CHAPTER II

GENERAL CHARACTERISTICS OF CONGRUENCE-FREE COMMUTATIVE SEMIRINGS WITH 1.

SECTION 1 BASIC THEOREMS

The purpose of this chapter is to investigate properties which are common to all types of congruence-free commutative semirings with 1. As will become apparent later, Theorem 2.1.1 is a key element in many of the proofs in this thesis.

<u>2.1.1 Theorem</u>. Let S be a congruence-free commutative semiring with 1. Then S is MC.

Proof : Choose an $x \in S$ which is not a multiplicative zero. Define a relation $\sim$ on S as follows:

For $y, z \in S$, $y \sim z$ iff $xy = xz$. $\sim$ is trivially an equivalence relation. Suppose $y \sim z$. Then $xy = xz$, so for all $a \in S$, $xy + xa = xz + xa$. Thus $x(y + a) = x(z + a)$. Therefore $(y + a) \sim (z + a)$. Also, $a(xy) = a(xz)$, so $x(ay) = x(az)$. Thus $ay \sim az$. Therefore $\sim$ is an congruence relation. Thus $\sim = S \times S$ or $\sim = \Delta$. Suppose $\sim = S \times S$. Then $xa = xb$ for all $a, b \in S$. Therefore $x.1 = xb$, for all $b \in S$. Thus $x = xb$ for all $b \in S$, and so x is a multiplicative zero which contradicts our initial assumption. Therefore $\sim = \Delta$ i.e. S is MC. #

2.1.2 Corollary. Every congruence-free commutative semiring with 1 which has a multiplicative zero can be embedded in a quotient semifield. Every congruence-free commutative semiring with 1 which has no multiplicative zero can be embedded in a quotient division semiring.

Proof : Follows from Theorem 2.1.1. and Chapter I. #

2.1.3 Theorem. Let S be a congruence-free commutative semiring w.1 and let QS be its quotient semifield or quotient division semiring. Then QS is congruence-free.

Proof : Suppose that $\sim'$ is a nontrivial congruence relation on QS. Since we can consider S to be a subsemiring of QS we can define $\sim$ on S by $x \sim y$ iff $x \sim' y$ in QS. $\sim$ is clearly a congruence relation on S.

Therefore $\sim = \Delta$ or $S \times S$. Since $\sim'$ is not trivial, there exist $\frac{x}{y}, \frac{x'}{y'} \in QS$ such that $\frac{x}{y} \sim' \frac{x'}{y'}$ and $\frac{x}{y} \neq \frac{x'}{y'}$. Therefore $xy' \sim' yx'$, and thus $xy' \sim yx'$. But since $\frac{x}{y} \neq \frac{x'}{y'}$, $xy' \neq yx'$. Therefore $\sim = S \times S$ since S is congruence-free. But there exist $\frac{a}{b}, \frac{a'}{b'} \in QS$ such that $\frac{a}{b} \not\sim' \frac{a'}{b'}$. Therefore $ab' \not\sim' a'b$, and so $ab' \not\sim a'b$. Thus $\sim \neq S \times S$ which is a contradiction. Thus QS is congruence-free.

The converse of the theorem above is in general false. Z^+ is not congruence-free but Q^+ is congruence-free. However, with the help of the following concept we can obtain a partial converse to Theorem 2.1.3.

2.1.1 Definition. Let S be a commutative semiring and $\sim$ a congruence on S. $\sim$ is said to be MC iff for all nonzero x and for all $a,b \in S$, $xa \sim xb$ implies $a \sim b$. S is said to be MC congruence-free iff the only possible MC congruences on S are $S \times S$ and Δ.

Thus in a division semiring or a semifield S, S is congruence-free iff S is MC congruence-free. In a commutative semiring with 1, congruence-free implies MC congruence-free. If S is a commutative semiring and $\sim$ an MC congruence on S claim that $\frac{S}{\sim}$ is MC. Suppose that $x,y \in S$ and $[x][y] = [x][z]$ in $\frac{S}{\sim}$. Then $[xy] = [xz]$. Thus $xy \sim xz$ so if x is nonzero then $y \sim z$ since $\sim$ is MC. Thus $\frac{S}{\sim}$ is MC. The following theorem is a partial converse to Theorem 2.1.3.

