CHAPTER III
MODELING

The incident of annular rise, which was inquired in this work, can be used to ascertain the liquid surface tension and suggested to be one way further of the accurate methods. The liquid rise in an annular tube is illustrated in Figure 3.1, an annular tube is dipped into a liquid that wets the tube. The capillary force in annular tube and force of surface tension between the liquid and the tube are the reason of liquid rises in an annular tube.

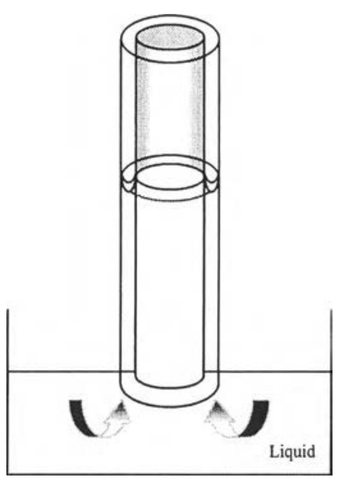

Figure 3.1 Capillary rise in an annular tube with vertical orientation.

A principle of surface tension definition is the line of force ($\gamma = F/l$). The phenomenon of annular rise can be similarly treated as the capillary rise. Understanding the annular rise phenomenon allows us to clarify correlation between any performing conditions or material properties and interfacial tension, which can be determined by

$$\gamma = \frac{\rho g H (r_o - r_i)}{2\cos\theta} \tag{3.1}$$

where γ is the surface tension of liquid, (mN/m).

ρ is the liquid density, (kg/m^3).

θ is the contact angle between annular tube wall, air, and liquid.

H is the height of the rising liquid, (m).

g is the gravitational acceleration, (m/s^2).

r_o is the outer tube radius of annular tube, (m).

r_I is the inner tube radius of annular tube, (m).

A liquid film is formed by surface tension that is balanced by some equal and contrary forces and an excess pressure comes from stresses within the particle. The difference in pressure that occurs across a curved interface has many important consequences.

The meniscus formed at the junction between a liquid surface (chemical used in this work is water) and wall of a glass tube is an indication of the spreading or wetting tendency of liquid on glass. The liquid spreading up the tube wall and is limited in its ascent only by the gravitational force.

The mathematics describing the shape of the meniscus is, in principle, very simple. The hydrostatic pressure at any point on the meniscus surface is balanced by the pressure due to the curvature at that point as expressed by Equation (2.18), it can be related to the theory of static pressure as follows

$$\rho gh = \gamma\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \tag{3.2}$$

when R_1, R_2 and L are manifested in terms of Cartesian coordinates by analytical geometry, It becomes apparent that Equation (3.2) is deceptively simple. The full equation becomes (Hunter, 1986)

$$\frac{(\rho''-\rho')g(z-z_0)}{\gamma} = \frac{\left[1+\left(\frac{\partial z}{\partial x}\right)^2\right]\frac{\partial^2 z}{\partial y^2}+\left[1+\left(\frac{\partial z}{\partial y}\right)^2\right]\frac{\partial^2 z}{\partial x^2}-2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x \partial y}}{\left[1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]^{3/2}} \tag{3.3}$$

where $(\rho''-\rho')$ is the difference in densities between the two phases that form the interface, and gravity is taken as acting in the z direction along with the height of meniscus rising.

Providing the surface has axial symmetry, (about z say) as is often the case for surfaces that will interest us, Equation (3.3) becomes Equation (3.4) (Hunter, 1986) as shown below

$$\frac{(\rho''-\rho')gh}{\gamma} = \frac{\frac{d^2h}{dr^2}}{\left[1+\left(\frac{dh}{dr}\right)^2\right]^{3/2}} + \frac{\frac{dh}{dr}}{r\left[1+\left(\frac{dh}{dr}\right)^2\right]^{1/2}} \tag{3.4}$$

where r, a radius of meniscus region, is measured axially.

Equation (3.4), which has some approximate analytical solutions, was derived by Young in 1805. Generally, though, it can be solved numerically. This way is a relatively easy task for computer era using extensive spreadsheet solving to get solutions, accompanying the computer program to generate them.

