#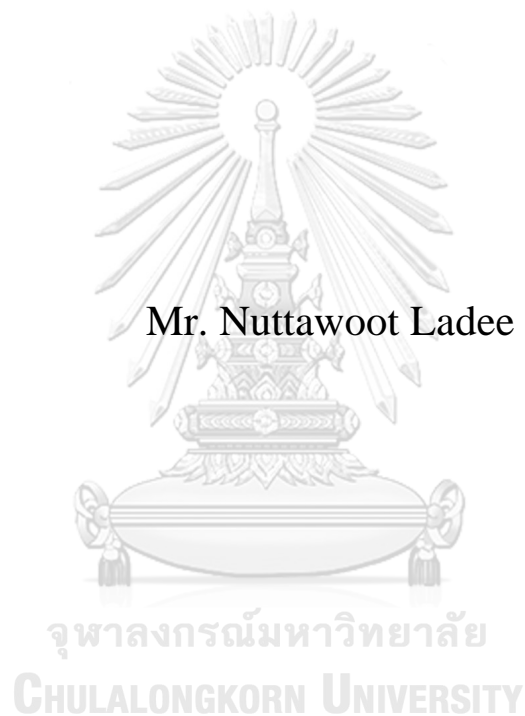 DETERMINING THE TRACKING ERROR AND VALUE-AT-RISK OF AN ACTIVE PORTFOLIO WHEN COMBINED WITH A PASSIVE PORTFOLIO WITH VALUE-AT-RISK CONSTRAINT

Mr. Nuttawoot Ladee

An Independent Study Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science in Financial Engineering
Department of Banking and Finance
FACULTY OF COMMERCE AND ACCOUNTANCY
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การกำหนดค่าตามรอยคลาดเคลื่อนและมูลค่าความเสี่ยงของพอร์ตการลงทุนเชิงรุก เมื่อรวมกับ
พอร์ตการลงทุนเชิงรับ ภายใต้ข้อจำกัดมูลค่าความเสี่ยงของพอร์ตการลงทุนรวม

นายณัฐวุฒิ ลาดี

Independent Study Title	**DETERMINING THE TRACKING ERROR AND VALUE-AT-RISK OF AN ACTIVE PORTFOLIO WHEN COMBINED WITH A PASSIVE PORTFOLIO WITH VALUE-AT-RISK CONSTRAINT**
By	**Mr. Nuttawoot Ladee**
Field of Study	**Financial Engineering**
Thesis Advisor	**Associate Professor SIRA SUCHINTABANDID, Ph.D.**

Accepted by the FACULTY OF COMMERCE AND ACCOUNTANCY, Chulalongkorn University in Partial Fulfillment of the Requirement for the Master of Science

INDEPENDENT STUDY COMMITTEE

.... **Chairman**
(Associate Professor THAISIRI WATEWAI, Ph.D.)

.... **Advisor**
(Associate Professor SIRA SUCHINTABANDID, Ph.D.)

.... **Examiner**
(Tanawit Sae-Sue, Ph.D.)

จุฬาลงกรณ์มหาวิทยาลัย
CHULALONGKORN UNIVERSITY

ณัฐวุฒิ ลาดี **:** การกำหนดค่าตามรอยคลาดเคลื่อนและมูลค่าความเสี่ยงของพอร์ตการลงทุนเชิงรุก เมื่อรวมกับพอร์ตการลงทุนเชิงรับ ภายใต้ข้อจำกัดมูลค่าความเสี่ยงของพอร์ตการลงทุนรวม**. (DETERMINING THE TRACKING ERROR AND VALUE-AT-RISK OF AN ACTIVE PORTFOLIO WHEN COMBINED WITH A PASSIVE PORTFOLIO WITH VALUE-AT-RISK CONSTRAINT)** อ.ที่ปรึกษาหลัก **:** รศ. ดร.สิระ สุจินตะบัณฑิต

เมื่อผู้จัดการกองทุนมีการบริหารพอร์ตการลงทุนใดๆ ก็มักจะมีการกำหนดข้อจำกัดของค่าตามรอยคลาดเคลื่อนและมูลค่าความเสี่ยงเพื่อควบคุมความเสี่ยงของพอร์ตการลงทุน ทั้งนี้พอร์ตการลงทุนสามารถแบ่งออกเป็นพอร์ตการลงทุนย่อย ซึ่งรายงานวิจัยฉบับนี้จะเน้นให้ความสนใจไปที่พอร์ตการลงทุนที่ประกอบด้วยส่วนของการลงทุนเชิงรุกและส่วนของการลงทุนเชิงรับ โดยจะเสนอวิธีการกำหนดข้อจำกัดของค่าตามรอยคลาดเคลื่อนและมูลค่าความเสี่ยงของพอร์ตการลงทุนเชิงรุกเมื่อมีการกำหนดมูลค่าความเสี่ยงของพอร์ตการลงทุนรวม นอกจากนี้จะมีการแสดงตัวอย่างการคำนวนโดยใช้ข้อมูลจริง และแสดงปัญหาที่เกิดขึ้นเมื่อสมมติฐานในภาคทฤษฎีต่างจากภาคปฏิบัติ เช่น (1) หุ้นที่สามารถลงทุนได้แตกต่างจากหุ้นในดัชนีชี้วัด (2) มีข้อจำกัดในการขายชอร์ต นอกจากนี้ค่าความเสี่ยงต่างๆในภาคทฤษฎียังเป็นการคำนวนแบบ "ก่อนเหตุการณ์" แต่การคำนวนค่าความเสี่ยงในทางปฏิบัติเป็นแบบ "หลังเหตุการณ์" ดังนั้นการนำวิธีการกำหนดค่าตามรอยคลาดเคลื่อนและมูลค่าความเสี่ยงของพอร์ตการลงทุนเชิงรุกที่เสนอไปใช้จริง จึงต้องคำนึงถึงข้อจำกัดดังกล่าวด้วย

สาขาวิชา	วิศวกรรมการเงิน	ลายมือชื่อนิสิต	
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ปีการศึกษา	**2563**	ลายมือชื่อ อ.ที่ปรึกษาหลัก	
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Nuttawoot Ladee : DETERMINING THE TRACKING ERROR AND VALUE-AT-RISK OF AN ACTIVE PORTFOLIO WHEN COMBINED WITH A PASSIVE PORTFOLIO WITH VALUE-AT-RISK CONSTRAINT. Advisor: Assoc. Prof. SIRA SUCHINTABANDID, Ph.D.

When an asset manager manages a portfolio, he usually imposes a TE or a VaR limit on his portfolio to control portfolio risks. However, a portfolio may be divided into different portions. In our research, we consider a portfolio that consists of an active portion and a passive portion. Having a VaR budget of an entire portfolio, we propose methods to determine TE and VaR limit on the active portfolio and illustrate them by using real data. In the empirical part, some problems can occur when we relax certain theoretical assumptions such as (1) stocks in an investment universe can be different from stocks in a benchmark (2) there is a restriction on short-selling. Moreover, risk measures derived from the analysis part are "ex-ante," but calculation methods for monitoring are "ex-post." Therefore, we need to consider these limitations when using the proposed limit on TE and VaR of an active portfolio in real life.

Field of Study:	**Financial Engineering**	**Student's Signature** ....
Academic Year:	**2020**	**Advisor's Signature** ....

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Section 1: Introduction

When investors assign an asset manager to manage a portfolio for them, they usually set Value-at-Risk (VaR) to control the portfolio risk since it is easy to understand. However, investors also want to motivate asset managers to make more profit by giving some incentive for asset managers who make more profit than the benchmark. Nevertheless, this may make asset managers too aggressive so that they choose to invest in the portfolio that is too different from the benchmark; consequently, the portfolio return also has a big difference. So, investors need to impose a constraint on the Tracking Error (TE) as well.

From the research paper conducted by (Riccetti, 2016), he considers a manager who manages an active portfolio. His research question is, "How should a risk manager set TE and VaR constraints in asset management?". To answer this question, he suggests that the risk manager chooses TE limit ($\widehat{TE}$) from a range $[TE^{min}, TE^{max}]$ and chooses a VaR limit ($\hat{V}$) from a range $(-\infty, V^{max}]$. His objective is to find a good set of constraints that can help asset managers achieve good performance. His definition for a good set of constraints is to make a feasible region contain as many efficient portfolios (in the mean-variance sense) and, at the same time, as few inefficient portfolios as possible. In his method (as Figure1), from the data that was given such as stock universe, benchmark, confidence level (θ), and commission fee (com), he uses them to derive the suggested TE^{min} and TE^{max} first. After choosing TE limit ($\widehat{TE}$) from a range $[TE^{min}, TE^{max}]$, he uses this number to determine $\hat{V}^{max}$. Then, a risk manager can choose to set a VaR limit ($\hat{V}$). Now, an asset manager can choose to invest in a portfolio from a feasible region which has tracking error of a portfolio (TE_p) less than TE limit ($\widehat{TE}$) and Value-at-risk of a portfolio (V_p) less than VaR limit ($\hat{V}$).

Figure 1: *Riccetti research methodology*
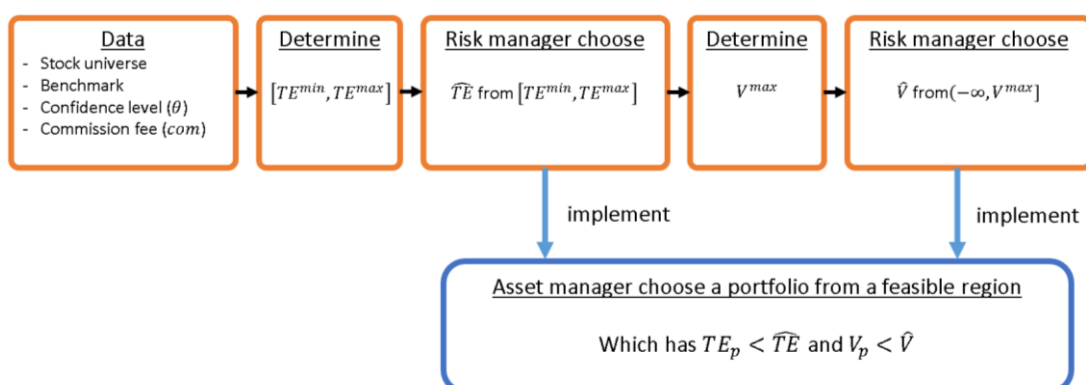
In contrast to Riccetti's work, this paper considers a different situation: while Riccetti's objective is to find an appropriate range of TE and VaR for a single portfolio, we, on the other hand, consider a portfolio that consists of an active portion and a passive portion. The risk budget of the entire portfolio is usually given in terms of the overall Value-at-Risk budget, while the active portion is typically managed under a TE budget (and also possible for a VaR budget). In this paper, we assume that we have a given constraint on the Value-at-Risk of the entire portfolio (V_G) and the proportion W_A allocated to the active part of the portfolio. And we are interested in determining the TE limit of the active portion of the portfolio. Therefore, our research question is, "What is the appropriate way to set TE limit of the active portfolio that is also consistent with the VaR constraint on the entire portfolio?". In our opinion, the appropriate TE constraint should make the feasible region contain as many efficient portfolios as possible while ruling out the portfolios that are inefficient or violate the overall VaR limit. In so doing, we will derive the upper limit on VaR of an active portfolio (V_A^{max}), which is limited by the given Value-at-Risk of the entire portfolio (V_G). The proposed V_A^{max} will provide us the largest feasible region of active portfolios that do not make the entire portfolio VaR exceed the given limit. For the lower limit on VaR of an active portfolio (V_A^{min}), we will find the least VaR limit that can be set from an unconstrained portfolio region. There will be no feasible region when setting a VaR limit less than this proposed V_A^{min}. Therefore, we get a range $\left[V_A^{min}, V_A^{max}\right]$ and then we derive a range $\left[TE_A^{min}, TE_A^{max}\right]$ in which the TE limit of an active portfolio ($\widehat{TE_A}$) should be set. The proposed TE_A^{max} will provide us the largest feasible region that does not violate the overall VaR in case the given overall VaR (V_G) more than the VaR at benchmark portfolio (V_B). And the proposed TE_A^{min} will provide us at least a portfolio's expected return that can compensate for the commission fees. The outline of our method is shown in Figure 2 below. After a risk manager chooses TE limit of an active portfolio ($\widehat{TE_A}$) and VaR limit of an active portfolio ($\widehat{V_A}$) from the suggested range, the active manager can follow these TE and VaR mandates while the combined portfolio is guaranteed to satisfy the overall VaR limit (V_G).

Figure 2: Our research methodology
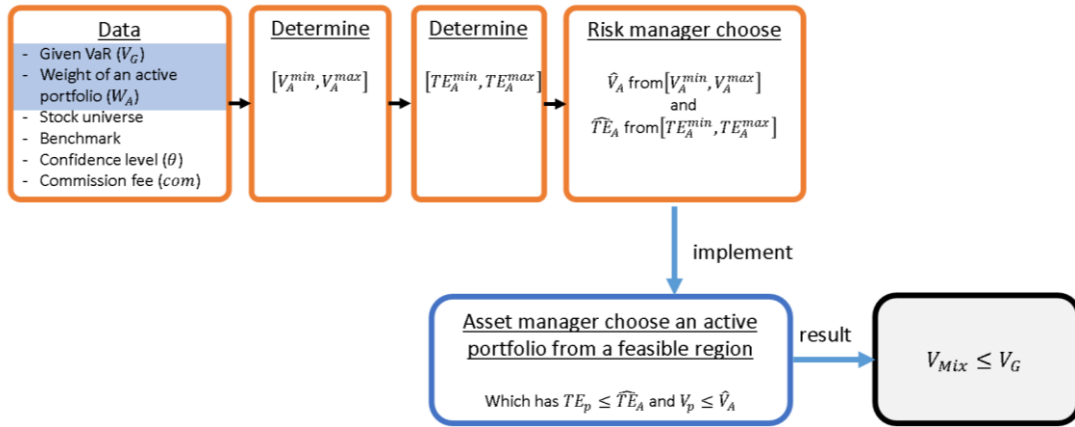
In sum, we extend Riccetti's framework, which identifies risk limits of a stand-alone portfolio. Our framework covers the situation in which risk limits are identified for a portion of a part-passive-part-active portfolio whose overall risk is itself controlled by the introduction of an overarching VaR constraint. In our method, we first use the overall VaR constraint to imply the VaR range and the feasible region of the active portion of the portfolio. Once the region of active portfolios is determined, we then identify the range of active portfolio's TE.

Beside from our main contribution, we extend some parts of Riccetti's work that he did not cover. In so doing, we propose new methods to determine a TE limit and a VaR limit of a single portfolio in various cases:

- We propose the method to determine a lower limit on TE (TE^{min}) when VaR is given (V_G). The TE^{min} proposed by Riccetti can be contradicted with the given VaR which causes empty feasible region.
- We propose the method to determine the lower limit on VaR (V^{min}) which Riccetti's work did not mention. It is the minimum value that can be achieved from an unconstrained portfolio region.
- We propose the method to determine the upper limit on TE (TE^{max}) in the case of the expected return of the benchmark is less than an expected return of the minimum variance portfolio (called inefficient benchmark). We will provide clearer instructions on how to calculate TE^{max} base on Riccetti's work.

Based on these modifications of Riccetti's work, we will extend the analysis further to determine a TE limit and a VaR limit of the active portion of a part-passive-part-active portfolio (main objective of this paper)

To implement the suggested risk limits in practice, we need to estimate the risk-return characteristics of the portfolio constituents in order to compute the TE limit. However, while in-sample data is used for estimation, the measurement of realized TEs and VaRs is done "out-of-sample." More specifically, our limits are computed based on "ex-ante" measures (ex-ante VaRs, ex-ante TEs, etc.), but the measurement in practice is usually performed in ex-post (ex-post VaRs, ex-post TEs, etc.) Furthermore, stocks in an investment universe is the same as stocks in a benchmark in theory, while it can be different in real life. Moreover, short-selling is hard to implement in practice. Therefore, it is interesting to test whether the realized risk measures truly stay within the given constraints, provided that re-estimation and re-balancing are carried out at reasonable frequency. This experiment is the objective of our second part, which implements the suggested risk limits using real market data and investigates the ex-post risk measures in the following issues:

- an effect on having stocks in an investment universe different from stocks in a benchmark,
- an effect on short-selling restriction, and
- an effect on yearly re-estimate parameters and re-balance portfolio.

This paper will proceed as follows. Section 2 reviews portfolio frontier and reviews how Riccetti (2016) determined a TE limit ($\widehat{TE}$) and a VaR limit ($\hat{V}$). If we consider only one active portfolio, section 3 will explain how to determine the TE limit when the VaR is given. This section also describes how can we determine the lower limit on TE (TE^{min}), the lower limit on VaR(V^{min}), and the upper limit on TE (TE^{max}) in the case of inefficient benchmark specifically value of α which Riccetti (2016) did not cover. We will develop the methods of determining TE limit and a VaR limit of a single portfolio from section 3 into combined portfolio in section 4. We will determine TE of an active portfolio ($\widehat{TE}_A$) in Section 4 in the case that we are given with two portfolios, which the one is a passive portfolio, and the other is an active portfolio. Section 5 is an empirical experiment to observe the practical issues in implementing the proposed limits. Finally, the conclusion is in section 6.

Section 2: Literature review

2.1 Review portfolio frontier

This section reviews some portfolio frontiers and constraints that have been proposed in the literature, which will be useful in this paper. Mean-variance frontier (MVF), Constrained-TE frontier (CTF) and Value-at-Risk (VaR) are shown in Figure3. The analysis is conducted in a variance-expected return space (σ^2, μ), but some figures are presented in the usual standard deviation-expected return space (σ, μ).

Figure 3: *Mean-variance frontier (MVF), Constrained-TE frontier (CTF), Value-at-Risk (VaR) and Benchmark (B) in (σ, μ) space*
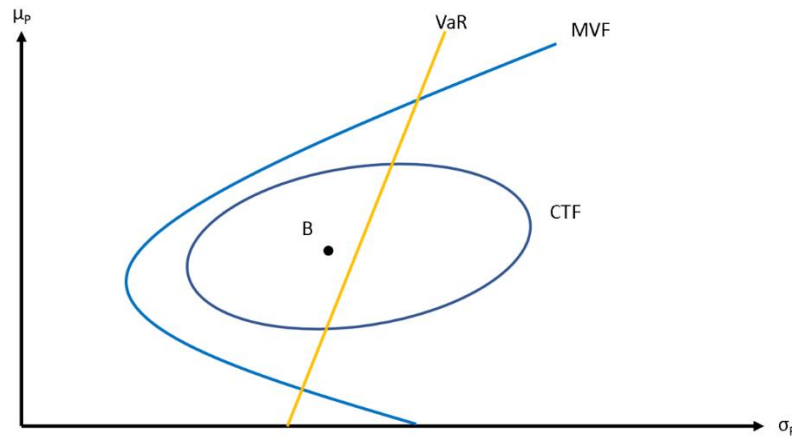
2.1.1 Mean-variance frontier (MVF)

From the given stock universe, we can have an infinite number of portfolios on the right side of the Mean-variance frontier (MVF) (Markowitz, 1952) as show in Figure3. As the derivation by (Merton, 1972), MVF can be constructed from the portfolios which has minimum variance with the given expected return. When there is no risk-free asset, the optimization can be set as:

$$
\left[
\begin{array}{lll}
\textbf{Min} & \sigma_p^2 & = \frac{1}{2}w'\Omega w \\
\textbf{s.t.} & \mu_p & = w'E \\
& i'w & = 1,
\end{array}
\right.
\tag{1}
$$

where each notation is defined as:

$$\sigma_p^2 \quad = \quad \text{variance of a portfolio return,}$$
$$\mu_p \quad = \quad \text{expected return of a portfolio,}$$
$$\Omega \quad = \quad \text{variance-covariance matrix of asset returns,}$$
$$w \quad = \quad \text{vector of portfolio weights,}$$
$$E \quad = \quad \text{vector of expected returns,}$$
$$i \quad = \quad \text{vector of ones.}$$

The Mean-variance frontier is given by:

$$\sigma_p^2 = \frac{1}{d}\left(\mu_p - \frac{b}{a}\right)^2 + \frac{1}{a}, \qquad (2)$$

with $a = i'\Omega^{-1}i$, $b = i'\Omega^{-1}E$, $c = E'\Omega^{-1}E'$ and $d = c - \frac{b^2}{a}$. In the (σ^2, μ) space, equation (2) represents a parabola which minimum variance portfolio (C) mean and variance are $\mu_C = \frac{b}{a}$ and $\sigma_C^2 = \frac{1}{a}$ respectively. Therefore, it is a hyperbola in the (σ, μ) space.

2.1.2 Constrained TE frontier (CTF)

When we impose a limit on tracking error (TE), the feasible portfolios for the asset manager are inside an elliptical area (for relatively low TE values) that contains the benchmark (B). (Palomba & Riccetti, 2012) called this ellipse the constrained TE frontier (CTF). The shape of constant TE frontier (CTF) in the original mean-variance space is explained by (Jorion, 2003) which was derived from a maximization (or equivalently a minimization) over x:

$$
\begin{cases}
\textbf{Max} & x'E \\
\textbf{s.t.} & x'i \quad = \quad 0 \\
& x'\Omega x \quad = \quad T^2 \\
& (q + x)'\Omega(q + x) \quad = \quad \sigma_p^2,
\end{cases}
\qquad (3)
$$

where

$$q \quad = \quad \text{vector of benchmark weights,}$$
$$x \quad = \quad \text{vector of weights that deviate from the benchmark,}$$
$$q + x \quad = \quad \text{vector of portfolio weights,}$$
$$T^2 \quad = \quad \text{variance of excess returns.}$$

By setting up the Lagrangian and solving the equation, we get the relationship between y and z as:

$$dy^2 + 4\Delta_2 z^2 - 4\Delta_1 yz - 4T^2(d\Delta_2 - \Delta_1^2) = 0, \tag{4}$$

where

$$
\begin{aligned}
y &= \sigma_p^2 - \sigma_B^2 - T^2, \\
z &= \mu_p - \mu_B, \\
\sigma_B^2 &= \text{variance of the benchmark return}, \\
\mu_B &= \text{expected return on the benchmark}, \\
\Delta_1 &= \mu_B - \mu_C, \\
\Delta_2 &= \sigma_B^2 - \sigma_C^2.
\end{aligned}
$$

This equation is a quadratic equation of the type $Ay^2 + Bz^2 + cyz + F = 0$ which represents an ellipse when the term:

$$
\begin{aligned}
AB - \left(\frac{1}{4}\right)C^2 &= d(4\Delta_2) - \left(\frac{1}{4}\right)(-4\Delta_1)^2 \\
&= 4(d\Delta_2 - \Delta_1^2)
\end{aligned}
\tag{5}
$$

is strictly positive. Therefore, the center of the ellipse is at $(\sigma_B^2 + T^2, \mu_B)$ in the (σ^2, μ) space and the shape of ellipse will somewhat distort in (σ, μ) space.

From the ellipse equation (equation (4)), we provide derivation more in detail in the Appendix. We can find the weight of a portfolio on the CTF (i.e., "an optimal weight") from:

$$x = -\frac{1}{\lambda_2 + \lambda_3}\Omega^{-1}(E + \lambda_1 i + \lambda_3 \Omega q), \tag{6}$$

where each notation is defined as:

$$
\begin{aligned}
\lambda_1 &= -\frac{b+\lambda_3}{a}, \\
\lambda_3 &= -\frac{\Delta_1}{\Delta_2} \pm \sqrt{\frac{\Delta_1^2}{\Delta_2^2} - \frac{dy^2 - 4T^2\Delta_1^2}{\Delta_2 y^2 - 4T^2\Delta_2^2}}, \\
\lambda_2 + \lambda_3 &= \pm(-2)\sqrt{\frac{\Delta_1^2 - d\Delta_2}{y^2 - 4T^2\Delta_2}}.
\end{aligned}
$$

The portfolio constructed from an optimal weight will be considered as "an optimal portfolio."

