CHAPTER IV

Ginzburg Ψ-Theory of High T_c Superconductivity

The Ginzburg - Landau theory is enormously successful in explaining the properties of conventional bulk superconductors. In this chapter, we shall present the free energy density from the Ginzburg Ψ- theory and apply for the properties of high-T_c superconductors.

Based on various experimental observations on the high - T_c superconducting oxides to date , one gets several surprises. It is of course too early to reach a consensus on the microscopic understanding of various phenomena in these materials. Before we go to model in section 4.1, we present a brief account of our preliminary attempts at a macroscopic phenomenological level for some of the electrodynamical and transport properties of the oxide materials.

The superconducting materials are inhomogenous with defects and oxygen deficiencies. It also possesses certain amount of structural disorder. Spectroscopic and transport measurements reveal in favor of the granularity of the materials. Besides , the materials are poor metals at high temperature and are insulating (semiconducting) antiferromagnetic (53 - 59) with short - range interaction (60) in the intermediate range of temperature.

4.1 The Model

In this section an expansion is made of free energy function. The assumed coexistance of high - T_c superconductivity and antiferromagnetism. Let Ψ is the order parameter , then in the absence of a magnetic field the free energy is

$$F_s = F_{no} + A|\Psi|^2 + \frac{1}{2}B|\Psi|^4 + \frac{1}{3}C|\Psi|^6 + ---- \tag{4.1}$$

where F_{no} is the free energy of the normal state in zero magnetic field , and the coefficients A , B and C are to be taken as functions of temperature , T.

4.1.1 The Coefficients Temperature Dependence

The state of complete thermodynamic equilibrium corresponds to a mimimum of the free energy F_s with respect to Ψ^* i.e., to the condition

$$\left(\frac{\partial F_s}{\partial \Psi^*}\right)_T = 0 \tag{4.2}$$

We now find the temperature dependence of A , B and C. Equation (4.2) for the value , Ψ_o , of Ψ at the minimum gives

$$A\Psi + B|\Psi|^2\Psi + C|\Psi|^4\Psi = 0 \tag{4.3}$$

Below the critical temperature we have $\Psi \neq 0$, whence the equilibrium value

$$|\Psi_o|^2 = \frac{-B + \sqrt{B^2 - 4AC}}{2C} \tag{4.4}$$

We chose the plus sign in front of the square root , and the coefficient C is small and positive. This choice ensures satisfaction, below T_c, of the stability condition (the minimum of the free energy) in the equilibrium state

$$|\Psi_o|^2 = \frac{-B + B\,(1-4AC/B^2)^{1/2}}{2C}$$

$$= \frac{-B + B\,(1 - 4AC/2B^2 + ---)}{2C}$$

where $4\,AC/B^2 < 1$

$$\lim_{c \longrightarrow 0^+} |\Psi_o|^2 = -A/B \qquad (4.5)$$

Substituting Eq. (4.5) into (4.1) the equilibrium free energy given by

$$F_s - F_{no} = -\frac{A^2}{2B} \qquad (4.6)$$

At T_c we have $|\Psi|^2 = 0$, thus it is obvious that $A(T_c) = 0$, Below the T_c , $|\Psi|^2 > 0$ and $F_s - F_{no} < 0$, consequently $A < 0$ and $B > 0$. In the self - consistent theory of second - order phase transitions , the coefficients A,B and C are regarded as expandable in power of $(T_c - T)$ and , with the scaling thory (see Appendix) , we have

$$A = -A_o \left(\frac{T_c - T}{T_c}\right)^{4/3} \qquad (4.7a)$$

$$B = B_o \left(\frac{T_c - T}{T_c}\right)^{2/3} \tag{4.7b}$$

$$C = C_o \tag{4.7c}$$

where A_o , B_o and C_o are positive phenomenological parameters.

4.1.2 Gingzburg - Theory and External Magnetic Field

The free energy density function of the high - T_c superconductors system in a given external magnetic field $\vec{H}$ as a function of Ψ and $\vec{A}$ has the following form

$$F_s = F_{no} + A|\Psi|^2 + \frac{1}{2}B|\Psi|^4 + \frac{1}{3}C|\Psi|^6$$

$$+ \frac{1}{2m^*}\left|(-i\hbar\nabla - \frac{e}{c}\vec{A})\Psi\right|^2 + \frac{h^2}{8\pi} \tag{4.8}$$

where $\vec{h} = \nabla \times \vec{A}$, the coefficients A , B and C in Eq. (4.7) are the same as Ginzburg's proposal (33). Hence , we called the Ginzburg Ψ - theory for high - T_c superconductivity.

