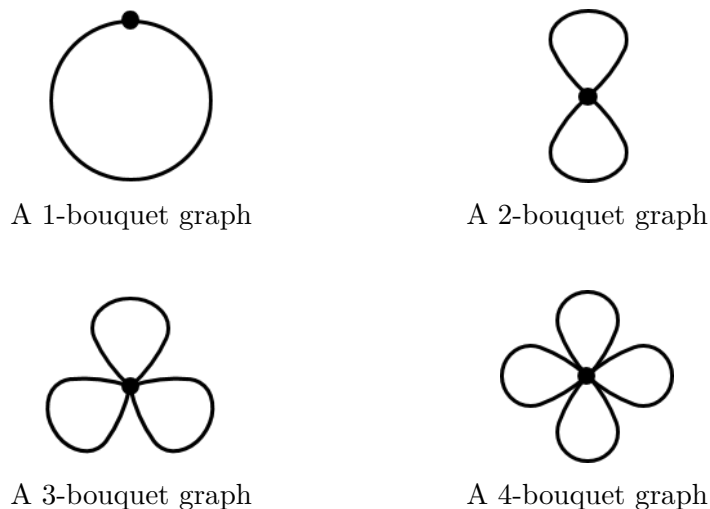
A 4-bouquet graph

Figure 2.2: Examples of bouquet graphs

Definition 2.15. [4] Let $k \in \mathbb{N}$. A *path graph* or *k-path graph* is a graph having exactly $k + 1$ vertices v_0, v_1, v_2, ..., v_k and k edges e_1, e_2, e_3, ..., e_k such that for all $i \in \{1, 2, 3, ..., k\}$, e_i is incident to v_{i-1} and v_i.

Figure 2.3 shows examples of path graphs.

A 1-path graph

A 2-path graph

A 3-path graph

A 4-path graph

Figure 2.3: Examples of path graphs

Definition 2.16. [4] Let $k \in \mathbb{N}$. A graph G with $k + 1$ vertices is a *pseudopath graph* or *k-pseudopath graph* if G contains a k-path subgraph P such that for each edge e of G, e is either a loop of G or an edge of P.

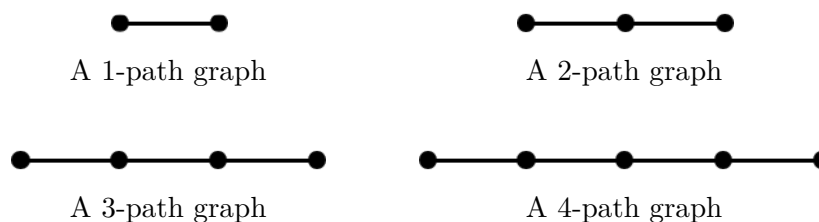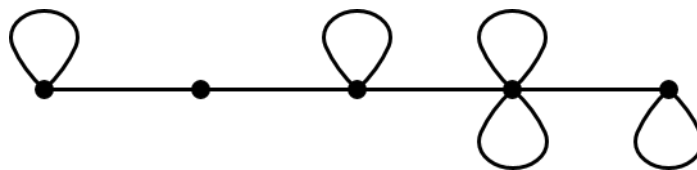Figure 2.4 shows an example of pseudopath graph.

Figure 2.4: An example of pseudopath graph

Definition 2.17. [4] Let $k \in \mathbb{N}$. A *cycle graph* or *k-cycle graph* is a graph having exactly k vertices v_1, v_2, v_3, ..., v_k and k edges e_1, e_2, e_3, ..., e_k such that e_1 is incident to v_1 and v_k, and for all $i \in \{2, 3, 4, ..., k\}$, e_i is incident to v_{i-1} and v_i.

A cycle graph C is called a simple cycle graph if C is a simple graph. Figure 2.5 shows examples of cycle graphs.
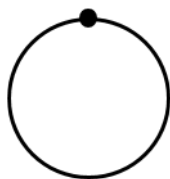
A 1-cycle graph
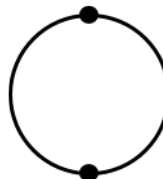
A 2-cycle graph
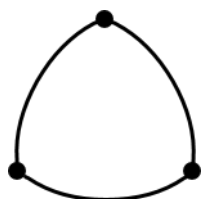
A 3-cycle graph
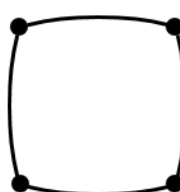
A 4-cycle graph

Figure 2.5: Examples of cycle graphs

Lemma 2.18. *[6] Let G be a connected graph such that for each vertex v of G, $d_G(v) = 2$. Then, G is a cycle graph.*

Definition 2.19. Let G be a graph with k edges and e be an edge of G. A subgraph $G - e$ of G is a graph with $k - 1$ edges such that $V(G - e) = V(G)$, $E(G - e) \subseteq E(G)$, and the number of times that e occurs in $E(G - e)$ is less than the number of times that e occurs in $E(G)$ by 1.

Definition 2.20. Let G be a graph. An edge e of G is a *cut-edge* if the number of components of $G - e$ is greater than the number of components of G.

Lemma 2.21. *[4] Let G be a graph. An edge e of G is a cut-edge if and only if there is no cycle subgraph C of G such that e is an edge of C.*

Lemma 2.22. *[4] Let G be a connected graph and e be a cut-edge of G. Then, $G - e$ has exactly 2 components.*

Definition 2.23. [7] A connected graph G is a *tree* if G contains no cycle subgraphs.

Figure 2.6 shows an example of tree.
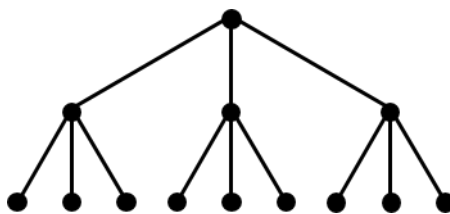
Figure 2.6: An example of Tree

Lemma 2.24. *[7] Every non-trivial tree has at least 2 leaves.*

Theorem 2.25. *Let T be a non-trivial tree and e be an edge of T. Then, $T - e$ has exactly 2 components such that each component is a tree.*

Proof. By Lemma 2.21, e is a cut-edge of T. By Lemma 2.22, $T - e$ has exactly 2 components. Since T contains no cycle subgraphs, $T - e$ contains no cycle subgraphs. Hence, each component of $T - e$ is a tree. $\qquad\square$

2.2 Combinatorial Game Theory

This thesis constructs a new combinatorial game by modifying the existing one. Thus, the following definitions and examples help the reader understand more about it.

Definition 2.26. [3] A *combinatorial game* is a game determined by a finite set called a set of *states*, and the game satisfies the following conditions.

    (1) There are two players called *Player I* and *Player II*, and Player I is the *opponent* of Player II and vice versa.

    (2) The set of all possible states is finite.

    (3) Rules of playing game are specified for actions of both players, each action is called a *turn*.

    (4) Two players alternate turning from one state to another.

    (5) The game ends when there is no possible turn satisfying the rules of playing game, and either one player wins and the other loses or two players draw.

    Note that S_0, S_1, S_2, ..., S_t denote states of a game $\mathcal{G}$ with $t+1$ states.

    By (4) and (5) of Definition 2.26, if $\mathcal{G}$ is a combinatorial game with $t+1$ states, then there is a list $\tau_1, \tau_2, \tau_3, ..., \tau_t$ of turns such that for all $i \in \{1, 2, 3, ..., t\}$, τ_i is a turn from S_{i-1} to S_i, τ_i is Player I's turn if i is odd, τ_i is Player II's turn if i is even, and there is no possible turn from S_t.

Definition 2.27. For any combinatorial game $\mathcal{G}$ with $t+1$ states, S_0 and S_t of $\mathcal{G}$ are called the *initial state* and the *terminal state*, respectively.

Definition 2.28. [3] A combinatorial game $\mathcal{G}$ is an *impartial game* if the rules of $\mathcal{G}$ make no distinction between players, otherwise $\mathcal{G}$ is called a *partizan game*.

Example 2.29. Any Rock-Paper-Scissors game is not a combinatorial game, the game does not satisfy (4) of Definition 2.26 because both players reveal their choices at the same time.

Example 2.30. *Tic-Tac-Toe game* is a game for two players starting with a 3×3 array of spaces. Two players alternately turn by Player I marks sign $\bigcirc$ and Player II marks sign $\times$ in a space. The game ends when there is a vertical, horizontal or diagonal row that is marked by the same sign, or every spaces is marked by a sign. The player who marks three of their sign in a vertical, horizontal, or diagonal row wins.

By the rule of the game, we obtain that any Tic-Tac-Toe game is a partizan combinatorial game.

Let $\mathcal{G}$ be a Tic-Tac-Toe game that is shown in Figure 2.7. Then, $\mathcal{G}$ has 9 states and 8 turns, and Player I loses and Player II wins.

Figure 2.7: A Tic-Tac-Toe game $\mathcal{G}$

Example 2.31. Let $m, n \in \mathbb{N}$. *Dots and Boxes game* [1] or $m \times n$ *Dots and Boxes game* is a game for two players starting with an $(m + 1) \times (n + 1)$ array of dots and 0 points for both players. Two players alternately *turn* by the following rules.

(1) A player draws one vertical or horizontal line between two adjacent dots.

(2) A player who draws the fourth line of k square boxes of size 1×1 earns k points and draws one more line.

The game ends when every two adjacent dots has a line, and the player having the most points wins.

By the rule of the game, we obtain that any Dots and Boxes game is an impartial combinatorial game.

Let $\mathcal{G}$ be a 2×2 Dots and Boxes game that is shown in Figure 2.8. Then, $\mathcal{G}$ has 11 states and 10 turns, and Player I loses and Player II wins.