<u>2.1.4 Theorem</u>. Let S be an MC commutative semiring wiht 1 and QS its quotient semifield or quotient division semiring. Then S is MC congruence-free if QS is congruence-free.

Proof: If $||S|| = 1$ then $S = \{1\} = QS$ so the result is trivial. So suppose that $||S|| > 1$. Let $\sim$ be a nontrivial MC congruence on S. Define a relation $\sim'$ on QS by saying that for $\frac{x}{y}$ and $\frac{x'}{y'} \in QS$, $\frac{x}{y} \sim' \frac{x'}{y'}$ iff $xy' \sim yx'$. Claim that $\sim'$ is well defined. Suppose that $\frac{a}{b} = \frac{x}{y}$ and $\frac{a'}{b'} = \frac{x'}{y'}$ and that $\frac{x}{y} \sim' \frac{x'}{y'}$. Thus $xy' \sim yx'$, $ay = bx$ and $a'y' = b'x'$. Since $xy' \sim yx'$, $axy' \sim ayx'$. Thus $axy' \sim (bx)x'$. Thus $a'(axy') \sim a'(bxx')$ so $(ax)(a'y') \sim (bx)(a'x')$ Thus $(ab')(xx') \sim (ba')(xx')$.

Case 1: Both x and x' are not multiplicative zeros. Then since $\sim$ is MC $ab' \sim ba'$ so $a/b \sim' a'/b'$.

Case 2: Both x and x' are multiplicative zeros. Then both a and a' are multiplicative zeros so $ab' = ba'$ i.e. $ab' \sim ba'$.

Case 3: x is a multiplicative zero but x' is not. But then $x'y \sim y'x = x$. Thus $1(x'y) \sim x(x'y)$. Thus since $x'y$ is nonzero and $\sim$ is MC, $1 \sim x$. Thus $\sim = S \times S$ which is a contradiction.

Case 4: x' is a multiplicative zero but x is not. Just reverse the roles of x and x' in Case 3.

In sum, $\sim'$ is well defined. Clearly $\frac{x}{y} \sim' \frac{x}{y}$, for all $\frac{x}{y} \in QS$. Also $\frac{x}{y} \sim' \frac{a}{b}$ implies $\frac{a}{b} \sim' \frac{x}{y}$.

Suppose that $\frac{x_1}{y_1} \sim' \frac{x_2}{y_2}$ and $\frac{x_2}{y_2} \sim' \frac{x_3}{y_3}$. Then $x_1y_2 \sim y_1x_2$ and $x_2y_3 \sim y_2x_3$. Therefore $x_1y_2y_3 \sim y_1x_2y_3$ and $x_2y_3y_1 \sim y_2x_3y_1$. Thus $x_1y_2y_3 \sim y_2x_3y_1$. Since $\sim$ is MC and y_2 is nonzero we get $x_1y_3 \sim x_3y_1$. Thus $\frac{x_1}{y_1} \sim' \frac{x_3}{y_3}$. Therefore $\sim'$ is transitive and thus $\sim'$ is an equivalence relation on QS. Claim that $\sim'$ is a congruence on QS. Suppose that $\frac{x_1}{y_1} \sim' \frac{x_2}{y_2}$. Choose $\frac{a_1}{b_1} \in QS$. We want to show that $\frac{a_1}{b_1} + \frac{x_1}{y_1} \sim' \frac{a_1}{b_1} + \frac{x_2}{y_2}$ This is true iff $(a_1y_1 + b_1x_1)y_2b_1 \sim (x_2b_1 + a_1y_2)b_1y_1$. Since $x_1y_2 \sim y_1x_2$ we have $x_1y_2b_1 \sim y_1x_2b_1$. By multiplying again by b_1 and by commutivity we get that $b_1x_1y_2b_1 \sim x_2b_1b_1y_1$ (1) Now $a_1y_1y_2b_2 = a_1y_2b_2y_1$ so $a_1y_1y_2b_1 = a_1y_2b_1y_1$ (2). Thus by adding both sides of equation (2) to equation (1) we get that $b_1x_1y_2b_1 + a_1y_1y_2b_1 \sim x_2b_1b_1y_1 + a_1y_2b_1y_1$. Thus by the distributive law we get that $(b_1x_1 + a_1y_1)y_2b_1 \sim (x_2b_1 + a_1y_2)b_1y_1$. or in other words $\frac{a_1}{b_1} + \frac{x_1}{y_1} \sim' \frac{x_2}{y_2} + \frac{a_1}{b_1}$. Since the selection of these elements was arbitrary we have proven that the equivalence relation preserves addition.