2.1.3 Value-at-Risk (VaR)

A portfolio's VaR (V_p) is the maximum potential loss over a time period at a given confidence level. In (σ, μ) space, the VaR constraint is represent by a line with intercept $-V_p$. Assuming that security returns have a multivariate normal distribution, the portfolio's VaR at the $100\theta\%$ confidence level is:

$$V_p = z_\theta \sigma_p - \mu_p, \tag{7}$$

where z_θ is the critical value obtained from the inverse cumulative distribution of a standardized normal with the confidence level $0.5 < \theta < 1$. In (σ, μ) space, when we imposed the limit on Value-at-Risk (VaR), the feasible portfolios will be on the left side of a straight line which is the VaR limit and has a slope z_θ (see, e.g., (Alexander & Baptista, 2008)).

2.2 Review the limit on TE and VaR proposed by Riccetti (2016)

Considering the MVF and CTF, Riccetti (2016) defined portfolios which have specific characteristic (in Figure 4) as follows: (1) B: benchmark (2) J_2: ellipse portfolio with minimum variance (3) J_1: ellipse portfolio with maximum expected return and (4) C: minimum variance portfolio. Noted that Riccetti (2016) only considers "optimal portfolios" on the CTF.

Figure 4: Portfolios defined by Riccetti(2016)
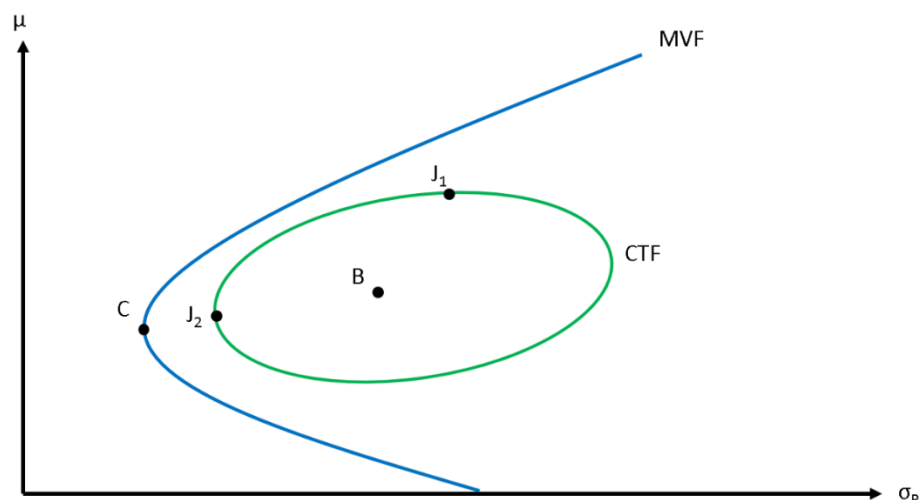
The closed-form solutions of mean and variance in each portfolio are explained in Jorion (2003). The mean, variance, and VaR of each portfolio are described in Table 1.

Table 1: Calculation of relevant portfolio.

Portfolio	Mean	Variance	VaR
B	μ_B	σ_B^2	$V_B = Z_\theta \sigma_B - \mu_B$
J_2*	$\mu_B - \dfrac{\Delta_1 \sqrt{T}}{\sqrt{\Delta_2}}$	$\sigma_{J_2} = \sigma_B^2 + T - 2\sqrt{T\Delta_2}$	$V_{J_2} = z_\theta \left(\sigma_B^2 + T - 2\sqrt{T\Delta_2}\right)^{\frac{1}{2}} - \mu_B + \dfrac{\Delta_1 \sqrt{T}}{\sqrt{\Delta_2}}$
J_1	$\mu_B + \sqrt{dT}$	$\sigma_{J_1} = \sigma_B^2 + T + 2\Delta_1 \sqrt{\dfrac{T}{d}}$	$V_{J_1} = z_\theta \left(\sigma_B^2 + T + 2\Delta_1 \sqrt{\dfrac{T}{d}}\right)^{\frac{1}{2}} - \mu_B - \sqrt{dT}$
C	$\mu_C = \dfrac{b}{a}$	$\sigma_C^2 = \dfrac{1}{a}$	-

*Note that: starting from a small number of TE, portfolio J_2 will be move to the left side as CTF expanding when TE increases. It will coincide with portfolio C when TE reaches to Δ_2 (it will be derived in section 3.3). However, when TE increases more than Δ_2, portfolio J_2 will be move to the right side of portfolio C. It does not make sense when portfolio J_2 which is the minimum variance portfolio given TE still have variance more than portfolio C in this case. Therefore, we conclude that if $T \geq \Delta_2$, portfolio J_2 will be the same as portfolio C.

2.2.1 Lower limit on TE (TE^{min})

As explained in Riccetti (2016), his objective is to determine the limit on TE ($\widehat{TE}$) for an amount large enough to give asset managers a non-empty and not too narrow set of feasible portfolios. For the lower limit on TE (TE^{min}), it should be increasing function of the commission fee so at least investors can get the return after fees equal to investing in benchmark from a portfolio (as Figure 5). The lower limit on TE should be $TE^{min} = \dfrac{com^2}{d}$ which was analytically explained in (Luca, 2012), where com is commission fee that includes all management fees. If the limit on TE ($\widehat{TE}$) is set at $\dfrac{com^2}{d}$, portfolio J_1 which is the maximum expected return portfolio on ellipse will have expected return equal to $\mu_B + com$. If the limit on TE ($\widehat{TE}$) is less than TE^{min}, an asset manager cannot invest in a portfolio that have an expected return after fee more

than investing on a benchmark. Therefore, the limit on TE ($\widehat{TE}$) should be set more than or equal to $\frac{com^2}{d}$.

Figure 5: *A feasible portfolio region when* $\widehat{TE} = \frac{com^2}{d}$
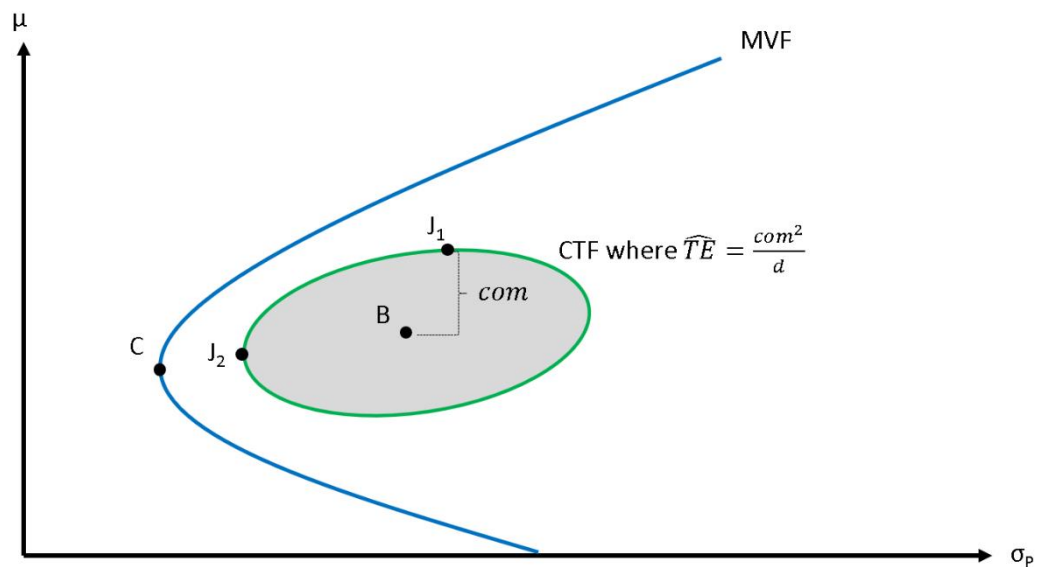
2.2.2 Upper limit on TE (TE^{max})

2.2.2.1 Upper limit on TE (TE^{max}) in case $\Delta_1 > 0$

Riccetti (2016) considers a general case where the expected return of a benchmark is greater than of a minimum variance portfolio ($\Delta_1 > 0$). For the upper limit on TE (TE^{max}), it should provide a feasible region that contains as many efficient portfolios and, at the same time, as few inefficient portfolios as possible. The ellipse area of constrained TE Frontier is expanded from point B (where is a benchmark portfolio) as TE increases. Suppose we set the limit on TE ($\widehat{TE}$) for a small number (e.g. $\widehat{TE}$ which is less than $\Delta_2 = \sigma_B^2 - \sigma_C^2$), a feasible portfolio region will be on the gray area, which its shape is an ellipse (as Figure 6).

Figure 6: *A feasible portfolio region when* $\widehat{TE} < \Delta_2$ *in case* $\Delta_1 > 0$
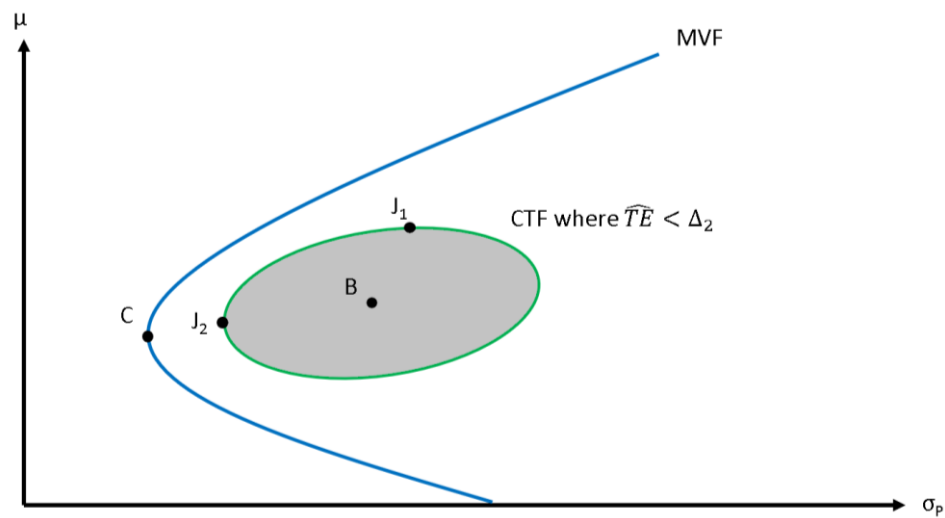
If TE increases to Δ_2, the ellipse area will be expanded so that the minimum variance portfolio on ellipse (J_2) coincides with the minimum variance portfolio (C). This value is derived by Jorion (2003). Since some parts of an ellipse are not investable, the feasible portfolio region will be on the gray area in Figure 7.

Figure 7: *A feasible portfolio region when* $\widehat{TE} = \Delta_2$ *in case* $\Delta_1 > 0$
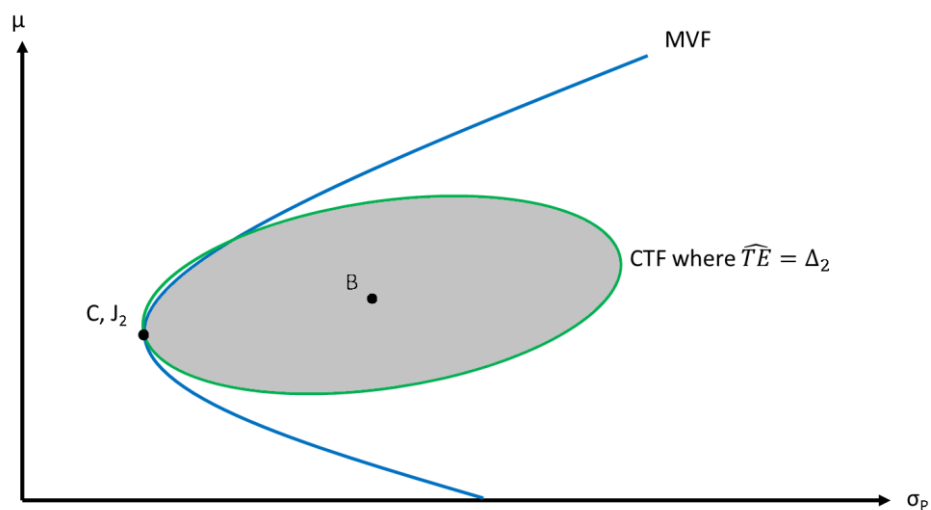
If TE increases more than Δ_2, the ellipse area will be expanded from the benchmark. As Figure 8, even though the ellipse area is expanded to cover some part of efficient portfolios on the MVF, it also covers portfolios with high absolute risk (σ) and cover more portfolio that cannot be invests. As a result, the feasible portfolio region will look like the gray area in Figure 8. Therefore, Riccetti (2016) proposes that TE should not more than Δ_2 in the case $\Delta_1 > 0$.

Figure 8: A feasible portfolio region when $\widehat{TE} > \Delta_2$ in case $\Delta_1 > 0$
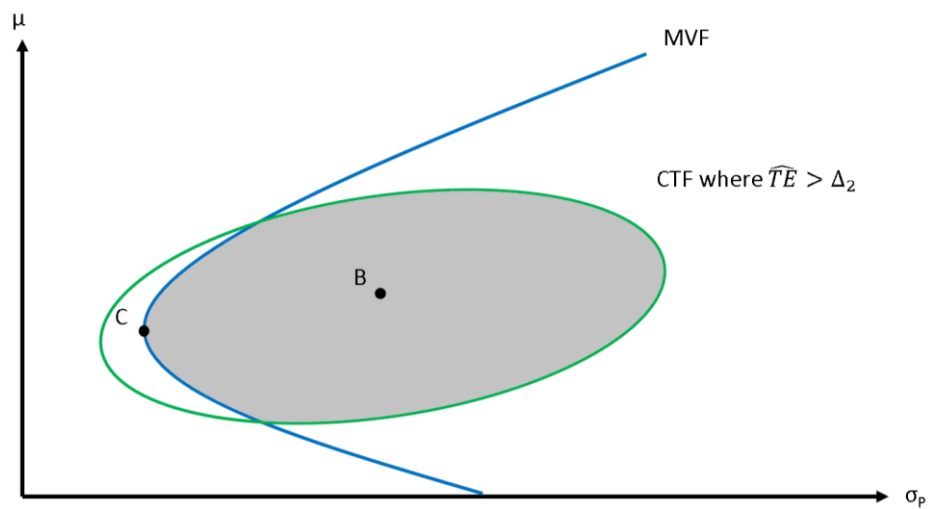
2.2.2.2 Upper limit on TE (TE^{max}) in case $\Delta_1 \leq 0$

In the case where expected return of a benchmark is less than or equal to a minimum variance portfolio ($\Delta_1 \leq 0$). When TE increases to Δ_2, the ellipse area is expanded so that the minimum variance portfolio on ellipse (J_2) coincides with minimum variance portfolio (C). However, most parts of the ellipse area are on the lower side of MVF which contains a lot of inefficient portfolios (as Figure 9). So, Riccetti (2016) proposed to increase TE limit ($\widehat{TE}$) more than Δ_2 to expand the ellipse covering more part on the upper side on MVF (as Figure 10).

Figure 9: A feasible portfolio region when $\widehat{TE} = \Delta_2$ in case $\Delta_1 < 0$
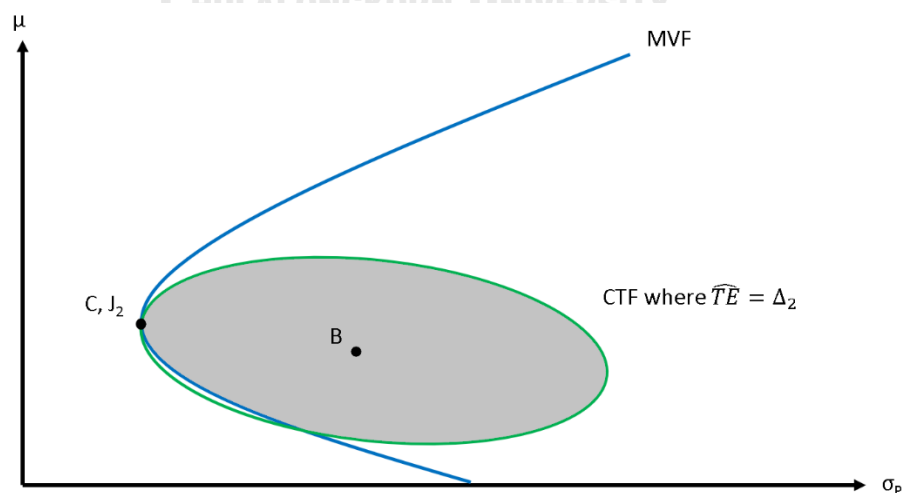
Figure 10: *A feasible portfolio region when $\widehat{TE} > \Delta_2$ in case $\Delta_1 < 0$*
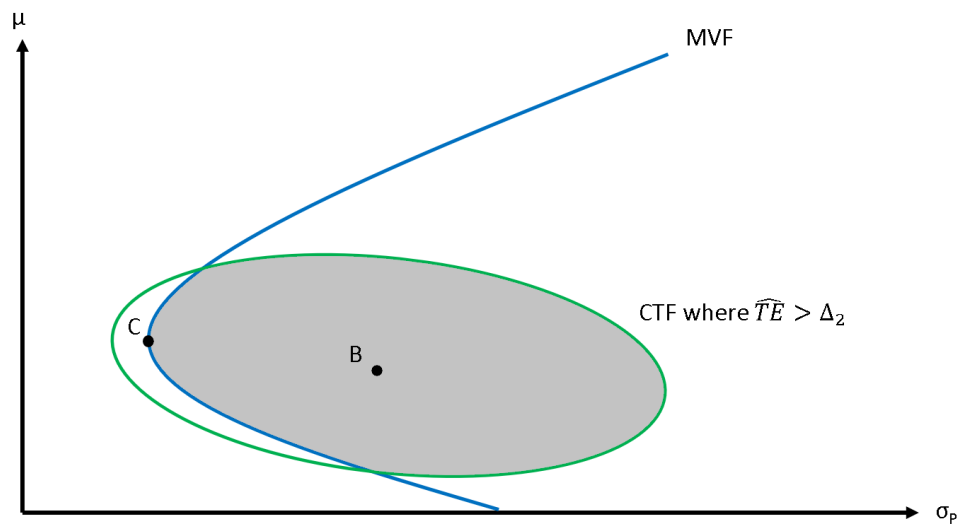
The situation when $\Delta_1 < 0$ means expected return of the benchmark is less than expected return of the minimum variance portfolio (Riccetti (2016) called it as an inefficient benchmark). This benchmark is far from the efficient part of the Markowitz frontier. So, asset managers can improve the portfolio performance with a very active strategy to move portfolio allocation away from the benchmark, which requires a high TE. So, in this case, the TE limit ($\widehat{TE}$) should be more than Δ_2. Therefore, Riccetti concluded that the upper limit on TE ($\widehat{TE}^{max}$) should be $\alpha\Delta_2$ where $\alpha = 1$ when $\Delta_1 > 0$ and $\alpha > 1$ when $\Delta_1 \leq 0$. Since he did not derive the calculation in this value and suggested investors change the benchmark to a new one that has a higher expected return than the minimum variance portfolio (C), we will propose the calculation of α in section 3.3.

2.2.3 Upper limit on VaR (V^{max})

For the upper limit on VaR (V^{max}), as same as the objective of finding the TE limit, Riccetti suggests that it should provide a feasible region that contains as many efficient portfolios and, at the same time, as few inefficient portfolios as possible. As shown in Figure 11, J_2 represents the portfolio with minimum variance, while K represents the portfolio with the minimum VaR. There will be a trade-off between VaR and variance reduction. If VaR is set at V_K, we can only invest at point K. However, if we set VaR at $\hat{V}_{J_2}$, we will have many choices to invest. Therefore, Palomba and Riccetti (2012) proposed that the minimum range of upper limit on VaR should be at V_{J_2}.

Figure 11: Minimum VaR portfolio (K) and minimum variance portfolio (J_2)
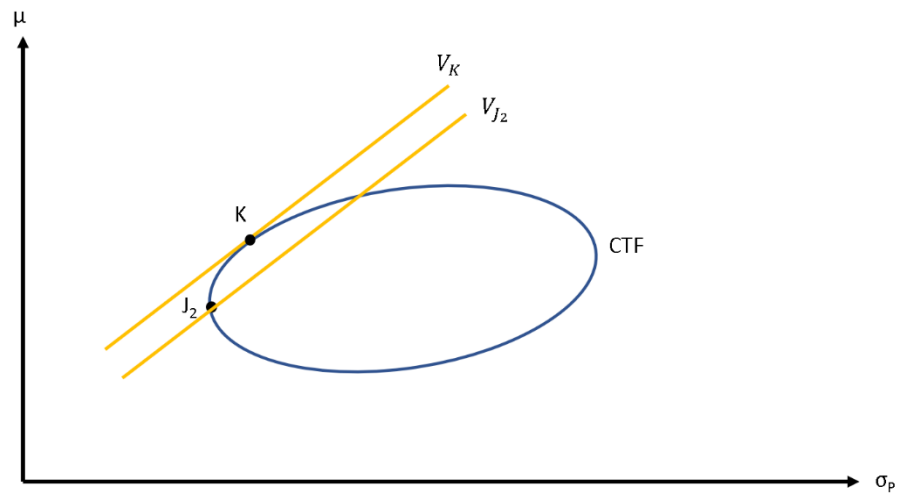
Figure 12: VaR constraints at portfolio level V_{J_1} and V_B in case $V_{J_1} < V_B$
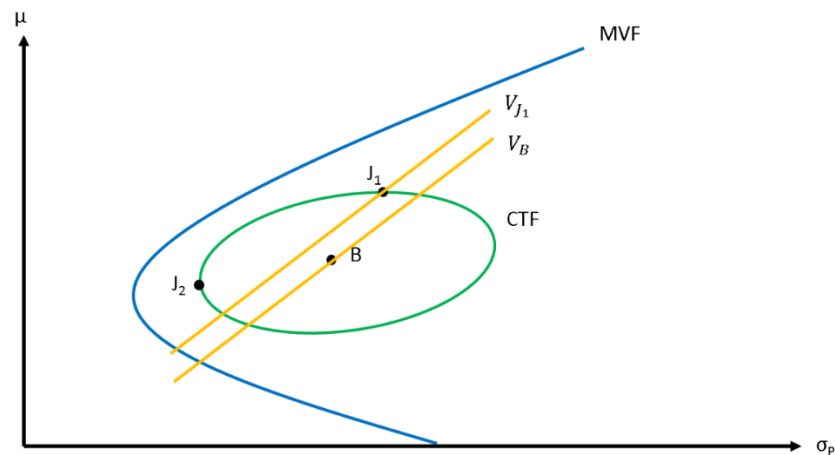
Considering the maximum range of the upper limit on VaR (V^{max}) in Figure 12, we compare a VaR of the maximum expected return portfolio on the CTF (V_{J_1}) and a VaR of the benchmark portfolio. It is obvious that if we choose VaR more than at J_1, it will only increase an inefficient set of portfolios. Therefore, Riccetti (2016) proposes that the upper limit on VaR should not more than V_{J_1}. However, V_{J_1} may be too risky compare to at benchmark (V_B), so he divided the proposed maximum range of the upper limit on VaR (V^{max}) into three scenarios:

1. In case $V_{J_2} \leq V_{J_1} < V_B$, if $V_{J_1} < V_B$, then V_{J_1} is still a reasonable choice to be upper limit on VaR.

2. In case $V_{J_2} \leq V_B < V_{J_1}$, if $V_B < V_{J_1}$ then V_B should be upper limit on VaR so that some feasible region is not too risky (as seen in Figure 13 below).

3. In case $V_{J_1} < V_{J_2}$, Setting the upper limit on VaR at V_{J_1} (as shown in Figure 14 below), the feasible set will be relatively small. Increasing the VaR limit more than that would increase the feasible set, but also increase the inefficient set as well.