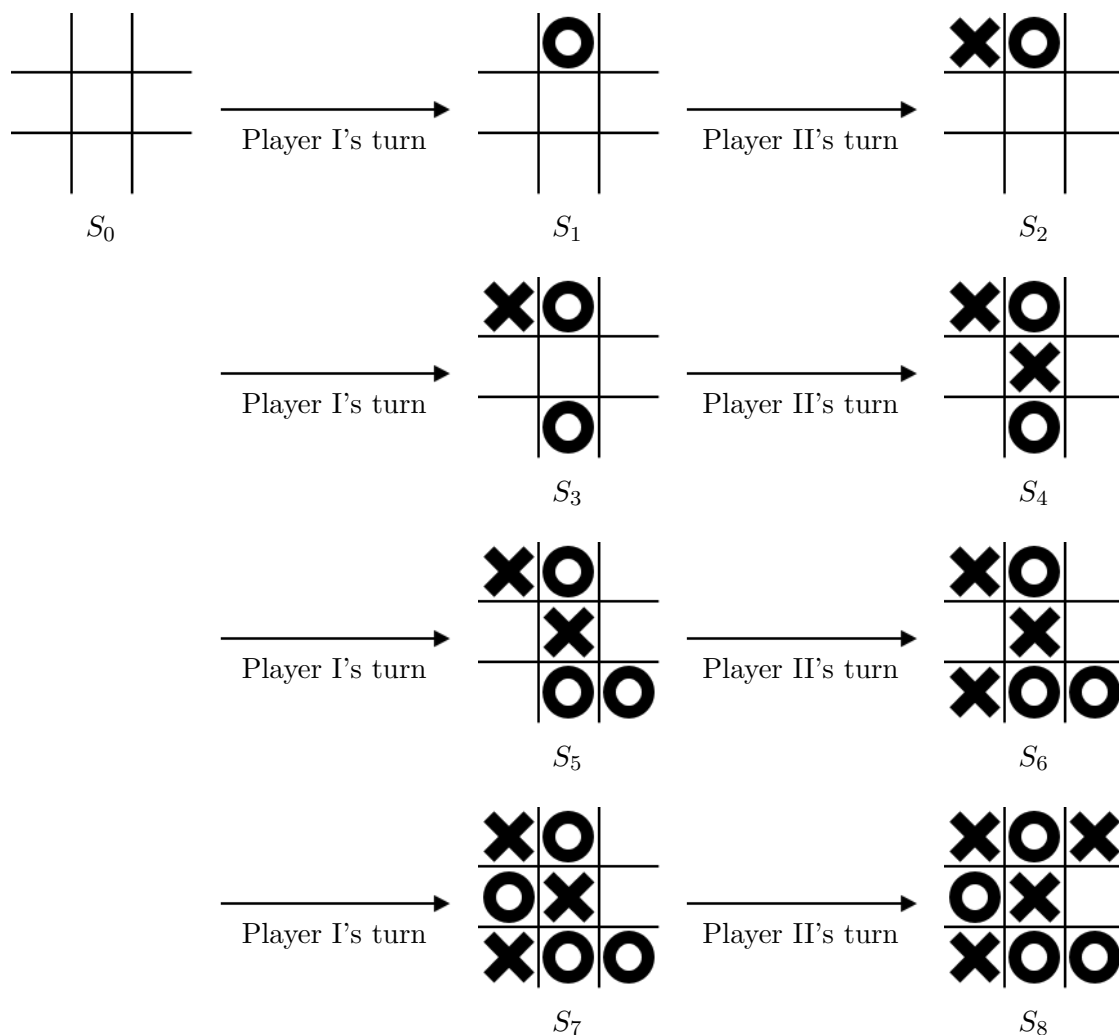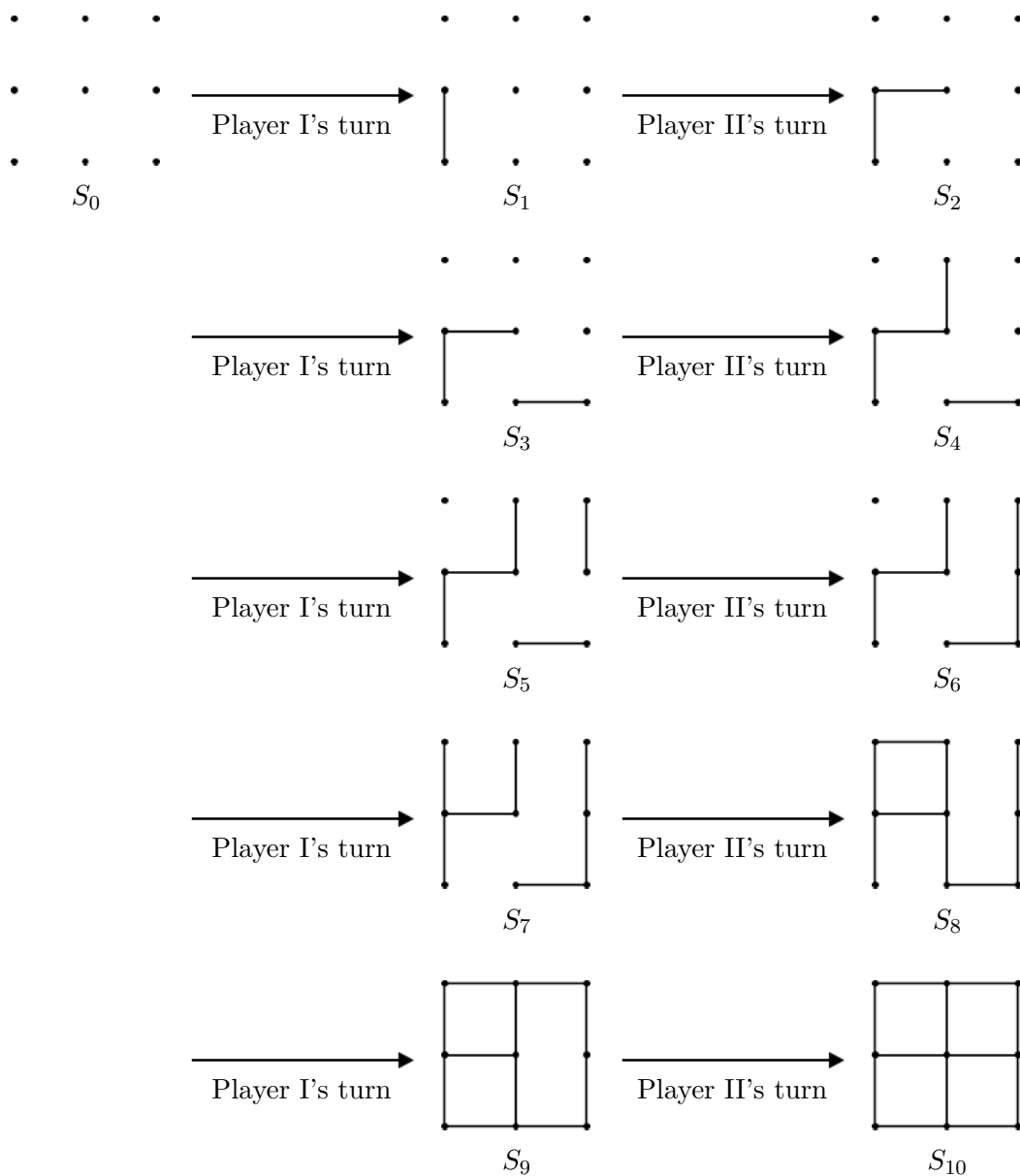Figure 2.8: A 2×2 Dots and Boxes game $\mathcal{G}$

CHAPTER III
CLOSING OCTAGONS GAME

In this chapter, we introduce the definition of Closing Octagons game and formulate this game into the new game using graph. Several terminologies involving our Closing Octagons game are given and some analysis of the game are provided.

3.1 Definition of Closing Octagons Game

In this section, we give a definition of our Closing Octagons game and their relevant terminologies.

First, let us introduce a Closing Octagons game of type I.

Definition 3.1. Let $m, n \in \mathbb{N}$. *Closing Octagons game of type I* or $m \times n$ *Closing Octagons game of type I* is a game for two players starting with an $m \times n$ array of octagons such that every two adjacent octagons has one common side and 0 points for both players. Two players alternately *turn* by the following rules.

(1) A player *move* by coloring of one side of an octagon.

(2) A player who colors the eighth side of k octagons earns k points and takes one more move.

The game ends when every side of octagons has been colored, and the player having the most points wins.

By Definition 3.1, a move is a coloring one side of an octagon and a turn is a possible list of consecutive move(s) by one player that satisfies the rules of playing the game.

Definition 3.2. For any $m \times n$ Closing Octagons game $\mathcal{G}$ of type I, $m \times n$ is called the *size* of $\mathcal{G}$.

Definition 3.3. For any Closing Octagons game $\mathcal{G}$ of type I, a *state* of $\mathcal{G}$ is a triple (A, p_1, p_2) where A, p_1 and p_2 are an array of octagons with colored-sides and points of Player I and Player II that are changed by turns, respectively, including $(A_0, 0, 0)$ where A_0 is an array of octagons with no colored-sides.

To be specific, we use $A(S)$ to denote the array of octagons with colored-sides of a state S of a Closing Octagons game of type I.

Remark 3.4. Any Closing Octagons game of type I is an impartial combinatorial game.

Example 3.5. Let $\mathcal{G}$ be a 2×2 Closing Octagons game of type I that is shown in Figure 3.2. Then, $\mathcal{G}$ has 26 states and 25 turns.

A turn from S_{19} to S_{20} gives Player II 2 points, and a turn from S_{24} to S_{25} gives Player I 2 points. Then, two players draw.
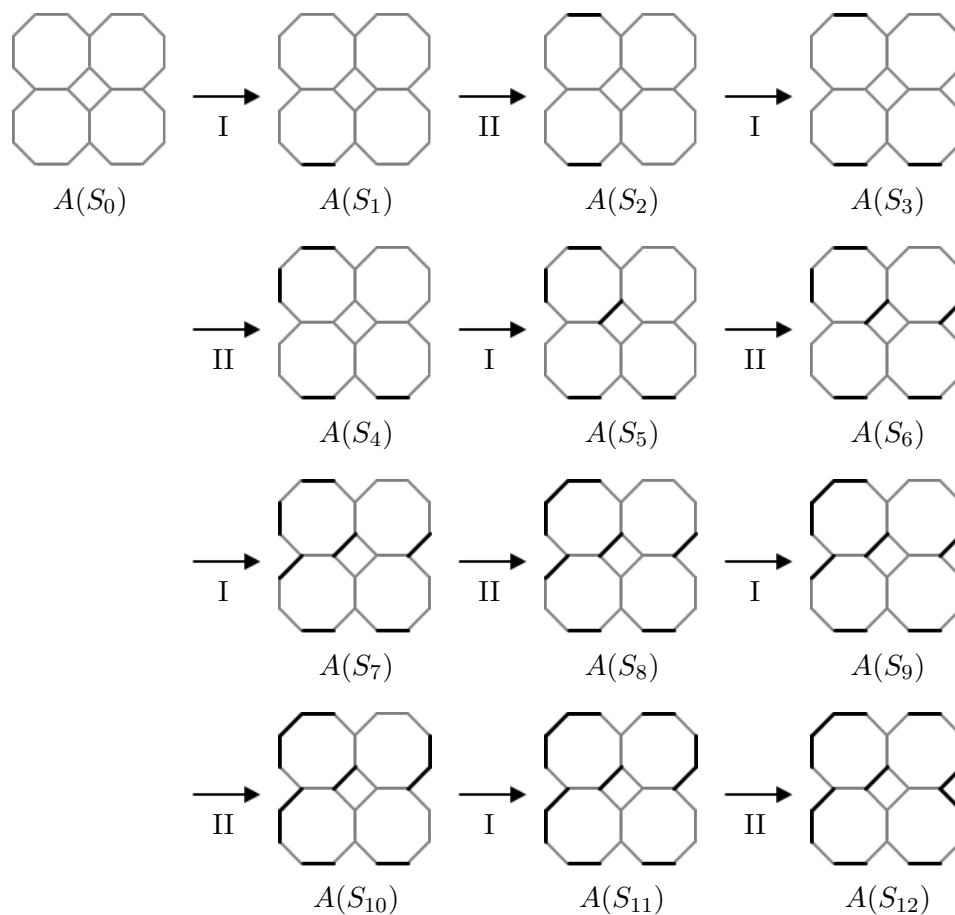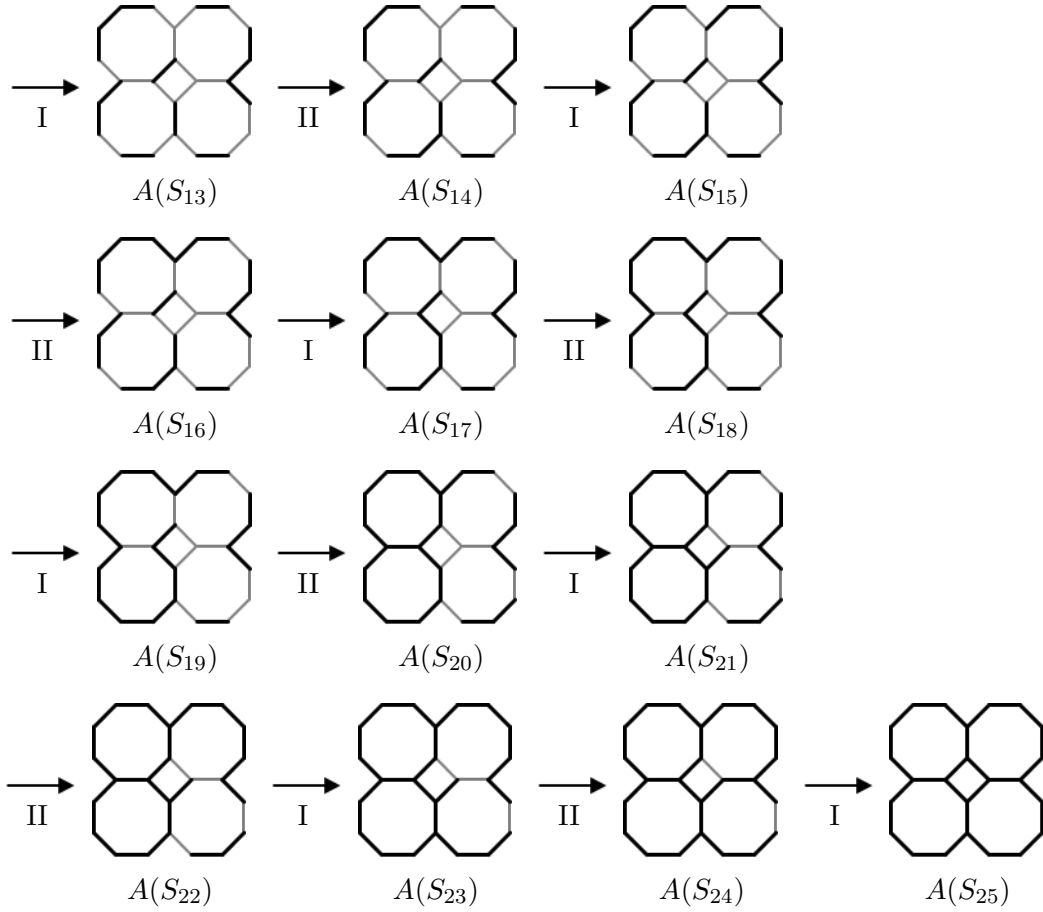
Figure 3.2: A 2×2 Closing Octagons game $\mathcal{G}$ of type I

Next, we formulate an $m \times n$ Closing Octagons game of type I into the new game using graphs. We construct the graph $(G_0)_{m \times n}$ representing the $m \times n$ array of octagons with no colored-sides. Let us regard the set of octagons as the vertex set of $(G_0)_{m \times n}$, and the edge set is the set of sides of octagons such that e is a simple edge incident to vertices u and v if e is a common side of octagons u and v, and e is a loop incident to a vertex v if e is an in-common side of an octagon v. Thus, we get the new game starting with $(G_0)_{m \times n}$ called an $m \times n$ Closing Octagons game of type II.

Definition 3.6. Let $m, n \in \mathbb{N}$. *Closing Octagons game of type II* or $m \times n$ *Closing Octagons game of type II* is a game for two players starting with the graph $(G_0)_{m \times n}$ and 0 points for both players. Two players alternately *turn* by the following rules.

(1′) A player *move* by removing one edge of the graph.

($2'$) A player who removes the last edge of k vertices earns k points and takes one more move.

The game ends when the graph contains no edges, and the player having the most points wins.

By Definition 3.6, a move is a removal of one edge and a turn is a possible list of consecutive move(s) by one player that satisfies the rules of playing the game.

Definition 3.7. For any $m \times n$ Closing Octagons game $\mathcal{G}$ of type II, $m \times n$ is called the *size* of $\mathcal{G}$.