In case of the left-hand side term in Equation (3.4) is zero (no pressure across the interface), the solution is still an ordinary differential equation in pure mathematics (Nitsche, 1975) and has only newly been analyzed in full (Almgren and Taylor, 1976). If the left-hand side is non-zero, numerical methods of integration are the only source of recourse.

The complicated differential equation relates to the shape of an axially symmetrical interface and liquid rising height, permits us to understand the shape assumed by mobile interfaces and suggests that the height of rising liquid might be calculated through a study of these shapes.

3.1 The Vertical Capillary Tube

3.1.1 The Meniscus Shape with Hemispherical Shape

A capillary tube is dipped vertically into a liquid that wets the tube. We observe that the liquid rises in the tube, above the level of the free surface. The mathematical model was derived to relate the height of liquid rising with the surface tension.

This model will be solved by using a numerical method called "shooting method", with measuring the surface tension.

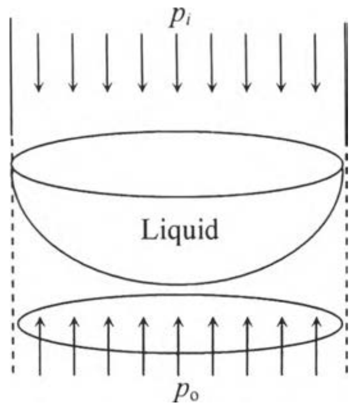

Figure 3.2 Pressure difference across a curved surface.

If we presume the meniscus as a hemisphere, then the two radii of curvature are equal to each other, and are given by $R_1 = R_2 = R$.

As the Equation (2.18), Laplace Equation, we can find the pressure difference across the meniscus as

$$\Delta p = p_i - p_o = \frac{2\gamma}{R} \tag{3.5}$$

(Note the positive sign. The pressure on the liquid side of the meniscus p_o is below atmospheric. The pressure is always higher on the concave side of the surface p_i, which in this case is in the atmosphere.)

From Equation (3.4), although the fact that the local curvature can be approximated, can also be presented by general form with connecting to Equation (3.2), that is

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{\frac{d^2h}{dr^2}}{\left[1 + \left(\frac{dh}{dr}\right)^2\right]^{3/2}} + \frac{\frac{dh}{dr}}{r\left[1 + \left(\frac{dh}{dr}\right)^2\right]^{1/2}} \tag{3.6}$$

The following expressions from analytical geometry is a general function for R_1^{-1} and R_2^{-1} for surfaces with an axis of symmetry. (Fennell and Wennerstrom, 1994) There are,

$$R_1^{-1} = \frac{d^2h/dr^2}{\left[1+(dh/dr)^2\right]^{3/2}} \tag{3.7}$$

and

$$R_2^{-1} = \frac{dh/dr}{r\left[1+(dh/dr)^2\right]^{1/2}} \tag{3.8}$$

To predict a hemispherical surface, the Equation (3.7) which is hemispherical term (Wilkes, 1999) is taken to characterize that shape and is rearranged to be

$$\frac{d^2h}{dr^2} = \frac{1}{2R_1}\left[1+\left(\frac{dh}{dr}\right)^2\right]^{3/2} \tag{3.9}$$

Equation (3.7) is a second-order nonlinear differential equation. Therefore, the assumption of differential terms is needed to convert the second-order differential equation to the first-order differential equation by assuming,

$$\phi = \frac{dh}{dr} \tag{3.10}$$

combining Equations (3.9) and (3.10) gives

$$\frac{d\phi}{dr} = \frac{1}{2R_1}\left(1+\phi^2\right)^{3/2} \tag{3.11}$$

Meniscus attribute is predicted starting from the meniscus tip location up to where the meniscus surface touched with the capillary tube wall. In this regard, we can know the parameters at the beginning of estimation, the meniscus tip, there are

$$\phi = \frac{dh}{dr} = h = 0 \qquad \text{at } r = 0 \tag{3.12}$$

On the other hand, the point of meniscus surface contacted with the capillary tube wall also present the parameter at the end of estimation, there are

$$\frac{dh}{dr} = \cot\theta \qquad \text{at } r = \text{capillary diameter}/2 \tag{3.13}$$

Applying a numerical method which is shooting method with Euler's method solving the first derivative over the step size dr. Further, because of the repetitive nature of the calculations, Euler's method is used to implement by spreadsheet as shown in Appendix A.