Thus to show that $\sim'$ is a congruence on QS we must now show that $\sim'$ preserves multiplication.

To show that $\sim'$ preserves multiplication suppose $\frac{x}{y} \sim' \frac{x_1}{y_1}$ and choose $\frac{a}{b} \in QS$. Thus $xy_1 \sim yx_1$, so $abxy_1 \sim abyx_1$. Thus $\frac{a}{b}\frac{x}{y} \sim' \frac{a}{b}\frac{x_1}{y_1}$.

Thus $\sim'$ is a congruence relation on QS, so $\sim' = \Delta$ or $QS \times QS$.

Suppose that $\sim' = \Delta$.

Thus $\sim = \Delta$ since if $x \sim y$ and $x \neq y$ then $\frac{x}{1} \neq \frac{y}{1}$ but $\frac{x}{1} \sim' \frac{y}{1}$ which is a contradiction. Suppose that $\sim' = QS \times QS$. Then $\frac{x}{1} \sim' \frac{y}{1}$ forall $x,y \in S$. Thus $1.x \sim 1.y$ for all $x, y \in S$. Thus $x \sim y$ for all $x, y \in S$. Therefore $\sim = S \times S$.

Thus as $\sim$ was an arbitrary MC congruence relation on S we have the desired result. #

Thus in particular $\mathbb{Z}^+$ is MC congruence-free since $\mathbb{Q}^+$ is congruence-free. The converse of Theorem 2.1.3 is in general false. $S = \mathbb{Q}^+ \cup \{\infty\}$ with the trivial structure is MC congruence-free because S is an ∞-semifield but S = QS is not congruence free. As an example of a nontrivial MC congruence consider $S = Z^+ \times Z^+$ with the usual addition and multiplication. Then define a relation $\sim$ on S by saying that $(x,y) \sim (a,b)$ iff $x = a$. $\sim$ is obviously a congruence on S, and supposing that $(a,b)(x,y) \sim (a,b)(x_1,y_1)$ then $(ax,by) \sim (ax_1,by_1)$. Thus $ax = ax_1$, so $x = x_1$. Therefore $(x,y) \sim (x_1,y_1)$. Thus $\sim$ is MC.

In an arbitrary commutative semiring with 1, a multiplicative zero may neither be an additive zero nor an additive identity. For example let $S = Z_0^+ \times Z_\infty^+$ with the usual addition and multiplication. Then $(0, \infty)$ is a multiplicative zero but not an additive zero or an additive identity. In fact many interesting pathologies may occur. For example let $S = \{\text{open intervals } (a, \infty) \mid a \in \mathbb{R}^+\}$ Let multiplication be set intersection and addition be set union. Then S is a commutative semiring. The open interval $(0, \infty)$ is both the multiplicative identity and the additive zero. But in a congruence-free semiring with 1 we get the following :

2.1.5 Theorem. Let S be a congruence-free commutative semiring with 1 which has a multiplicative zero a. Then a is either an additive identity or an additive zero.

Proof : Define a relation $\sim$ on S as follows. For $y, x \in S$, say that $x \sim y$ iff $x + a = y + a$. Clearly $x \sim x$ and $x \sim y \Rightarrow y \sim x$. Suppose $x \sim y$ and $y \sim z$. Then $x + a = y + a$ and $y + a = z + a$. Thus $x + a = z + a$ so $x \sim z$. Thus $\sim$ is an equivalence relation. Suppose $x, y, b \in S$ and $x \sim y$. Then $x + a = y + a$, so $(x + b) + a = (y + b) + a$. Therefore $x + b \sim y + b$. Also $b(x + a) = b(y + a)$. Thus $bx + ba = by + ba$. But $ba = a$, so $bx + a = by + a$. Thus $bx \sim by$, so $\sim$ is a congruence relation on S. Thus $\sim = S \times S$ or Δ. Suppose $\sim = S \times S$. Then for all $y \in S$, $a + a = y + a$. $(1 + 1)a = y + a$. Therefore $a = y + a$. Thus a is an additive zero. Suppose that $\sim = \Delta$ and $x \in S$. $a + a = (1 + 1)a = a$ and $x + a + a = x + a + a$. Thus we get $x + (1 + 1)a = x + a + a$, so $x + a = x + a + a$. Thus $x + a = x$. Thus a is an additive identity. #