Definition 3.8. For any Closing Octagons game $\mathcal{G}$ of type II, a *state* of $\mathcal{G}$ is a triple (G, p_1, p_2) where G, p_1 and p_2 are a graph and points of Player I and Player II that are changed by turns, respectively, including $((G_0)_{m \times n}, 0, 0)$.

To be specific, we use $G(S)$ to denote the graph of a state S of a Closing Octagons game of type II.
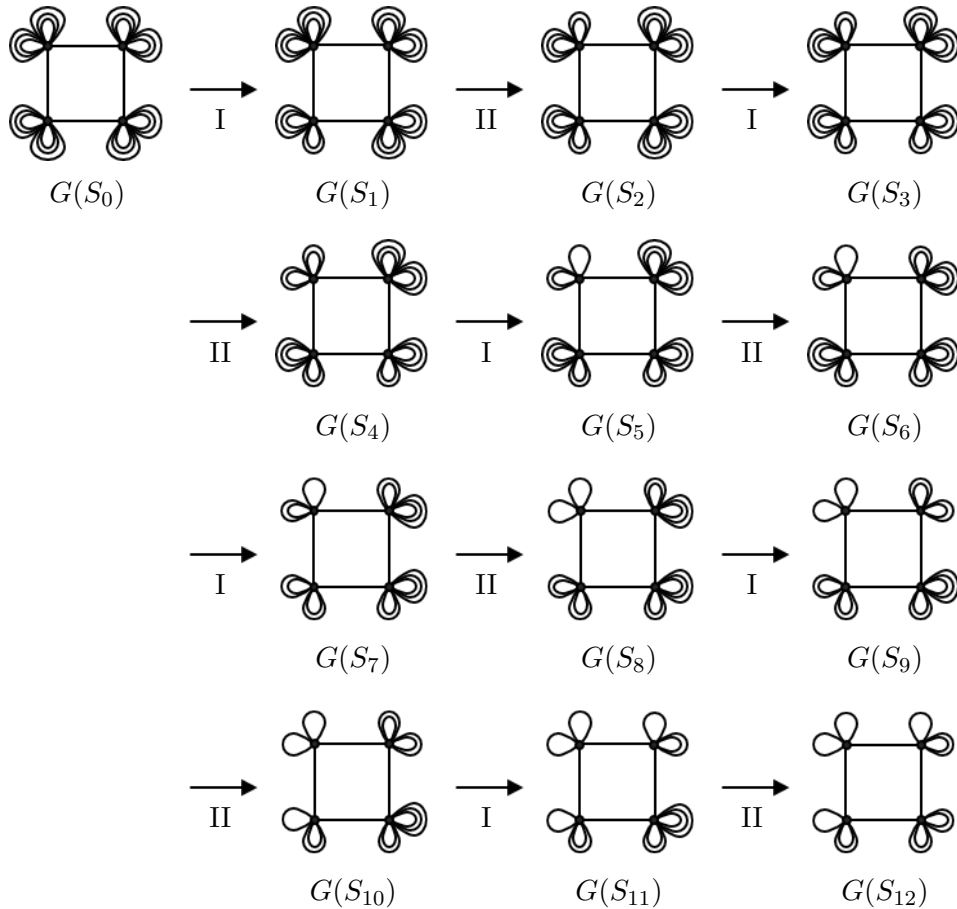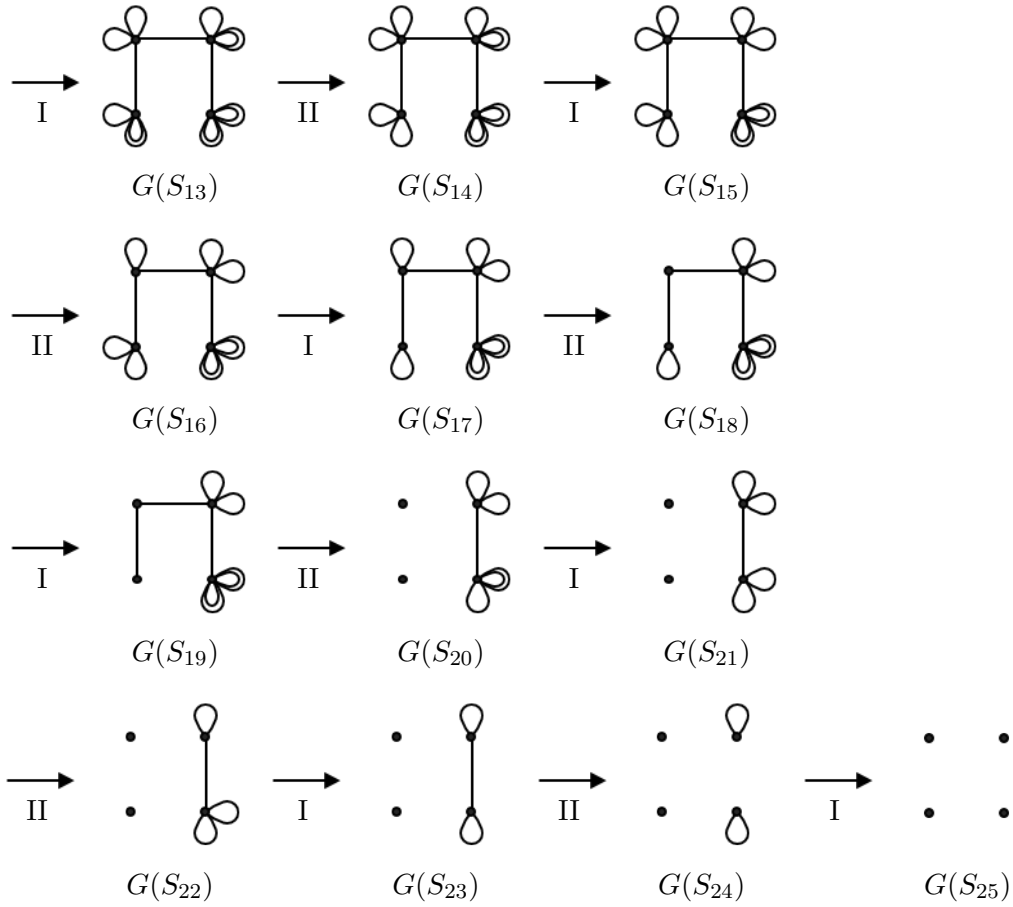
Figure 3.4: A 2×2 Closing Octagons game $\mathcal{G}$ of type II

Remark 3.9. Any Closing Octagons game of type II is an impartial combinatorial game.

Example 3.10. Let $\mathcal{G}$ be a 2×2 Closing Octagons game of type II that is shown in Figure 3.4. Then, $\mathcal{G}$ has 26 states and 25 turns, and two players draw.

A Closing Octagons game of type II that is played on a graph theoretic formulation is easier to consider and we can inherit some terminologies in graph theory to use with the game. From now on, as we talk about a *Closing Octagons game*, we refer to the Closing Octagons game of type II.

Definition 3.11. Let G be a graph. A move μ of G is a *winning move* if μ is a removal of an edge e of G such that the number of isolated vertices of $G - e$ is greater than the number of isolated vertices of G.

By Definition 3.11, we obtain that if $\mathcal{G}$ is a Closing Octagons game, a move μ of $\mathcal{G}$ is a winning move if μ gives points to a player.

Remark 3.12. For any Closing Octagons game $\mathcal{G}$, if $\tau = \mu_1, \mu_2, \mu_3, ..., \mu_k$ is a turn having at least 2 moves of $\mathcal{G}$, then

(1) if τ is a turn from some state to the terminal state, then μ_1, μ_2, μ_3, ..., μ_k are winning moves, and

(2) if τ is a turn from some state to a non-terminal state, then μ_1, μ_2, μ_3, ..., μ_{k-1} are winning moves and μ_k is not a winning move.

Definition 3.13. Let $\mathcal{G}$ be a Closing Octagons game. A non-terminal state S of $\mathcal{G}$ is a *normal state* if there is no possible turn containing a winning move from S to another state, otherwise S is called a *strategic state*.

By Definition 3.13, we obtain that if $\mathcal{G}$ is a Closing Octagons game, then S_0 is a normal state of $\mathcal{G}$, and a turn from a normal state to another state has exactly one move.

Example 3.14. According to the Closing Octagons game in Example 3.10, S_1, S_2, S_3, ..., S_{18}, S_{20}, S_{21}, S_{22} and S_{23} are normal states, and S_{19} and S_{24} are strategic states.

Definition 3.15. Let S be a state of a Closing Octagons game. A component K of $G(S)$ is a *weak component* if there is a possible turn from S to another state such that all edges of K are removed, otherwise K is called a *strong component*.

Definition 3.16. Let S be a state of a Closing Octagons game. A strong component K of $G(S)$ is a *chain* if for each edge e of K, all components of $K - e$ are weak components.

In general, if there is a possible turn from a normal state to another normal state, then players often make the turn. Because, if some player turns from a normal state to a strategic state, then the opponent can earn some points from this strategic state. Moreover, if all components of the graph of a strategic state

are weak components, then players often remove all edges of the graph. In order to analyze a winning strategy, we add more rules into this game and the game with these additional rules is called a normal game.

Definition 3.17. A Closing Octagons game $\mathcal{G}$ is a *normal game* if two players turn by the following rule.

(3′) If there is a possible turn τ from a state to a normal state, then a player has to make the turn τ.

(4′) If all components of the graph G of a state are weak components, then a player has to remove all edges of G.

Example 3.18. Let $\mathcal{G}$ be a normal 2×2 Closing Octagons game that is shown in Figure 3.6. Then, S_0, S_1, S_2, ..., S_{22} are normal states, and S_{23} and S_{24} are strategic states.
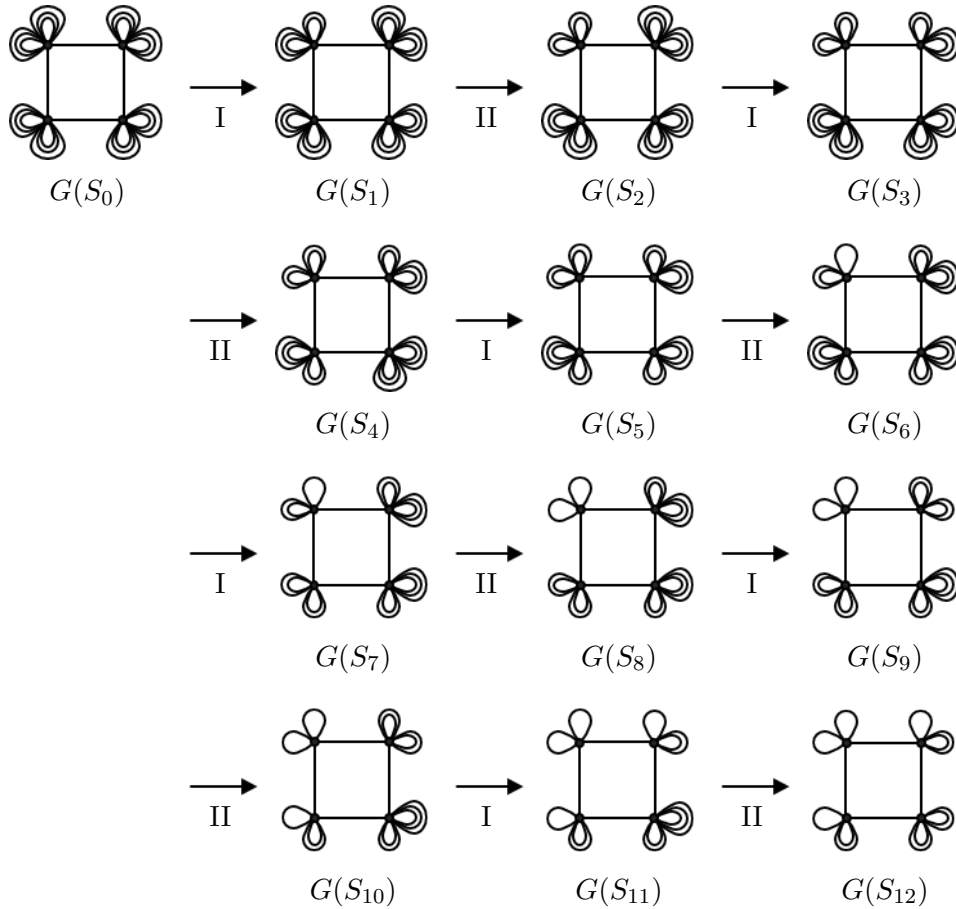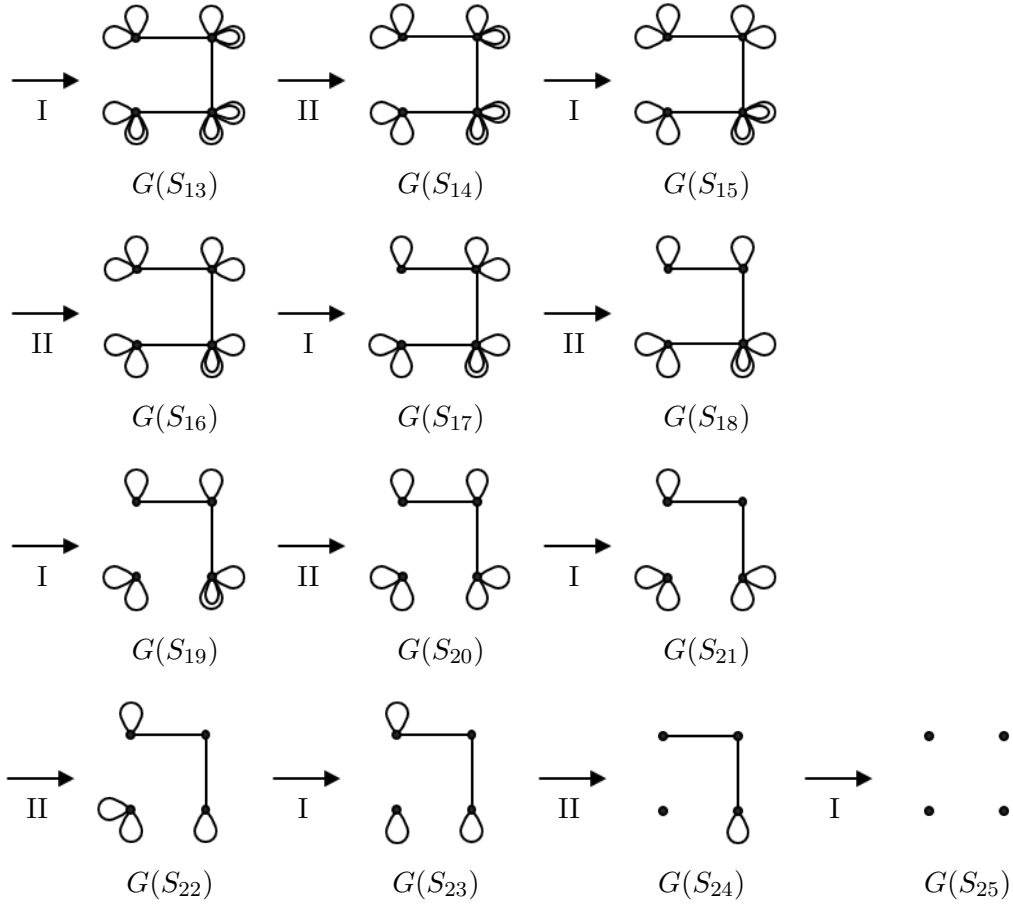
Figure 3.6: A normal 2×2 Closing Octagons game $\mathcal{G}$