4.2 Minimization and Results

4.2.1 The Ginzburg - Landau equations

We must now minimize the Gibbs free energy density G_s with respect to the order parameter Ψ , and vector potentail $\vec{A}$. We set

$$\int d^3x \, [\, F_s - (4\pi)^{-1}\, \vec{h}.\vec{H}] = \int d^3 G_s \tag{4.9}$$

where the integral is extended over the volume of the sample. This variational problem leads, by standard methods (see Sec. 3.1.2), given the following field equations

$$A\Psi + B|\Psi|^2\Psi + C|\Psi|^4\Psi + \frac{1}{2m^*}\left(-i\hbar\nabla - \frac{e\vec{A}}{c}\right)^2\Psi = 0 \qquad (4.10)$$

$$\vec{J} = \frac{c}{4\pi}\nabla \times \vec{h} = \frac{-ie^*\hbar}{2m^*}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) - \frac{(e^*)^2|\Psi|^2}{m^*c}\vec{A} \qquad (4.11)$$

Equations (4.10) and (4.11) are the fundamental Ginzburg - Landau equations. The first gives the order parameter and the second gives the currents, that is, the diamagnetic response of a superconductor.

Equations (4.11) has thus given for the penetration depth,

$$\lambda^2 = \frac{m^*c^2}{4\pi(e^*)^2|\Psi|^2} \qquad (4.12)$$

as obtained in any standard text (45). If the dimension of the specimen are much greater than λ, then we have $\vec{B} = 0$, absence of a magnetic field, inside specimen, Eq. (4.10) reduces to Eq. (4.3) $|\Psi|^2$ can be replaced by its equilibrium value in absence of magnetic field $|\Psi_0|^2$ defined by Eq. (4.5). with A and B given by Eq. (4.7) lead to an explicit form for temperature depence of λ,

$$\lambda^2 = \frac{m^*c}{4\pi(e^*)^2(|A|/B)} \qquad (4.13a)$$

$$\lambda(T) = \left(\frac{m^* c^2 B_o}{4\pi(e^*)^2 A_o}\right)^{1/2} \left(1 - \frac{T}{T_c}\right)^{-1/2} \qquad (4/13b)$$

For the purpose of calcutaling a upper critical field H_{c2} of high - T_c superconductivity in a bulk sample we have considered the simplest case which there are no surface involved , so we assume a uniform applied $\vec{H}$ along the z axis

$$\vec{H} = H\hat{z} . \qquad (4.14)$$

A convenient choice (38) a vector potential $\vec{A}$ is

$$\vec{A} = H X \hat{y} \qquad (4.15)$$

With Eq. (4.15), the nonlinearized form of Eq. (4.10) becomes

$$-\frac{\hbar^2 \partial^2\Psi}{2m^*\partial x^2} - \frac{\hbar^2\partial^2\Psi}{2m^*\partial z^2} + \frac{1}{2m^*}\left(-i\hbar\frac{\partial}{\partial y} - \frac{e^* H x}{c}\right)^2\Psi = |A|\Psi - B|\Psi|^2\Psi - C|\Psi|^4\Psi \qquad (4.16)$$

Since the effective potential depends only on x , which implies that the corresponding of the eigenfunction are plane waves

$$\Psi(\vec{X}) = f\ (x)\ e^{ik_y y}\ e^{ik_z z} \qquad (4.17)$$

Substituting this into Eq. (4.16) and rearranging terms give

$$-\frac{\hbar^2}{2m^*}\frac{d^2 f}{dx^2} + \frac{(e^* H)^2}{2m^* c^2}(x-x_o)^2 f = \left(|A| - \frac{\hbar^2 k_z^2}{2m^{*2}}\right) f - B f^3 - Cf^5 \qquad (4.18)$$

$$\text{where} \qquad x_o = \frac{k_y \hbar c}{e^* H} \qquad (4.19)$$

We can solve Eq. (4.18) immediately by noting that it is the Schrodinger equation for a particle of mass m^* bound in a harmonic oscillator potential well centred at x_o with force constant $(e^*H)^2 / m^*c^2$. This problem is formally the same as that of finding the quantized states of a normal charged particle in a magnetic field separated by the cyclotron energy $\hbar\omega_c$. The resulting harmonic oscillator eigenvalues at ground state are