Notation : Let S be a semiring. An element in S which is a multiplicative zero and an additive identity is called a zero element of S and we shall always denote it by 0. An element in S which is both a multiplicative zero and an additive zero is called an infinity element of S and we shall always denote it by ∞.

This property that the multiplicative zero acts as an additive identity or an additive zero is also true for semifields which is another indication of the similarity of congruence-free semirings with multiplicative zero and semifields. The next theorem concerning additive cancellativity is another exact analogue of the semifield case.

2.1.6 Theorem: Let S be a congruence-free commutative semiring with 1, such that $||S|| > 2$. Then if S has one AC element, S is AC.

Proof : Let $S^{1} = \{ x \in S \mid x \text{ is not AC} \}$. Define $\sim\; = \Delta \cup (S' \times S')$. $\sim$ is clearly an equivalence relation on S. Suppose $x \in S'$. Then there exist $a \neq b \in S$, such that $x + a = x + b$. Therefore for all $s \in S$, $s + x + a = s + x + b$. Thus $s + x \in S'$ Thus for all $x, y, s \in S$, $x \sim y$ implies $x + s \sim y + s$. Now suppose that $x \sim y$ and s is a multiplicative zero. Thus $sx = sy = s$ so $sx \sim sy$. Suppose s is not a mult. zero. If $x \sim y$ and $x = y$ then $sx = sy$ so $sx \sim sy$. Suppose $x \neq y$ Then $x, y \in S'$. Thus there exist $a \neq b \in S$ such that $x + a = x + b$. Therefore $s(x + a) = s(x + b)$, so $sx + sa = sx + sb$. But $sa \neq sb$ since S is AC. Thus $sx \in S'$. Similarly $sy \in S'$, so $sx \sim sy$. Thus $\sim$ is a congruence on S. Thus $S' = \phi$, $S' = \{ x_1 \}$ for some $x_1 \in S$, or $S' = S$. $S' = S$ is impossible since by assumption S has one AC element. If $S' = \phi$ we are done, so suppose $S' = \{x_1\}$ for some $x_1 \in S$. We distinguish two cases :

Case 1 : S has no multiplicative zero

Case 2 : S has a multiplicative zero

In Case 1 suppose $S' = \{x_1\}$. Then there exist $a \neq b \in S$ such that $x_1 + a = b + x_1$. Without loss of generality assume that $a \neq x_1$. Then $ax_1 \neq x_1^2$. But $ax_1 + a^2 = ab + ax_1$. Also $a^2 \neq ab$ since S is MC. Thus $S' \supseteq \{x_1, ax_1\}$, so $ax_1 = x_1$. Therefore $a = 1$, so $x_1 + 1 = b + x_1$ where $b \neq 1$. But $b(x_1 + b) = b(x_1 + 1)$. Thus $bx_1 + b = b^2 + bx_1$. but $bx_1 \in S'$, since $b \neq a = 1$ and thus $b^2 \neq b$. Also $bx_1 \neq x_1$ since $b \neq 1$. Thus $||S^1|| > 1$ which is a contradiction. Thus in Case 1, $S' = \phi$ and we are done.