Figure 13: *VaR constraints at portfolio level* $\boldsymbol{V_{J_1}}$ *and* $\boldsymbol{V_B}$ *in case* $\boldsymbol{V_{J_1} > V_B}$
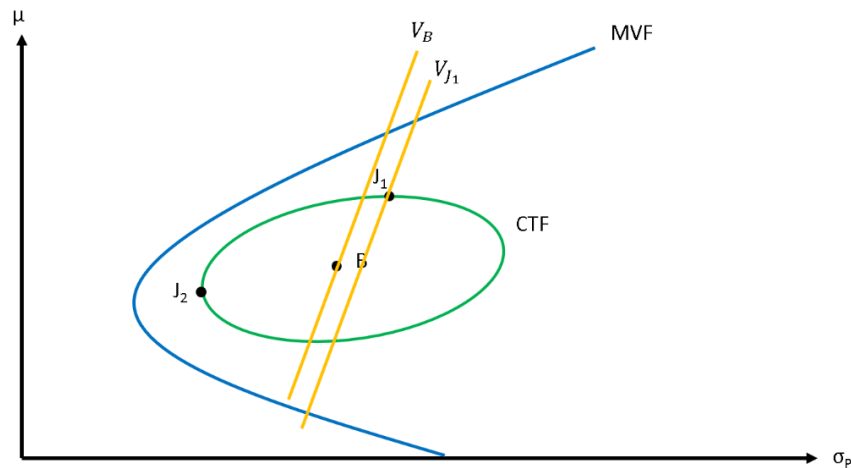
Figure 14: *VaR constraints at portfolio level* $\boldsymbol{V_{J_1}}$ *and* $\boldsymbol{V_{J_2}}$ *in case of* $\boldsymbol{V_{J_1} < V_{J_2}}$
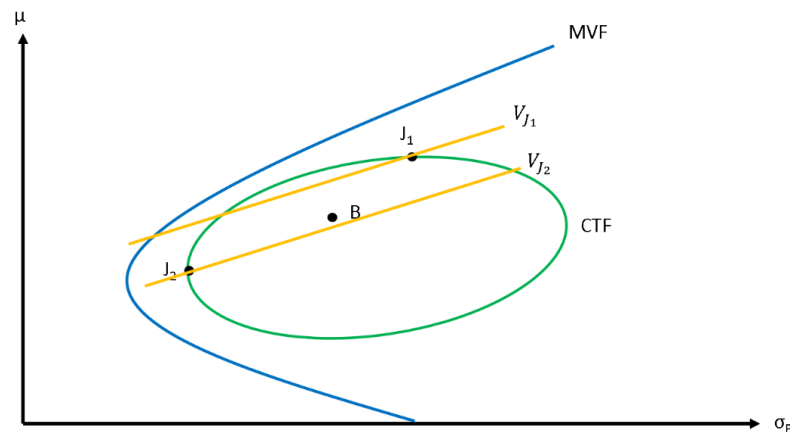
If we define the efficient constraint TE portfolios, they are portfolios on the CTF only the upper part of the minimum variance portfolio (J_2) until the maximum expected return portfolio (J_1). As Figure15, the efficient constraint TE portfolios are sets $\widehat{J_2 J_1}$ on an ellipse. Comparing constraint on VaR at V_B with constraint on standard deviation at B (σ_B) (or variance limit), if we choose to constraint on VaR at $\hat{V}_B$, it will cut efficient sets $\widehat{J_2 K_2}$ but increase efficient sets $\widehat{B_1 J_1}$ and also increase inefficient set in area BB_1K_1. Choosing constraint on standard deviation at B (σ_B) will cut efficient sets $\widehat{B_1 J_1}$ but increase efficient sets $\widehat{J_2 K}$ and increase inefficient set in area $BK_2J_2B_2$. It is a

free choice for the risk manager to choose VaR limit or variance limit (standard deviation limit). However, the limit on VaR implies a riskier set of a feasible portfolio than the limit in variance term (standard deviation term), so Riccetti (2016) suggests using constraint on variance (standard deviation) at σ_B rather than constraint on VaR at $\hat{V}_B$.

Figure 15: *Feasible region when we set VaR or standard deviation at B in case of* $V_{J_1} < V_{J_2}$
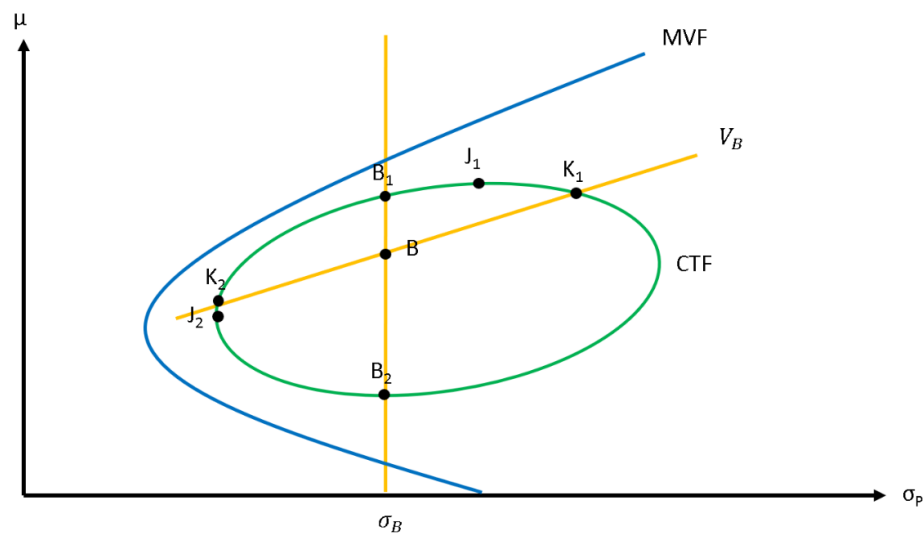
In sum, the limits proposed by Riccetti (2016) are suggested base on the limits that provide a feasible region that contains as many efficient portfolios and, at the same time, as few inefficient portfolios as possible. As summarized in Table 2 below, the TE limit should not be less than $\frac{com^2}{d}$ because there will be no portfolio's expected return that high enough to compensate for the commission fees. The TE limit should not be more than $\alpha\Delta_2$ because it would only increase a few efficient portfolio regions while increasing a lot of inefficient portfolio regions. For the VaR limit, in general, if it were less than V_{J_2}, we would unnecessarily forgo some portfolios on the efficient CTF. On the other hand, if the VaR limit is more than V_{j_1}, it will only increase inefficient feasible regions, which will contradict Riccetti's objective of setting the limits. The suggested VaR limit is also changed depends on the value among V_{J_2}, V_{J_1}, and V_B.

Table 2: *Proposed limits by Riccetti(2016)*

	Limit	Case	Value
TE	Lower		$T_{min} = \dfrac{com^2}{d}$
	Upper(range)		$\dfrac{com^2}{d} < T_{max} \leq \alpha\Delta_2$
VaR	Range in general		$V_{J_2} \leq V_{max} \leq V_{j_1}$
	Upper	$V_{J_2} \leq V_{J_1} < V_B$	$V_{max} = V_{J_1}$
		$V_{J_2} \leq V_B < V_{J_1}$	$V_{max} = V_B$
		$V_{J_1} < V_{J_2}$	Prefer limit on variance to VaR at σ_B

Section 3: Extension from Riccetti (2016)

This section will modify and complete some of Riccetti's works that determine the TE limit and the VaR limit of a single portfolio. Then we will extend the modified work to determine the TE limit and the VaR limit of the active portion of the entire portfolio in the next section. If we were given the VaR limit, the TE limit that we should set should not make any parts of portfolio region exceed the given VaR limit. In general, if the VaR limit (V_G) is more than the VaR at portfolio B (V_B), the TE should be set at the level that the ellipse of the CTF tangent with V_G at portfolio G (as shown in Figure16). We define that level of TE as TE_G.

Given TE limit ($\widehat{TE}$) = TE_G, the portfolio that has highest VaR is at portfolio G and the portfolio that has the lowest VaR is at portfolio K.

Figure 16: *The ellipse tangent with the given VaR ($\boldsymbol{V_G}$) and the resulting portfolio K on the ellipse with the lowest VaR*
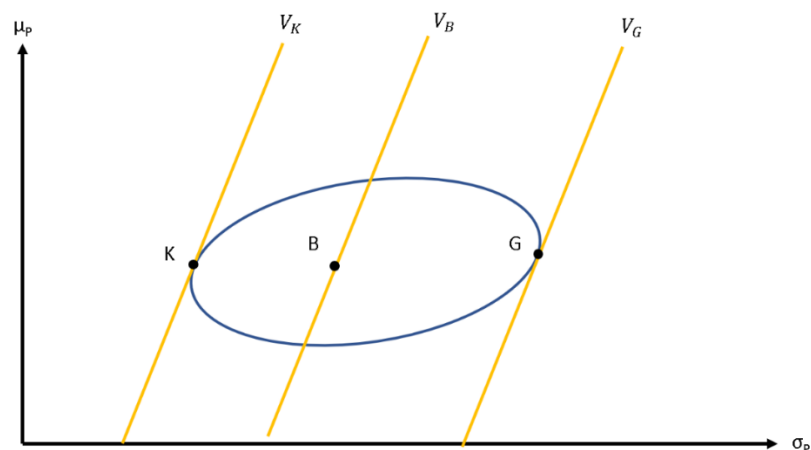
To find a portfolio on the CTF with a specific slope, we take the partial derivative with respect to y on equation (4), we get the slope:

$$z' = \frac{2\Delta_1 z - dy}{4\Delta_2 z - 2\Delta_1 y}. \tag{8}$$

When we set $z' = 0$, we can find the point of an ellipse that has maximum / minimum expected return as:

Table 3 : Calculation on maximum/minimum expected return point

Points	Calculations
Maximum expected return point (portfolio J_1)	$\left(\mu_B + \sqrt{T^2 d}, \sigma_B^2 + T^2 + 2\Delta_1\sqrt{\frac{T^2}{d}}\right)$
Minimum expected return point	$\left(\mu_B - \sqrt{T^2 d}, \sigma_B^2 + T^2 - 2\Delta_1\sqrt{\frac{T^2}{d}}\right)$

And by taking the partial derivative with respect to z on equation (4) we get the slope:

$$y' = \frac{2\Delta_1 y - 4\Delta_2 z}{dy - 2\Delta_1 z}. \tag{9}$$

Then we set $y' = 0$, we can find the point of an ellipse that has maximum / minimum variance as:

Table 4 : Calculation on maximum/minimum expected variance point

Points	Calculations
Maximum variance point	$\left(\mu_B + \Delta_1\sqrt{\frac{T^2}{\Delta_2}}, \sigma_B^2 + T^2 + 2\sqrt{T^2\Delta_2}\right)$
Minimum variance point (portfolio J_2)	$\left(\mu_B - \Delta_1\sqrt{\frac{T^2}{\Delta_2}}, \sigma_B^2 + T^2 - 2\sqrt{T^2\Delta_2}\right).$

Considering a VaR of a portfolio, it can be computed from $V_p = Z_\theta \sigma_p - \mu_p$. To find the slope of VaR in the mean-variance space in term of z, we get:

$$\mu_p = Z_\theta \sigma_p - V_p$$
$$z = Z_\theta \sqrt{y + \sigma_B^2 + T^2} - V_p - \mu_B \tag{10}$$
$$z' = \frac{Z_\theta}{2\sqrt{y + \sigma_B^2 + T^2}}. \tag{11}$$

From equation (8), we can find z in term of z' as:

$$
\begin{aligned}
z' &= \frac{2\Delta_1 z - dy}{4\Delta_2 z - 2\Delta_1 y} \\
4\Delta_2 zz' - 2\Delta_1 yz' &= 2\Delta_1 z - dy \\
dy - 2\Delta_1 yz' &= 2\Delta_1 z - 4\Delta_2 zz' \\
z &= \frac{dy - 2\Delta_1 yz'}{2\Delta_1 - 4\Delta_2 z'} .
\end{aligned}
\tag{12}
$$

Therefore, the tangent point of VaR and the ellipse is the point that:

$$
z = \frac{dy - \dfrac{\Delta_1 Z_\theta y}{\sqrt{y + \sigma_B^2 + T}}}{2\Delta_1 - \dfrac{2\Delta_2 Z_\theta}{\sqrt{y + \sigma_B^2 + T}}}
$$

$$
z = \frac{dy\sqrt{y + \sigma_B^2 + T} - \Delta_1 Z_\theta y}{2\Delta_1 \sqrt{y + \sigma_B^2 + T} - 2\Delta_2 Z_\theta} .
\tag{13}
$$

The value of y can be found by substituting the value of z from equation (13) to the ellipse equation (4), so we get:

$$
dy^2 + 4\Delta_2 \left(\frac{dy\sqrt{y + \sigma_B^2 + T} - \Delta_1 Z_\theta y}{2\Delta_1 \sqrt{y + \sigma_B^2 + T} - 2\Delta_2 Z_\theta} \right)^2 - 4\Delta_1 y \left(\frac{dy\sqrt{y + \sigma_B^2 + T} - \Delta_1 Z_\theta y}{2\Delta_1 \sqrt{y + \sigma_B^2 + T} - 2\Delta_2 Z_\theta} \right) - 4T(d\Delta_2 - \Delta_1^2) = 0,
\tag{14}
$$

Since we cannot find the closed-form solution by plugging-in the value of z to the ellipse equation (4), we will use the numerical search to get the value of y, which should have two values. To get the value of z, we plug-in those values of y back to equation (12). Then we revert them to the value of μ_p and σ_p^2. The higher variance portfolio will be the highest VaR portfolio given the TE limit and vice versa. as Figure16, if we set $T = TE_G$, the higher variance portfolio will be portfolio G and the lower variance portfolio will be portfolio K.

3.1 Determining the lower limit on TE (TE^{min}) when VaR is given (V_G)

When a VaR limit was given before, if we set the TE limit to $\frac{com^2}{d}$ as Riccetti suggest, there may be the case that there is no feasible region. So, we propose the lower limit on TE (TE^{min}) as to at least give us a portfolio to invest in. Considering the case where $V_G < V_B$, In this case, there are only portfolios on the left side of V_G can have VaR less than V_G. Since we cannot invest in a portfolio on the right side of V_G, the lower

limit on TE (TE^{min}) should be set as tangential CTF is on the right side of V_G (as shown in Figure 17 below). To find such a value of TE that tangent with V_G, we start by assigning a small number of TE. We then find a tangential portfolio on CTF by searching for the pair of z and y that give the lowest VaR portfolio equal to V_G. Considering z and y in equation (13) and (14), we want to find the value of TE that satisfies an equation:

$$Z_\theta\sqrt{y + \sigma_B^2 + T} - z - \mu_B - V_G = 0, \tag{15}$$

where $Z_\theta\sqrt{y + \sigma_B^2 + T} - z - \mu_B$ is the VaR of the lowest VaR portfolio given a TE limit. If $Z_\theta\sqrt{y + \sigma_B^2 + T} - z - \mu_B - V_G > 0$, we will increase the value of TE until it becomes zero (vice versa). The TE that satisfies equation (15) will be the lower limit on TE (TE^{min}).

Figure 17: *The lower limit on TE ($\boldsymbol{TE^{min}}$) when VaR is given ($\boldsymbol{V_G}$) in the case $V_G < V_B$*
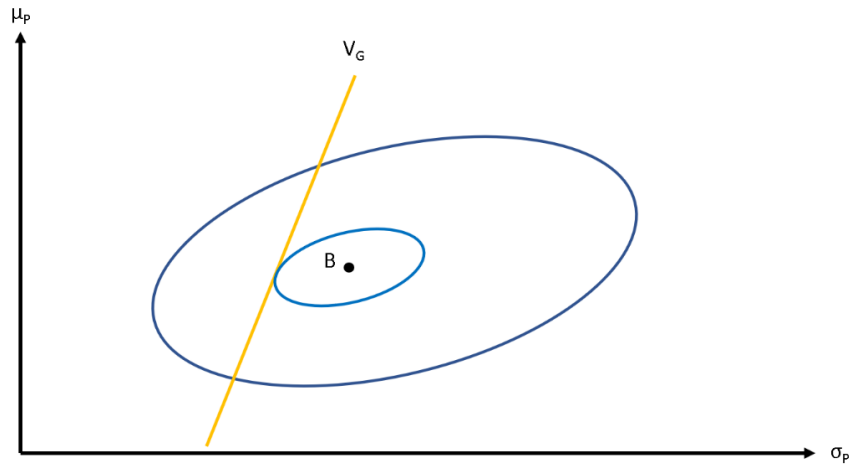
3.2 Determining the lower limit on VaR (V^{min}) that can be set

From Riccetti's work, there is no lower limit on VaR (V^{min}) in his suggestion, so we will propose it in this section. We propose that the lower limit on VaR should provide us at least a portfolio to invest in regardless of how large a feasible region.

Figure 18: *Minimum VaR (V^{min}) that can be set*
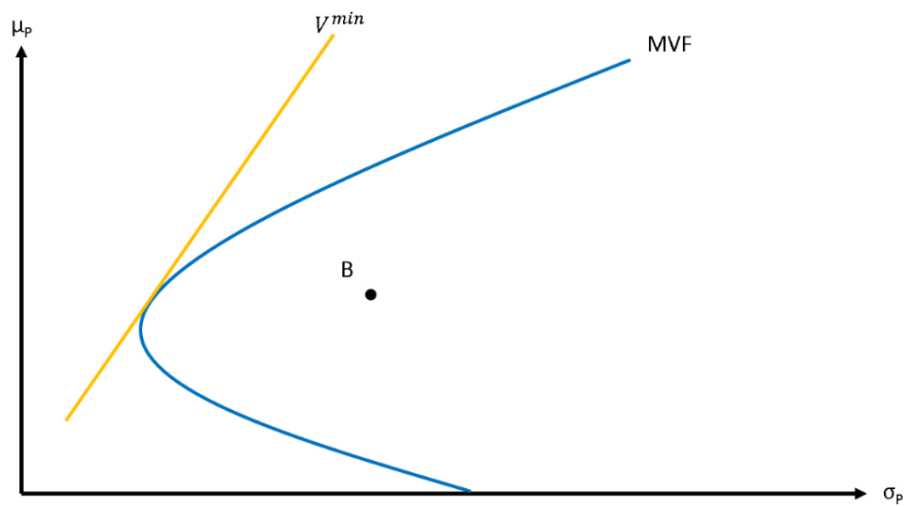
Considering the minimum value of V_G that can be set, it should be the VaR of the lowest VaR portfolio in MVF (as shown in Figure 18). So, we propose the lower limit on VaR (V^{min}) to be the VaR of the portfolio at the tangential point of the MVF and the VaR line. Given the stock universe that we can invest, the mean-variance efficient frontier (MVF) from equation (2) can be rewritten as:

$$\sigma_p^2 = \frac{1}{d}(\mu_p - \mu_C)^2 + \sigma_C^2$$
$$\sigma_p^2 = \frac{1}{d}(\mu_p^2 - 2\mu_p\mu_C + \mu_C^2) + \sigma_C^2 . \tag{16}$$

The slope of this MVF is:

$$2\sigma_p = \frac{1}{d}(2\mu_p{\mu_p}' - 2{\mu_p}'\mu_C)$$
$$d\sigma_p = \mu_p{\mu_p}' - {\mu_p}'\mu_C$$
$$\mu_p' = \frac{d\sigma_p}{\mu_p - \mu_C} . \tag{17}$$

From $V_p = Z_\theta\sigma_p - \mu_p$, the slope of VaR line is:

$$0 = Z_\theta - {\mu_p}'$$
$$\mu_p' = Z_\theta. \tag{18}$$

Therefore, the point that has lowest value of VaR is at:

$$
\begin{aligned}
Z_\theta &= \frac{d\sigma_p}{\mu_p - \mu_C} \\
\sigma_p &= \frac{Z_\theta(\mu_p - \mu_C)}{d}.
\end{aligned}
\tag{19}
$$

Substitute the value of σ_p from equation (19) to equation (16), we get:

$$
\begin{aligned}
\left(\frac{Z_\theta(\mu_p - \mu_C)}{d}\right)^2 &= \frac{1}{d}(\mu_p - \mu_C)^2 + \sigma_C^2 \\
Z_\theta^2(\mu_p - \mu_C)^2 &= d(\mu_p - \mu_C)^2 + d^2\sigma_C^2 \\
\mu_p - \mu_C &= \pm\sqrt{\frac{d^2\sigma_C^2}{Z_\theta^2 - d}} \\
\mu_p &= \mu_C \pm \frac{d\sigma_C}{\sqrt{Z_\theta^2 - d}}.
\end{aligned}
\tag{20}
$$

Since the efficient frontier is the line above minimum variance portfolio (C) on MVF, then we will consider only:

$$
\mu_p = \mu_C + \frac{d\sigma_C}{\sqrt{Z_\theta^2 - d}},
\tag{21}
$$

for $\dfrac{d\sigma_C}{\sqrt{Z_\theta^2 - d}}$ which should have a positive value. So, substituting μ_p into equation (19),

we get:

$$
\begin{aligned}
\sigma_p &= \frac{Z_\theta\left(\mu_C + \frac{d\sigma_C}{\sqrt{Z_\theta^2 - d}} - \mu_C\right)}{d} \\
\sigma_p &= \frac{Z_\theta\sigma_C}{\sqrt{Z_\theta^2 - d}}
\end{aligned}
\tag{22}
$$

Therefore, the minimum Value at Risk is:

$$
\begin{aligned}
V^{min} &= \frac{Z_\theta^2\sigma_C}{\sqrt{Z_\theta^2 - d}} - \mu_C - \frac{d\sigma_C}{\sqrt{Z_\theta^2 - d}} \\
V^{min} &= \frac{(Z_\theta^2 - d)\sigma_C}{\sqrt{Z_\theta^2 - d}} - \mu_C \\
V^{min} &= \sigma_C\sqrt{Z_\theta^2 - d} - \mu_C.
\end{aligned}
\tag{23}
$$

3.3 Proposed value of α when benchmark is inefficient

In Riccetti (2016), the upper limit on TE (TE^{max}) is proposed at $\alpha\Delta_2$ where Δ_2 is the value at which portfolio J_2 coincides with C. While $\alpha \geq 1$ when $\Delta_1 \leq 0$, and $\alpha = 1$ when $\Delta_1 > 0$. When $\Delta_1 \leq 0$, the CTF will be on the lower side of the MVF. So, we suggest that the TE should be increased until we can at least invest in the minimum VaR portfolio (as shown in Figure 19). Therefore, we will find $\widehat{TE}$ that make CTF coincide with MVF at point M where point M indicates the minimum VaR portfolio. In this case, our proposed α will provide a value that is more than one (concordant with Riccetti's suggestion), which will increase the efficient portfolio region. Moreover, we can invest in the lowest risk portfolio in terms of VaR.