This mathematical model may remain in error if we assume that the meniscus is a hemisphere, but this is a good assumption for liquids that wet the inside of the tube wall with a contact angle near zero, in very small capillary tubes.

3.1.2 The Meniscus Shape with Circular Shape of Capillary Tube

In case we reconsider the meniscus as a circular, which is used to characterize any general liquid surfaces in capillary tube. Therefore, the two radii of curvature of this circular surface are dissimilar. ($R_1 \neq R_2$)

We ought to predict the meniscus shape by using both R_1 and R_2 simultaneously. From the Equation (3.6), we diminish the left-hand side of that equation with supposing

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1 + R_2}{R_1 R_2} = \frac{1}{RR} \tag{3.14}$$

where $RR = \dfrac{R_1 R_2}{R_1 + R_2}$

Take the place of Equation (3.15) into Equation (3.6). Hence, we gain

$$\frac{1}{RR} = \frac{\dfrac{d^2h}{dr^2}}{\left[1+\left(\dfrac{dh}{dr}\right)^2\right]^{3/2}} + \frac{\dfrac{dh}{dr}}{r\left[1+\left(\dfrac{dh}{dr}\right)^2\right]^{1/2}} \tag{3.15}$$

Since the above equation is the second-order linear differential equation, which is still complex to solve the problem. In consequence, the assumption of differential terms are needed to simplify that obstacle as it was done as Equation (3.10).

Assuming,

$$\phi = \frac{dh}{dr}$$

Dealing Equations (3.10) with (3.15) gives

$$\frac{1}{RR} = \frac{d\phi / dr}{\left[1+\phi^2\right]^{3/2}} + \frac{\phi}{r\left[1+\phi^2\right]^{1/2}} \tag{3.16}$$

So,

$$\frac{d\phi}{dr} = \left[\left(\frac{1}{RR}\right) - \frac{\phi}{r\left(1+\phi^2\right)^{1/2}}\right]\left(1+\phi^2\right)^{3/2} \tag{3.17}$$

At this place, we can manipulate Equation (3.17) to predict the meniscus shape that rise in vertical capillary tube. (The inspection of other positions of capillary tube and another geometry is considered in next sections.) The numerical methods, shooting method with Euler's method, are used to provide a direct estimation of meniscus shape as well as the liquid rising height as following relationship, from Equations (3.4), (3.6) and (3.14).

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{\frac{d^2h}{dr^2}}{\left[1+\left(\frac{dh}{dr}\right)^2\right]^{3/2}} + \frac{\frac{dh}{dr}}{r\left[1+\left(\frac{dh}{dr}\right)^2\right]^{1/2}} = \frac{1}{RR} = \frac{\rho g h}{\gamma} \tag{3.18}$$

Likewise, we can know the parameters at the beginning of estimation, the meniscus tip, there are

$$\phi = \frac{dh}{dr} = h = 0 \qquad \text{at } r = 0 \tag{3.19}$$

While, the point of meniscus surface contacted with the capillary tube wall also present the parameter at the end of estimation, there are

$$\frac{dh}{dr} = \cot\theta \qquad \text{at } r = \text{capillary diameter}/2 \tag{3.20}$$

Considering especially for last two terms of the above equation, we are able to find for the liquid rising height, that is

$$h = \frac{\gamma}{\rho g(RR)} \tag{3.21}$$

3.2 The Inclined Annular Tube

This section was intended to display the techniques that are available to investigate the annular tube in any positions, the complicated trace. In the same way, the Laplace Equation is valuable to clear up the obstacle.

The Equation (3.18), circular surface analysis, was taken to characterize the meniscus in any inclined annular tubes. The numerical methods, shooting method with Euler's method, are used to provide a direct estimation of meniscus shape and the liquid rising height.