In Case 2 let ∞ be the multiplicative zero in S. Suppose $S' = \{x_1\}$ and suppose $x_1 = \alpha$. Then $\alpha \neq 0$ since 0 is AC so $\alpha = \infty$. Now consider the set $S \setminus S' = S \setminus \{\infty\}$. Define a relation p on S by xpy iff $x = y$ or $x,y \in S \setminus S'$. Clearly p is an equivalence relation on S. Suppose xpy and $x \neq y$. Then $x,y \in S \setminus S'$. Choose $z \in S$, $z \neq \infty$. Then $zx \neq \infty$ since S is MC. Thus $zx, zy \neq \infty$ and $zx, zy \in S \setminus S'$, so $zxpzy$. Clearly $\infty xp\infty y$. Now $x + \infty = \infty$ and $y + \infty = \infty$ so $x + \infty py + \infty$. Suppose $x + z = \infty$ where z is chosen as above. Then $x + z = \infty + z$. But $z \in S \setminus S'$. Thus $x = \infty$ which is a contradiction, so $x + z \neq \infty$ Similarly $y + z \neq \infty$. Thus $x + z$, $y + z \in S \setminus S'$, and so $x + z \, py + z$. Thus p is a congruence on S, so $p = \Delta$ or $S \times S$. But $||S \setminus S'|| \geq 2$ since $||S'|| = 1$ and $||S|| > 2$. Thus $p \neq \Delta$. Clearly $p \neq S \times S$ a contradiction. Thus $x_1 \neq \infty = \alpha$.

Now suppose $x_1 \neq \infty$. Then there exist $a \neq b \in S$ such that $x_1 + a = x_1 + b$. Thus $x_1(x_1 + a) = x_1(x_1 + b)$ and so $x_1^2 + x_1a = x_1^2 + x_1b$. Now $a \neq b$ and S is MC, so $x_1a \neq x_1b$. Thus $x_1^2 \in S'$ so $x_1^2 = x_1$ Therefore $x_1 = 1$, so we get that $1 + a = 1 + b$. Without any loss of generality, we can assume that $a \neq \infty$. Then $a(1 + a) = a(1 + b)$, so $a + a^2 = a + ab$.

Since $a \neq \infty$ $a^2 \neq ab$, so $a \in S'$ and thus $a = 1$. Thus $1 + 1 = 1 + b$. But b is AC since $b \neq 1$ and $\{1\} = S'$. But $b(1 + 1) = b(1 + b)$. Therefore $b + b^2 = b + b$, so $b^2 = b$. Thus since $b \neq 1$, $b = \infty$. Choose $y \neq \infty = b \in S \setminus S'$. Then $y(1 + 1) = y(1 + b)$, so $y + y = y + yb$. Since y is AC $y = yb = \infty$, a contradiction. Thus $x_1 \neq \infty$ is impossible and we have already proved that $x_1 = \infty$ is impossible. Thus for Case 2 as in Case 1 $S' = \phi$ so S is AC. #

The following example shows that the assumption that $||S|| > 2$ is necessary.

<u>2.1.1 Example</u> : Let $S = \{0,1\}$. Define + by $1 + 1 = 1$, $0 + 1 = 1 + 0 = 1$ and $0 + 0 = 0$. Define multiplication by $1.0 = 0.1 = 0.0 = 0$ and $1.1 = 1$. It is easy to verify that S is a congruence-free commutative semiring with 1. 0 is AC but 1 is not AC in S, since $1 + 0 = 1 + 1$ but $1 \neq 0$. This algebraic structure is called the Boolean 0-semifield.

In Theorem 2.1.5 we have shown that if S is a congruence-free commutative semiring with 1 then either S has no multiplicative zero or if S has a multiplicative zero a then $a = \infty$ or $a = 0$. It turns out that congruence-free commutative semirings with 1 have radically different structures depending on whether or not they have a multiplicative zero and the type of multiplicative zero. We will study each case separately in chapters III,IV and V . To simplify things we introduce the following notation. If S is a congruence-free commutative semiring with 1 then we say that S is :

"Type I" if S has a multiplicative zero which is an additive identity

"Type II" if S has a multiplicative zero which is an additive zero

"Type III" If S has no multiplicative zero.

From now on we shall call a congruence-free commutative semiring with 1 which is Type I a "Type I semiring" etc.

Section 2.2 Double Ideals

2.2.1 Definition : Let S be a semiring. Then $\phi \neq D \subseteq S$ is a double ideal iff $x + d$, $d + x$, dx and $xd \in D$, for all $d \in D$, $x \in S$,

Thus S is always a double ideal in S. $\{\infty\}$ is a double ideal in a semiring with ∞ . $\{x \geq 3 | x \in Z^+\}$ is a double ideal in Z^+.

Definition. Let S be a semiring with an infinity element, Then S is said to be double ideal free iff the the only double ideals of S are S and $\{\infty\}$.