Definition 3.19. Let $\mathcal{G}$ be a normal Closing Octagons game. A normal state S of $\mathcal{G}$ is a *critical state* if there is no possible turn from S to another normal state.

Lemma 3.20. *Every normal Closing Octagon game has exactly one critical state.*

Proof. Let $\mathcal{G}$ be a normal Closing Octagons game. Suppose that S_{k_1} and S_{k_2} are critical states of $\mathcal{G}$ where $k_1 < k_2$. Then, there is a strategic state S_l where $k_1 < l < k_2$ such that S_{l+1} is a normal state. Consider a turn which is a list $\mu_1, \mu_2, \mu_3, ..., \mu_k$ of consecutive moves from S_l to S_{l+1} such that μ_k is a removal of an edge e incident to u and v (If e is a loop, then $u = v$.). Then, μ_k is not a winning move and $G(S_{l+1})$ contains no e. Then, $G(S_{l+1})$ is a subgraph of $G(S_{k_1}) - e$. Since S_{l+1} is a normal state, each removal of an edge of $G(S_{l+1})$ incident to u or v is not a winning move. Then, each removal of an edge of $G(S_{k_1}) - e$ incident to u or v is not a winning move. Since S_{k_1} is a normal state, each removal of an edge of

$G(S_{k_1})$ is not a winning move. Then, each removal of an edge of $G(S_{k_1}) - e$ is not a winning move. This implies that there is a possible turn μ_k from S_{k_1} to another normal state. This contradicts the assumption that S_{k_1} is a critical state of $\mathcal{G}$. $\qquad \square$

By Lemma 3.20, we obtain that if $\mathcal{G}$ is a normal Closing Octagons game with $t + 1$ states such that S_k is the critical state, then S_0, S_1, S_2, ..., S_k are normal states and S_{k+1}, S_{k+2}, S_{k+3},..., S_{t-1} are strategic states of $\mathcal{G}$.

Example 3.21. According to the normal Closing Octagons game in Example 3.18, S_{22} is the critical state.

3.2 Analysis of Closing Octagons Game

In this section, we analyze our game in terms of moves and turns, normal states and critical states, strong components, weak components and chains.

Theorem 3.22. *The number of moves of an $m \times n$ Closing Octagons game is $6mn + m + n$.*

Proof. Let $\mathcal{G}$ be an $m \times n$ Closing Octagons game. Then, $G(S_0)$ of $\mathcal{G}$ has 4 vertices of degree 14, $2(m-2) + 2(n-2)$ vertices of degree 13, and $(m-2)(n-2)$ vertices of degree 12. Thus, the sum of degree of all vertices of $G(S_0)$ equals $14(4) + 13(2(m-2) + 2(n-2)) + 12(m-2)(n-2) = 12mn + 2m + 2n$. By Lemma 2.8, the number of edges of $G(S_0)$ is $6mn + m + n$. Therefore, the number of moves of $\mathcal{G}$ is $6mn + m + n$. $\qquad \square$

Theorem 3.23. *Let S be a normal state of a normal $m \times n$ Closing Octagons game such that $G(S)$ has exactly k edges.*
    (1) If $k - m - n$ is even, then a turn from S to another state is Player I's.
    (2) If $k - m - n$ is odd, then a turn from S to another state is Player II's.

Proof. Let $\mathcal{G}$ be a normal $m \times n$ Closing Octagons game. Then, S_0, S_1, S_2, ..., S are normal states. By Theorem 3.22, the number of all moves of $\mathcal{G}$ is $6mn + m + n$. Since $G(S)$ has exactly k edges, the number of moves from S_0 to S is $6mn + m + n - k$.

Since a turn from a normal state to another state has exactly 1 move, the number of turns from S_0 to S is $6mn + m + n - k$. Obviously, $6mn + m + n - k$ and $k - m - n$ have the same parity. Therefore,

(1) if $k - m - n$ is even, then a turn from S to another state is Player I's, and

(2) if $k - m - n$ is odd, then a turn from S to another state is Player II's. $\square$

The following theorems involving states of the normal game.

Lemma 3.24. *Let S be a normal state of a normal Closing Octagons game. Then, $G(S)$ contains no isolated vertices.*

Proof. Let $\mathcal{G}$ be a normal Closing Octagons game. Then, S_0, S_1, S_2, ..., $S_k = S$ are normal states. Thus, for all $i \in \{0, 1, 2, ..., k\}$, there is no possible turn from S_i to S_{i+1} giving points to a player. This implies that $G(S)$ contains no isolated vertices. $\square$

Theorem 3.25. *Let $\mathcal{G}$ be a normal Closing Octagons game. A state S of $\mathcal{G}$ is a normal state if and only if for each vertex v of $G(S)$, $w_{G(S)}(v) \geq 2$.*

Proof. Assume that S is a normal state of $\mathcal{G}$. By Lemma 3.24, $G(S)$ contains no isolated vertices. Then, for each vertex v of $G(S)$, $w_{G(S)}(v) \geq 1$.

To show that for each vertex v of $G(S)$, $w_{G(S)}(v) \geq 2$, suppose that there is a vertex v_0 of $G(S)$ such that $w_{G(S)}(v_0) = 1$. Let e_0 be an edge incident to v_0 and μ_0 be a removal of e_0. Then, $w_{G(S)-e_0}(v_0) = 0$. Thus, μ_0 is a winning move of $G(S)$. This implies that there is a possible turn containing μ_0 from S to another state. This contradicts the assumption that S is a normal state. Hence, for each vertex v of $G(S)$, $w_{G(S)}(v) \geq 2$.

Conversely, assume that S is a state of $\mathcal{G}$ such that for each vertex v of $G(S)$, $w_{G(S)}(v) \geq 2$. Let e be arbitrary edge of $G(S)$ and μ is a removal of e. Let v_1 and v_2 be vertices of $G(S)$ incident to e. If e is a loop, then $v_1 = v_2$. If e is a simple edge, then $v_1 \neq v_2$. Then, $w_{G(S)}(v_1) \geq 2$ and $w_{G(S)}(v_2) \geq 2$. Thus, $w_{G(S)-e}(v_1) \geq 1$ and $w_{G(S)-e}(v_2) \geq 1$. Then, μ is not a winning move of $G(S)$. This implies that there is no possible turn containing a winning move from S to another state. Hence, S is a normal state. $\square$

Theorem 3.26. *Let $\mathcal{G}$ be a normal Closing Octagons game. A state S of $\mathcal{G}$ is the critical state if and only if for each vertex v of $G(S)$,*

(1) $w_{G(S)}(v) \geq 2$, and

(2) if $w_{G(S)}(v) \geq 3$, then each edge incident to v is a simple edge, and the weight of each vertex adjacent to v is 2.

Proof. Assume that S is a critical state of $\mathcal{G}$. Let v be arbitrary vertex of $G(S)$. By Theorem 3.25, (1) holds.

To show (2), assume that v is a vertex of $G(S)$ such that $w_{G(S)}(v) \geq 3$.

First, we show that each edge incident to v is a simple edge. Suppose that there is a loop l_0 of $G(S)$ incident to v. Let μ_1 be a removal of l_0. Then, $w_{G(S)-l_0}(v) \geq 2$. Thus, μ_1 is a possible turn from S to a normal state. This contradicts the assumption that S is a critical state. Hence, each edge incident to v is a simple edge.

Next, we show that the weight of each vertex adjacent to v is 2. Suppose that there is a vertex v_0 of $G(S)$ adjacent to v such that $w_{G(S)}(v_0) \geq 3$. Let e_0 be a simple edge of $G(S)$ incident to v and v_0 and μ_2 be a removal of e_0. Since $w_{G(S)}(v) \geq 3$ and $w_{G(S)}(v_0) \geq 3$, $w_{G(S)-e_0}(v) \geq 2$ and $w_{G(S)-e_0}(v_0) \geq 2$. Then, μ_2 is a possible turn from S to a normal state. This contradicts the assumption that S is a critical state. Hence, the weight of each vertex adjacent to v is 2.

Conversely, assume that S is a state of $\mathcal{G}$ such that (1) and (2) hold. By Theorem 3.25, S is a normal state. Let e be arbitrary edge of $G(S)$ and μ be a removal of e. Let v_1 and v_2 be vertices of $G(S)$ incident to e. By (1), $w_{G(S)}(v_1) \geq 2$. If $w_{G(S)}(v_1) = 2$, then $w_{G(S)-e}(v_1) = 1$. If $w_{G(S)}(v_1) \geq 3$, then by (2), $w_{G(S)}(v_2) = 2$, and then $w_{G(S)-e}(v_2) = 1$. Thus, μ is a possible turn from S to a strategic state. Hence, S is a critical state. $\square$

Corollary 3.27. *Let $\mathcal{G}$ be a normal $1 \times n$ or 2×2 Closing Octagons game. A state S of $\mathcal{G}$ is the critical state if and only if for each vertex v of $G(S)$, $w_{G(S)}(v) = 2$.*

Proof. For each vertex v of $G(S_0)$ of $\mathcal{G}$, the number of simple edges incident to v is at most 2. Then, Theorem 3.26 implies that a state S of $\mathcal{G}$ is the critical state if and only if for each vertex v of $G(S)$, $w_{G(S)}(v) = 2$. $\square$

Next, we consider strong components in the graph of states of the game.

Theorem 3.28. *Let S be a state of a Closing Octagons game and K be a component of $G(S)$ such that K is not a 1-cycle component. If K contains no leaves, then K is a strong component.*

Proof. Assume that K contains no leaves. Then, the weight of each vertex of K is at least 2. Thus, for each edge e of K, $K - e$ contains no isolated vertices. This implies that there is no possible turn from S to another state such that all edges of K are removed. Therefore, K is a strong component. $\qquad\square$

By Theorem 3.28, we obtain that if S is a state of a Closing Octagons game and K is a weak component of $G(S)$, then either K is a 1-cycle component or K contains a leaf.

Theorem 3.29. *Let S be a state of a Closing Octagons game and K be a component of $G(S)$. If K contains a simple cycle subgraph, then K is a strong component.*

Proof. Assume that K contains a simple cycle subgraph C. To show that there is no possible turn from S to another state such that all edges of K are removed, suppose that there is a possible turn τ_0 from S to another state such that all edges of K are removed. Let μ_0 be a move of τ_0 such that μ_0 is the first removal of some simple edge e_0 of C. Since e_0 is a simple edge of C, the weight of each vertex of K incident to e_0 is at least 2. Then, μ_0 is not a winning move of K. This implies that μ_0 is the last move of τ_0, and some edges of K are not removed. This contradics the statement that τ_0 is a possible turn from S to another state such that all edges of K are removed. Therefore, K is a strong component. $\qquad\square$

The following facts are concerning weak components in the graph of states of the game.

Lemma 3.30. *Let S be a state of a Closing Octagons game. A component K of $G(S)$ is a weak component if and only if there is a list of consecutive winning moves of $G(S)$ such that all edges of K are removed.*

Proof. By Definition 3.15, a component K of $G(S)$ is a weak component if and only if there is a possible turn from S to another state such that all edges of K are removed. Obviously, a component K of $G(S)$ is a weak component if and only if there is a list of consecutive winning moves such that all edges of K are removed. $\qquad\square$

Theorem 3.31. *Let S be a state of a Closing Octagons game and K be a component of $G(S)$. If K is a tree, then K is a weak component.*

Proof. Obviously, a trivial tree is a weak component. Assume that K is a non-trivial tree with k edges. To prove that there is a list of consecutive winning moves such that all edges of K are removed, we use the mathematical induction on k.