$$A - B/\sqrt{2} + C/\sqrt{3} - \frac{\hbar k_z^2}{2m^*} = \frac{1}{2}\hbar\omega_c \quad (4.20)$$

where $$\omega_c = \left(\frac{e^*H}{m^*c}\right) \quad (4.21)$$

In view of Eqs. (4.20) and (4.21) , thus

$$H = \frac{2m^*c}{\hbar e^*}\left(|A| - B/\sqrt{2} - C/\sqrt{3} - \frac{\hbar^2 k_z^2}{2m^*}\right) \quad (4.22)$$

The highest value of H is upper critical field H_{c2} , and it is obviously given by $k_z = 0$.

$$H_{c2} = \frac{2m^*c(|A| - B/\sqrt{2} - C/\sqrt{3})}{\hbar e^*} \quad (4.23)$$

Where the corresponding eigenfunction is the ground state wave function of the harmonic oscillator

$$f(x) = \exp[-(x-x_o)^2/\xi^2(T)] \quad (4.24)$$

where (61 , 62) , $$\xi^2(T) = \frac{\hbar^2}{2m^*(|A| - B/\sqrt{2} - C/\sqrt{3})} \qquad (4.25)$$

The coherence length $\xi(T)$ is the characteristic distance for variations in the order parameter. We can rewrite Eq.(4.23) as

$$H_{c2} = \frac{\Phi_o}{2\pi\,\xi^2(T)} \qquad (4.26)$$

where $$\Phi_o = \frac{hc}{e^*}$$

When we have neglected B and C , Eqs (4.23) and (4.25) reduces to

$$\xi^2(T) = \frac{\hbar^2}{2m^*|A|} \qquad (4.27a)$$

or $$\xi(T) = \left(\frac{\hbar^2}{2m^*A_o}\right)^{1/2} \left(1 - \frac{T}{T_c}\right)^{-2/3} \qquad (4.27b)$$

and
$$H_{c2} = \frac{2m^*c\,|A|}{\hbar e^*} \qquad (4.28a)$$

$$= \frac{2m^*c|A_o|}{\hbar e^*} \left(1 - \frac{T}{T_c}\right)^{4/3} \qquad (4.28b)$$

This means that the solutions of the linearized Ginzburg - Landau equation have been obtained by dropping the terms $B|\Psi|^2\Psi$ and $C|\Psi|^4\Psi$ 3 in Eq. (4.6),corresponding to dropping $\frac{1}{2}B|\Psi|^4$ and $\frac{1}{3}C|\Psi|^6$ in Eq. (4.8). These omission will be justified only if

$|\Psi|^2 << |\Psi_0|^2 = - A/B$. The linearized theory will be appropriate only when the magnetic field has reduced to a value much smaller than Ψ_0.

4.2.2 Thermodynamic Field and Specific Heat

The thermodynamic critical field H_c is the field at which the Gibbs free energies G_s and G_n in the superconducting and normal phase are equal. $\vec{B}$ is zero in the superconducting phase , and is zero in the normal phase , with $|\Psi|^2 = |A|/B$ Eq. (4.9) therefore gives for the energies at applied field $\vec{H}$

$$G_s = v(F_{no} - \frac{A^2}{2B} + \frac{C|A|^3}{3B^3} + \frac{H^2}{8\pi}) \qquad (4.31a)$$

$$G_n = v\, F_{no} \qquad (4.32b)$$

where v is the volume of the specimen. The thermodynamic critical field is given by $G_s = G_n$, neglect C , (C is small) and combining with Eqs. (4.13) and, (4.27) , we find

$$H^2_c = \frac{4\pi\,|A|^2}{B} = \left(\frac{4\pi\,A^2}{Bo}\right)\left(1 - \frac{T}{T_c}\right)^2 \qquad (4.32a)$$

or

$$H_c = \frac{\Phi}{2\sqrt{2}\,\pi\,\lambda\,\xi} \qquad (4.32b)$$

We recall from Eq.(4.6) that a free energy density the superconducting oxide and normal state in absence magnetic field is then