2.2.3 Definition Let S be a semiring without an infinity element. Then S is said to be double ideal free iff S is the only double ideal of S. Thus every ring, division semiring and semifield is double ideal free. Z^+ is not double ideal free.

2.2.1 Proposition. Let S be a congruence-free commutative semiring with 1. Then S is double ideal free

Proof : First suppose S is type III and let $M \subseteq S$ be a proper double ideal. Then $\sim \; = (M \times M) \cup \Delta$ is a congruence on S. Thus $M = \phi$, $M = S$ or $M = \{x_1\}$ for some $x_1 \in S$. $M = \phi$ is excluded by definition 2.2.1. $M = S$ is excluded by assumption. Thus $M = \{x_1\}$ for some $x_1 \in S$. But $x_1 . a \in M$ for all $a \in S$. Therefore x_1 is a multiplicative zero which contradicts the fact that S is type III. Therefore S is double ideal free.

Now suppose S is type I or type II. Then define M and $\sim$ as before. As above $M = \{x_1\}$ for some $x_1 \in S$.

We showed that x_1 is a multiplicative zero. $x_1 = 0$ is impossible since then $M = S$. Thus by Definition 2.2.2, S is double ideal free. #

The converse to the theorem above is in general false. $\mathbb{Q}^+ \times \mathbb{Q}^+$ is double ideal free but if we define $(x,y) \sim (a,b)$ iff $x = a$ then $\sim$ is a nontrivial congruence on $\mathbb{Q}^+ \times \mathbb{Q}^+$.

2.2.2 Proposition : If S is a congruence-free commutative semiring with 1 then if S has no infinity element then $S + S = S$. If S has an infinity element then either $S + S = S$ or $S + S = \{\infty\}$.

Proof : $S + S$ is a double ideal. Therefore if S is type III or type I, $S + S = S$. Suppose that S is type II. Then since S is double ideal free $S + S = S$ or $S + S = \{\infty\}$. #

2.2.3 Proposition : Let S be a commutative semiring with 1 or with an additive identity a. Then S has a maximum proper double ideal or

S has no proper double ideal.

Proof : Suppose S has a proper double ideal. Let $M = \cup\{S' \mid S' \subset S$ such that S' is a proper double ideal$\}$. M is obviously a double ideal. Now suppose S has 1. Then $1 \notin S'$ for all proper double ideals S'. Thus $1 \notin M$, so $M \subset S$. Similarly if $a \in S$ then $a \notin S'$, for all proper double ideals S'. Thus $a \notin M$. Thus $M \subset S$. Clearly $M \supseteq S'$ for any proper double ideal S'. #

<u>2.2.4 Corollary</u>: If S is a commutative semiring with ∞ $(||S||>1)$ which has an additive identity or a multiplicative identity then S has a maximum proper double ideal.

Proof : $\{\infty\}$ is a proper double ideal. Now apply Proposition 2.2.3. #

<u>2.2.4 Definition</u> : Let S be a semiring and J a double ideal in M. Then J is said to be <u>prime</u> iff $xy \in J \Rightarrow x \in J$ or $y \in J$ for all $x,y \in S$

<u>2.2.5 Theorem</u> : Let S be a commutative semiring with 1 which has a proper double ideal. Then the maximum proper double ideal M is prime.

Proof : Suppose for $x,y \in S$, $xy \in M$. Suppose also that $x,y \notin M$. Then $y^{-1} \notin S$ since if it were then $x = (xy)y^{-1} \in M$. Then $ky \neq 1$, for all $k \in S$. Suppose that $ky + d = 1$ for some $k,d \in S$, Then $x = x(yk + d) = xyk + xd \in M$ since $xy \in M$ and M is a double ideal. Thus $x \in M$ which is a contradiction. Thus for all $d,k \in S$, $ky + d \neq 1$. Thus the set $S^1 = M \cup \{ky \mid k \in S\} \cup \{ky + d \mid k,d \in S\}$ is a double ideal which does not contain 1. i.e. S^1 is a proper double ideal of S. But S^1 properly contains M since $y \in S^1$ and $y \notin M$. This contradicts the maximality of M. Thus x or $y \in M$ i.e. M is prime. #

The converse of this theorem is false. In $\mathbb{Z}^+$ with xy defined as $\max(x,y)$ and $x+y$ defined as $\max(x,y)$, $\{x \geq 3 \mid x \in Z^+\}$ is prime but not maximal.