Basic step: $k = 1$. K has exactly 1 edge. Then, there is a winning move which is a removal of the edge of K.

Inductive step: $k \geq 2$. Assume that for each non-trivial tree T such that the number of edges less than k, there is a list of consecutive winning moves such that all edges of T are removed. By Lemma 2.24, K has at least 2 leaves. Let v_0 be a leaf of K, e_0 be an edge of K incident to v_0 and μ_1 is a removal of e_0. By Corollary 2.25, $K - e_0$ has exactly 2 components such that one component is a trivial component and the other component K' is a non-trivial tree with $k - 1$ edges. Then, μ_1 is a winning move of K. By the induction hypothesis, there is a list $\mu_2, \mu_3, \mu_4, ..., \mu_k$ of consecutive winning moves such that all edges of K' are removed. This implies that $\mu_1, \mu_2, \mu_3, ..., \mu_k$ is a list of consecutive winning moves such that all edges of K are removed.

Therefore, K is a weak component. $\qquad\square$

Corollary 3.32. *Let S be a state of a Closing Octagons game and K be a component of $G(S)$. If K is a path component, then K is a weak component.*

Proof. Assume that K is a path component. Then, K contains no cycle subgraphs. This implies that K is a tree. By Theorem 3.31, K is a weak component. $\qquad\square$

Theorem 3.33. *Let S be a state of a Closing Octagons game and K be a component of $G(S)$. If K contains a loop l such that $K - l$ is a tree, then K is a weak component.*

Proof. Assume that K has exactly k edges and contains a loop l_0 such that $K - l_0$ is a tree. Then, l_0 is only one loop of K. To prove that there is a list of consecutive winning moves such that all edges of K are removed, we use the mathematical induction on k.

Basic step: $k = 1$. There is a winning move which is a removal of l_0 of K.

Inductive step: $k \geq 2$. Assume that for each component G containing l_0 such that $G - l_0$ is a tree and the number of edges of G less than k, there is a list of consecutive winning moves such that all edges of G are removed. By Lemma 2.24, $K - l_0$ has at least 2 leaves. Then, K has at least 1 leaf. Let v_0 be a leaf of K, e_0 be a simple edge of K incident to v_0 and μ_1 is a removal of e_0. Then, v_0 and $e - 0$ are a leaf and a simple edge of $K - l_0$, respectively. Since K has eaxctly k edges, $K - l_0$ has exactly $k - 1$ edges. By Corollary 2.25, $(K - l_0) - e_0$ has exactly 2 components such that one component is a trivial component and the other is a tree with $k - 2$ edges. Then, $K - e_0$ has exactly 2 components such that one component is a trivial component and the other component K' has exactly $k - 1$ edges and contains l_0 such that $K' - l_0$ is a tree. Then, μ_1 is a winning move of K. By the induction hypothesis, there is a list $\mu_2, \mu_3, \mu_4, ..., \mu_k$ of consecutive winning moves such that all edges of K' are removed. This implies that $\mu_1, \mu_2, \mu_3, ..., \mu_k$ is a list of consecutive winning moves such that all edges of K are removed.

Therefore, K is a weak component. $\qquad\square$

Corollary 3.34. *Let S be a state of a Closing Octagons game and K be a component of $G(S)$. If K is a pseudopath component having exactly 1 loop, then K is a weak component.*

Proof. Assume that K is a pseudopath component having exactly 1 loop l_0. Then, $K - l_0$ is a path component. Then, $K - l_0$ contains no cycle subgraphs. This implies that $K - l_0$ is a tree. By Theorem 3.33, K is a weak component. $\qquad\square$

Finally, we consider the chains in the graph of states of our game.

Lemma 3.35. *Let S be a state of a Closing Octagons game and K be a strong component of $G(S)$. If K contains a leaf, then K is not a chain.*

Proof. Assume that K contains a leaf v_0. Suppose that K is a chain. Let e_0 be a simple edge incident to v_0 and μ_1 be a removal of e_0. Then, $w_{K-e_0}(v_0) = 0$. Thus, μ_1 is a winning move of K. Since K is a chain, all components of $K - e_0$ are weak components. Then, there is a list $\mu_2, \mu_3, ..., \mu_k$ of consecutive winning moves such that all edges of $K - e_0$ are removed. This implies that $\mu_1, \mu_2, \mu_3, ..., \mu_k$ is a list of consecutive winning moves such that all edges of K are removed. Then, K is a weak component. This contradicts the statement that K is a chain. Therefore, K is not a chain. $\qquad\square$

Theorem 3.36. *Let S be a state of a Closing Octagons game. A pseudopath component K of $G(S)$ is a chain if and only if K has exactly 2 loops and no leaves.*

Proof. Assume that K is a pseudopath chain of $G(S)$. By Lemma 3.35, K contains no leaves. Then, K has at least 2 loops.

To show that K has exactly 2 loops. Suppose that K contains at least 3 loops. Then, there is a loop l_0 of K such that $K - l_0$ contains no leaves. By Theorem 3.28, $K - l_0$ is a strong component. This contradicts the assumption that K is a chain. Hence, K has at least 2 loops.

Conversely, assume that K is a pseudopath component of $G(S)$ such that K has exactly 2 loops and no leaves. Then, K is isomorphic to a graph that is shown in Figure 3.7. By Theorem 3.28, K is a strong component. Let e be arbitrary edge of K. If e is a loop, then $K - e$ is a pseudopath component having exactly 1 loop. If e is a simple edge, then $K - e$ has exactly 2 components such that each component is a pseudopath component having exactly 1 loop. By Corollary 3.34, all components of $K - e$ are weak components. Hence, K is a chain. $\qquad\square$

By Theorem 3.36, we can conclude that a k-pseudopath chain is a graph isomorphic to a graph that is shown in Figure 3.7.
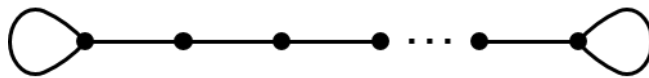

Figure 3.7: A pseudopath chain

Lemma 3.37. *Let S be a state of a Closing Octagons game. If K is a strong component of $G(S)$ such that for each vertex v of K, $w_K(v) = 2$. Then, K is a 2-bouquet component, a pseudopath chain or a simple cycle component.*

Proof. Assume that K is a strong component of $G(S)$ such that for each vertex v of K, $w_K(v) = 2$. Then, either K is a simple component or K contains a loop.

Case 1. K is a simple component. Then, for each vertex v of K, $d_K(v) = 2$. By Lemma 2.18, K is a simple cycle component.

Case 2. K contains a loop. Then, K is not a simple cycle component. To show that K is a 2-bouquet component or a pseudopath chain, assume that K is not a 2-bouquet component. Then, K has at least 1 simple edge.

First, we show that K contains no simple cycle subgraphs. Suppose that K contains a simple cycle subgraph C. Since K is not a simple cycle component, there is an edge e_1 of K incident to some vertex v_1 of C such that e_1 is not an edge of C. Then, $w_K(v_1) \geq 3$. This contradicts the assumption that for each vertex v of K, $w_K(v) = 2$. Hence, K contains no simple cycle subgraphs.

Let P be a maximal path subgraph of K and u_1 and u_2 be distinct leaves of P. Since K contains no simple cycle subgraphs, each of u_1 and u_2 is incident to a loop of K.

Next, we show that K is a pseudopath component. Suppose that K is not a pseudopath component. Then, there is a simple edge e_2 of K incident to some vertex v_2 of P such that e_2 is not an edge of P and v_2 is not a leaf of P. Then, $w_K(v_2) \geq 3$. This contradicts the assumption that for each vertex v of K, $w_K(v) = 2$. Hence, K is a pseudopath component such that P is a maximal path subgraph of K.

Finally, we show that K has exactly 2 loops. Suppose that K has at least 3 loops. Since each of u_1 and u_2 is incident to a loop, there is a loop l_0 of K incident to some vertex v_3 of P such that v_3 is not a leaf of P. Then, $w_K(v_3) \geq 3$. This contradicts the assumption that for each vertex v of K, $w_K(v) = 2$. Hence, K has exactly 2 loops.

Consequently, K is a pseudopath chain. $\qquad\qquad\square$

Theorem 3.38. *Let S be a state of a Closing Octagons game and K be a component of $G(S)$. Then, the following are equivalent.*

(1) K is a chain.

(2) For each vertex v of K, $w_K(v) = 2$.

(3) K is a 2-bouquet component, a pseudopath chain or a simple cycle component.

Proof. To show that (1) implies (2), assume that K is a chain of $G(S)$. Obviously, K is neither a trivial component nor a 1-cycle component. By Lemma 3.35, K contains no leaves. Then, for each vertex v of K, $w_K(v) \geq 2$.

To show that for each vertex v of K, $w_K(v) = 2$, suppose that there is a vertex v_0 of K such that $w_K(v_0) \geq 3$. Then, either there is a loop of K incident to v_0 or all edges of K incident to v_0 are simple edges.

Case 1. There is a loop l_0 of K incident to v_0. Then, $w_{K-l_0}(v_0) \geq 2$. This implies that $K - l_0$ contains no leaves. By Theorem 3.28, $K - l_0$ is a strong component. Then, K is not a chain.

Case 2. All edges incident to v_0 are simple edges. Let u_0 be a vertex of K adjacent to v_0 and e_1 be an edge of K incident to u_0 and v_0. Then, $K - e_1$ has at most 2 components.

Case 2.1. $K - e_1$ has exactly 1 component. Then, there is a list $v_0, v_1, v_2, ..., v_k = u_0$ of vertices of $K - e_1$ such that for all $i \in \{1, 2, 3, ..., k\}$, v_{i-1} and v_i are adjacent. Since u_0 and v_0 are adjacent in K, K contains a simple cycle subgraph C such that $V(C) = \{v_0, v_1, v_2, ..., v_k = u_0\}$. Since $w_K(v_0) \geq 3$, there is a simple edge e_2 of K incident to v_0 such that e_2 is not an edge of C. Then, $K - e_2$ contains C. By Theorem 3.29, $K - e_2$ is a strong component. Then, K is not a chain.

Case 2.2. $K - e_1$ has exactly 2 component. Then, one component of $K - e_1$ contains u_0 and the other component K' contains v_0. Since $w_K(v_0) \geq 3$, $w_{K-e_1}(v_0) \geq 2$. This implies that K' contains no leaves. By Theorem 3.28, $K - l_0$ is a strong component. Then, K is not a chain.

So by 2 cases, K is not a chain. This contradicts the assumption that K is a chain. Hence, for each vertex v of K, $w_K(v) = 2$.

To show (2) implies (3), we use Lemma 3.37 directly.

To show (3) implies (1), we first observe that a pseudopath chain is a chain. If K is a 2-bouquet component, then by Theorem 3.28, K is a strong component, and then for each edge e of K, $K - e$ is a 1-bouquet component which is a weak component. If K is a simple cycle component, then by Theorem 3.29, K is a strong component, and then for each edge e of K, $K - e$ is a path component which is a weak component. Hence, K is a chain. $\square$

Example 3.39. Let S be the critical state of a normal 2×2 Closing Octagons game. By Corollary 3.27, for each vertex v of $G(S)$, $w_{G(S)}(v) = 2$. By Theorem 3.38, each component of $G(S)$ is a 2-bouquet component, a pseudopath chain or a simple cycle component. We obtain that there are 6 cases of $G(S)$ up to isomorphism shown in Figure 3.8.
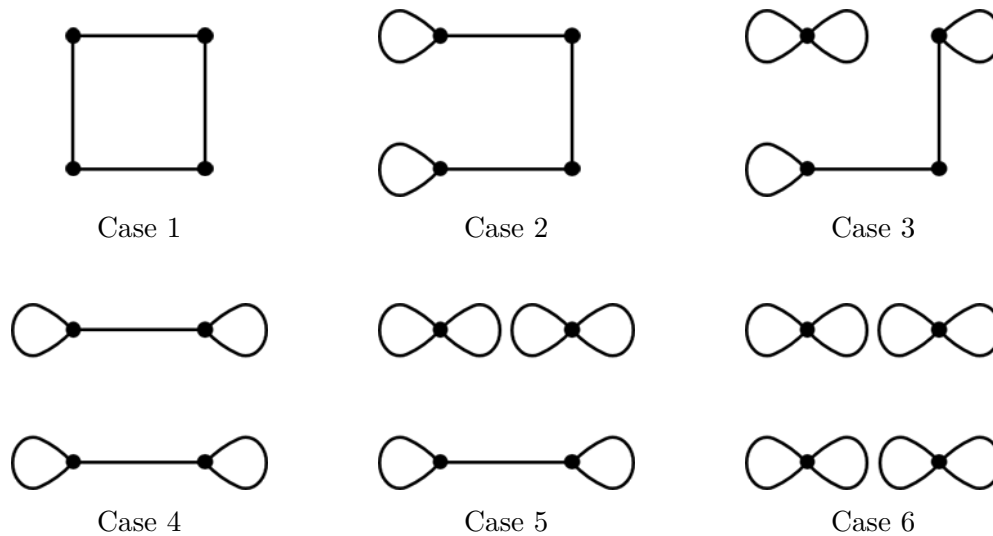

Figure 3.8: Six cases of the graph of the critical state of a normal 2×2 game

CHAPTER IV
STRATEGIES FOR PLAYING THE GAME

In this chapter, we give strategies of playing the normal Closing Octagons game for each player to win or draw or get the most possible points in some situations and some games of size $1 \times n$, $2 \times n$ and 3×3. A *strategy* is a plan of playing the game that is constructed for moves or turns of a player. However, the strategy must be usable and does not contradict the main rules of normal game. A strategy is called a *winning strategy* for a player if the player wins when the player plays according to the plan of the strategy, no matter how the opponent plays.