$$F_s - F_{no} = -\frac{A^2}{2B}$$

$$= \frac{-A^2}{2B}\left(1 - \frac{T}{T_c}\right)^2$$

and the specific heat has a discontinuity at transition given by

$$C = -T \; \partial^2 F/\partial T^2 ,$$

$$C_s - C_n = \Delta C = \frac{A_o^2}{B_o T_c} \tag{4.33}$$

4.2.3 The Lower Critical Field

We shall consider a high - T_c superconductor in a low magnetic field $\vec{H}$, the sample exhibits a complete Meissner effect , and $\vec{B}$ vanishes. As $\vec{H}$ is increased to the Lower critical field H_{c1} , the penetration of magnetic flux becomes favorable , and quantized flux lines (vortices) are formed parallel to the field $\vec{H} = H\hat{z}$. By definition , when $H = H_{c1}$ the Gibbs free energy must have the same value whether the first vortex is in or out of the sample. Thus , at H_{c1}

$$G_s \text{ (no flux)} = G_s \text{ (first vortex)}$$

or , since

$$G_s = F_s - (H/4\pi) \int h \, d^3x ,$$

$$F_s = F_s + \epsilon_1 L - \frac{H_{c1} \Phi_o L}{4\pi}$$

Where ϵ_1 is the free energy per unit length of a vortex filament , and L is the length of the vortex line in the sample. Thus,

$$H_{c1} = \frac{4\pi \epsilon_1}{\Phi_o} \tag{4.34}$$

We can now calculate the free energy per unit length ϵ_1, Neglecting the core , we have only the contribution from the field energy and kinetic energy of the currents

$$\epsilon_1 = \int d^2s \left\{ \frac{h^2}{8\pi} + \frac{1}{2m^*} \left| \left(-i\hbar\nabla - \frac{e^*}{c}\vec{A} \right) \Psi \right|^2 \right\} \tag{4.35}$$

We consider the extreme type II limit , in which $K = \lambda / \xi >> 1$, because useful analytic results can be obtained. The simplification results Ψ can rise from zero to limiting value within a core region of radius $\sim \xi$. Thus , over most of the vortex (of radius $\lambda >> \xi$) the high - T_c superconductor will act like and ordinary London superconductor. In the London model , $|\Psi|^2$ remains constant at large distances $r \sim \lambda >> \xi$. Thus essentially the whole of magnetic flux passes through the region outside the core. Let , $\Psi = |\Psi| \exp(i\phi(x))$, the order parameter can very through its phase $\phi(x)$, and the second term of Eq. (4.35) may be rewritten as

$$\left| \left(-i\hbar\nabla - \frac{e^*}{c}\vec{A} \right) \Psi \right|^2 = \frac{1}{2m^*} \left| \hbar \nabla \phi - \frac{e^*}{c}\vec{A} \right|^2 \Psi^2 \tag{4.36}$$

Comparison with Eq. (4.11) shows that this expression is related to the supercurrent $\vec{J}s$, and Eq. (4.35) becomes

$$\epsilon_1 = \int d^2s \left\{ \frac{h^2}{8\pi} + \frac{m^* J^2_s}{2(e^*)^2 n^*_s} \right\} \tag{4.37}$$

where $n^*_s = |\Psi|^2$. Eq. (4.37) may be rewritten in several ways. If the superfluid velocity field $\vec{v}$ is defined by the equation

$$\vec{J}_s = n^*_s e^* \vec{v}_s \tag{4.38}$$

then ϵ_1 assumes the intuitive from

$$\epsilon_1 = \int d^2s \left\{ \frac{h^2}{8\pi} + \frac{1}{2} m^* v_s^2 n^*_s \right\} \tag{4.39}$$

expressed as the sum of the magnetic field energy and the electronic kinetic energy. Alternatively , We may use Maxwell's equation , $\nabla \times \vec{h} = (4\pi/c)\, J_s$, to find

$$\epsilon_1 = \frac{1}{8\pi} \int d^2s \left\{ h^2 + \lambda^2 \,|\nabla \times h|^2 \right\} \tag{4.40}$$

where

$$\lambda^2 = \frac{m^* c^2}{4\pi (e^*)^2 n^*_s}$$

In order to calculate ϵ_1 , we must have solution for magnetic field $\vec{h}(\vec{r})$, with a singlurity at $\vec{r} \to 0^+$ In the absence of vortices , outside the core , the Eq. (4.11) is

$$\vec{J}_s = - \frac{(e^*)^2 |\Psi|^2}{m^* c} \vec{A}$$

or

$$\left(\frac{4\pi^2}{c}\right) \nabla \times \vec{J}_s + \vec{h} = 0 \tag{4.41}$$