Figure 19: The upper limit on TE ($\boldsymbol{TE^{max}}$) in case of inefficient benchmark
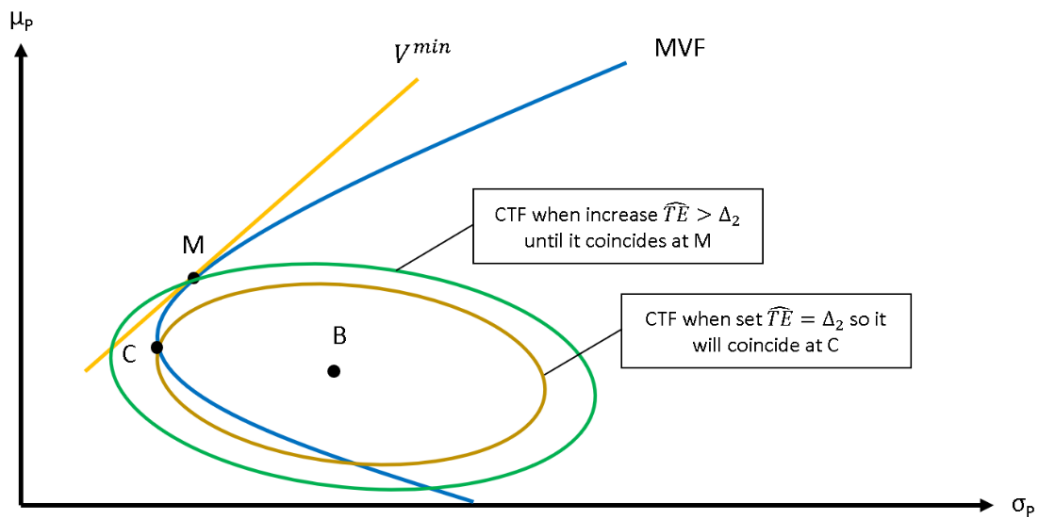
Considering σ_p^2 in the mean-variance efficient frontier (MVF) from equation (16) and put this value into y, we get:

$$y = \frac{1}{d}\left(\mu_p^2 - 2\mu_p\mu_C + \mu_C^2\right) + \sigma_C^2 - \sigma_B^2 - T^2$$
$$y = \frac{1}{d}\left(\mu_p - \mu_C\right)^2 - \Delta_2 - T^2$$
$$y = \frac{1}{d}(z + \Delta_1)^2 - \Delta_2 - T^2. \tag{24}$$

To find the points which the mean-variance efficient frontier (MVF) contacts with the constrained TE frontier (CTF). We substitute y from equation (24) into equation (4), so we get:

$$0 = dy^2 + 4\Delta_2 z^2 - 4\Delta_1 yz - 4T^2(d\Delta_2 - \Delta_1^2)$$

$$0 = d\left[\frac{1}{d}(z+\Delta_1)^2 - \Delta_2 - T^2\right]^2 + 4\Delta_2 z^2 - 4\Delta_1\left[\frac{1}{d}(z+\Delta_1)^2 - \Delta_2 - T^2\right]z -$$
$$4T^2(d\Delta_2 - \Delta_1^2)$$

$$0 = d\left[\frac{1}{d^2}(z+\Delta_1)^4 - \frac{2}{d}(z+\Delta_1)^2(\Delta_2 + T^2) + (\Delta_2 + T^2)^2\right] + 4\Delta_2 z^2 -$$
$$4\Delta_1 z\left[\frac{1}{d}(z^2 + 2z\Delta_1 + \Delta_1^2) - \Delta_2 - T^2\right] - 4T^2 d\Delta_2 + 4T^2\Delta_1^2$$

$$0 = d^2\left[\frac{1}{d^2}(z+\Delta_1)^4 - \frac{2}{d}(z+\Delta_1)^2(\Delta_2 + T^2) + (\Delta_2 + T^2)^2\right] + 4d\Delta_2 z^2 -$$
$$4d\Delta_1 z\left[\frac{1}{d}(z^2 + 2z\Delta_1 + \Delta_1^2) - \Delta_2 - T^2\right] - 4T^2 d^2\Delta_2 + 4T^2 d\Delta_1^2$$

$$0 = [(z^4 + 4z^3\Delta_1 + 6z^2\Delta_1^2 + 4z\Delta_1^3 + \Delta_1^4) - 2d(z^2 + 2z\Delta_1 + \Delta_1^2)(\Delta_2 + T^2) +$$
$$d^2(\Delta_2^2 + 2\Delta_2 T^2 + T^4)] + 4d\Delta_2 z^2 - [4\Delta_1 z^3 + 8z^2\Delta_1^2 + 4\Delta_1^3 z - 4d\Delta_1\Delta_2 z -$$
$$4d\Delta_1 T^2 z] - 4T^2 d^2\Delta_2 + 4T^2 d\Delta_1^2$$

$$0 = [(z^4 + 4z^3\Delta_1 + 6z^2\Delta_1^2 + 4z\Delta_1^3 + \Delta_1^4) - 2d(\Delta_2 z^2 + 2\Delta_2\Delta_1 z + \Delta_2\Delta_1^2 +$$
$$T^2 z^2 + 2T^2\Delta_1 z + T^2\Delta_1^2) + d^2\Delta_2^2 + 2d^2\Delta_2 T^2 + d^2 T^4] + 4d\Delta_2 z^2 -$$
$$4\Delta_1 z^3 - 8z^2\Delta_1^2 - 4\Delta_1^3 z + 4d\Delta_1\Delta_2 z + 4d\Delta_1 T^2 z - 4T^2 d^2\Delta_2 + 4T^2 d\Delta_1^2$$

$$0 = z^4 + 4z^3\Delta_1 + 6z^2\Delta_1^2 + 4z\Delta_1^3 + \Delta_1^4 - 2d\Delta_2 z^2 - 4d\Delta_2\Delta_1 z - 2d\Delta_2\Delta_1^2 -$$
$$2dT^2 z^2 - 4dT^2\Delta_1 z - 2dT^2\Delta_1^2 + d^2\Delta_2^2 + 2d^2\Delta_2 T^2 + d^2 T^4 + 4d\Delta_2 z^2 -$$
$$4\Delta_1 z^3 - 8z^2\Delta_1^2 - 4\Delta_1^3 z + 4d\Delta_1\Delta_2 z + 4d\Delta_1 T^2 z - 4T^2 d^2\Delta_2 + 4T^2 d\Delta_1^2$$

$$0 = z^4 - 2z^2\Delta_1^2 + \Delta_1^4 + 2d\Delta_2 z^2 - 2d\Delta_2\Delta_1^2 - 2dT^2 z^2 + 2dT^2\Delta_1^2 + d^2\Delta_2^2 -$$
$$2d^2\Delta_2 T^2 + d^2 T^4$$

$$0 = z^4 - 2z^2(\Delta_1^2 - d\Delta_2 + dT^2) + (\Delta_1^4 - 2d\Delta_2\Delta_1^2 + 2dT^2\Delta_1^2 + d^2\Delta_2^2 -$$
$$2d^2\Delta_2 T^2 + d^2 T^4)$$

$$0 = z^4 - 2z^2(\Delta_1^2 - d\Delta_2 + dT^2) + (\Delta_1^2 - d\Delta_2 + dT^2)^2$$

$$0 = [z^2 - (\Delta_1^2 - d\Delta_2 + dT^2)]^2 . \tag{25}$$

From the explanation is provided by (Palomba, 2008). there are three cases for solutions in equation (25):

1) No solution: when $\Delta_1^2 - d\Delta_2 + dT^2 < 0$ or T is very small, the CTF will have no contact point with the MVF.

2) 1 solution: when $\Delta_1^2 - d\Delta_2 + dT^2 = 0$ or $T^2 = \Delta_2 - \frac{\Delta_1^2}{d}$, the CTF will have a contact point with the MVF when $z = 0$ or $\mu_p = \mu_B$ which means the first contact occurs on the horizontal from the benchmark.

3) 2 solutions: when $\Delta_1^2 - d\Delta_2 + dT^2 > 0$ or T^2 is large enough, the CTF will have two contact points with the MVF when:

$$\begin{aligned}
z &= \pm\sqrt{\Delta_1^2 - d\Delta_2 + dT^2} \\
\mu_p - \mu_B &= \pm\sqrt{\Delta_1^2 - d\Delta_2 + dT^2} \\
\mu_p &= \mu_B \pm \sqrt{\Delta_1^2 - d\Delta_2 + dT^2}.
\end{aligned} \tag{26}$$

The ellipse of the CTF will coincide with the MVF at the minimum variance portfolio (C) as shown in Figure 7 when $\mu_p = \mu_C$ or:

$$
\begin{aligned}
\mu_C &= \mu_B \pm \sqrt{\Delta_1^2 - d\Delta_2 + dT^2} \\
\mu_C - \mu_B &= \pm\sqrt{\Delta_1^2 - d\Delta_2 + dT^2} \\
(-\Delta_1)^2 &= \Delta_1^2 - d\Delta_2 + dT^2 \\
dT^2 &= d\Delta_2 \\
T^2 &= \Delta_2 .
\end{aligned}
\tag{27}
$$

To find the value of TE which makes the CTF coincide with the MVF at portfolio M, we substitute $\mu_p = \mu_C + \dfrac{d\sigma_C}{\sqrt{Z_\theta^2 - d}}$ from (20) into equation (26). Since we consider only on the upper part on efficient frontier, we get:

$$
\begin{aligned}
\mu_C + \frac{d\sigma_C}{\sqrt{Z_\theta^2 - d}} &= \mu_B + \sqrt{\Delta_1^2 - d\Delta_2 + dT^2} \\
\frac{d\sigma_C}{\sqrt{Z_\theta^2 - d}} - \Delta_1 &= \sqrt{\Delta_1^2 - d\Delta_2 + dT^2} \\
\frac{d^2\sigma_C^2}{Z_\theta^2 - d} - \frac{2d\sigma_C\Delta_1}{\sqrt{Z_\theta^2 - d}} + \Delta_1^2 &= \Delta_1^2 - d\Delta_2 + dT^2 \\
\frac{d^2\sigma_C^2}{Z_\theta^2 - d} - \frac{2d\sigma_C\Delta_1}{\sqrt{Z_\theta^2 - d}} + d\Delta_2 &= dT^2 \\
T^2 &= \frac{d\sigma_C^2}{Z_\theta^2 - d} - \frac{2\sigma_C\Delta_1}{\sqrt{Z_\theta^2 - d}} + \Delta_2 .
\end{aligned}
\tag{28}
$$

Therefore, in the case of the inefficient benchmark, α from Riccetti (2016) should have the value proportional to Δ_2, so we get:

$$
\begin{aligned}
\alpha &= \frac{T^2}{\Delta_2} \\
&= \frac{d\sigma_C^2}{\Delta_2(Z_\theta^2 - d)} - \frac{2\sigma_C\Delta_1}{\Delta_2\sqrt{Z_\theta^2 - d}} + 1 .
\end{aligned}
\tag{29}
$$

By an assumption that

1. The benchmark portfolio (B) is inside an efficient frontier (MVF), so Δ_2 is positive.
2. To get value of α to be a real number, $Z_\theta^2 - d$ must be positive.
3. Since we consider the case of the inefficient benchmark, Δ_1 is negative.
4. By assuming the CTF to be an ellipse, d must be positive (described in Appendix). And σ_C is standard deviation of the minimum variance portfolio, which is positive.

If we consider $\frac{d\sigma_C^2 - 2\sigma_C\Delta_1\sqrt{z_\theta^2 - d}}{\Delta_2(z_\theta^2 - d)}$, it will have a positive value. Therefore, α will have a value of more than 1 when the benchmark is inefficient ($\mu_B - \mu_C < 0$ or $\Delta_1 < 0$), which is concordant with Riccetti (2016).

Section 4: Determining the Tracking Error ($\widehat{TE}_A$) and VaR ($\widehat{V}_A$) of an active portfolio when combined with a passive portfolio

Considering portfolios in (σ_p, μ_p) space, as shown in Figure 20, the relationship between expected return and volatility of the mixed portfolios will be a straight line when the correlation of two portfolios equal to 1. the shape of the mixed portfolios will be like the dashed line when two portfolios have correlation less than 1. If we compare those two lines with the same expected return, the dashed line will have variance less than the straight line. This relationship also applies to VaR. So, we will first consider the case correlation equal to 1, which causes a VaR of a mixed portfolio to be highest. Let portfolio A is the active portfolio with the highest VaR given a value of Tracking error (TE_A) and portfolio B is a passive portfolio which has expected return and variance equal to the benchmark.

Figure 20: Mixed portfolio with correlation 1 and less than 1
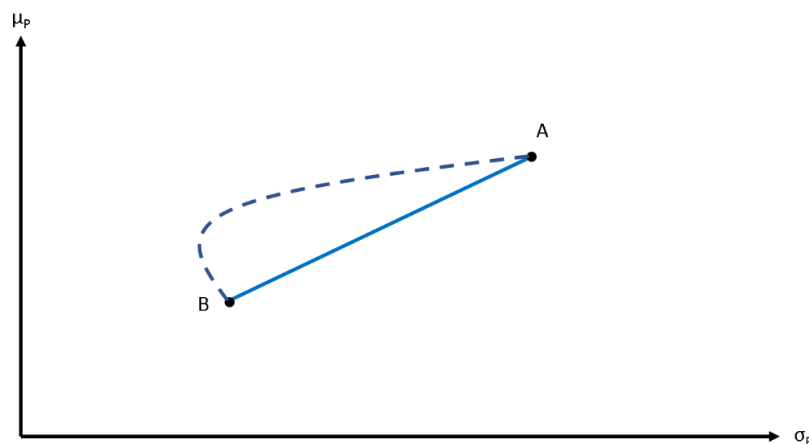
4.1 Case $V_B < V_G$

In the case $V_B < V_G$, the TE limit and VaR limit of an active portfolio will be proposed as the highest value that will not make any part of the portfolio region breach the given VaR of the entire portfolio. Setting the active portfolio's TE limit and VaR limit beyond the proposed values will introduce the chance that the entire portfolio's VaR breaches the VaR limit (V_G). Suppose we invest in portfolio A with weight W_A and portfolio B (Benchmark) with weight $(1 - W_A)$, the VaR of the mixed portfolio (V_{pMix}) will be equal to V_G. This portfolio A is the portfolio with the highest VaR on the CTF from setting TE limit equal to TE_A. With this value of TE, we can choose any portfolio in CTF which will not cause the VaR of the mixed portfolio greater than V_G (as shown in Figure 21). So, TE_A is the upper limit on TE given V_A (called TE_A^{max}).

Figure 21: An active portfolio which has the highest value of VaR with the given TE in case of ρ=1
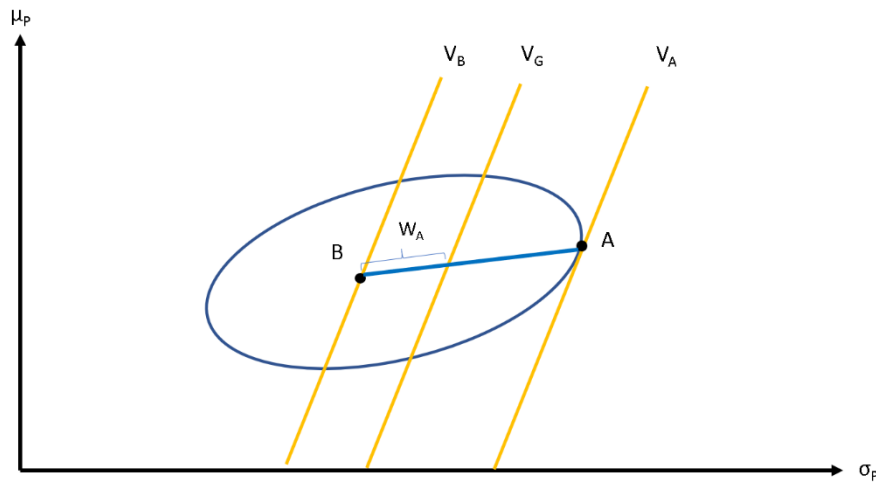
Assuming correlation between A and B (ρ) equal to 1, the VaR of the mixed portfolio can be computed from weighted average VaR of the two portfolios. So, it can be expressed by:

$$V_{pMix} = W_A V_A + (1 - W_A)V_B.$$ (30)

With the given V_G and W_A, we want to find a mixed portfolio that $V_{pMix} = V_G$. Since we know V_B, we will solve equation (30) for V_A. So, we get:

$$V_A = \frac{(V_G - (1 - W_A)V_B)}{W_A}.$$ (31)

We can find TE that tangent with V_A as the same approach as explained in section 3.1. Start from assigning a small number of TE, we can find a pair of z and y that gives the highest value of VaR. We will increase the value of TE until it satisfies

$$V_A - Z_\theta\sqrt{y + \sigma_B^2 + T} + z + \mu_B = 0. \tag{32}$$

The TE we get from equation (32) should be greater than $\frac{com^2}{d}$ as Riccetti's suggestion for an active manager to get more return than the commission he pays. Supposing we set the TE limit at $\frac{com^2}{d}$, the VaR limit should be more than the minimum VaR obtained from the TE limit at $\frac{com^2}{d}$ ($V_{TE=com^2/d}^{min}$). So, $V_{TE=com^2/d}^{min}$ will be represented as the lower limit on VaR (V_A^{min}). The calculation of $V_{TE=com^2/d}^{min}$ can be adapted from the instructions in section 3.1. In sum, the set of limits of an active portfolio in case $V_B < V_G$ are summarized in Table 5 below.

Table 5: *Set of limits of an active portfolio in case* $\boldsymbol{V_B < V_G}$

Set of limits	Lower limit	Upper limit
VaR	$V_A^{min} = V_{TE=com^2/d}^{min}$	$V_A^{max} = V_A$
TE	$TE_A^{min} = \frac{com^2}{d}$	$TE_A^{max} = TE_A$

Figure 22: *An active portfolio which has the highest value of VaR with the given TE in case of ρ<1*
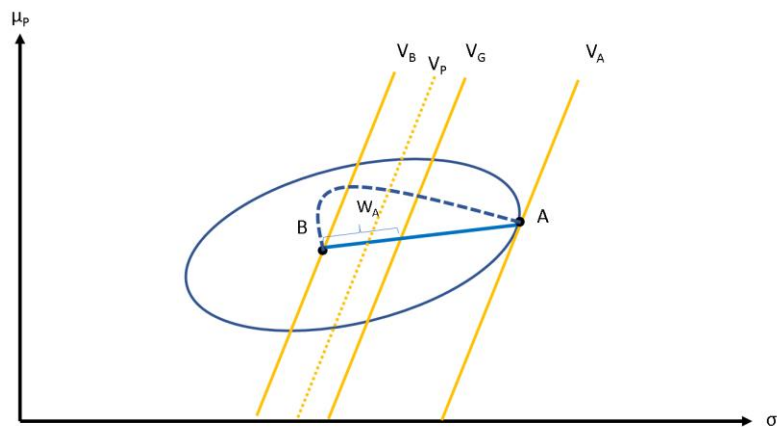
However, the correlation between A and B is not equal to one in reality. With the same parameters such as W_A, V_A, V_B, etc. except $\rho < 1$, this situation will make VaR of an entire portfolio (V_P) less than V_G (as seen in Figure 22 above). Therefore, we are

still able to increase V_A until V_{pMix} equal to V_G. If we relax a correlation assumption, a VaR of an entire portfolio can be express by:

$$
\begin{aligned}
V_{pMix} &= z_\theta \sigma_{pMix} - \mu_{pMix} \\
&= z_\theta \sqrt{W_A^2 \sigma_A^2 + (1 - W_A)^2 \sigma_B^2 + 2\rho W_A W_B \sigma_A \sigma_B} - (W_A \mu_A + \\
&\quad (1 - W_A)\mu_B).
\end{aligned}
\tag{33}
$$

Given a VaR of an entire portfolio equal to V_G, we will get:

$$
\begin{aligned}
V_G &= z_\theta \sqrt{W_A^2 \sigma_A^2 + (1 - W_A)^2 \sigma_B^2 + 2\rho W_A(1 - W_A)\sigma_A \sigma_B} - \\
&\quad (W_A \mu_A + (1 - W_A)\mu_B) \\
V_G + (W_A \mu_A + (1 - W_A)\mu_B) &= z_\theta \sqrt{W_A^2 \sigma_A^2 + (1 - W_A)^2 \sigma_B^2 + 2\rho W_A(1 - W_A)\sigma_A \sigma_B} \\
(V_G + W_A \mu_A + (1 - W_A)\mu_B)^2 &= z_\theta^2(W_A^2 \sigma_A^2 + (1 - W_A)^2 \sigma_B^2 + 2\rho W_A(1 - W_A)\sigma_A \sigma_B) \\
(V_G + (1 - W_A)\mu_B)^2 + 2(V_G + &= z_\theta^2 W_A^2 \sigma_A^2 + z_\theta^2(1 - W_A)^2 \sigma_B^2 + 2z_\theta^2 \rho W_A(1 - W_A)\sigma_A \sigma_B \\
(1 - W_A)\mu_B)W_A \mu_A + W_A^2 \mu_A^2 & \\
(V_G + (1 - W_A)\mu_B)^2 &= z_\theta^2 W_A^2 \sigma_A^2 + 2z_\theta^2 \rho W_A(1 - W_A)\sigma_A \sigma_B - 2(V_G + \\
&\quad (1 - W_A)\mu_B)W_A \mu_A - W_A^2 \mu_A^2 + z_\theta^2(1 - W_A)^2 \sigma_B^2 .
\end{aligned}
\tag{34}
$$

Consider $V_A = z_\theta \sigma_A - \mu_A$, so $\sigma_A = \frac{V_A + \mu_A}{z_\theta}$. Therefore, we get:

$$
\begin{aligned}
(V_G + (1 - W_A)\mu_B)^2 &= W_A^2(V_A + \mu_A)^2 + 2z_\theta \rho W_A(1 - W_A)(V_A + \mu_A)\sigma_B - \\
&\quad 2(V_G + (1 - W_A)\mu_B)W_A \mu_A - W_A^2 \mu_A^2 + z_\theta^2(1 - W_A)^2 \sigma_B^2 \\
(V_G + (1 - W_A)\mu_B)^2 &= W_A^2(V_A^2 + 2V_A \mu_A) + 2z_\theta \rho W_A(1 - W_A)(V_A + \mu_A)\sigma_B - \\
&\quad 2(V_G + (1 - W_A)\mu_B)W_A \mu_A + z_\theta^2(1 - W_A)^2 \sigma_B^2 \\
0 &= W_A^2(V_A^2 + 2V_A \mu_A) + 2z_\theta \rho W_A(1 - W_A)(V_A + \mu_A)\sigma_B - \\
&\quad 2(V_G + (1 - W_A)\mu_B)W_A \mu_A + z_\theta^2(1 - W_A)^2 \sigma_B^2 - (V_G + \\
&\quad (1 - W_A)\mu_B)^2.
\end{aligned}
\tag{35}
$$

Equation (35) will leave us two unknown parameters (i.e. V_A and μ_A). From the main objective is to find a TE limit in equation (32) with the given ρ, we can solve equations (13), (14), (32), (35) by following this:

1. Assume a number of V_A which should be more than V_A in case of $\rho = 1$.
2. Find TE that tangent with V_A as the same approach as explained in section 3.1 so that the TE limit we get is satisfied equation (32).
3. With the TE limit we find from step 2, we can also find z from equation (13) which leads us to know μ_A.

4. If the assumed V_A in step 1 and μ_A from step 3 does not satisfy equation (35), we will change a number of V_A and repeat step 1 – 4 again.

In this paper, with the capability of MATLAB on solving equations, we use the function "vpasolve" to solve equations (13), (14), (32), (35) simultaneously. Then we get parameters y, z, T^2, and V_A (an illustration will be shown in section 5). Therefore, T^2 and V_A will be TE limit and VaR limit on an active portfolio, respectively.

4.2 Case $V_B \geq V_G$

In this case, since $V_B \geq V_G$, there are always some parts of CTF exceeding the given entire portfolio's VaR limit regardless of the TE limit we set. Unlike the $V_B < V_G$ case, we cannot find a TE limit that makes all portfolio in the feasible region satisfy the entire portfolio's VaR limit. Therefore, we will propose the active portfolio's TE limit and VaR limit that makes the feasible region contain as many efficient portfolios as possible while ruling out the portfolios that are inefficient or violate the overall VaR limit.