4.1 Some Strategies

In this section, we give examples of some strategies and analyze results of playing the normal Closing Octagons game in some situations.

First, we give a definition of maximum turn and introduce the first strategy, namely, Strategy A.

Definition 4.1. Let S be a state of a Closing Octagons game. A possible turn τ from S to another state is a *maximum turn* if for each passible turn τ' from S to another state, the number of moves of τ' is less than or equal to the number of moves of τ.

Strategy 4.2 (Strategy A). Let S be a state of a Closing Octagons game. A player has to make a maximum turn from S to another state.
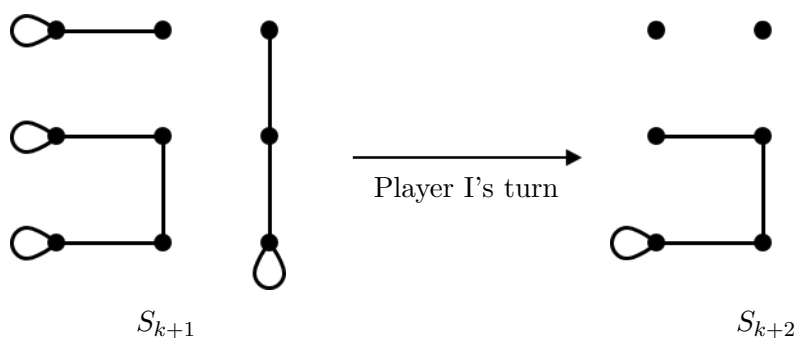
Example 4.3. Consider a playing normal 3×3 Closing Octagons game such that the graph of the critical state S_k is shown in Figure 4.1. Since $G(S_k)$ has 11 edges, Theorem 3.23 implies that a turn from S_k to S_{k+1} is Player II's.
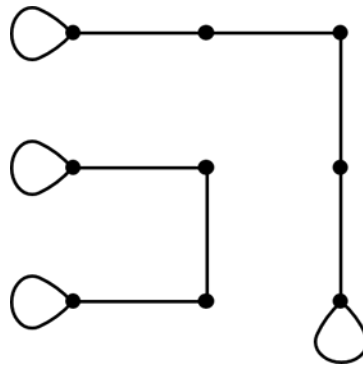
Figure 4.1: The graph of the critical state S_k

If Player II makes a turn from S_k to S_{k+1} that is shown in Figure 4.2, then $G(S_{k+1})$ has exactly 2 weak components and 1 chain.
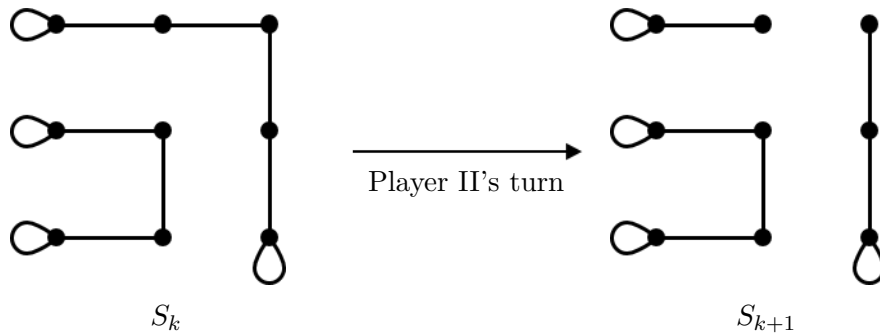
Figure 4.2: Player II's turn from S_k to S_{k+1}

If Player I makes a turn from S_{k+1} to S_{k+2} by using Strategy A that is shown in Figure 4.3, then Player I earns 5 points.

Figure 4.3: Player I's turn from S_{k+1} to S_{k+2}

Next, Player II has to make a turn from S_{k+2} to the terminal S_{k+3} by removing all edges of $G(S_{k+2})$, and then Player II earns 4 points. Therefore, Player I wins.

By Example 4.3, Strategy A makes Player I to win. However, if Player II does not make a turn from S_k to S_{k+1} as shown in Figure 4.2, then Player I may not win, as we can see in Example 4.4.

Example 4.4. Consider a playing normal 3×3 Closing Octagons game such that the graph of the critical state S_k is shown in Figure 4.1. If Player II makes a turn from S_k to S_{k+1} that is shown in Figure 4.4, then $G(S_{k+1})$ has exactly 2 weak components and 1 chain.
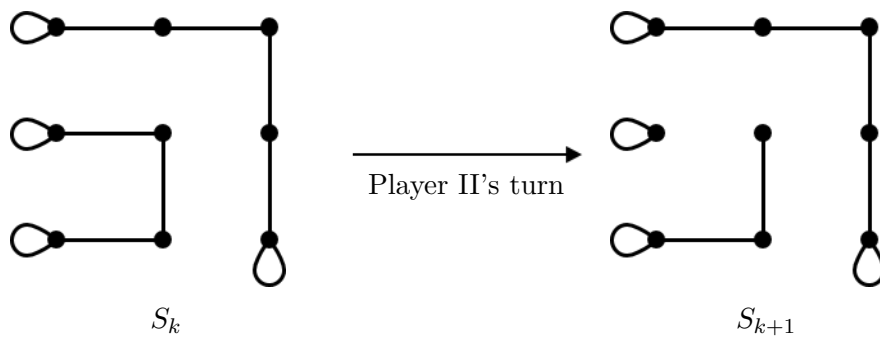

Figure 4.4: Player II's turn from S_k to S_{k+1}

If Player I makes a turn from S_{k+1} to S_{k+2} by using Strategy A that is shown in Figure 4.5, then Player I earns 4 points.
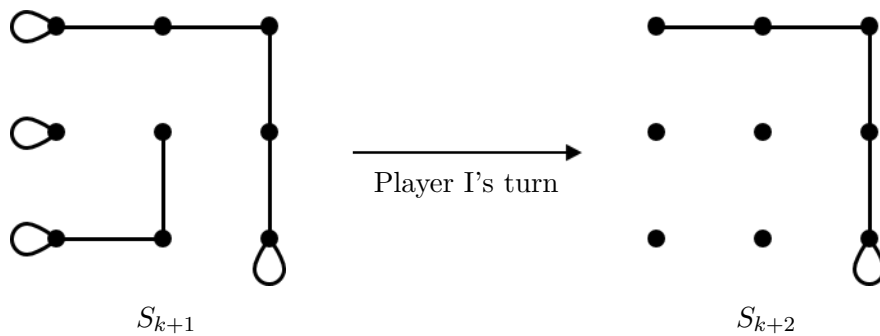

Figure 4.5: Player I's turn from S_{k+1} to S_{k+2}

Next, Player II has to make a turn from S_{k+2} to the terminal S_{k+3} by removing all edges of $G(S_{k+2})$, and then Player II earns 5 points. Therefore, Player I loses.

By Example 4.4, Strategy A makes Player I to lose. We conclude that for playing normal Closing Octagons game, if the graph of the critical state is isomorphic

to a graph that is shown in Figure 4.1, then Strategy A is not a winning strategy for Player I.

Next, we introduce Strategy B and show examples of the difference between Strategy A and Strategy B.

Strategy 4.5 (Strategy B). Let S be a state of a Closing Octagons game. If τ is a maximum turn from S to another non-terminal state such that $\mu_1, \mu_2, \mu_3, ..., \mu_k$ are all winning moves of τ, and there is another possible turn τ_0 from S to another state such that

(1) all moves of τ_0 are elements of the set $\{\mu_1, \mu_2, \mu_3, ..., \mu_k\}$, and

(2) for each possible turn τ' from S to another state such that all moves of τ' are elements of the set $\{\mu_1, \mu_2, \mu_3, ..., \mu_k\}$, the number of moves of τ' is less than or equal to the number of moves of τ_0,

then a player has to make a turn τ_0 from S to another state.

Example 4.6. Consider a playing normal 3×3 Closing Octagons game such that the graph of the critical state S_k is shown in Figure 4.1, and Player II makes a turn from S_k to S_{k+1} that is shown in Figure 4.2.

If Player I makes a turn from S_{k+1} to S_{k+2} by using Strategy B that is shown in Figure 4.6, then Player I earns 3 points.

Figure 4.6: Player I's turn from S_{k+1} to S_{k+2}

Next, Player II can earn at most 2 points from S_{k+2} to S_{k+3} and Player I can earn at least 4 more points from S_{k+3} to the terminal state S_{k+4}. Therefore, Player I wins.

Example 4.7. Consider a playing normal 3×3 Closing Octagons game such that the graph of the critical state S_k is shown in Figure 4.1, and Player II makes a turn from S_k to S_{k+1} that is shown in Figure 4.4.

If Player I makes a turn from S_{k+1} to S_{k+2} by using Strategy B that is shown in Figure 4.7, then Player I earns 2 points.
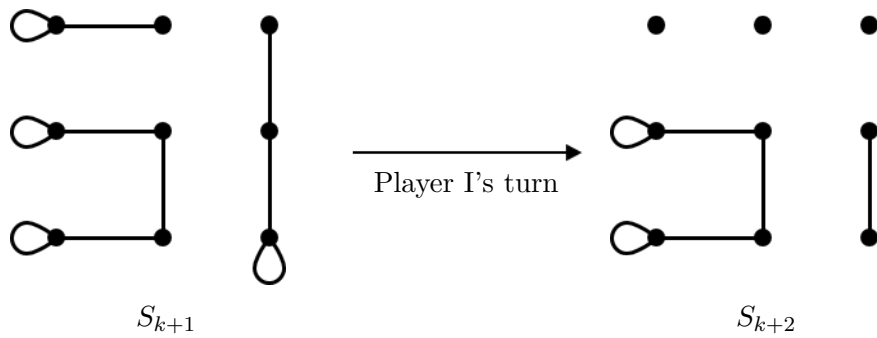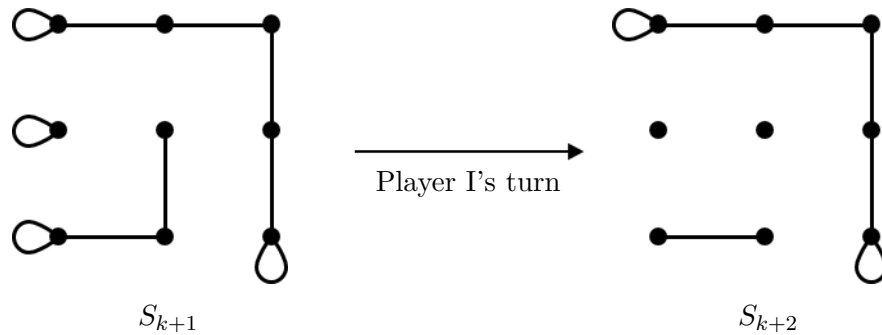
S_{k+1} Player I's turn S_{k+2}

Figure 4.7: Player I's turn from S_{k+1} to S_{k+2}

Next, Player II can earn at most 2 points from S_{k+2} to S_{k+3} and Player I can earn at least 5 more points from S_{k+3} to the terminal state S_{k+4}. Therefore, Player I wins.

By Example 4.6 and Example 4.7, Strategy B makes Player I to win. However, Strategy B may not always give more points than Strategy A, as we can see in Example 4.8.

Example 4.8. Consider a playing normal 2×3 Closing Octagons game such that the graph of the critical state S_k is shown in Figure 4.8. Since $G(S_k)$ has 7 edges, Theorem 3.23 implies that a turn from S_k to S_{k+1} is Player I's.
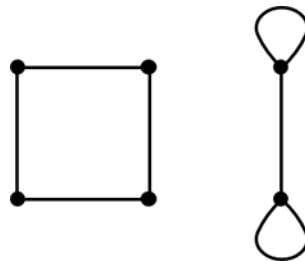
Figure 4.8: The graph of the critical state S_k

If Player I makes a turn from S_k to S_{k+1} that is shown in Figure 4.9, then $G(S_{k+1})$ has exactly 1 weak component and 1 chain.
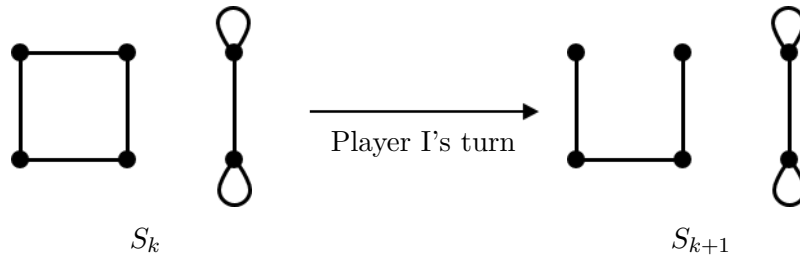

S_k $\qquad\qquad$ S_{k+1}

Figure 4.9: Player I's turn from S_k to S_{k+1}

Case 1. Player II makes a turn from S_{k+1} to S_{k+2} by using Strategy A that is shown in Figure 4.10. Then, Player II earns 4 points.
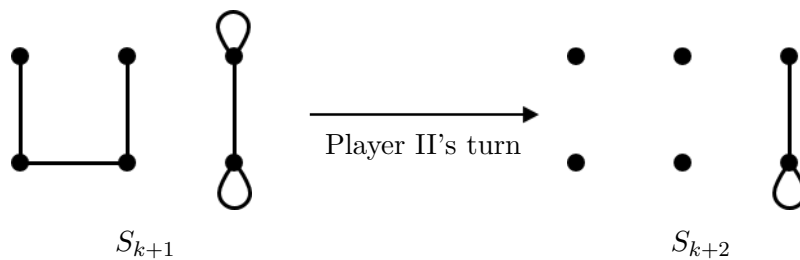

S_{k+1} $\qquad\qquad$ S_{k+2}

Figure 4.10: Player II's turn from S_{k+1} to S_{k+2}, using Strategy A

Next, Player I has to make a turn from S_{k+2} to the terminal S_{k+3} by removing all edges of $G(S_{k+2})$, and then Player I earns 2 points. Therefore, Player II wins.