If this relation held everywhere, the fluxoid for any path would be zero. We correct this by inserting a term to take acount for singular behavior at the vortex core, so that Eq. (4.41) becomes

$$\left(\frac{4\pi}{c}\right) \nabla \times \vec{J}_s + \vec{h} = \hat{z}\phi_o \delta_2(\vec{r}) \qquad (4.42)$$

where $\hat{z}$ is a unit vector along the vortex and $\delta_2(\vec{r})$ is a two - dimensional delta function at the Maxwell equation , $\nabla \times \vec{h} = (4\pi/c)\vec{J}$, we obtain

$$\lambda^2 \nabla \times (\nabla \times \vec{h}) + \vec{h} = \hat{z}\,\phi\delta_2(\vec{r}) \qquad (4.43)$$

since $\nabla.\vec{h} = 0$, Eq (4.43) can be written

$$\nabla^2 \vec{h} - \frac{\vec{h}}{\lambda^2} = -\hat{z}\,\frac{\phi_o}{\lambda^2}\,\delta_2(\vec{r}) \qquad (4.44)$$

at $\vec{r} \neq 0$, Eq (4.44) becomes ,

$$\nabla^2 \vec{h} - \frac{\vec{h}}{\lambda^2} = 0$$

$$\frac{d^2 h}{dr^2} + \frac{1}{r}\frac{dh}{dr} - \left(\frac{1}{\lambda^2}\right)h = 0 \qquad (4.45)$$

with boundary condition $h(\vec{r}) = 0$, at ∞ . Thus the appropriate solution is

$$h(r) = C_1 K_0 (r/\lambda) \qquad (4.46)$$

where K_0 is a zero - order Hankel function imaginary , C_1 is arbitrary constant evaluate C_1 , using

Eq. (3.40) . and Eq. (4.46)

$$\int \vec{h}.d\vec{s} = \phi_o$$

or , $2\pi\int$ hrdr $= \phi_o$

$2\pi\ C_1 \int_o K_o\ (r/\lambda)\ rdr = \phi_o$

so we obtain , $C_1 = \phi_o / 4\pi \lambda^2$, the complete solution of Eq. (4.45) is

$$h(r) = \frac{\phi_o}{4\pi^2} K(r/\lambda) \quad (4.47)$$

Eq. (4.40) , using the vector identity

$$|\nabla \times \vec{h}|^2 = (\nabla \times \vec{h}) . (\nabla \times \vec{h})$$

$$\nabla . (\vec{u} \times \vec{v}) = (\nabla \times \vec{u}).\vec{v} - (\nabla \times \vec{v}).\vec{u}$$

and combining Eq. (4.43) , may be written

$$\epsilon_1 = \frac{1}{8\pi} \int d^2s \left\{ h^2 + \lambda^2 (\nabla \times \nabla \times \vec{h}).\vec{h} + \nabla\cdot(\vec{h} \times (\nabla \times \vec{h}) \right.$$

$$= \frac{1}{8\pi} \int d^2s\ \vec{h}. \left\{\vec{h} + \lambda^2\ \nabla \times \nabla \times \vec{h}\right\} + \frac{1}{8\pi} \oint d^2s \left\{\vec{h} \times \nabla \times \vec{h}\right\}$$

$$= \frac{1}{8\pi} \int d^2s\ \vec{h}.\hat{z}\ \phi_o \delta(\vec{r}) + \frac{1}{8\pi} \oint d^2s \left\{ \vec{h} \times \nabla \times \vec{h} \right\} \quad (4.48)$$

where the line integrals are around the inner and outer perimeter of the integration area. Since the integration excludes the core , the first term contributes nothing. The second term goes to zero at infinty , but gives a finite contribution in encircling the core, namely

$$\epsilon_1 = \left(\frac{\phi_o}{4\pi}\right)^2 \int d^2s\ \delta_2(r)\ K\ (r/\lambda) \quad (4.49)$$