Figure 23: An active portfolio which has the lowest value of VaR with the given TE in case of ρ=1
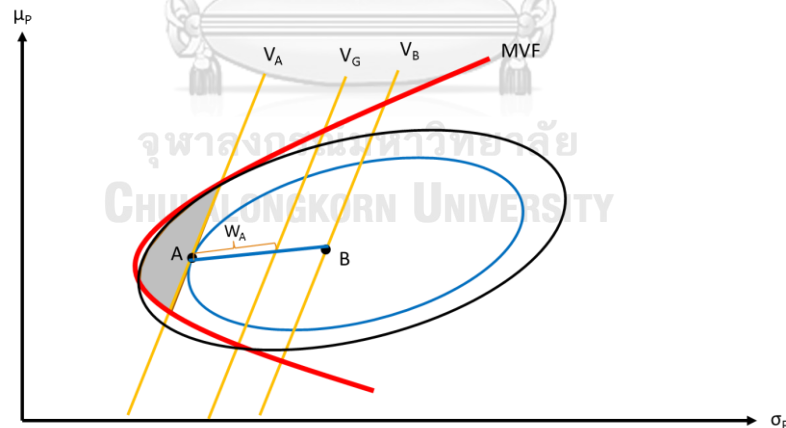
To give a clear explanation in the case of $V_B > V_G$, the illustration is shown in case of $\rho = 1$ as Figure 23 above. We can find a VaR limit and a TE limit of an active portfolio as the same method described in section 4.1. Even though, the VaR limit in this case is the upper limit on VaR (V_A^{max}) as same as the case of $V_B < V_G$, the TE whose CTF tangent with V_A on the right side is the lower limit on TE (TE_A^{min}). The V_A^{max} should not be more than V_A computed from the four equations (described in section 4.1) because some portfolios can breach the overall VaR limit. The TE_A^{min}

should not be less than TE_A computed from the four equations because there will be no feasible region. However, the TE_A^{min} still should be more than $\frac{com^2}{d}$ which will allow investors to get expected return more than the commission they pay. For the upper limit on TE (TE_A^{max}), we can set it as $\alpha\Delta_2$ as Riccetti proposed because it provides the most efficient portfolios while ruling out the inefficient portfolios as much as possible. Setting the TE limit more than $\alpha\Delta_2$ would only increase a few efficient portfolio regions while increasing a lot of inefficient portfolio regions. For the lower limit on VaR (V_A^{min}), we can set it as V^{min} as we proposed in section 3.2. Setting the VaR limit less than V^{min} will cause no feasible region. Therefore, the set of limits of an active portfolio in case $V_B \geq V_G$ are summarized in Table 6. Although the VaR limit can be set as Table 6, there may be the case where $V_A < V^{min}$ which is impossible to set the VaR limit equal to V_A. Therefore, in the case of $V_B > V_G$, since we rely on an active portfolio to reduce VaR of an entire portfolio, we can increase the weight of an active portion (or allow V_G to be increase) to prevent $V_A < V^{min}$ which give us empty feasible region.

Table 6: *Set of limits of an active portfolio in case $V_B \geq V_G$*

Set of limits	Lower limit	Upper limit
VaR	$V_A^{min} = V^{min}$	$V_A^{max} = V_A$
TE	$\frac{com^2}{d} \leq TE_A^{min} = TE_A$	$TE_A^{max} = \alpha\Delta_2$

The proposed active portfolio's limits given the entire portfolio VaR are summarized in Table 7 below.

Table 7: *Summary of the proposed active portfolio's limits given the entire portfolio VaR*

Limit		Case	
		$V_B < V_G$	$V_B \geq V_G$
TE	Lower	$TE_A^{min} = \frac{com^2}{d}$	$\frac{com^2}{d} \leq TE_A^{min} = TE_A$
	Upper	$TE_A^{max} = TE_A$	$TE_A^{max} = \alpha\Delta_2$
VaR	Lower	$V_A^{min} = V_{TE=com^2/d}^{min}$	$V_A^{min} = V^{min}$
	Upper	$V_A^{max} = V_A$	$V_A^{max} = V_A$

Section 5: Empirical experiment: Practical issues in implementing the proposed limits

In this section, we will use real data to determine the TE limit of the active portfolio using the method from the previous section. There will be many portfolios we can choose from within this TE limit. To reflect a real-life situation, we will choose the maximum expected return portfolio as an asset manager is likely to select this portfolio. Using a numerical optimizer to find the manger's decision, we obtain a portfolio on the CTF and the portfolio's parameters and ex-ante risk measures (such as weights, expected return, variance, VaR, and TE). These parameters and ex-ante risk measures will be the same as those calculated using the method from the previous section (i.e., "theoretical value"), provided that the following assumptions hold: (1) stocks that we can invest are as same as stocks in a benchmark (2) no restriction on investment (such as stocks can be short-sold).

During the data period that we use to compute the TE limit (i.e., the "in-sample" period), investing in any active portfolio within this TE limit still makes the entire portfolio's ex-ante VaR be within the given value (V_G). However, in reality, stocks in an investment universe can be different from a benchmark, and short-selling is hard to implement, so the expected return, variance, VaR, and TE of the CTF portfolios might not be the same as the theoretical values. The conditions on the different stock universe and short-selling may cause CTF to change from what we derived in the previous section. Choosing portfolios based on a wrong CTF may make the entire portfolio's VaR exceed the given limit. Furthermore, as time pass, parameters that are used to determine a TE limit can be changed. Moreover, VaR and TE used to derive a TE limit from the analysis part are "ex-ante," but calculation methods of VaR and TE for monitoring are "ex-post." Therefore, even we invest in an active portfolio whose ex-ante TE and VaR are within the limit at first, but later on, the ex-post TE or VaR of the active portfolio and VaR of the mixed portfolio can be far from the given value. So, we may need to re-estimate parameters and re-balance the portfolio to ensure that the entire portfolio's VaR will not exceed the limit.

To observe the problems of determining an active portfolio's TE limit, we will try to imitate investment management in Thailand in this section. We will use SET as a benchmark. Moreover, since ex-post TE and ex-post VaR are commonly used for calculating TE and VaR, we will use them to compute TE and VaR for this section as well. With the assumptions given in Table 8. This section will illustrate the issues on:

1. What are the effects of having stocks in an investment universe different from stocks in a benchmark?
2. What are the effects of prohibiting short-sale compare with allowing short-sale?
3. Are yearly re-estimating parameters and re-balancing the portfolio enough to help us control an entire portfolio's VaR to be within a given limit?

Table 8: *Assumptions*

List	Detail
Stock universe	select 20 stocks from SET
Benchmark	SET index
Calculation of return	weekly return
Training period	1 years from 1 January 2014 – 31 December 2014
Testing period	5 years from 1 January 2015 – 31 December 2019
Given VaR(V_G)	V_B + 2% (calculated at 1 January 2015)
Correlation between an active and a passive portfolio	50%
Commission fee (*com*)	1.5% yearly
Given active portfolio weight (W_A)	40%
VaR confidence level	95%
Tracking error and VaR calculation method	- Ex-ante method when we determine a TE limit and a VaR limit in the in-sample period - Ex-post method when we monitor them in the out-sample period

5.1 The effects of having stocks in an investment universe different from stocks in a benchmark

Regarding the assumption in Table 8 above, we will use the SET index as a benchmark. However, to compare the effect of having a small number of stocks in an investment universe, we add an equal weight benchmark as another benchmark. To find an active portfolio's stock weights, we need to determine the TE limit and the VaR limit of an active portfolio. Then, we can find stocks' weights of the active portfolio using numerical search given the TE and VaR limit. In the empirical test, we will use MATLAB as a calculator. As we choose to invest in the maximum expected return

portfolio as an active portfolio, the portfolio's risk measures such as expected return, variance, VaR, and TE should be the same as theoretical values of portfolio J_1 (see the statistic calculation on Portfolio J_1 in Table 2). If we use a benchmark that consists of the same stocks as in the investment universe (in this case: equal weight benchmark), the weight of an active portfolio also can be calculated from equation (6). However, in the case of SET index as a benchmark, we do not have enough data on all stocks in SET index, so we cannot find weight of each stock in the benchmark. Moreover, investing in stocks in a benchmark is limited due to the small stock universe. Therefore, we cannot calculate the weight of an active portfolio using equation (6). So, we can only find stocks' weights of an active portfolio through numerical search.

In sum, this part will be divided into two cases: (1) equal weight benchmark and (2) SET index as a benchmark. We will compare statistics such as weight, expected return, variance, VaR, and TE with their theoretical values in each case. The first case will show the capability of MATLAB that can use numerical search to get the active portfolio's weight as the same value as calculated by using the method from section 2, and its parameters and risk measures have the same value as the theoretical portfolio. In contrast, the second case will show the effect of having a stock universe difference than a benchmark. A methodology of finding the weight of an active portfolio is proceeded as follows:

1. Determine an investment universe by selecting 20 stocks from SET and calculate weekly return from in-sample period price data on both an investment universe and a benchmark.
2. Compute expected returns and variance-covariance matrix of asset returns, then we can compute expected returns and variance of minimum variance portfolio and a benchmark portfolio. The calculation is described in equation (1) and (2).
3. With parameters calculated from step (2), we will determine an active portfolio's TE limit and VaR limit by finding 4 parameters: y, z, a TE limit ($\widehat{TE}$), and an active portfolio's VaR limit ($\widehat{V_A}$). We will solve four equations with "vpasolve" in MATLAB which are consists of equation (13), (14), (32), and (35):
   - equation (13) is used to find a portfolio that have the maximum VaR given TE limit or at the tangent point of a VaR line and an ellipse.
   - equation (14) is an ellipse equation which represented in the CTF in (σ^2, μ) space.

- equation (32) is used to find TE that makes VaR of the maximum VaR portfolio given TE limit equal to VaR of an active portfolio.
- equation (35) is used to find an active portfolio' VaR. It can be computed without involving the other three equations (13,14,32), if we assume a correlation of an active and a passive portfolio equal to one. So, the VaR of an active portfolio can be computed from equation (31). However, as we assume a correlation of an active and a passive portfolio to be less than one, we need parameter μ_A from the other three equations.

4. We will find the weights of the active portfolio by using "fmincon"[1] in MATLAB. Since we want to find weights of a maximum expected return, we set an objective function by minimizing $-w'E$ subject to tracking error calculated from step 3.

Table 9: portfolio weight and its statistics in case of equal weight benchmark

Stock Weights (1-20)										Mean*	Volatility*	VaR*	TE*
TE and VaR limit prescribed by Section 4												9.332%	3.018%
Theoretical active portfolio that maximizes expected return													
28.76%	68.58%	35.98%	37.36%	28.62%	70.84%	13.56%	7.18%	-53.23%	17.62%	3.981%	3.873%	2.390%	3.018%
-50.67%	39.09%	32.25%	18.52%	7.91%	-28.29%	-26.09%	-96.56%	-23.27%	-28.14%				
Value of numerical search from MATLAB													
28.76%	68.58%	35.98%	37.36%	28.62%	70.83%	13.56%	7.18%	-53.23%	17.62%	3.981%	3.873%	2.390%	3.018%
-50.67%	39.09%	32.25%	18.52%	7.92%	-28.30%	-26.09%	-96.56%	-23.26%	-28.15%				

* Note that mean, volatility, VaR, and TE are computed from weekly return

As a result shown in Table 9 above, stock weights of the maximum expected return portfolio found by MATLAB's numerical search are equal to the theoretical value (can be calculated from equation (6 Portfolio statistics, including mean, variance, VaR, and TE, also have the same value as portfolio J_1 as characterized by Riccetti (2016) (see Table 1). It shows that MATLAB's numerical search is capable of finding the true value of stock weights of the maximum expected return portfolio given TE limit, which is the weights of a portfolio on the CTF (i.e., "the optimal weights"). So, the portfolio's parameters and the risk measures such as expected return, variance, VaR, and TE are equal to the theoretical value of portfolio J_1.

From the CTF's derivation, an optimal portfolio is a portfolio constructed from an optimal weight that is derived from maximizing expected return with respect to the portfolio's TE, portfolio's variance, and the sum of different weights from the

---

[1] "fmincon" in MATLAB is commonly used to find parameters that minimize a value of a function subject to non-linear constrain.

benchmark equal to zero (see equation 3). So, optimal portfolios are on the CTF. However, we cannot conclude that all portfolios on the CTF are the optimal portfolios. There are many portfolios constructed from different sets of weights having the same mean and variance as the optimal portfolio. However, their TEs are higher than the TE of the optimal portfolio (or the TE limit). From the fact that the more TE limit, the more area of the CTF expanded from the benchmark portfolio. We can conclude that a higher TE portfolio than the TE limit must be on or outside of the CTF. Therefore, we can imply that any portfolio that has TE less than the TE limit (regardless of being an optimal or un-optimal portfolio) must be inside the CTF.

The result also proves that the risk measures of the portfolio found by MATLAB, such as expected return, variance, VaR, and TE, follows the ellipse equation (see equation 4). Therefore, as long as optimal portfolios' statistic follows the points on CTF, if we set a TE limit as proposed in section 4, it can be guaranteed that an active portfolio which has TE less than or equal to the proposed limit will have VaR less than or equal to the active portfolio's VaR limit ($\widehat{V_A}$) in the in-sample period. As a result, the entire portfolio's VaR will also be less than or equal to the given value.

Table 10: *active portfolio weight and its statistics in case of SET index as a benchmark*

Stock Weights (1-20)										Mean*	Volatility*	VaR*	TE*
TE and VaR limit prescribed by Section 4												9.105%	2.889%
Theoretical active portfolio that maximizes expected return **													
N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	3.649%	3.545%	2.181%	2.889%
N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A				
Value of numerical search from MATLAB													
47.32%	63.64%	28.31%	25.81%	26.92%	71.67%	7.78%	5.09%	-54.77%	13.91%	3.493%	3.377%	2.062%	2.889%
-33.11%	37.51%	32.83%	13.25%	8.84%	-29.40%	-30.11%	-92.84%	-18.78%	-23.86%				

* Note that mean, volatility, VaR, and TE are computed from weekly return
** the theoretical active portfolio weights cannot be computed by using equation (6) in the case of SET index as a benchmark. Because equation (6) is derived from the maximization problem (equation (3)) without the constraint on stock universe. Moreover, we do not have enough data on all stocks in the benchmark.

In the case of using SET index as a benchmark, we cannot compute the optimal weight because we do not have enough data on all stocks in SET index, and a number of stocks in the investment universe is also less than that of the benchmark. From the derivation in Appendix A, it requires all stocks in a benchmark to be investable and we need to know weights in each stock that are used to calculate

benchmark. Therefore, we cannot directly calculate optimal portfolio's weights from equation (6). we can still find portfolio's risk measures that include mean, variance, VaR, and TE from the calculation method provided in Table 1 (i.e., "theoretical value"). So, we can compare the result from MATLAB's numerical search with the theoretical value. As shown in Table 10, the parameters such as mean, variance, and VaR of the maximum expected return portfolio found by MATLAB is less than the theoretical values. It can be implied that the maximum expected return portfolio given the TE limit moves to the upper-right side of the theoretical portfolio on (σ^2, μ) space. In the case of SET index as a benchmark, the result in Table 9 considers being better than the equal weight benchmark case in terms of VaR The VaR of the active portfolio in the case of SET index as a benchmark lower than that of the equal weight benchmark case. However, it also proves that portfolios can be moved from theoretical portfolios if an investment universe differs from a benchmark. There may be the case that portfolios on the CTF move to the lower-right side of the theoretical portfolios. So, even we invest in an active portfolio that TE is within the limit, the active portfolio's VaR can be more than the limit. Consequently, the entire portfolio's VaR will be more than the given limit.

However, the entire portfolio's VaR is unlikely to be more than the given limit when an investment universe is different from a benchmark. Because of the three reasons:

(1) There is a large gap to increase the VaR of an active portfolio when we choose to invest in the maximum expected return portfolio (J_1). Because V_{J_1} is far less than V_M (VaR of the maximum VaR portfolio given TE limit)

(2) Suppose a new active portfolio is moved from a theoretical portfolio with any directions with equal probability. There is only 25% chance that a new active portfolio is moved to the lower-right side of from a theoretical portfolio.

(3) After choosing an active portfolio, if the correlation between the active portfolio and the passive portfolio is less than the assumption, the entire portfolio's VaR will be reduced more from a high diversification effect.

Therefore, it is still safe to use the proposed TE limit from section 4 on determining an active portfolio's TE limit, even if stocks in an investment universe are different from a benchmark.

5.2 Effects of prohibit short-sale vs. allowing short-sale

From the previous part, we conclude that we can use the proposed TE limit from section 4 on determining an active portfolio's TE limit when stocks in an investment universe are different from stocks in a benchmark. However, putting more restrictions on finding a portfolio to invest in may make the portfolio's TE and VaR deviate more from theoretical values. Since short-selling is hard to implement in practice, we will impose a short-selling restriction into this part. The test will be divided into two parts: (1) Allow short sell, and (2) Not allow short sell. Then we will observe the chosen active portfolio's statistics (expected return, variance, VaR, and TE) on the first day of the testing period and the movement of the TE and the VaR of the active portfolio and VaR of the entire portfolio for the whole testing period.

Table 11: *active portfolio weight and its statistics in case of Allow/Not allow short sell with SET index as a benchmark*

Stock Weights (1-20)										Mean*	Volatility*	VaR*	TE*
TE and VaR limit prescribed by Section 4												9.105%	0.083%
Theoretical active portfolio that maximizes expected return**													
N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	3.649%	3.545%	2.181%	2.889%
N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A				
Value of the active portfolio from MATLAB's numerical search (Case: Allow short selling)													
47.32%	63.64%	28.31%	25.81%	26.92%	71.67%	7.78%	5.09%	-54.77%	13.91%	3.493%	3.377%	2.062%	2.889%
-33.11%	37.51%	32.83%	13.25%	8.84%	-29.40%	-30.11%	-92.84%	-18.78%	-23.86%				
Value of the active portfolio from MATLAB's numerical search (Case: Not allow short selling)													
0.01%	31.60%	0.01%	4.14%	14.08%	0.02%	22.13%	0.00%	0.00%	5.43%	1.693%	3.418%	3.930%	2.889%
0.01%	1.29%	0.01%	21.22%	0.02%	0.00%	0.00%	0.00%	0.00%	0.00%				

* Note that mean, volatility, VaR, and TE are computed from weekly return
** the theoretical active portfolio weights cannot be computed by using equation (6) in the case of SET index as a benchmark. Because equation (6) is derived from the maximization problem (equation (3)) without the constraint on the stock universe. Moreover, we do not have enough data on all stocks in the benchmark.

As we find a maximum expected return portfolio's weight given the TE limit using MATLAB's numerical search, the result is shown in Table 11 above. For the case where short-selling is prohibited, the active portfolio's expected return is less than the case where short-selling is allowed as we impose a restriction on portfolio weights. Since the active portfolio variance does not change much, a significant decrease in the portfolio's mean makes the portfolio's VaR significantly increase. However, the portfolio's VaR is still very far from the active portfolio's VaR limit. It is safe to invest in this active portfolio in the case where short-selling is not allowed.

We will invest in an active portfolio and a passive portfolio for five years for testing portfolio statistics. The entire portfolio consists of 60% of a passive portfolio and 40% of an active portfolio with the weight as Table 11 on 1 January 2015. Without a re-estimating/re-balancing portfolio, the results of TE and VaR of the active portfolio and VaR of the mix portfolio from 1 January 2015 to 31 December 2019 in the case where short-selling is allowed is shown in Figure 24 and in the case where short-selling is not allowed is shown in Figure 25.

As the result shown in Figures 24 below, if we allow short-selling on an active portfolio, the ex-post TE and the ex-post VaR of the active portfolio are very high in the middle of 2016 to the end of 2017. Consequently, it causes the ex-post VaR of the mix portfolio to exceed the given limit. However, in the case where short-selling is prohibited as shown in Figures 25, all of the ex-post TE, the ex-post VaR of the active portfolio and the ex-post VaR of the mix portfolio are in the limit for all testing period. The portfolio weights become lower leverage than the allowed short-selling case, so the portfolio's returns will be less volatile. Although the portfolio without short-selling causes the risk measures to deviate from the theoretical portfolio (which should increase the chance of the risk measures exceeding the limit), the portfolio's risk measures do not tend to exceed the limit in the out-of-sample period.

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Figure 24: TE, VaR of active portfolio and VaR of mix portfolio in the case where short-selling is allowed
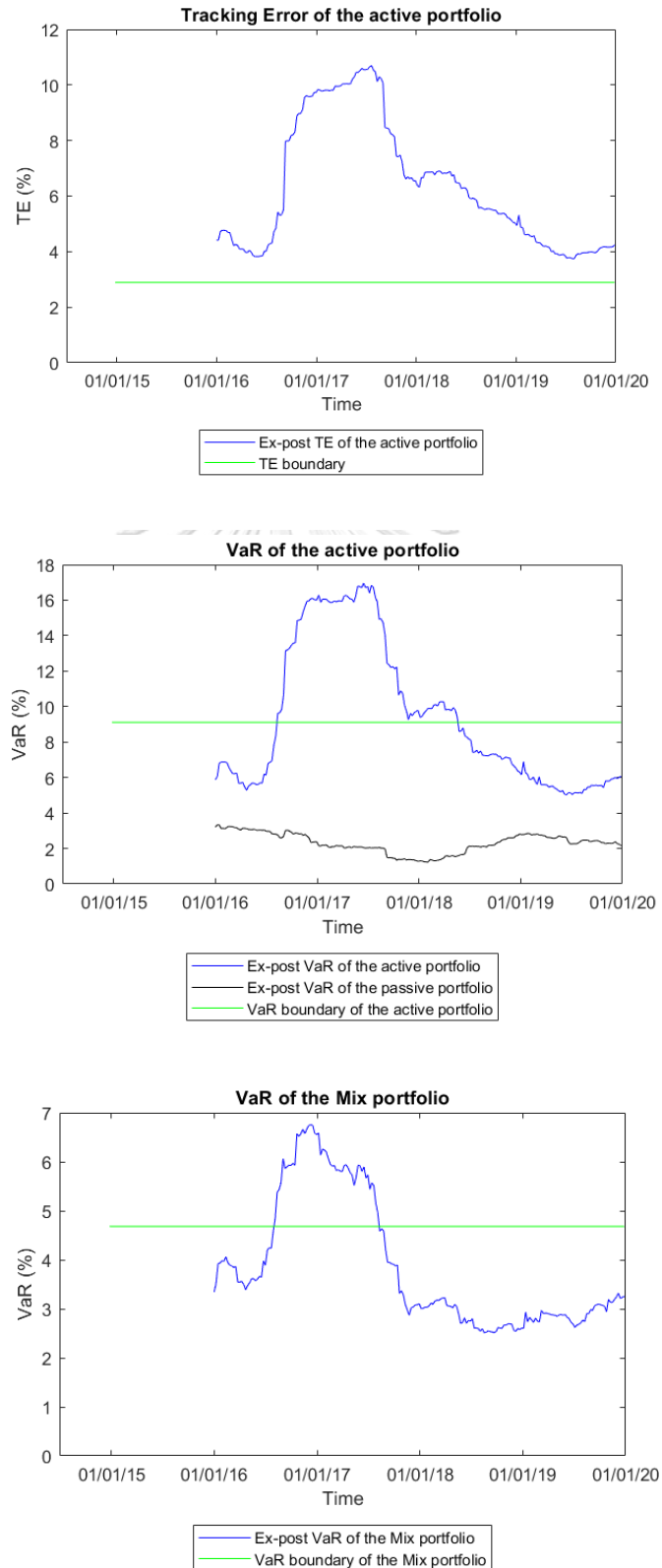
Figure 25: *TE, VaR of active portfolio and VaR of mix portfolio in case of not allow short selling*
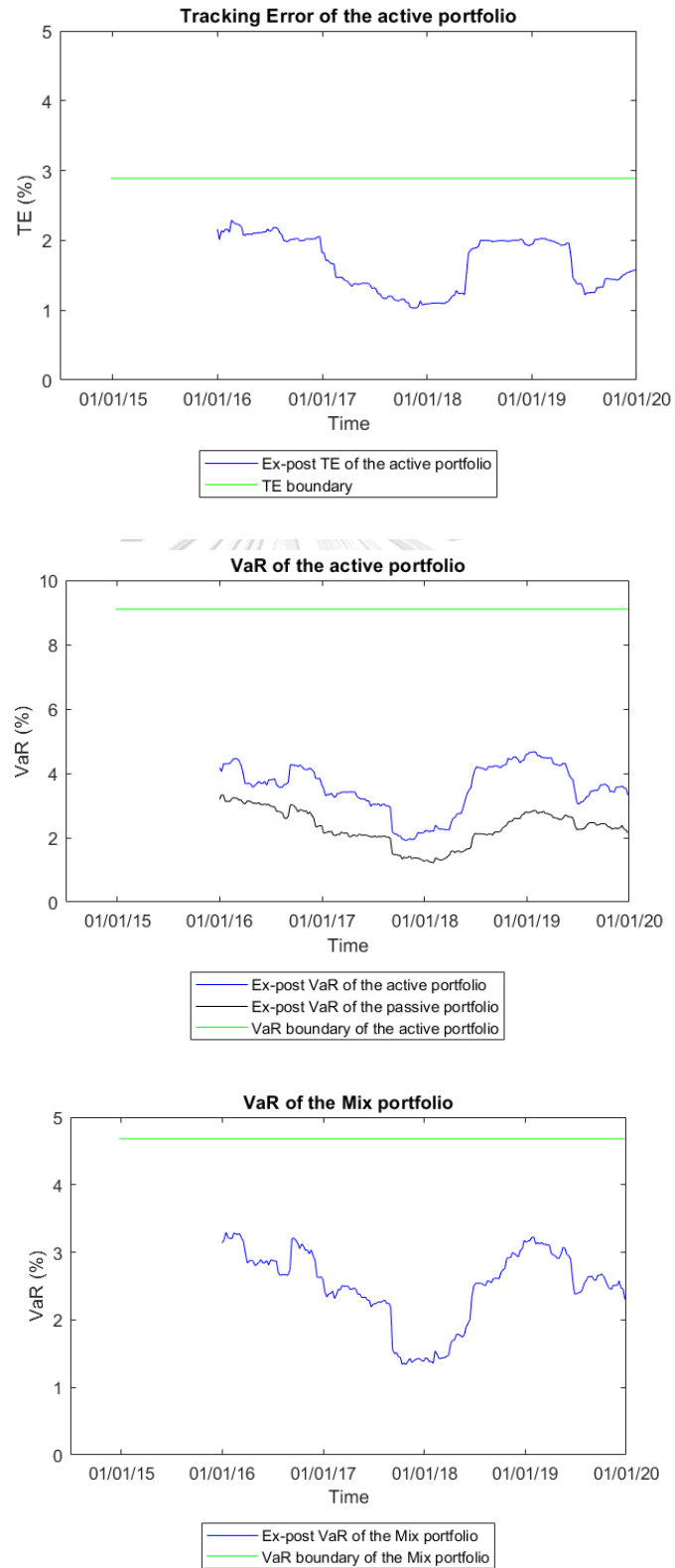
Due to the restriction on short-selling, the portfolio's returns are less volatile. It is safe to set the TE limit to the maximum as we proposed in section 4 while maximizing the opportunity for return. However, if we allow short-selling, the portfolio could be highly leveraged, and the entire portfolio's VaR might breach the given limit. So, without re-estimating/re-balancing portfolios, we recommend setting the TE limit at a lower level than the TE_A^{max}.