Case 2. Player II makes a turn from S_{k+1} to S_{k+2} by using Strategy B that is shown in Figure 4.11. Then, Player II earns no points.
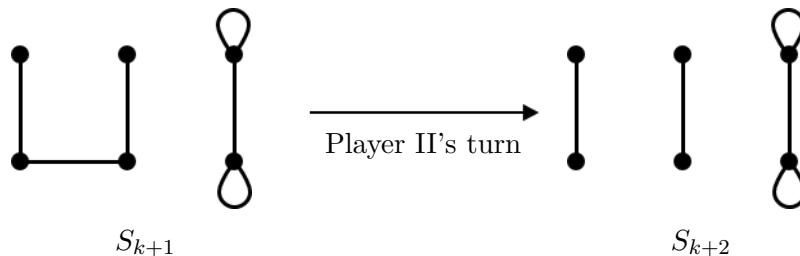

S_{k+1} $\qquad\qquad$ S_{k+2}

Figure 4.11: Player II's turn from S_{k+1} to S_{k+2}, using Strategy B

If Player I makes a turn from S_{k+2} to S_{k+3} that is shown in Figure 4.12, then Player I earns 4 points.
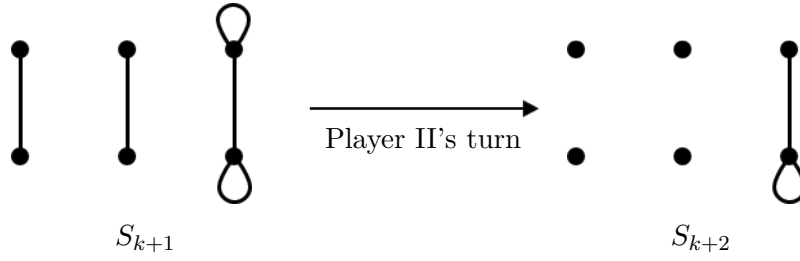
Figure 4.12: Player I's turn from S_{k+1} to S_{k+2}

Then, Player II has to make a turn from S_{k+3} to to the terminal S_{k+4} by removing all edges of $G(S_{k+3})$, and then Player II earns 2 points. Therefore, Player II loses.

Theorem 4.9. *For any normal Closing Octagons game such that the graph of the critical state is a chain, a player who turns from the first strategic state to another state wins.*

Proof. Let S_k be the critical state of a normal Closing Octagons game such that $G(S_k)$ is a chain. Then, for each edge e of $G(S_k)$, a removal of e is not a winning move of $G(S_k)$ and all components of $G(S_k) - e$ are weak components. Thus, S_{k+1} is the first strategic state such that all components of $G(S_{k+1})$ are weak components. This implies that a turn from S_{k+1} to another state is a removal of all edges of $G(S_{k+1})$. Therefore, the player who turns from S_{k+1} to another state wins. $\square$

The following theorems involving Strategy A and normal games of size $m \times n$ such that all components of the graph of the critical state are 2-bouquet components.

Theorem 4.10. *For playing normal $m \times n$ Closing Octagons game where m and n are odd, if all components of the graph of the critical state are 2-bouquet components, then Strategy A is a winning strategy for Player II.*

Proof. Let S_k be the critical state of a normal $m \times m$ Closing Octagons game where m and n are odd such that all components of $G(S_k)$ are 2-bouquet components and Player II plays according to Strategy A. Then, $G(S_k)$ has exactly $2mn$ edges.

Obviously, $2mn - m - n$ is even. By Theorem 3.23, a turn from S_k to S_{k+1} is Player I's.

Since all mn components of $G(S_k)$ are 2-bouquet components, Player I turns from S_k to S_{k+1} by removing a loop of a 2-bouquet component. Then, $G(S_{k+1})$ has exactly 1 1-bouquet component and $mn - 1$ 2-bouquet components.

By Strategy A, Player II turns from S_{k+1} to S_{k+2} by removing a loop of a 1-bouquet component and a loop of a 2-bouquet component, respectively. Then, Player II earns 1 point and $G(S_{k+2})$ has exactly 1 1-bouquet component and $mn - 2$ 2-bouquet components.

Next, it easy to see that two players alternately turn from S_{k+2} to S_{k+mn} such that Player I either turns by removing a loop of a 2-bouquet component or turns by removing a loop of a 1-bouquet component and a loop of a 2-bouquet component, respectively, and earns at most 1 more point, and Player II turns by removing all edges of 1-bouquet components and a loop of a 2-bouquet component, respectively, and earns at least 1 more point.

Now, Player II turns from S_{k+mn} to the terminal state S_{k+mn+1} by removing all edges of 1-bouquet components. Then, Player II earns at least 1 more point.

Since mn is odd and the first point is of Player II, Player II can earn at least $\frac{mn+1}{2}$ points and Player I can earn at most $\frac{mn-1}{2}$ points. Therefore, Player II wins. $\qquad\square$

Theorem 4.11. *For playing normal $m \times n$ Closing Octagons game where m or n is even, if all components of the graph of the critical state are 2-bouquet components, then a player who uses Strategy A wins or draws.*

Proof. Similar to the proof of Theorem 4.10, a player who uses Strategy A can earn at least $\frac{mn}{2}$ points and the opponent can earn at most $\frac{mn}{2}$ points. Therefore, the player who uses Strategy A wins or draws. $\qquad\square$

4.2   Strategy for Playing Normal $1 \times n$ and $2 \times n$ Games

In this section, we consider only the games of size $1 \times n$ and $2 \times n$ and analyze the strategy for playing these games.

Theorem 4.12. *For any normal 1×2 Closing Octagons game, Player II wins or draws.*

Proof. Let $\mathcal{G}$ be a normal 1×2 Closing Octagons game and S_k be the critical state of $\mathcal{G}$. Then, there are 2 cases of $G(S_k)$ up to isomorphism, shown in Figure 4.13.

Case 1   Case 2

Figure 4.13: Two cases of the graph of the critical state of a normal 1×2 game

Case 1. $G(S_k)$ is a 1-pseudopath chain. By Theorem 3.23, a turn from S_k to S_{k+1} is Player I's. Then, a Player I's turn from S_k to S_{k+1} is a removal of an edge of $G(S_k)$. Since $G(S_k)$ is a chain, all components of $G(S_{k+1})$ are weak components. Thus, a Player II's turn from S_{k+1} to S_{k+2} is removal of all edges of $G(S_{k+1})$. This implies that Player II earns 2 points.

Case 2. All components of $G(S_k)$ are 2-bouquet components. By Theorem 3.23, a turn from S_k to S_{k+1} is Player II's. Then, a Player II's turn from S_k to S_{k+1} is a removal of a loop of a 2-bouquet component. Thus, $G(S_{k+1})$ has exactly 1 1-bouquet component and 1 2-bouquet component. Then, a Player I's turn from S_{k+1} to S_{k+2} is either a removal of a loop of a 2-bouquet component or removal of a loop of a 1-bouquet component and a loop of a 2-bouquet component, respectively. Then, Player I can earn at most 1 point and all components of $G(S_{k+2})$ are weak components. Now, a Player II's turn from S_{k+2} to S_{k+3} is removal of all edges of $G(S_{k+2})$. This implies that Player II can earn at least 1 point.

Therefore, by both cases, Player II wins or draws. $\square$

Theorem 4.13. *For playing normal 1×2 Closing Octagons game, there is a strategy for Player I to draw.*

Proof. Let $\mathcal{G}$ be a normal 1×2 Closing Octagons game and S_k be the critical state of $\mathcal{G}$. We construct a strategy for Player I as the following.

(A) Player I's has to turn from S_0 to S_1 by removing a simple edge of $G(S_0)$.

(B) Player I's has to turn from S_{k+1} to S_{k+2} by removing a loop of a 1-bouquet component and a loop of a 2-bouquet component, respectively.

Then, (A) implies that all components of $G(S_k)$ are 2-bouquet components. By Case 2 of the proof of Theorem 4.12, (B) implies that Player I earns 1 point. Therefore, Player I draws. □

Strategy 4.14. Let S_i be a state of a $1 \times n$ or $2 \times n$ Closing Octagons game, a player turns according to the following plan.

(A) If S_i is the initial state, then the player has to remove a simple egde.

(B) If S_i is a normal state such that $i \neq 0$, then

(B1) if the opponent's turn from S_{i-1} to S_i is a removal of a loop incident to a vertex v and $G(S_i)$ contains a simple egde e incident to v, then the player has to remove e,

(B2) if the opponent's turn from S_{i-1} to S_i is a removal of a loop incident to a vertex v and $G(S_i)$ contains no simple edges incident to v but $G(S_i)$ contains a simple edge e', then the player has to remove e',

(B3) if the opponent's turn from S_{i-1} to S_i is a removal of a simple edge and $G(S_i)$ contains a simple edge e', then the player has to remove e', and

(B4) if $G(S_i)$ contains no simple edges but $G(S_i)$ contains a loop l such that a removal of l is a turn from S_i to a normal state, then the player has to remove l.

(C) If S_i is a strategic state, then the player has to turn by using Strategy A.

Theorem 4.15. *For playing normal $1 \times n$ Closing Octagons game where n is odd, Strategy 4.14 is a winning strategy for Player II.*

Proof. Let $\mathcal{G}$ be a normal $1 \times n$ Closing Octagons game (where n is odd) such that Player II plays according to Strategy 4.14. Then, $G(S_0)$ of $\mathcal{G}$ has 2 vertices incident to 7 loops and 1 simple edge, and $n - 2$ vertices incident to 6 loops and 2 simple edges, shown in Figure 4.14.
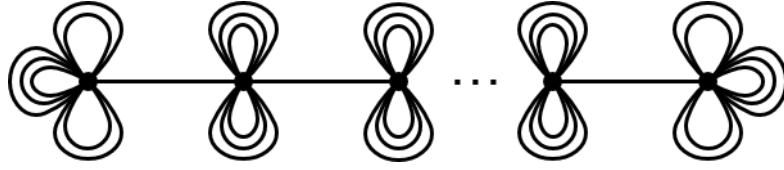
Figure 4.14: The graph of the initial state of a normal $1 \times n$ game $\mathcal{G}$

For each vertex v of $G(S_0)$, the number of loops of $G(S_0)$ incident to v is greater than the number of simple edges of $G(S_0)$ incident to v by at least 4. By (A), (B1), (B2) and (B3) of Strategy 4.14, each removal of a simple edge is a turn from a normal state to another normal state, and all simple edges has to be removed before the critical state is reached.

Then, we obtain that all components of the graph of the critical state of $\mathcal{G}$ are 2-bouquet components. By Theorem 3.23, a turn from the critical state to the first strategic state is Player I's. By (C) of Strategy 4.14, Player II has to turn from each strategic state to another state by using Strategy A. By Theorem 4.10, Player II wins. $\qquad \square$

Theorem 4.16. *For playing normal $1 \times n$ Closing Octagons game where n is even, a player who uses Strategy 4.14 wins or draws.*

Proof. Let $\mathcal{G}$ be a normal $1 \times n$ Closing Octagons game where n is even such that there is a player who plays according to Strategy 4.14. Similar to the proof of Theorem 4.15, all components of the graph of the critical state of $\mathcal{G}$ are 2-bouquet components. We can see that (B4) of Strategy 4.14 satisfies Strategy A. By Theorem 4.11, the player who uses Strategy 4.14 wins or draws. $\qquad \square$

Corollary 4.17. *For playing normal $1 \times n$ Closing Octagons game where n is even, there is no winning strategy for both players.*

Proof. Suppose that there is a winning strategy σ for some players and the player plays by using σ. Then, the player wins, no matter how the opponent plays. If the opponent plays by using Strategy 4.14, then Theorem 4.16 implies that the opponent wins or draws. This contradicts the assumption that σ is a winning strategy for the player. Therefore, there is no winning strategy for both players. $\qquad \square$

Theorem 4.18. *For playing normal $2 \times n$ Closing Octagons game, a player who uses Strategy 4.14 wins or draws.*

Proof. Let $\mathcal{G}$ be a normal $2 \times n$ Closing Octagons game such that there is a player who plays according to Strategy 4.14. Then, $G(S_0)$ of $\mathcal{G}$ has 4 vertices incident to 6 loops and 2 simple edges, and $2n - 4$ vertices incident to 5 loops and 3 simple edges, shown in Figure 4.15.
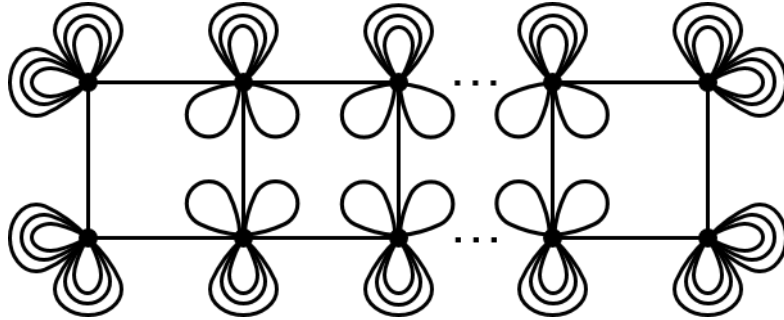

Figure 4.15: The graph of the initial state of a normal $2 \times n$ game $\mathcal{G}$

For each vertex v of $G(S_0)$, the number of loops of $G(S_0)$ incident to v is greater than the number of simple edges of $G(S_0)$ incident to v by at least 2. By (A), (B1), (B2) and (B3) of Strategy 4.14, each removal of a simple edge is a turn from a normal state to another normal state, and all simple edges has to be removed before the critical state is reached.