Eq. (4.49) , consider near the singlurity ,

$$K_o(r/\lambda) \simeq -\ln(r/\lambda) = \ln(\lambda/r) \quad (4.50)$$

substituting Eq. (4.50) into the integrand , using the properties of delta function and it is natural to cut off the integral at $r \approx \xi$, we get ,

$$\epsilon_1 = \left(\frac{\Phi_o}{4\pi}\right)^2 \int d^2s \, \delta_2(\vec{r}) \ln(\lambda/r)$$

$$= \left(\frac{\Phi_o}{4\pi\lambda}\right)^2 \ln(\lambda/\xi) \quad (4.51)$$

substituting Eq. (4.51) into Eq. (4.34) , we find the lower critical field

$$H_{c1} = \frac{\Phi_o}{4\pi\lambda^2} \ln\left(\frac{\lambda}{\xi}\right) \quad (4.52)$$

$$= \frac{\Phi_o}{4\pi\lambda^2(0)} \left(1-\frac{T}{T_c}\right)^{2/2} \ln\left[K(0)\left(1-\frac{T}{T_c}\right)^{1/2}\right] \quad (4.53)$$

where $K(0) = \lambda(0) / \xi(0)$. Eqs. (4.26), (4.32) (4.52) can be written in the form

$$H_{c2} = \left(\frac{2K^2}{\ln k}\right) H_{c1} \quad (4.55)$$

$$H_{c1} = \frac{H_c \ln K}{\sqrt{2}\, K} \quad (4.56)$$

where $K = \lambda / \xi$. Thus , apart frome the ln K term ,

$$\frac{H_c}{H_{c1}} = \frac{H_{c2}}{H_c} = \sqrt{2}\, K \qquad (4.57)$$

or
$$H_c = (H_{c1}\ H_{c2})^{1/2} \qquad (4.58)$$

so that H_c is approximately the geometric mean of H_{c1} and H_{c2}

4.2.4 Nucleation at Surfaces

The calculation that led to Eq. (4.26) is only valid in an infinite medium. It neglects boundary effects. Since real superconductors are finite in size , the behavior of surfaces must be consider. We now turn to the calculation of critical fields in the presence of surfaces and called the surface critical field or surface necleation field H_{c3} .

The simplest case is when we have an applied field parallel to the surface of a bulk specimen. In our chosen gauge , $A_x = A_n = 0$, so the boundary condition becomes simply

$$\frac{\partial \Psi}{\partial x} = \frac{\partial f}{\partial x} = 0 \text{ . at } x=0 \qquad (4.59)$$

A very simple variational approach gives quite a good approximation and illustrates the usefulness of variational methods in working with the Ψ - theory. We outline the calcutation here , leaving the details as an exercise. Motivated by Eq. (4.28) , we take our trial function to be

$$\Psi(x) = f(x)\, e^{ik_y y} = e^{-ax^2}\, e^{ik_y y} \qquad (4.60)$$

With x measured from sample surface and is bounded as x , this function automatically satisfies the boundary condition of Eq. (4.59)

Since $B|\Psi|^4$ and $C|\Psi|^6$ terms are small compared with the other terms in the right hand ride of Eq. (4.16) they can thus be neglected , at near T_c. Substituting Eq. (4.60) into Eq. (4.16) then yields

$$-\frac{\hbar^2}{2m^*}\frac{d^2f}{dx^2} + \frac{(e^*H)^2}{2m^*c^2}(x-x_o)^2 f = |A|f \qquad (4.61)$$

Therefore , due to the boundary condition for the surface of bulk specimen

$$|A| = \frac{\int f(x)\mathcal{H}f(x)dx}{\int f(x)\ f(x)dx} \qquad (4.62)$$

where $$\mathcal{H} = \frac{\hbar^2}{2m^*}\frac{d^2}{dx^2} + \frac{(e^*H)^2}{2m^*c^2}(x-x_o)^2 \qquad (4.63)$$

consider term $\int f(x)\mathcal{H}f(x)\ dx = I_1$

$$I_1 = -\frac{\hbar^2}{2m^*}\int\frac{d^2f}{dx^2}dx + \int\frac{(e^*H)^2}{2m^*c^2}(x-x_o)^2 f^2dx$$

$$= -\frac{\hbar^2}{2m^*}\int_0^\infty f\frac{d}{dx}\left(\frac{df}{dx}\right)dx + \int_0^\infty\frac{(e^*H)^2}{2m^*c^2}(x-x_o)^2f^2dx$$

$$= -\frac{\hbar^2}{2m^*}\cdot f\frac{df}{dx}\Big|_{x=0}^{\infty} - \int_0^\infty\left(\frac{df}{dx}\right)\left(\frac{df}{dx}\right)dx + \int_0^\infty\frac{(e^*H)^2}{2m^*c^2}(x-x_o)^2f^2dx$$

$$= -\frac{\hbar^2}{2m^*}\int_0^\infty \left(\frac{df}{dx}\right)^2 dx + \int_0^\infty \frac{(e^* H)^2}{2m^* c^2}(x-x_0)^2 f^2 dx$$