From Table 5, we suggest the range of the TE limit on the active portfolio as $[TE_A^{min}, TE_A^{max}] = [\frac{com^2}{d}, TE_A]$. The active portfolio's TE limit can be set as $(1 - \gamma)TE_A^{min} + \gamma TE_A^{max}$ where γ is the weight on TE_A^{max} whose value is between 0 to 1. We will illustrate TE, VaR, NAV of the active portfolio, and VaR of the mix portfolio in the cases of setting γ at 100%, 75%, 50%, 25%, 0% with and without re-estimating/re-balancing portfolio. Setting γ at 100% means that we set TE limit at TE_A^{max}. And Setting γ at 0% means that we set TE limit at TE_A^{min}.

Table 12: *Value of the active portfolio from MATLAB's numerical search in different TE limits*

Stock Weights (1-20)										Mean*	Volatility*	VaR*	TE*
Setting active portfolio's TE limit at 100%of TE^{max}													
47.32%	63.64%	28.31%	25.81%	26.92%	71.67%	7.78%	5.09%	-54.77%	13.91%	**3.493%**	**3.377%**	**2.062%**	**2.889%**
-33.11%	37.51%	32.83%	13.25%	8.84%	-29.40%	-30.11%	-92.84%	-18.78%	-23.86%				
Setting active portfolio's TE limit at 75%of TE^{max}													
44.21%	55.31%	24.26%	21.57%	23.83%	63.05%	6.65%	4.80%	-47.14%	12.25%	**3.035%**	**3.043%**	**1.970%**	**2.502%**
-25.82%	33.05%	29.25%	11.48%	8.45%	-25.04%	-26.04%	-79.53%	-15.08%	-19.51%				
Setting active portfolio's TE limit at 50%of TE^{max}													
40.49%	45.34%	19.40%	16.49%	20.13%	52.73%	5.31%	4.46%	-38.02%	10.27%	**2.488%**	**2.665%**	**1.895%**	**2.043%**
-17.09%	27.70%	24.97%	9.36%	8.00%	-19.81%	-21.16%	-63.61%	-10.65%	-14.32%				
Setting active portfolio's TE limit at 25%of TE^{max}													
35.51%	32.04%	12.91%	9.72%	15.18%	38.95%	3.52%	4.00%	-25.83%	7.63%	**1.758%**	**2.217%**	**1.889%**	**1.445%**
-5.44%	20.57%	19.28%	6.53%	7.39%	-12.85%	-14.65%	-42.36%	-4.74%	-7.38%				
Setting active portfolio's TE limit at minimum value													
25.00%	3.89%	-0.81%	-4.61%	4.73%	9.80%	-0.27%	3.04%	-0.05%	2.04%	**0.212%**	**1.710%**	**2.601%**	**0.549%**
19.21%	5.48%	7.22%	0.54%	6.10%	1.90%	-0.89%	2.60%	7.78%	7.29%				

* Note that mean, volatility, VaR, and TE are computed from weekly return

There will be a trade-off when we increase or decrease the TE limit. Setting the TE limit at the maximum allows us to invest in the higher expected return portfolio, but it also increases the chance of the entire portfolio VaR exceeding the limit. On the other hand, setting the TE limit at the minimum decreases the chance of the entire portfolio VaR exceeding the limit, but it will lower the portfolio's expected return. As shown in Table 12 above, high leverage stock weights become lower when we decrease the TE limit, resulting in a lower portfolio's expected return.

Figure 26: *Comparing TEs of the active portfolios in each level of the TE limit setting* **without** *re-estimating/re-balancing portfolio*
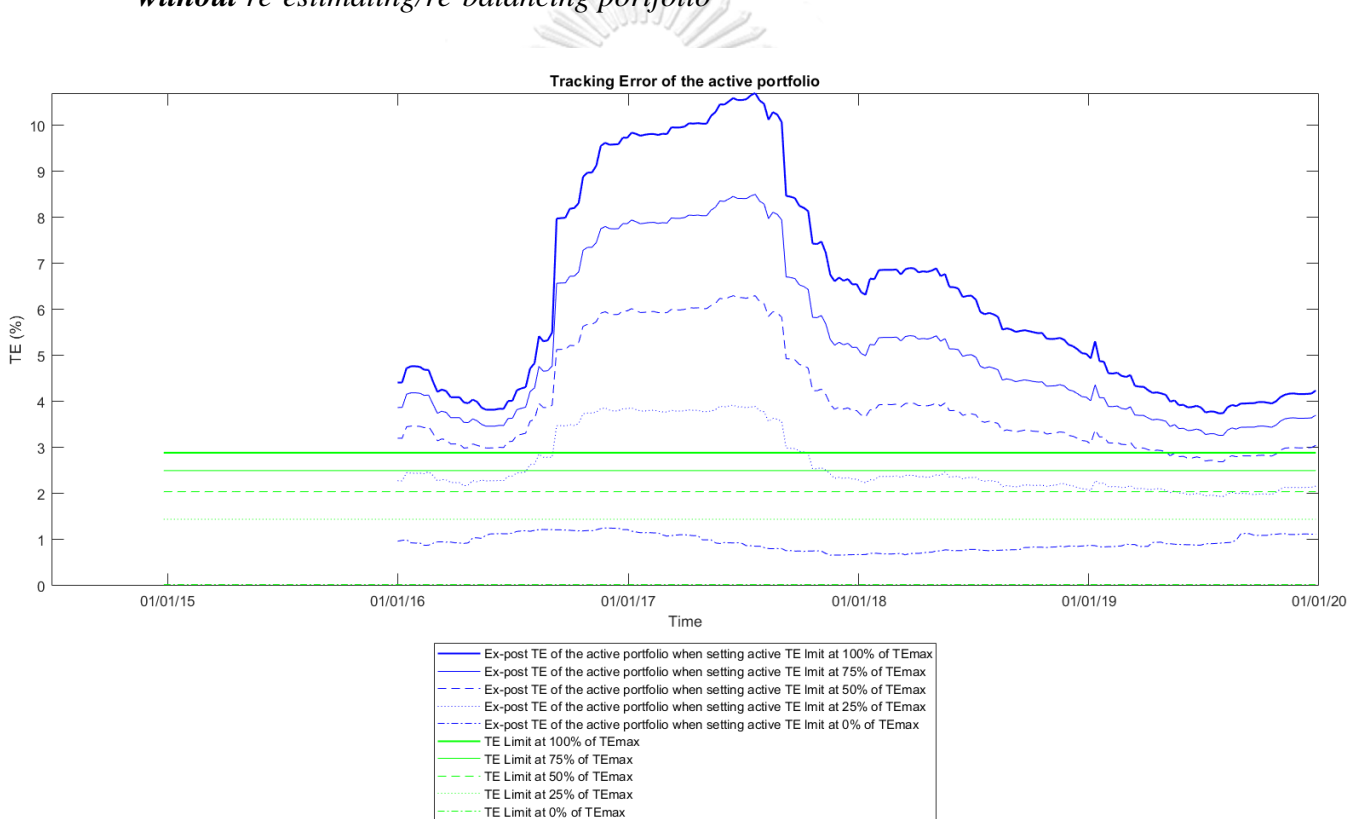
When we invest in the active portfolio with the weight in Table 7, we will observe the ex-post TE, the ex-post VaR, NAV of the active portfolio, and the ex-post VaR of the mix portfolio in the "out-sample" period. As shown in Figure 26 above, without re-estimating/re-balancing the portfolio, the active portfolio's TE becomes lower as we set the TE limit at a lower percentage of γ. However, all of the active portfolio's TE still be higher than the TE limit regardless of the TE limit level. So, it

can imply that if stocks in an active portfolio can be short-sold, it will be hard to control the ex-post active portfolio's TE to be within the limit. (Without re-estimating/ re-balancing a portfolio)

*Figure 27: Comparing VaRs of the active portfolios in each level of the TE limit setting **without** re-estimating/re-balancing portfolio*

As shown in Figure 27 above, when we compare the active portfolios' VaR in each level of the TE limit setting, they are also become lower as we set the TE limit at a lower percentage of $W_{TE_{max}}$. From our stocks' set, the active portfolios' VaR starts to be lower than the VaR limit when setting TE limit at 25% of the TE_A^{max}. If we set TE limit at the very low level such as TE^{min}, the active portfolio's VaR will become very close to the passive portfolio's VaR.
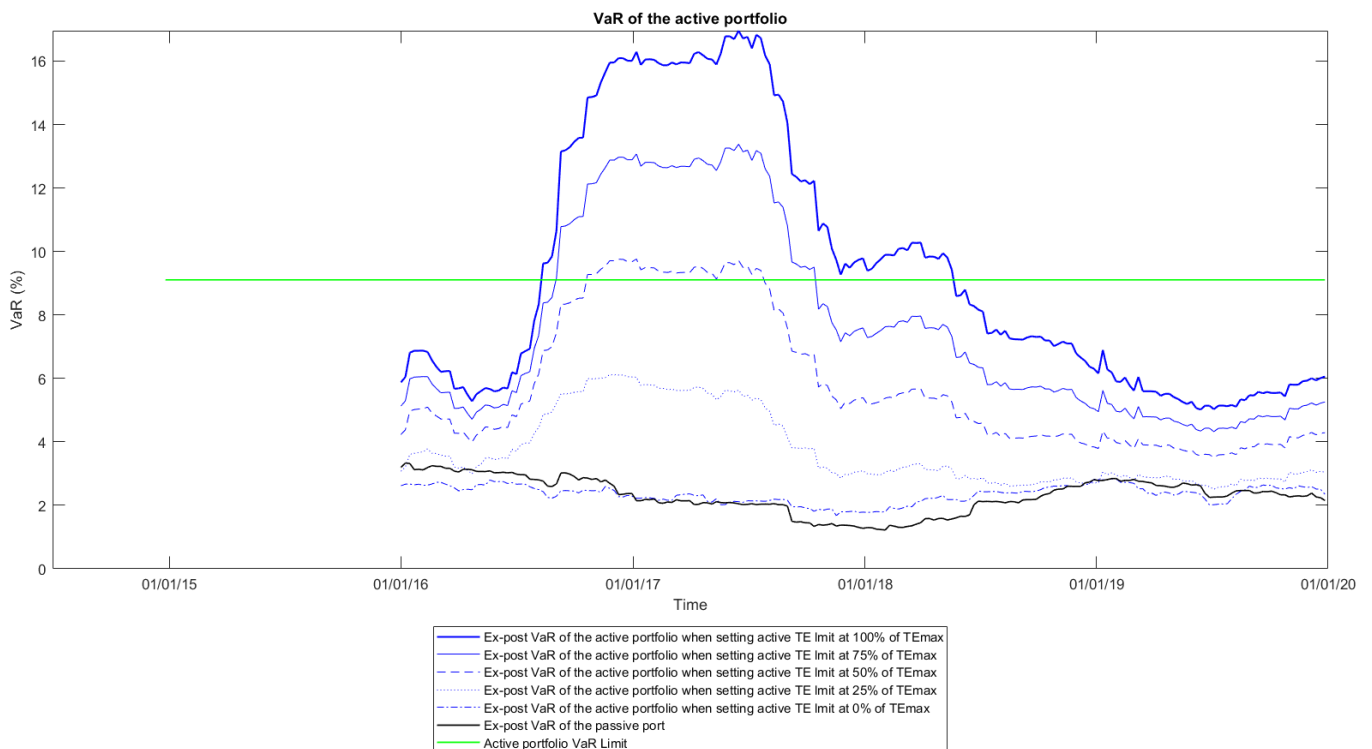
*Figure 28: Comparing VaRs of the mix portfolios in each level of the TE limit setting **without** re-estimating/re-balancing portfolio*
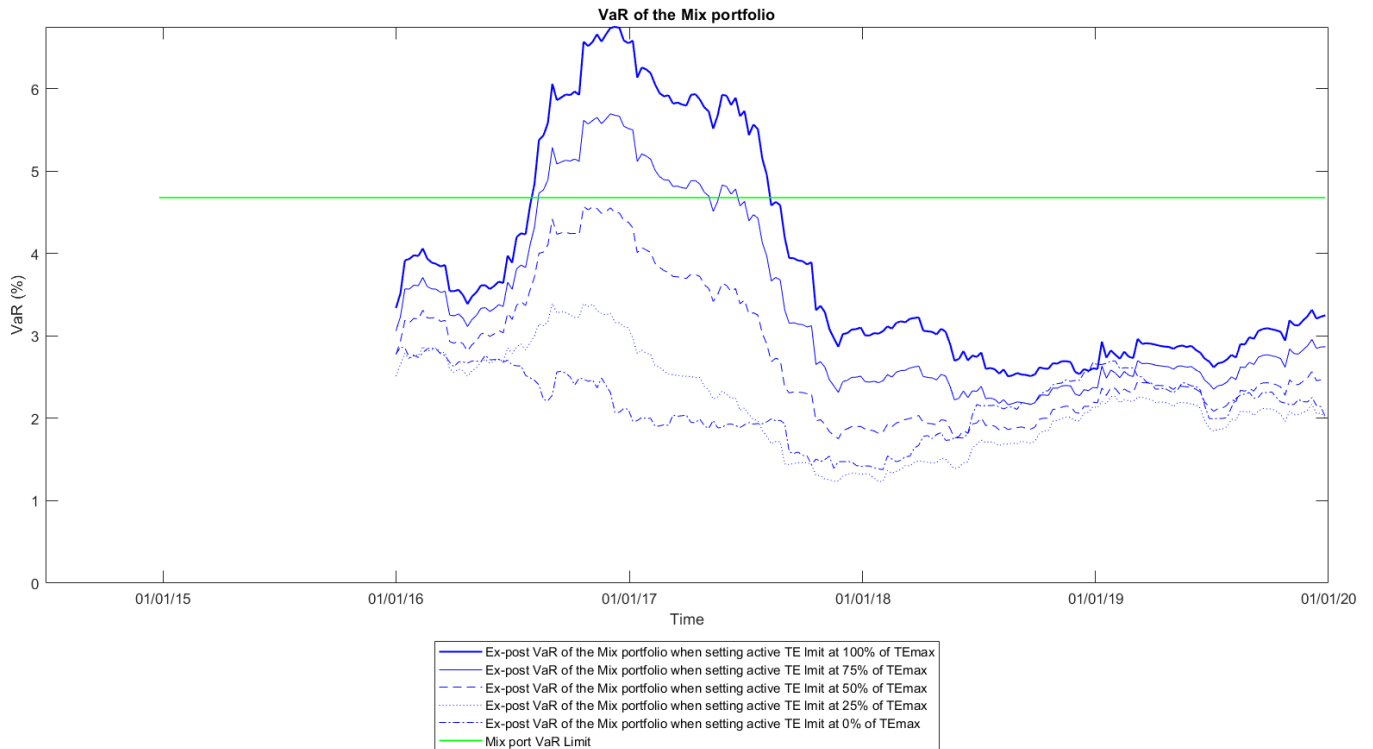
For the entire portfolio's VaR, since we invest in the passive portfolio for 60% and the active portfolio for 40%, the entire portfolio's VaR will be calculated from both portions. Because the passive portfolio's VaR always stays at a low level, the active portfolio's VaR will be the main factor that stimulates the entire portfolio's VaR. As shown in Figure 28 above, the entire portfolio's VaR starts to be lower than the given limit (V_G) when setting the TE limit at 50% of the TE_A^{max}. So, we conclude that we need to set the TE limit below TE_A^{max} to allow some buffer for the entire portfolio's VaR within the given limit.

Figure 29: *Comparing NAVs of the active portfolios in each level of the TE limit setting **without** re-estimating/re-balancing portfolio*
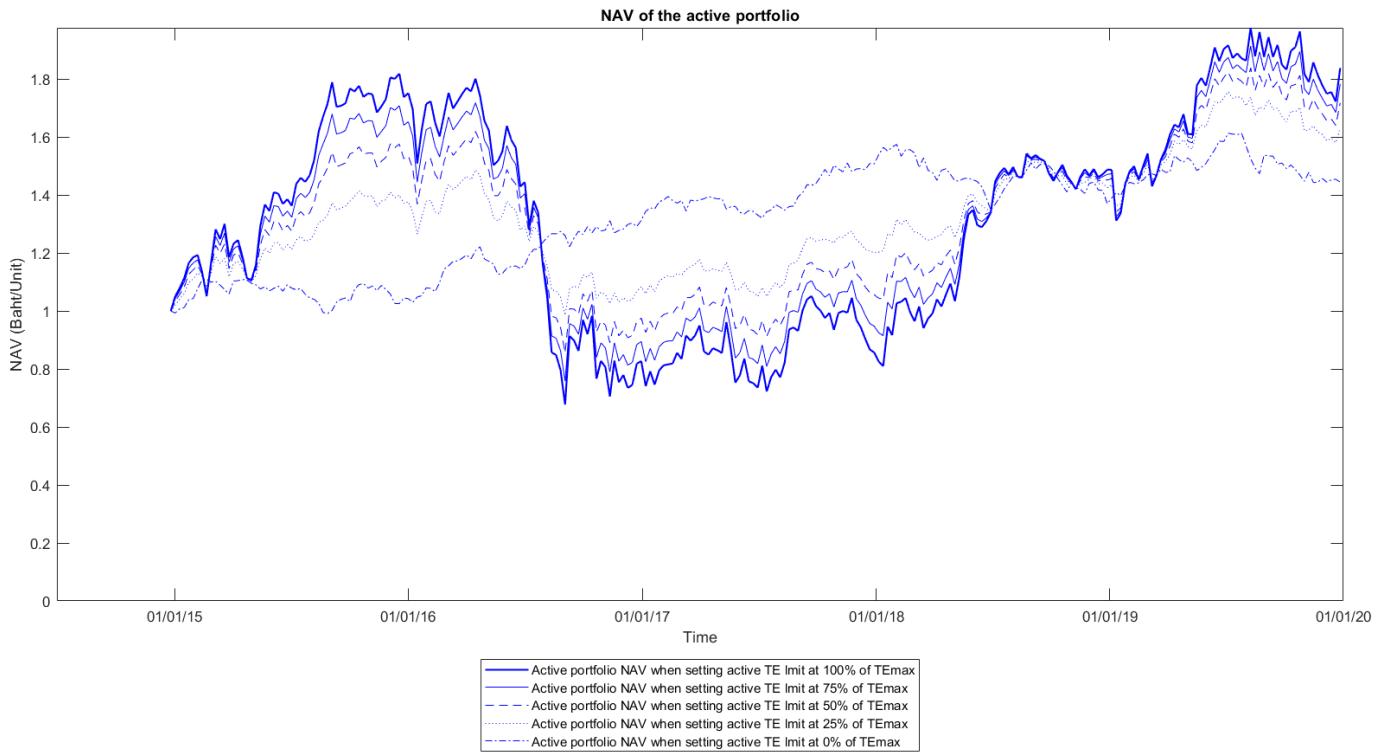
Even though an entire portfolio's VaR can be lower if we set a TE limit at a low level, the portfolio returns will also be lower. As shown in Table 12, the active portfolio's expected returns become lower when setting the TE limit at a low level. The active portfolio's actual returns presented in NAV, as Figure 29 above, are also lower if we set a TE limit at a lower level. Although setting TE at TE_A^{max} can make the portfolio's return more volatile, it still produces the highest return at the end of 2019.

However, since we prioritize the entire portfolio VaR rather than its expected return, we suggest setting the TE limit below TE_A^{max} when we do not re-estimate/re-balance the portfolio. From our example of the stocks' set, setting the TE limit at 75% of TE_A^{max} will be safe for the entire portfolio's VaR within the limit while having a high return opportunity.

5.3 effects on yearly re-estimating/re-balancing portfolio

As the result in part 5.2, to prevent VaR of the mix portfolio exceeds the given limit, we will illustrate the effect of yearly re-estimating/re-balancing portfolio in this part. According to best practices for portfolio re-balancing by (Zilbering, Jaconetti, & Kinniry Jr, 2015), they suggest that "re-balancing strategy based on reasonable monitoring frequencies (such as annual or semi-annual) and reasonable allocation thresholds (variations of 5% or so) is likely to provide sufficient risk control relative to the target asset allocation". From their research paper, the risk-adjusted returns are not meaningfully different whether a portfolio is re-balanced more frequently, but costs will be increase more as we re-balance more. Moreover, since parameters change as time pass, keeping only the same weight (re-balance), the active portfolio's TE and VaR may still exceed the limit. Therefore, we decide to illustrate both yearly re-estimate parameters and re-balance portfolio in this part to keep the active portfolio's TE and VaR within the limit at the beginning of each year.

In the case where short-selling is allowed as shown in Figure 30 below, the ex-post TE of the active portfolio still exceeds the limit even we re-estimating/re-balancing the portfolio (right-hand side of Figure 30). However, ex-post VaR of the active portfolio and VaR of the mix portfolio can still be within the limit when we re-estimating/re-balancing portfolio for most of the period. However, in the middle of 2019, a sudden price change in the high leverage stocks causes the ex-post TE, ex-post VaR of the active portfolio to significantly increases and very far from the limits. So, we re-estimate parameters and re-balance portfolios again. After re-estimating/re-balancing in the middle of 2019, the ex-post TE, the ex-post VaR of the active portfolio, and the ex-post VaR of the mix portfolio stop increasing. However, they start increasing at the end of 2019 again as the sudden price change in the high leverage stocks causing the ex-post VaR of the mix portfolio to exceed the limit. So, we need to re-estimate parameters and re-balance portfolios at the start of 2020, and it will be done as a yearly schedule. Even though we closely monitor the risk measures, it is still hard to reduce the risk measures to be within the limit. Because the risk measures are calculated in the "ex-post" method, which is based on one-year data, so it needs time to change ex-post risk measures.