Now let S be semiring and J a double ideal in S. Then we can define a natural congruence $\sim$ on S by saying for $x,y \in S$, $x \sim y$ iff $x = y$ or $x,y \in J$. To verify that $\sim$ is indeed a congruence note that $\sim \; = \; \Delta \cup (J \times J)$. Thus $\sim$ is an equivalence relation. But suppose $x \in J$. Then for all $b \in S$; xb, bx, $x + b$ and $b + x \in J$. Thus $\sim$ preserves addition and multiplication and is thus a congruence. Give S/J the natural algebraic structure (i.e. $[x + y] = [x+y]$ and $[x][y] = [xy]$ for all $x,y \in S$) It is asy to verify that these operations are well defined. S/J is a semiring with an additive zero $\alpha = [x]$ for any $x \in J$ which is also a multiplicative zero, since for all $x \in J$, $y \in S$, $[x] + [y] = [x + y] = [x]$ since $x + y \in J$. Similarly $[y] + [x]$, $[x][y]$ and $[y][x]$ are equal to $[x]$. Moreover if S has 1 then S/J has a multiplicative identity $[1]$.

2.2.6 Proposition : Let S be an MC commutative semiring with 1 and with a maximum proper ideal M. Then S/M is MC.

Proof : Since S is MC it is sufficient to show that for all $x,y \in S$, $xy \in M$ implies $x \in M$ or $y \in M$ i.e. that M is prime. This follows from Theorem 2.2.5. #

It is possible for S to be AC but for no element in S/M to be AC. For example let $S = Z^+ \times \mathbb{Q}^+$ with the usual addition and multiplication. Then $M = \{(x,y) \in S \mid x > 1\}$ Thus $\alpha + \beta = \infty$ for all $\alpha, \beta \in S/M$. Thus no element in S/M is AC.

Let S be a commutative semiring with 1 and J a double ideal It is possible that J is not prime. For example let $S = Z^+$ with the usual addition and multiplication. Then $J = \{x \geq 4 \mid x \in Z^+\}$ is a double ideal but $2.2 \in J$. Thus J is not prime. Moreover in S/J $[2] \cdot [2] = [2] \cdot [4]$ so S/J is not $\mathcal{M}C$ even though S is $\mathcal{M}C$. Thus the maximality of M is critical in the arguments above.

As is shown in the following example S/M might not be an ∞ - semifield. In fact S/M might not be congruence-free.

<u>2.1.2 Example</u> : Let $S = \{\frac{m}{2^n}, m,n \in Z^+\} \cup \{\infty\}$ with the usual addition and multiplication. Let D be a proper double ideal in S and suppose $x \neq \infty$ and $x \in D$. Then $x = \frac{m}{2^n}$ for some $m,n \in Z^+$. Thus we can choose $l \in Z^+$ such that $m+l=2^n 2^{n_1}$ for some $n_1 \in Z^+$. Thus

$$\frac{m}{2^n} + \frac{l}{2^n} = \frac{2^{n_1+n}}{2^n} = 2^{n_1} \text{. But } \frac{1}{2^{n_1}} \in S \text{, so } \frac{1}{2^{n_1}} \cdot 2^{n_1} \in D.$$

Thus $1 \in D$ so $D = S$. Therefore M above $= \{\infty\}$ and $S/M \cong S$. Thus S/M is not a semifield since 3 has no multiplicative inverse in S. Also $(S \setminus \{\infty\} \times S \setminus \{\infty\}) \cup \{(\infty,\infty)\}$ is a nontrivial congruence on S so S is not congruence-free. If R is a ring and M a maximal ideal then, of course, R/M is a field. The analogue to this does not hold for commutative semirings modulo the maximum proper double ideal. However by Proposition 2.2.6, if S is a commutative semiring with 1 which has a maximal proper double ideal $\mathcal{M}$, then S/M is embeddable in an ∞ - semifield. Using the results developed in Chapter IV on type II semirings it is possible to prove additional interesting results concerning S/M.