Then, we obtain that all components of the graph of the critical state of $\mathcal{G}$ are 2-bouquet components. We can see that (B4) of Strategy 4.14 satisfies Strategy A. By Theorem 4.11, the player who uses Strategy 4.14 wins or draws. $\square$

Corollary 4.19. *For playing normal $2 \times n$ Closing Octagons game, there is no winning strategy for both players.*

Proof. Similar to the proof of Corollary 4.17, there is no winning strategy for both players. $\square$

4.3   Strategy for Playing Normal 3×3 Game

In this section, we consider only the game of size 3×3 and analyze the strategy for playing this game.

Strategy 4.20. Let S_i be a state of a 3×3 Closing Octagons game such that $i \neq 0$, a player turns according to the following plan.

(A) If S_i is a normal state, then

(A1) if the opponent's turn from S_{i-1} to S_i is a removal of a loop incident to a vertex v and $G(S_i)$ contains a simple egde e incident to v such that a removal of e is a turn from S_i to another normal state, then the player has to remove e,

(A2) if the opponent's turn from S_{i-1} to S_i is a removal of a loop incident to a vertex v and $G(S_i)$ contains no simple edges e incident to v such that a removal of e is a turn from S_i to another normal state but $G(S_i)$ contains a simple edge e' such that a removal of e' is a turn from S_i to another normal state, then the player has to remove e',

(A3) if the opponent's turn from S_{i-1} to S_i is a removal of a simple edge and $G(S_i)$ contains a simple edge e' such that a removal of e' is a turn from S_i to another normal state, then the player has to remove e',

(A4) if $G(S_i)$ contains no simple edges e such that a removal of e is a turn from S_i to another normal state but $G(S_i)$ contains a loop l such that a removal of l is a turn from S_i to another normal state, then the player has to remove l, and

(A5) if $G(S_i)$ contains no edges e such that a removal of e is a turn from S_i to another normal state but $G(S_i)$ contains a 2-bouquet component K, then the player has to remove a loop of K.

(B) If S_i is a strategic state, then

(B1) if $G(S_i)$ contains a 2-bouquet component K, then the player has to turn by removing all edges of weak components and a loop of K, respectively, and

(B2) if all components of $G(S_i)$ are weak components, then the player has to turn by removing all edges.

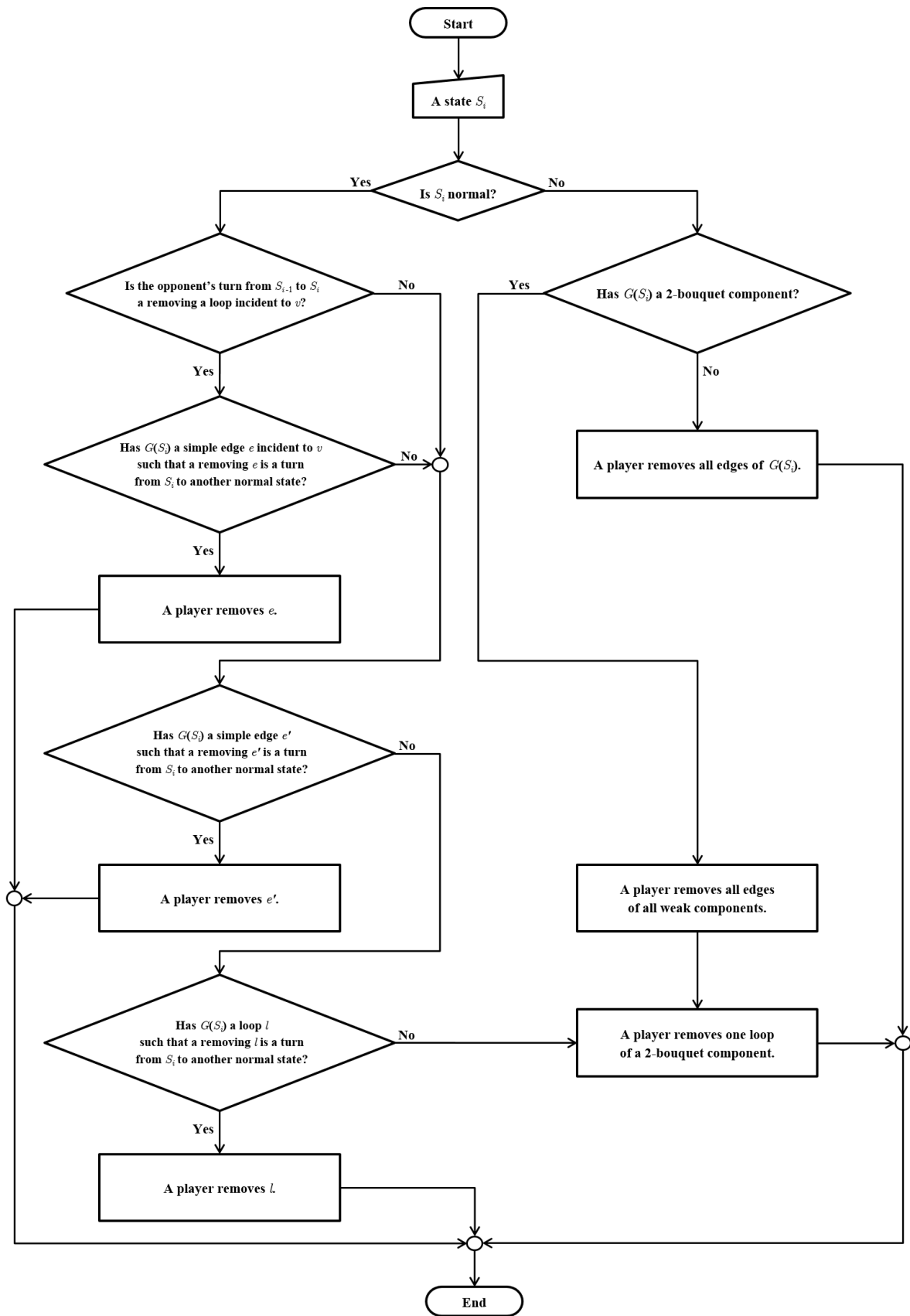For convenience, we illustrate Strategy 4.20 in flowchart, Figure 4.16.

Figure 4.16: Strategy 4.20

Because Strategy 4.20 is a strategy for Player II, plan for a turn of the initial state is not required.

Theorem 4.21. *For playing normal* 3×3 *Closing Octagons game, Strategy 4.20 is a winning strategy for Player II.*

Proof. Let $\mathcal{G}$ be a normal 3×3 Closing Octagons game such that Player II plays according to Strategy 4.20. Then, $G(S_0)$ of $\mathcal{G}$ has 4 vertices incident to 6 loops and 2 simple edges, 4 vertices incident to 5 loops and 3 simple edges, and 1 vertex v' incident to 4 loops and 4 simple edges, shown in Figure 4.17.
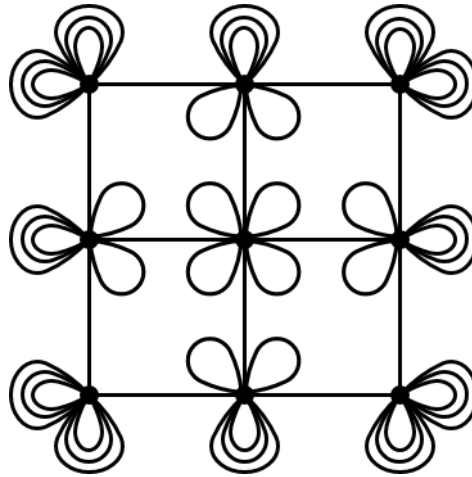

Figure 4.17: The graph of the initial state of a normal 3×3 game $\mathcal{G}$

For a vertex v', the number of loops and the number of simple edges incident to v' of $G(S_0)$ are equal. By (A1) of Strategy 4.20, 3 of simple edges incident to v' have to be removed when the number of loops incident to v' is at least 1. This implies that the last simple edge incident to v' may not be removed in turns from a normal state to another normal state

For each vertex $v \neq v'$, the number of loops incident to v of $G(S_0)$ is greater than the number of simple edges incident to v of $G(S_0)$ by at least 2. By (A1), (A2) and (A3) of Strategy 4.20, each simple edge e that is not incident to v' has to be removed when the number of loops incident to each of vertices $v \neq v'$ is at least 2. This implies that each removal of a simple edge that is not incident to v'

is a turn from a normal state to another normal state, and all simple edges that are not incident to v' has to be removed before the critical state is reached.

Then, we obtain that there are 2 cases of the graph of the critical state of $\mathcal{G}$ up to isomorphism, say G_1 and G_2, shown in Figure 4.18, respectively.
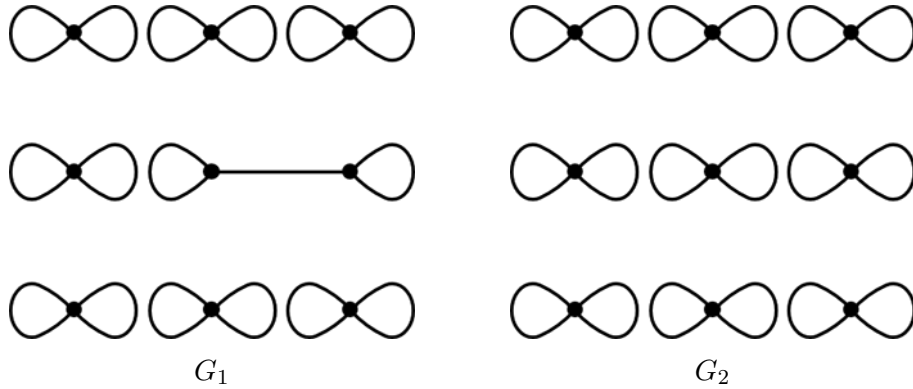

Figure 4.18: Two cases of the graph of the critical state of $\mathcal{G}$

Case 1. The graph of the critical state is G_1. By Theorem 3.23, a turn from the critical state to another state is Player I's. Since all components of G_1 are 2-bouquet components, a Player I's turn from the critical state to a strategic state is a removal of a loop of a 2-bouquet component. Then, the graph of the first strategic state has 1 1-bouquet component and 8 2-bouquet components. By (B1) and (B2) of Strategy 4.20, Player II can earn at least 5 points and Player I can earn at most 4 points.

Case 2. The graph of the critical state is G_2. By Theorem 3.23, a turn from the critical state to another state is Player II's. By (A5) of Strategy 4.20, a Player II's turn from the critical state to a strategic state is a removal of a loop of a 2-bouquet component. Then, the graph of the first strategic state has 1 1-bouquet component, 6 2-bouquet components, and 1 chain K having 2 vertices and 3 edges. By (B1) and (B2) of Strategy 4.20, Player II can earn 2 points from K and at least 3 points from the other. This implies that Player II can earn at least 5 points and Player I can earn at most 4 points.

Therefore, by both cases, Player II wins. $\square$

REFERENCES

[1] Berlekamp, E. R.: *Winning Ways for Your Mathematical Plays*, vol. II, Academic Press, London, 1982.

[2] Buzzard, K., Ciere, M.: Playing simple loony Dots and Boxes endgame optimally, *Integers* **14**, #G08 (2014).

[3] Ferguson, T. S.: Impartial combinatorial games [Online], University of California, Los Angeles, Available from `http://www.math.ucla.edu/~tom/Game_Theory/comb.pdf` [2018, May 31].

[4] Gross, J. L., Yellen, J.: *Graph Theory and Its Applications*, CRC Press, Boca Raton, 1999.

[5] Lenhardt, R.: *Closed $1 \times n$ Dots and Boxes*, Bachelor Thesis, Department of Computer Science, Comenius University, 2008.

[6] Tutte, W. T.: *Graph Theory*, Cambridge University Press, Waterloo, 2001.

[7] West, D. B.: *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, 2001.

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