Figure 30: *TE, VaR of active portfolio and VaR of mix portfolio in a case where short-selling is allowed with yearly re-estimating/re-balancing portfolio*

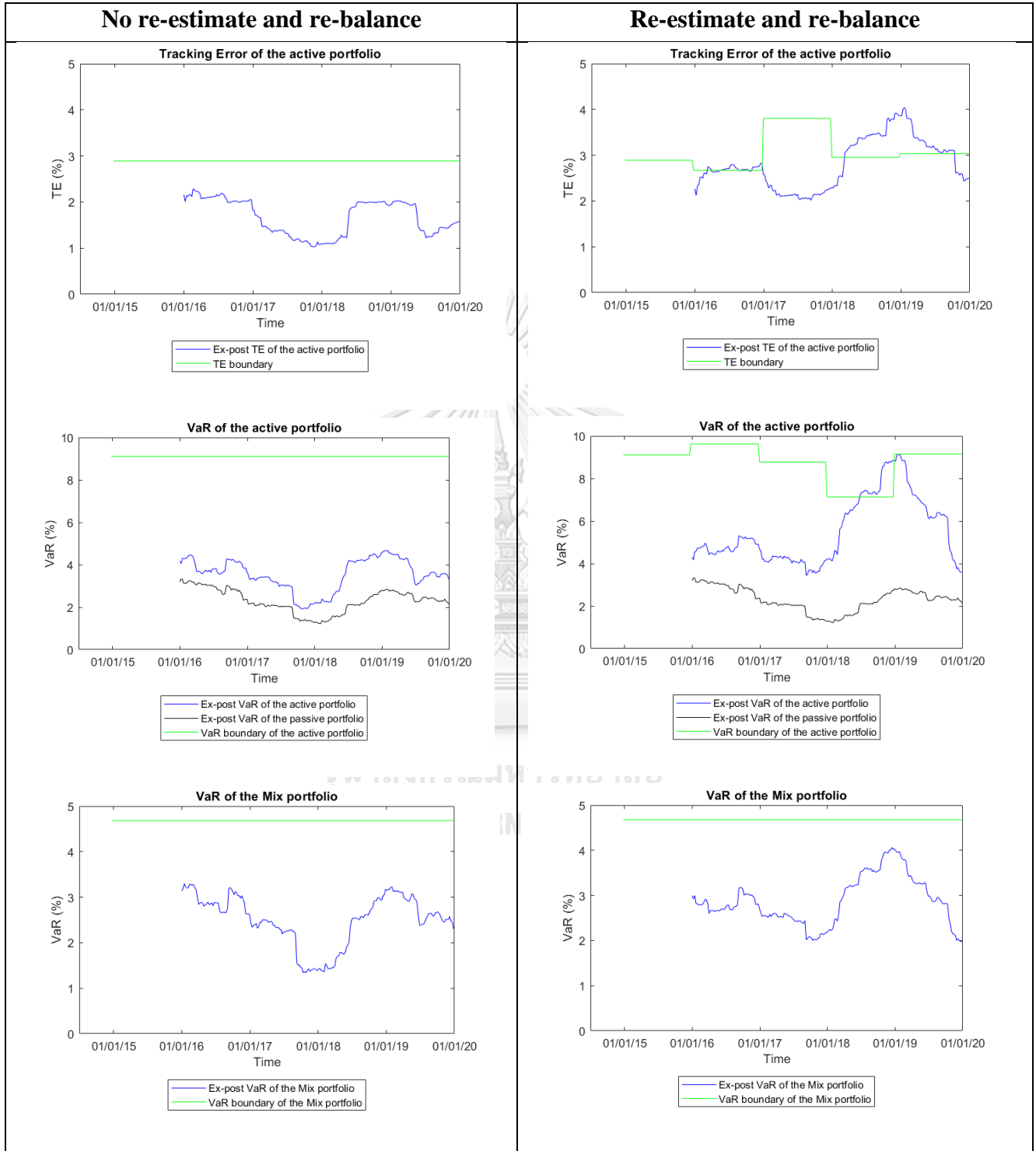| No re-estimate and re-balance	Re-estimate and re-balance
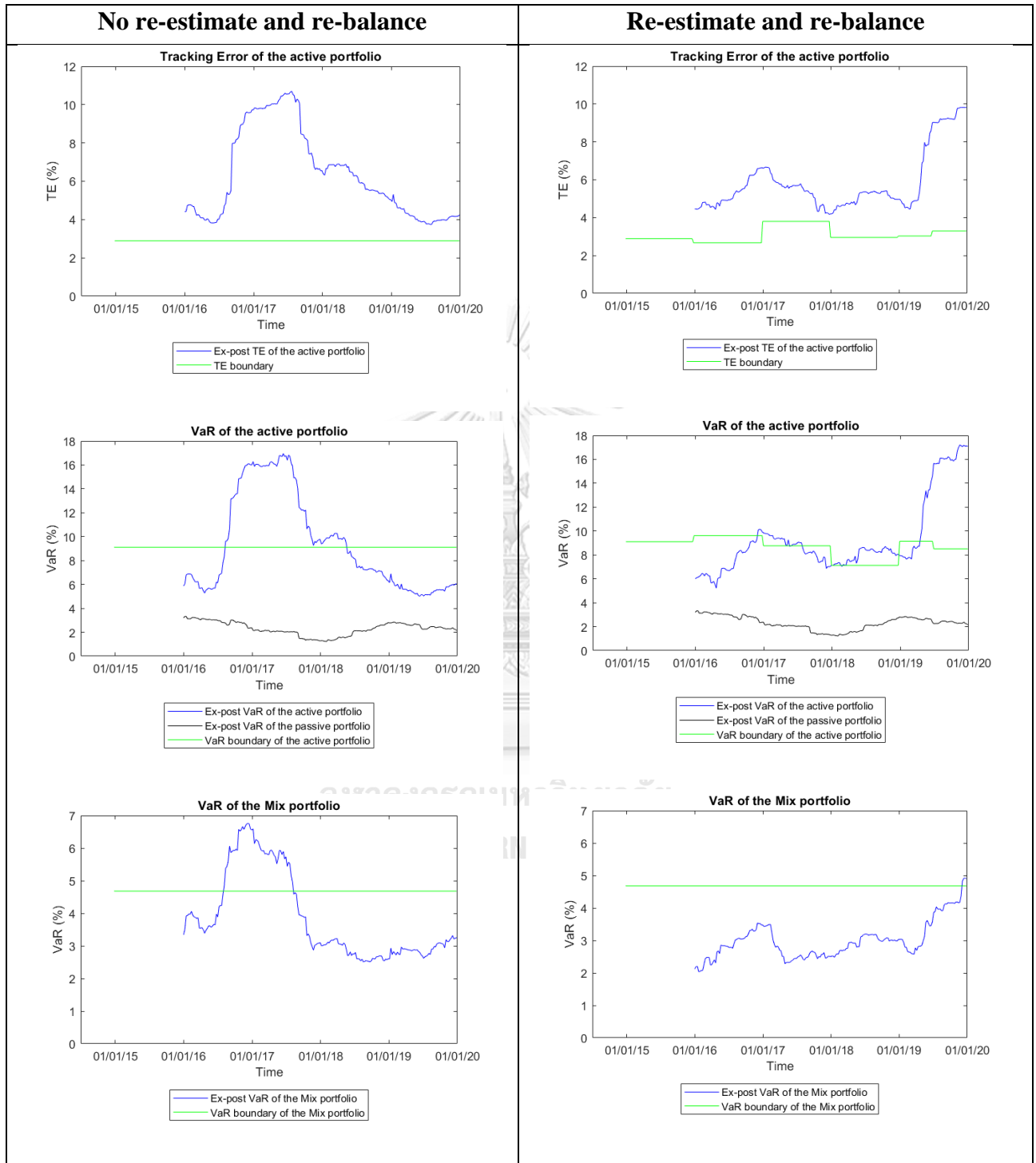
In sum, the result still shows the better result on yearly re-estimating/re-balancing portfolio than do nothing. Sometimes, re-estimating/re-balancing portfolio also helps an asset manager to increase risk limit. However, there may be some

occurrence that TE, VaR of the active portfolio, and VaR of the mix portfolio can exceed the limit, such as the second half of 2019. So, apart from the yearly re-estimate and re-balance portfolio, we still need to monitor the portfolio's risk and re-estimate/ re-balance the portfolio when TE and VaR of the active portfolio go too far from the limit.

In the case where short-selling is not allowed, as shown in Figure 27 below, even we do not need to re-estimating/re-balancing portfolio (left side of Figure 27), there are chances of ex-post TE, ex-post VaR of the active portfolio, and ex-post VaR of the mix portfolio exceeding the limit. With yearly re-estimating/re-balancing portfolio (right side of Figure 27), ex-post TE, ex-post VaR of the active portfolio is in the limit until 2018. However, they start to exceed the limit when choosing to invest in a newly re-balanced portfolio after 2018. This situation shows that there are cases that a chosen active portfolio in the case where short-selling is prohibited can make ex-post TE and ex-post VaR exceed the limit when time goes, even though they are less volatile than the case where short-selling is allowed. Re-estimating/re-balancing does not make ex-post TE and ex-post VaR drop suddenly because they are calculated from one-year returns which already happen in the past. Ex-post TE and ex-post VaR will only change a bit when the time goes. As we see the result after re-estimating/re-balancing portfolio and choosing a new active portfolio at the beginning of 2019, ex-post TE, ex-post VaR of the active portfolio starts to reduce. Even the ex-post VaR of the mix portfolio is not yet exceeding the limit. If we still insist on not re-estimate and re-balance the portfolio, the ex-post VaR of the mix portfolio can go further and exceed the limit. Therefore, a yearly re-estimating/re-balancing portfolio is still needed.

Figure 31: *TE, VaR of active portfolio and VaR of mix portfolio in the case where short-selling is not allowed with yearly re-estimating/re-balancing portfolio*

For the suggestion on setting the active portfolio's TE limit when we assume re-estimating and re-balancing portfolios regularly, the test will be as same as without re-estimating/ re-balancing portfolios in section 5.2. We will invest in the active portfolio with the weight in Table 7. Then we observe the ex-post TE, the ex-post VaR, NAV of the active portfolio, and the ex-post VaR of the mix portfolio in the "out-sample" period. As shown in Figure 32 below, the result can be concluded as the same as without re-estimating/re-balancing portfolio case. The ex-post active portfolio's TE becomes lower as we set the TE limit at a lower percentage of γ. And all of the active portfolio's TE still be higher than the TE limit regardless of the TE limit level.

Figure 32: *Comparing TEs of the active portfolios in each level of the TE limit setting* ***with*** *re-estimating/re-balancing portfolio*
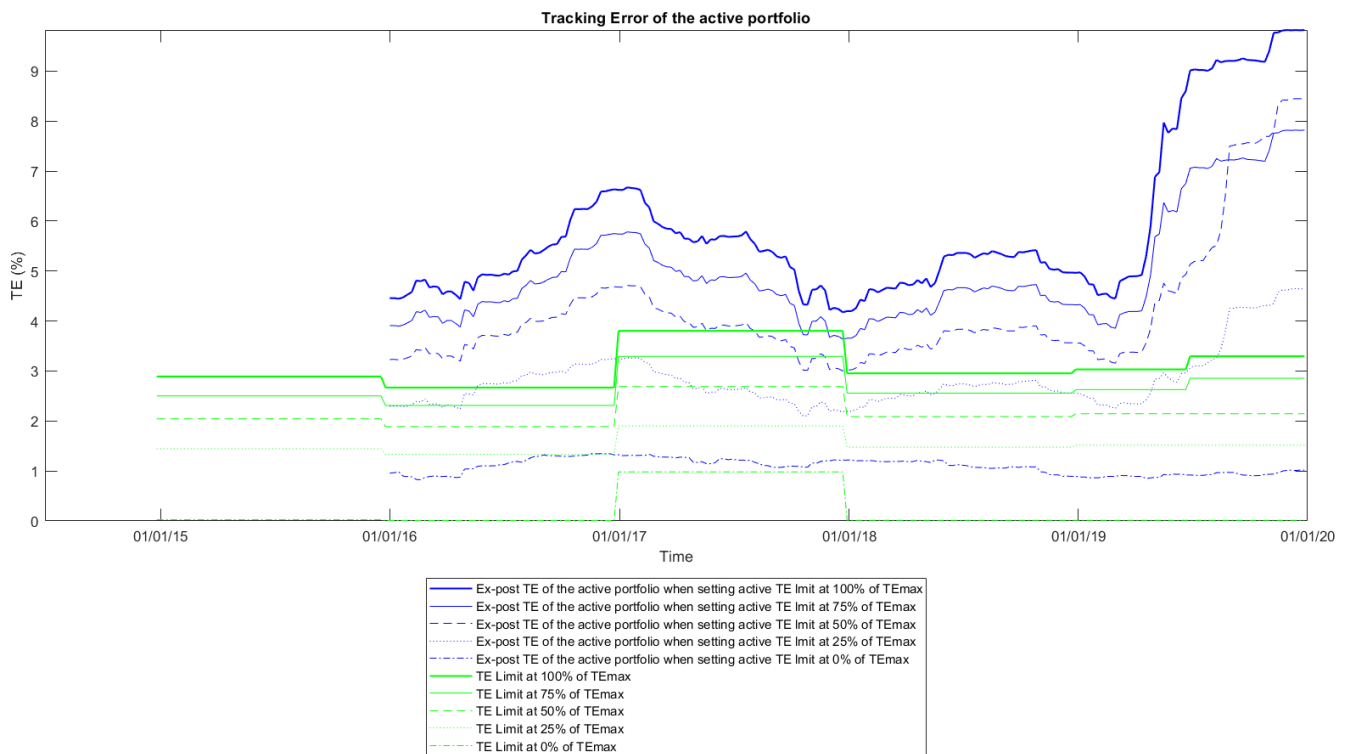
Figure 33: *Comparing VaRs of the active portfolios in each level of the TE limit setting* **with** *re-estimating/re-balancing portfolio*
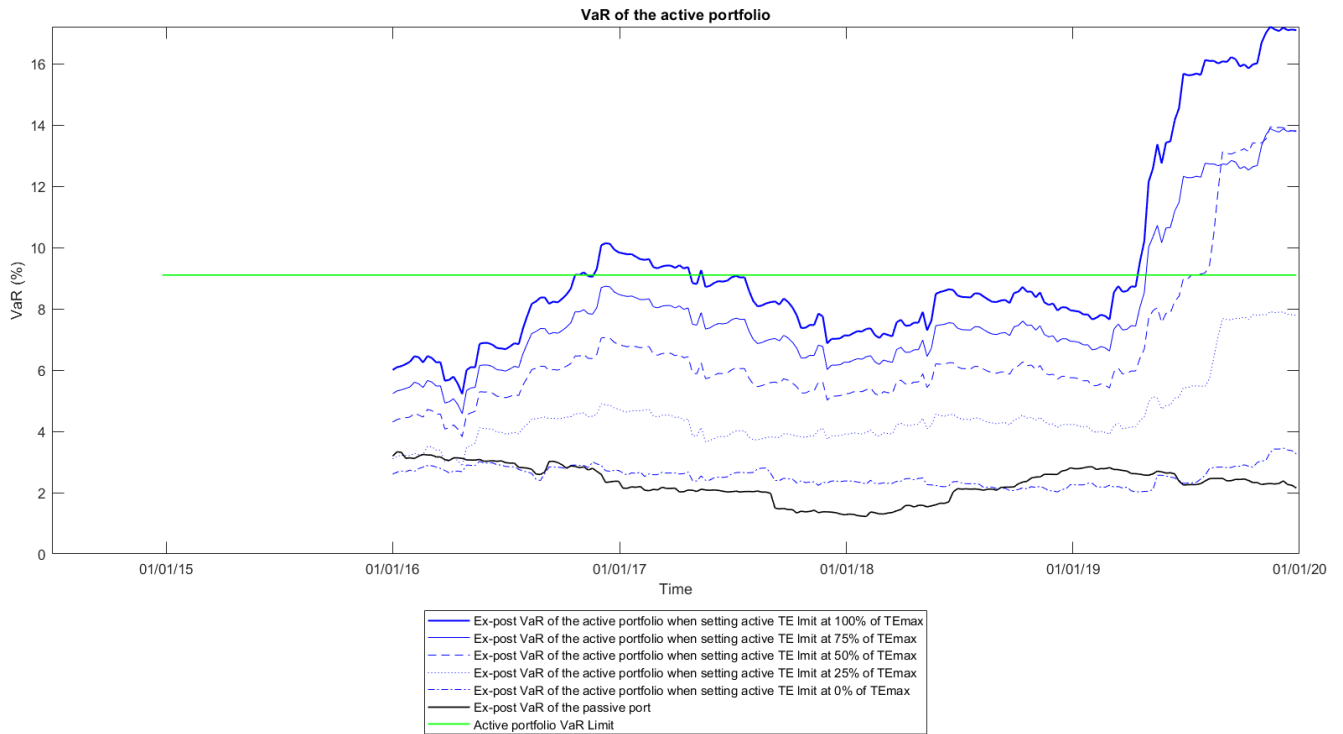
As shown in Figure 33 above, when comparing the active portfolios' VaR in each level of the TE limit setting, they become lower as we set the TE limit at a lower percentage of γ. As the same as the case of no re-estimating/re-balancing, the active portfolio's VaR starts to be lower than the VaR limit when setting the TE limit at 25% of the TE_A^{max}. If we set the TE limit at the TE^{min}, the active portfolio's VaR will become very close to the passive portfolio's VaR.

Figure 34: *Comparing VaRs of the active portfolios in each level of the TE limit setting* **with** *re-estimating/re-balancing portfolio*

For the entire portfolio's VaR, as shown in Figure 34 above, we do not require much buffer like the case of no re-estimating/re-balancing. The entire portfolio's VaR starts to be lower than the given limit (V_G) when setting the TE limit at 75% of the TE_A^{max}. Since we re-estimate/re-balance portfolios regularly, the risk measures of the active portfolio, including the entire portfolio's VaR can be reduced to be within the limit. Even though the ex-post risk measures cannot be reduced suddenly when we re-estimate/re-balance the portfolio, the movement of the ex-post risk measures can be adjusted in the long run.
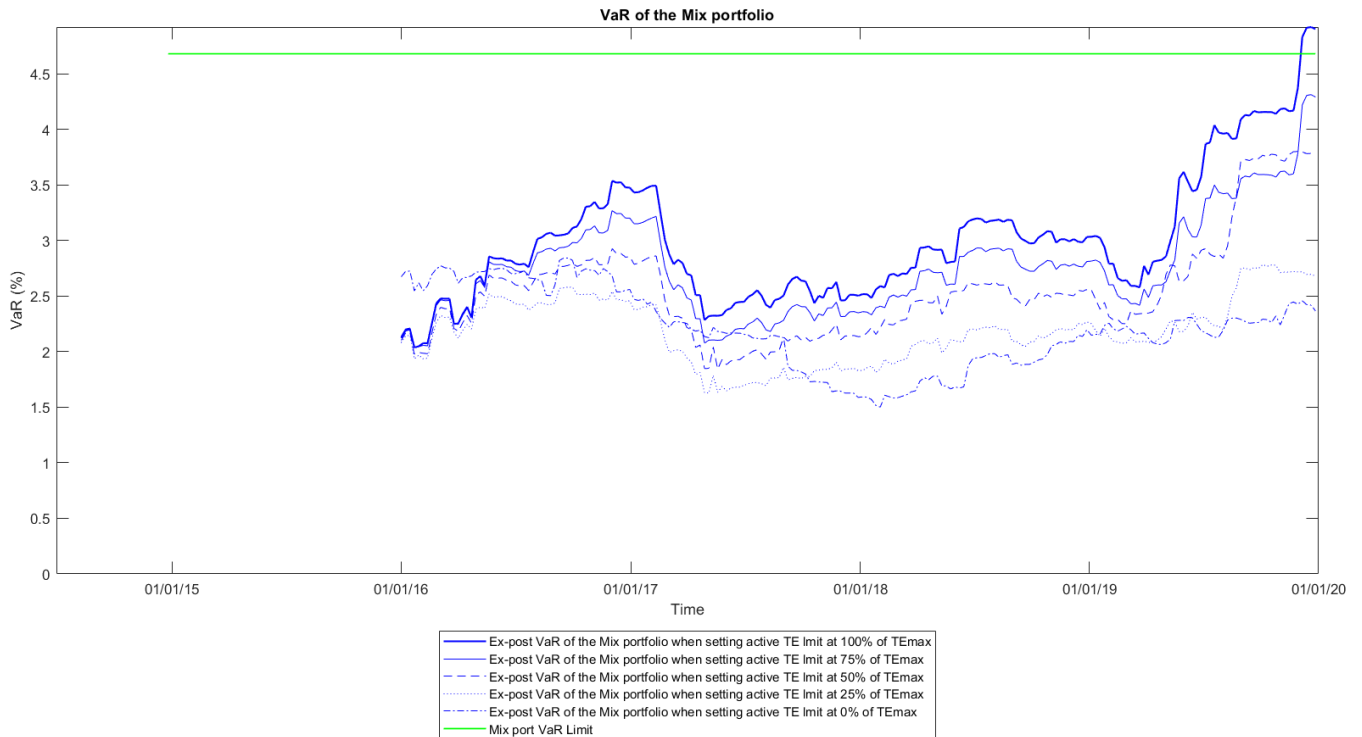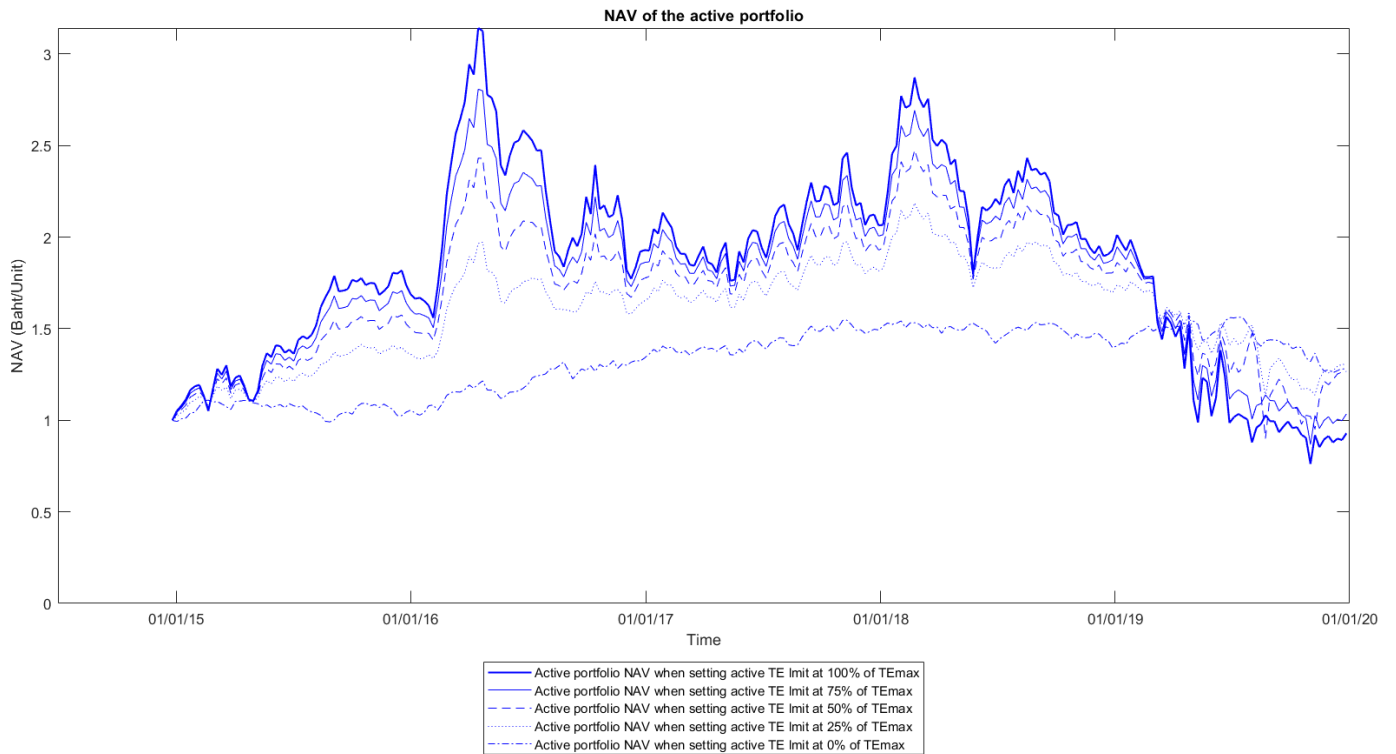
Figure 35: *Comparing NAVs of the active portfolios in each level of the TE limit setting* ***with*** *re-estimating/re-balancing portfolio*

As shown in Figure 35 above, there is still a trade-off between risk control and return opportunity. Even though the active portfolio's NAV of setting the TE limit at maximum is lowest among the other level at the end of 2019, setting the TE limit at maximum can still generate the most actual return in some period. The active portfolio's NAV in case of setting TE limit at maximum has a chance to reach 3 baht/unit. Although the portfolio construction with the highest TE limit yields higher NAV most of the time, it is accompanied by higher volatility and sometimes underperforms, as we can see in Figure 35 around the end of our test period. In other words, while setting a higher TE limit implies a higher expected return, the realized return also tends to be more volatile.