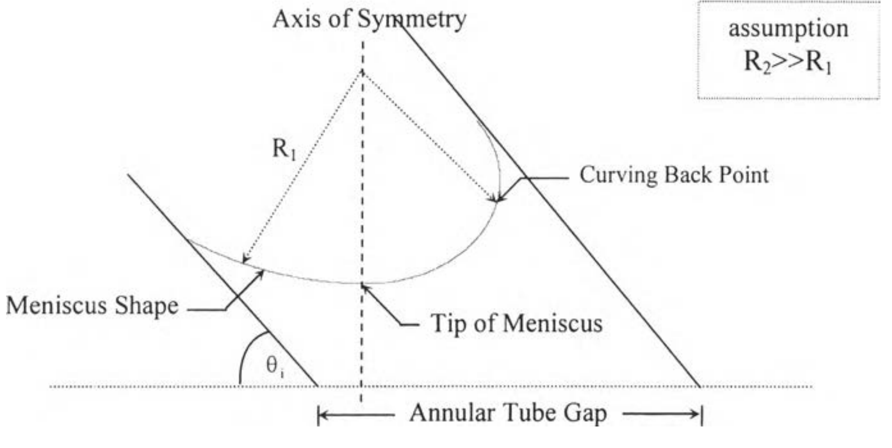

Figure 3.3 The meniscus in inclined annular tube illustration.

Meniscus shape can be predicted from inner tube wall to outer tube wall of an annular tube (annular tube gap). However, the parameters at the beginning and the end of estimation for any assigned position of annular tube (experimental data) can be proposed to identify the meniscus surface, there are

$$\frac{dh}{dr} = \cot 70° \text{ and } \frac{dh}{dr} = \cot 30° \qquad \text{at } 30° \text{ inclination} \tag{3.22}$$

$$\frac{dh}{dr} = \cot 55° \text{ and } \frac{dh}{dr} = \cot 25° \qquad \text{at } 45° \text{ inclination} \tag{3.23}$$

$$\frac{dh}{dr} = \cot 40° \text{ and } \frac{dh}{dr} = \cot 20° \qquad \text{at } 60° \text{ inclination} \tag{3.24}$$

The procedure of trial and errors of R_1 is also considerable to identify the meniscus shape that rises in inclined annular tube in view of Figure 3.4. In addition, this estimation can be applied under other characteristics, which are being in the same manner as inclined annular tube, for example, the investigation of water retained in annular cone.

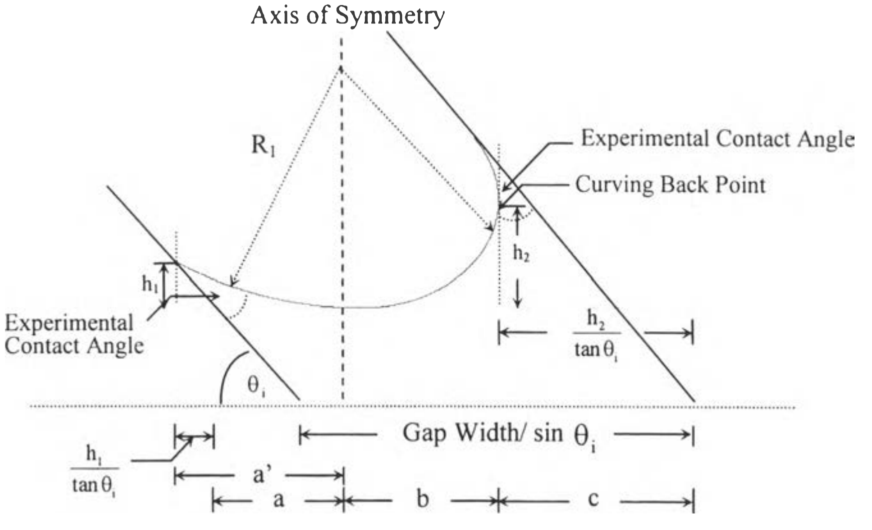

Figure 3.4 The schematic diagram of meniscus curve deliberation.