Inserting I_1, from the last Equation into Eq. (4.62) and using the trial function $f(x) = \exp(-ax^2)$, we find that

$$|A| = \frac{\hbar^2 a}{2m^*} + \frac{(e^* H)^2}{2m^* c^2}\left\{\frac{1}{4a} - x_0\left(\frac{2}{\pi a}\right)^{1/2} + x_0\right\} \quad (4.64)$$

Mininize this expression with respect to x_0, (i.e., $\partial|A|/\partial x_0 = 0$) and to a (i.e., $\partial|A|/\partial a = 0$) , we thus obtain

$$x_0 = (1/2\pi a)^{1/2}$$

$$a = \frac{e^* H}{2\hbar c}\left(1 - \frac{2}{\pi}\right)^{1/2}$$

Substituting x_0 and a from these last two equations into Eq. (4.64) gives

$$|A| = \left(\frac{\pi}{\pi - 2}\right)^{1/2} \frac{e^* \hbar H}{2m^* c} \quad (4.65)$$

The surface critical field H_{c3}, is given by Eq. (4.65) Thus,

$$H_{c3} = (3.32)\frac{m^* c|A|}{\hbar e^*} \quad (4.66)$$

the relation of H_{c3} to the upper critical field H_{c2} is clarified if we reexpreess H_{c3} in term of H_{c2} using Eq. (4.23). In this way , we arrive at the expression for H_{c3}

$$H_{c3} = 1.66\ H_{c2} \quad (4.67)$$

4.3 Application for Thin Films

4.3.1 Critical Current of a Thin Wire or Film

This will be the case if the sample is a thin wire or film so oriented with respect to any external field that any variation of $|\Psi|$ would need to occur in a thickness $d \ll \xi(T)$. In that case, the term in the free energy proportional to $(\nabla|\Psi|)^2$ would give an excessively large contribution if any substantial variation occured. As a result they do not, and we can approximate $\Psi(\vec{x})$ by $|\Psi| \exp(i\phi(\vec{x}))$, where $|\Psi|$ is constant. In this case, the expressions for the current density and free energy density can be written in the simple forms

$$J_x = -\frac{ie^*\hbar}{2m^*}|\Psi|^2\left(i\frac{\partial\phi}{\partial x}\right) - |\Psi|^2\left(-i\frac{\partial\phi}{\partial x}\right) - \left(\frac{e^{*2}}{m^*c}\right)|\Psi|^2 A_x$$

$$= \frac{e^*\hbar}{m^*}\frac{(\partial\phi)}{\partial x}|\Psi|^2 - \frac{(e^*)^2}{m^*c}|\Psi|^2 A_x$$

$$= \frac{e^*|\Psi|^2}{m^*}\left(\hbar\frac{\partial\phi}{\partial x} - \frac{e^*A_x}{c}\right) \tag{4.68}$$

or

$$\vec{J}_s = \frac{e^*|\Psi|^2}{m^*}\left(\hbar\nabla\phi - \frac{e^*\vec{A}}{c}\right) = e^*|\Psi|^2\,\vec{v}_s \tag{4.96a}$$

where

$$\vec{v}_s = \frac{1}{m^*}\left(\hbar\nabla\phi - \frac{e^*\vec{A}}{c}\right) \tag{4.96b}$$

and, we get

$$F_s = F_{no} + A|\Psi|^2 + \frac{1}{2}B|\Psi|^4 + \frac{1}{3}C|\Psi|^6 + \frac{1}{2}m^*\vec{v}_s^{\,2}|\Psi|^2 + \frac{h^2}{8\pi} \tag{4.70}$$

Let us now apply these equations to treat the case of a uniform current density through a thin film or wire. We can always neglect it for a sufficiently thin conductor. Then, for given V_s, we can minimize Eq. (4.70) to find the optimum value of $|\Psi|^2$.

$$\frac{\partial F_s}{\partial|\Psi|^2} = A + B|\Psi|^2 + C|\Psi|^4 + \frac{1}{2} m^* v_s = 0$$

We neglect term $|\Psi|^4 << |\Psi|^2$ the result is

$$|\Psi|^2 = |\Psi_o|^2 \left(1 - \frac{1}{2} \frac{m^* v_s^2}{|A|}\right) \qquad (4.71)$$

Substituting Eq. (4.71) into Eq. (4.69a), we get

$$J_s = e^*|\Psi_o|^2 \left(1 - \frac{m^* v_s^2}{2|A|}\right) v_s \qquad (4.72)$$

J_s. this has a maximum value when $\partial J_s / \partial v_s = 0$
We find that , $m^* v_s^2 / |A| = 2/3$. Substituting into Eq. (4.72), the appropiate the critical current is

$$J_c = \frac{2}{3} e^* |\Psi|^2 \left(\frac{2|A|}{3 m^*}\right)^{1/2} \qquad (4.73)$$

Using Eqs. (4.5) and (4.27a), we obtain

$$J_c = \frac{\sqrt{2}}{6\sqrt{3}\,\pi} \cdot \frac{c^2}{e^* \lambda^2} (m^*|A|)^{1/2}$$

$$= \frac{1}{6\sqrt{3}\pi} \frac{\hbar c^2}{e^* \lambda^2 \xi} = \frac{1}{6\sqrt{3}\pi^2} \frac{\Phi_o c}{\lambda^2 \xi} \qquad (4.74a)$$

Combining this with Eq. (4.32b) yields

$$J_c(T) = \frac{c H_c(T)}{3\sqrt{6}\,\pi\,\lambda(T)} \qquad (4.74b)$$

Finally, after some straightforward manipulation we find

$$J_c = \frac{2}{3}\left(\frac{A_o}{3 B_o}\right)\left(\frac{A_o}{m^*}\right)^{1/2}\left(1-\frac{T}{T_c}\right)^{3/2} \sim \left(1-\frac{T}{T_c}\right)^{3/2} \qquad (4.75)$$

4.3.2 <u>Parallel Critical Field of thin Films</u>

In order to calulation of the parallel critical field of thin films. We consider a film of thickness $d < \lambda$, in an external field $\vec{H}$ applied parallel to the plane of film and having the same value at both faces. In this case, we have to solve equation (4.61), now with the boundary conditions

$$\frac{d\Psi}{dx} = \frac{df}{dx} \quad \text{at } x = \pm\frac{d}{2} \qquad (4.76)$$

In this problem, we simplified with thin limit, f can hardly vary across the film, and we can intigrate Eq (4.61) across the thickness of the film, we have,

$$\frac{\hbar^2}{2m^*}\left[\left(\frac{df}{dx}\right)_{x=+\frac{d}{2}} - \left(\frac{df}{dx}\right)_{x=-\frac{d}{2}}\right] + \frac{(e^*H)^2}{2m^*c}\int_{-d/2}^{+d/2}(x-x_o)^2\, f\,dx = \int_{-d/2}^{+d/2}|A|\, f\, dx$$

the first term is zero because of the boundary conditions, the other terms , we can take f as a constant of, to get

$$\frac{(e^{*}H)^2}{2m^{*}c^2}\left(\frac{d^2}{12} + x_o^2 d\right) f_o = |A| f_o d \tag{4.77a}$$

or

$$H^2 = \frac{2 m^{*} c^2 |A|}{(e^{*})^2 (d^2/12 + x_o^2)} \tag{4.77b}$$

The largest value of H is H_{c11}. This obviously reguires taking $x_o = o$ in Eq.(4.77a), which means, not surprisingly,that the potential well for the problem is symmetric. Eq. (4.77b), combining with Eqs. (4.12), (4.25) and (4.32), we (62) find

$$H_{c11} = \left(\frac{24 m^{*} c^2 A_o}{(e^{*}) d^2}\right)^{1/2} \left(1-\frac{T}{T_c}\right)^{2/3} \tag{4.78}$$

$$= \frac{\sqrt{3}\,\phi_o}{\pi d\, \xi(T)} = 2\sqrt{6}\,\frac{H_c(T)\lambda(T)}{d} \tag{4.79}$$

This parallel critical field can exceed the thermodynamic critical field H_c by a large factor if d/λ is small enough . The physical reason for this is simply that the thin film, being largely penetrated by the field, has little diamagnetic energy for a given applied field in comparison to an equal volume of a bulk superconductor.