In sum, as we prioritize an entire portfolio VaR more than a portfolio expected return, we still recommend setting the TE limit below TE_A^{max} in case re-estimating/re-balancing portfolio regularly. From the example of stocks' set, setting the TE limit at 75% of the TE_A^{max} is the safe level for the entire portfolio's VaR to be within the limit while generating the high opportunity for return.

<u>Section 6: Conclusion</u>

When an asset manager manages a portfolio, he usually imposes a TE or a VaR limit on his portfolio to control portfolio risks. However, a portfolio may be divided into different portions. In our research, we consider a portfolio that consists of an active portion and a passive portion. Having a VaR budget of an entire portfolio, we propose methods to determine TE and VaR limit on the active portfolio. We adapt the methodology of determining TE and VaR limit on a portfolio proposed by Riccetti (2016) into this research paper. Moreover, in contrast to Riccetti (2016), we propose a new method to determine:

(1) the lower limit on TE (TE^{min}) when VaR is given (V_G),
(2) the lower limit on VaR (V^{min}) that can be set,
(3) the upper limit on TE (TE^{max}) in the case of an inefficient benchmark.

These methods will provide us with clearer instructions on determining the TE and the VaR limit on the active portfolio. We start from the idea of the portfolio on a CTF (an ellipse) tangent with the VaR line. The area inside the CTF represents a necessary (but not sufficient) condition for the TE to be within the limit. Therefore, provided that the tangential CTF is to the left-hand side of the VaR line, we can conclude that any portfolio whose tracking error is within the TE limit has value-at-risk that satisfies the VaR limit as well.

From the idea of determining a portfolio's TE limit, we apply this idea to determine a TE limit of an active portfolio when we have an entire portfolio consist of an active portion and a passive portion. We impose an assumption on a correlation of portfolios' return between an active and a passive portfolio. (The correlation equal to one will be most conservative when determining the TE and the VaR limit of the active portfolio.) We use the given VaR of the entire portfolio to determine the active portfolio's VaR. Then we determine the active portfolio's TE which makes the CTF tangent with the VaR line. Most parts of determining the TE and the VaR limit of an active portfolio can be easily computed as proposed in Table 3 and Table 4. However, the parts that we want to find active portfolio's TE which make the CTF tangent with the VaR line are involved with four equations (such as equation (13), (14), (32), and (35)). We need to solve those four equations to get the active portfolio's TE and the active portfolio's VaR, making the CTF tangent with the VaR line.

Based on actual market data, we try to determine an active portfolio's TE limit and an active portfolio's VaR limit and perform back-testing. In this empirical part, we

relax certain theoretical assumptions so that (1) stocks in an investment universe can be different from stocks in a benchmark (2) there is a restriction on short-selling. To determine the TE limit of the active portfolio, we use the method described in section 4. We then use MATLAB's numerical search to find an active portfolio weight: the maximum expected return portfolio with constraints on the TE limit and the VaR limit. Due to the difference in stocks between the investment universe and the benchmark, the expected return, variance, VaR, and TE of the active portfolio from MATLAB's numerical search deviate from the theoretical portfolio. In the case where the numerical search is carried out under the constraint of no short-sell, the active portfolio's VaR and the entire portfolio's VaR are even more likely to exceed the theoretical limit, at least in our in-sample period. Luckily, during the out-of-sample period, the results of the portfolio's risk measure do not tend to exceed the limit, because the active portfolio and the entire portfolio's returns are less volatile due to a low leverage in the case no-short-selling portfolio.

TEs and VaRs derived from the analysis part are "ex-ante" in the in-sample period, but we monitor them in the out-of-sample period in "ex-post." Ex-post TEs and ex-post VaRs likely to be more than the limit, especially in the case of having short positions in an active portfolio. The high entire portfolio's VaR usually comes from the active portfolio's VaR since the passive portfolio's VaR usually stays at the low level. Because the active portfolio returns are more volatile due to high leverage, we recommend re-estimating/re-balancing the portfolio regularly, such as annually or semi-annually, to control the risk measures. However, it is still hard to reduce the risk measures to be within the limit. Since the risk measures are calculated in "ex-post" based on one-year data in the empirical test, so it needs time for ex-post risk measures to change. Even though re-estimating/re-balancing the portfolio cannot reduce the ex-post TE and the ex-post VaR suddenly, it can change their direction to go up further.

In sum, this work presents formulas to compute the TE limit and VaR limit. We suggest setting the TE limit at a maximum to maximize the opportunity for return. However, in practice, when we monitor the risk measures in ex-post, there are issues. If short-selling is allowed, the portfolio could be highly leveraged, and the portfolio's VaR might breach the given limit. In that case, we recommend setting the TE limit below the TE_A^{max} when we do not re-estimate/re-balance the portfolio to allow some buffer. We do not require much buffer when assuming re-estimating/re-balancing portfolio regularly. So, we can set the TE limit higher than the case of no re-estimating/ re-balancing. However, if short-selling is not allowed, it quite safes to set the TE limit at TE_A^{max} as we proposed because the portfolio will be low leveraged enough to prevent

the entire portfolio's VaR from exceeding the limit. In this case, we can increase the TE limit to the maximum to gain more return opportunities.

APPENDIX

Constant TE frontier (CTF) can be derived from a maximization (or equivalently a minimization) over x:

$$
\begin{cases}
\textbf{Max} & x'E \\
\textbf{s.t.} & x'i \;=\; 0 \\
& x'\Omega x \;=\; T^2 \\
& (q+x)'\Omega(q+x) \;=\; \sigma_p^2,
\end{cases}
\tag{A1}
$$

where

$$
\begin{aligned}
q &= \text{vector of benchmark weights,} \\
x &= \text{vector of weights that deviate from the benchmark,} \\
q+x &= \text{vector of portfolio weights,} \\
T^2 &= \text{variance of excess returns.}
\end{aligned}
$$

We can set up Lagrangian function as:

$$
L = x'E + \lambda_1(x'i - 0) + 0.5\lambda_2(x'\Omega x - T^2) + 0.5\lambda_3(x'\Omega x + 2q'\Omega x + q'\Omega q - \sigma_p^2). \tag{A2}
$$

Taking partial derivatives with respect to $x, \lambda_1, \lambda_2, \lambda_3$ and set them equal to zero, we get:

$$
\frac{\partial L}{\partial x} \;=\; 0 \;=\; E + \lambda_1 i + \lambda_2 \Omega x + \lambda_3 \Omega x + \lambda_3 \Omega q, \tag{A3}
$$

$$
\frac{\partial L}{\partial \lambda_1} \;=\; 0 \;=\; x'i, \tag{A4}
$$

$$
\frac{\partial L}{\partial \lambda_2} \;=\; 0 \;=\; 0.5(x'\Omega x - T^2), \tag{A5}
$$

$$
\frac{\partial L}{\partial \lambda_3} \;=\; 0 \;=\; 0.5(x'\Omega x + 2q'\Omega x + q'\Omega q - \sigma_p^2). \tag{A6}
$$

From (A3), we get:

$$
x \;=\; -\frac{1}{\lambda_2+\lambda_3}\Omega^{-1}(E + \lambda_1 i + \lambda_3\Omega q). \tag{A7}
$$

Substitute x from (A7) to (A4), we get:

$$
\begin{aligned}
0 &= i'\Omega^{-1}(E + \lambda_1 i + \lambda_3 \Omega q) \\
0 &= i'\Omega^{-1}E + \lambda_1 i'\Omega^{-1}i + \lambda_3 i'\Omega^{-1}\Omega q \\
0 &= b + \lambda_1 a + \lambda_3 \\
\lambda_1 &= -\frac{b+\lambda_3}{a}
\end{aligned}
$$

(A8)

Substitute x from (A7) to (A5), we get

$$
\begin{aligned}
T^2 &= \left(\frac{1}{\lambda_2+\lambda_3}\right)^2 (E + \lambda_1 i + \lambda_3 \Omega q)'\Omega^{-1}\Omega\Omega^{-1}(E + \lambda_1 i + \lambda_3 \Omega q) \\
(\lambda_2+\lambda_3)^2 T^2 &= E'\Omega^{-1}E + \lambda_1 E'\Omega^{-1}i + \lambda_3 E'\Omega^{-1}\Omega q + \lambda_1 i'\Omega^{-1}E + \lambda_1^2 i'\Omega^{-1}i + \\
&\quad \lambda_1\lambda_3 i'\Omega^{-1}\Omega q + \lambda_3 q'\Omega\Omega^{-1}E + \lambda_1\lambda_3 q'\Omega\Omega^{-1}i + \lambda_3^2 q'\Omega\Omega^{-1}\Omega q \\
(\lambda_2+\lambda_3)^2 T^2 &= c + \lambda_1 b + \lambda_3 \mu_B + \lambda_1 b + \lambda_1^2 a + \lambda_1\lambda_3 + \lambda_3\mu_B + \lambda_1\lambda_3 + \lambda_3^2\sigma_B^2 \\
(\lambda_2+\lambda_3)^2 T^2 &= c + 2\lambda_1 b + 2\lambda_3\mu_B + \lambda_1^2 a + 2\lambda_1\lambda_3 + \lambda_3^2\sigma_B^2 \ .
\end{aligned}
$$

(A9)

Define $y = \sigma_p^2 - \sigma_B^2 - T^2$. If we consider (A5), we get $x'\Omega x = T^2$. Then we substitute x from (A7) to (A6), we get:

$$
\begin{aligned}
0 &= T^2 - \frac{2}{\lambda_2+\lambda_3}q'\Omega\Omega^{-1}(E + \lambda_1 i + \lambda_3 \Omega q) + q'\Omega q - \sigma_p^2 \\
T^2 + \sigma_B^2 - \sigma_p^2 &= \frac{2}{\lambda_2+\lambda_3}q'\Omega\Omega^{-1}(E + \lambda_1 i + \lambda_3 \Omega q) \\
-y &= \frac{2}{\lambda_2+\lambda_3}q'\Omega\Omega^{-1}(E + \lambda_1 i + \lambda_3 \Omega q) \\
-\frac{(\lambda_2+\lambda_3)y}{2} &= q'\Omega\Omega^{-1}E + \lambda_1 q'\Omega\Omega^{-1}i + \lambda_3 q'\Omega\Omega^{-1}\Omega q \\
-\frac{(\lambda_2+\lambda_3)y}{2} &= \mu_B + \lambda_1 + \lambda_3\sigma_B^2 \ .
\end{aligned}
$$

(A10)

Substitute λ_1 from (A8) to (A10), we get:

$$
\begin{aligned}
-\frac{(\lambda_2+\lambda_3)y}{2} &= \mu_B - \frac{b+\lambda_3}{a} + \lambda_3\sigma_B^2 \\
-\frac{(\lambda_2+\lambda_3)y}{2} &= \left(\mu_B - \frac{b}{a}\right) + \lambda_3\left(\sigma_B^2 - \frac{1}{a}\right) \\
-\frac{(\lambda_2+\lambda_3)y}{2} &= \Delta_1 + \lambda_3\Delta_2 \\
\lambda_2 + \lambda_3 &= -\frac{2}{y}(\Delta_1 + \lambda_3\Delta_2) \ .
\end{aligned}
$$

(A11)

Substitute λ_1 from (A8) and $\lambda_2 + \lambda_3$ from (A11) to (A9), we get:

$$\frac{4}{y^2}(\Delta_1 + \lambda_3\Delta_2)^2 T^2 = c - 2b\left(\frac{b+\lambda_3}{a}\right) + 2\lambda_3\mu_B + \left(\frac{b+\lambda_3}{a}\right)^2 a - 2\left(\frac{b+\lambda_3}{a}\right)\lambda_3 + \lambda_3^2\sigma_B^2$$

$$\frac{4T^2}{y^2}(\Delta_1^2 + 2\Delta_1\Delta_2\lambda_3 + \lambda_3^2\Delta_2^2) = c - \frac{2b^2}{a} - \frac{2b\lambda_3}{a} + 2\lambda_3\mu_B + \frac{b^2}{a} + \frac{2b\lambda_3}{a} + \frac{\lambda_3^2}{a} - \frac{2b\lambda_3}{a} - \frac{2\lambda_3^2}{a} + \lambda_3^2\sigma_B^2$$

$$\frac{4T^2}{y^2}(\Delta_1^2 + 2\Delta_1\Delta_2\lambda_3 + \lambda_3^2\Delta_2^2) = c - \frac{b^2}{a} + 2\lambda_3\mu_B - \frac{2b\lambda_3}{a} - \frac{\lambda_3^2}{a} + \lambda_3^2\sigma_B^2$$

$$\frac{4T^2}{y^2}(\Delta_1^2 + 2\Delta_1\Delta_2\lambda_3 + \lambda_3^2\Delta_2^2) = d + 2\lambda_3\left(\mu_B - \frac{b}{a}\right) + \lambda_3^2\left(\sigma_B^2 - \frac{1}{a}\right)$$

$$\frac{4T^2}{y^2}(\Delta_1^2 + 2\Delta_1\Delta_2\lambda_3 + \lambda_3^2\Delta_2^2) = d + 2\lambda_3\Delta_1 + \lambda_3^2\Delta_2$$

$$4T^2\Delta_1^2 + 8T\Delta_1\Delta_2\lambda_3 + 4T\lambda_3^2\Delta_2^2 = dy^2 + 2\lambda_3\Delta_1 y^2 + \lambda_3^2\Delta_2 y^2$$

$$0 = dy^2 + 2\lambda_3\Delta_1 y^2 + \lambda_3^2\Delta_2 y^2 - 4T^2\Delta_1^2 - 8T^2\Delta_1\Delta_2\lambda_3 - 4T^2\lambda_3^2\Delta_2^2$$

$$0 = (\Delta_2 y^2 - 4T^2\Delta_2^2)\lambda_3^2 + (2\Delta_1 y^2 - 8T^2\Delta_1\Delta_2)\lambda_3 + dy^2 - 4T^2\Delta_1^2$$

$$0 = (\Delta_2 y^2 - 4T^2\Delta_2^2)\lambda_3^2 + \frac{2\Delta_1}{\Delta_2}(\Delta_2 y^2 - 4T^2\Delta_2^2)\lambda_3 + dy^2 - 4T^2\Delta_1^2$$

$$0 = \lambda_3^2 + \frac{2\Delta_1}{\Delta_2}\lambda_3 + \frac{dy^2 - 4T^2\Delta_1^2}{\Delta_2 y^2 - 4T^2\Delta_2^2}. \tag{A12}$$

Solutions for the quadratic equation (A12) is:

$$\lambda_3 = \left[-\frac{2\Delta_1}{\Delta_2} \pm \sqrt{\frac{4\Delta_1^2}{\Delta_2^2} - 4\left(\frac{dy^2 - 4T^2\Delta_1^2}{\Delta_2 y^2 - 4T^2\Delta_2^2}\right)}\right]/2$$

$$\lambda_3 = -\frac{\Delta_1}{\Delta_2} \pm \sqrt{\frac{\Delta_1^2}{\Delta_2^2} - \frac{dy^2 - 4T^2\Delta_1^2}{\Delta_2 y^2 - 4T^2\Delta_2^2}}$$

$$\lambda_3 = -\frac{\Delta_1}{\Delta_2} \pm \sqrt{\frac{\Delta_1^2 y^2 - 4T^2\Delta_1^2\Delta_2}{(y^2 - 4T^2\Delta_2)\Delta_2^2} - \frac{d\Delta_2 y^2 - 4T^2\Delta_1^2\Delta_2}{(y^2 - 4T^2\Delta_2)\Delta_2^2}}$$

$$\lambda_3 = -\frac{\Delta_1}{\Delta_2} \pm \sqrt{\frac{y^2(\Delta_1^2 - d\Delta_2)}{(y^2 - 4T^2\Delta_2)\Delta_2^2}}$$

$$\lambda_3 = -\frac{\Delta_1}{\Delta_2} \pm \frac{y}{\Delta_2}\sqrt{\frac{\Delta_1^2 - d\Delta_2}{y^2 - 4T^2\Delta_2}}. \tag{A13}$$

Substitute λ_3 from (A13) to (A11), we get:

$$\lambda_2 + \lambda_3 = -\frac{2}{y}\left(\Delta_1 + \left(-\frac{\Delta_1}{\Delta_2} \pm \frac{y}{\Delta_2}\sqrt{\frac{\Delta_1^2 - d\Delta_2}{y^2 - 4T^2\Delta_2}}\right)\Delta_2\right)$$

$$\lambda_2 + \lambda_3 = -\frac{2}{y}\left(\pm y\sqrt{\frac{\Delta_1^2 - d\Delta_2}{y^2 - 4T^2\Delta_2}}\right)$$

$$\lambda_2 + \lambda_3 = \pm(-2)\sqrt{\frac{\Delta_1^2 - d\Delta_2}{y^2 - 4T^2\Delta_2}}. \tag{A14}$$

Define $z = \mu_p - \mu_B$. Substitute λ_1 from (A8), $\lambda_2 + \lambda_3$ from (A14), and λ_3 from (A13) to (A7) and compute $x'E$, we get:

$$x'E = -\frac{1}{\lambda_2 + \lambda_3}(E + \lambda_1 i + \lambda_3 \Omega q)' \Omega^{-1} E$$

$$x'E + q'E - q'E = -\frac{1}{\lambda_2 + \lambda_3}\left(E - \frac{b + \lambda_3}{a}i + \lambda_3 \Omega q\right)' \Omega^{-1} E$$

$$\mu_p - \mu_B = -\frac{1}{\lambda_2 + \lambda_3}\left(E'\Omega^{-1}E - \frac{b + \lambda_3}{a}i'\Omega^{-1}E + \lambda_3 q'\Omega\Omega^{-1}E\right)$$

$$z = -\frac{1}{\lambda_2 + \lambda_3}\left(c - \frac{b + \lambda_3}{a}b + \lambda_3 \mu_B\right)$$

$$= -\frac{1}{\lambda_2 + \lambda_3}\left(\left(c - \frac{b^2}{a}\right) + \left(\mu_B - \frac{b}{a}\right)\lambda_3\right)$$

$$= -\frac{1}{\lambda_2 + \lambda_3}(d + \Delta_1 \lambda_3)$$

$$= \pm\frac{1}{2}\sqrt{\frac{y^2 - 4T^2\Delta_2}{\Delta_1^2 - d\Delta_2}}\left(d + \Delta_1\left(-\frac{\Delta_1}{\Delta_2} \pm \frac{y}{\Delta_2}\sqrt{\frac{\Delta_1^2 - d\Delta_2}{y^2 - 4T^2\Delta_2}}\right)\right)$$

$$\pm 2z\sqrt{\frac{\Delta_1^2 - d\Delta_2}{y^2 - 4T^2\Delta_2}} = d - \frac{\Delta_1^2}{\Delta_2} \pm \frac{y\Delta_1}{\Delta_2}\sqrt{\frac{\Delta_1^2 - d\Delta_2}{y^2 - 4T^2\Delta_2}}$$

$$\pm 2z\Delta_2\sqrt{\frac{\Delta_1^2 - d\Delta_2}{y^2 - 4T^2\Delta_2}} = d\Delta_2 - \Delta_1^2 \pm y\Delta_1\sqrt{\frac{\Delta_1^2 - d\Delta_2}{y^2 - 4T^2\Delta_2}}$$

$$\pm(2z\Delta_2 - y\Delta_1)\sqrt{\frac{d\Delta_2 - \Delta_1^2}{4T^2\Delta_2 - y^2}} = d\Delta_2 - \Delta_1^2$$

$$\pm(2z\Delta_2 - y\Delta_1) = \sqrt{(d\Delta_2 - \Delta_1^2)(4T^2\Delta_2 - y^2)}$$

$$4\Delta_2^2 z^2 - 4\Delta_1\Delta_2 yz + \Delta_1^2 y^2 = 4dT^2\Delta_2^2 - d\Delta_2 y^2 - 4T^2\Delta_1^2\Delta_2 + \Delta_1^2 y^2$$

$$4\Delta_2^2 z^2 - 4\Delta_1\Delta_2 yz = 4dT^2\Delta_2^2 - d\Delta_2 y^2 - 4T^2\Delta_1^2\Delta_2$$

$$4\Delta_2 z^2 - 4\Delta_1 yz = 4dT^2\Delta_2 - dy^2 - 4T^2\Delta_1^2$$

$$0 = 4\Delta_2 z^2 - 4\Delta_1 yz - 4dT^2\Delta_2 + dy^2 + 4T^2\Delta_1^2$$

$$0 = dy^2 + 4\Delta_2 z^2 - 4\Delta_1 yz - 4T^2(d\Delta_2 - \Delta_1^2). \tag{A15}$$

For a quadratic equation of type $Ay^2 + Bz^2 + Cyz + F = 0$, equation (A15) represents and ellipse when the term:

$$AB - \frac{1}{4}C^2 = d(4\Delta_2) - \left(\frac{1}{4}\right)(-4\Delta_1)^2$$

$$AB - \frac{1}{4}C^2 = 4(d\Delta_2 - \Delta_1^2) \tag{A16}$$

is strictly positive. This term must be positive when the benchmark is within the efficient set which also imply d to be positive.

If we consider a minimization problem over x, it will also provide the same ellipse equation (equation (A15)) as the maximization problem.

Consider a minimization over x:

$$\left[\begin{array}{lrcl} \textbf{Max} & & -x'E & \\ \textbf{s.t.} & x'i & = & 0 \\ & x'\Omega x & = & T^2 \\ & (q + x)'\Omega(q + x) & = & \sigma_p^2. \end{array}\right. \tag{A17}$$

We can set up Lagrangian function as:

$$L = -x'E + \lambda_1(x'i - 0) + 0.5\lambda_2(x'\Omega x - T^2) + 0.5\lambda_3(x'\Omega x + 2q'\Omega x + q'\Omega q - \sigma_p^2). \tag{A18}$$

Taking partial derivatives with respect to x, λ_1, λ_2, λ_3 and set them equal to zero, we get:

$$\frac{\partial L}{\partial x} = 0 = -E + \lambda_1 i + \lambda_2\Omega x + \lambda_3\Omega x + \lambda_3\Omega q, \tag{A19}$$

$$\frac{\partial L}{\partial \lambda_1} = 0 = x'i, \tag{A20}$$

$$\frac{\partial L}{\partial \lambda_2} = 0 = 0.5(x'\Omega x - T^2), \tag{A21}$$

$$\frac{\partial L}{\partial \lambda_3} = 0 = 0.5(x'\Omega x + 2q'\Omega x + q'\Omega q - \sigma_p^2). \tag{A22}$$

From (A19), we get:

$$x = -\frac{1}{\lambda_2 + \lambda_3}\Omega^{-1}(-E + \lambda_1 i + \lambda_3\Omega q). \tag{A23}$$

Substitute x from (A23) to (A20), we get:

$$
\begin{aligned}
0 &= i'\Omega^{-1}(-E + \lambda_1 i + \lambda_3 \Omega q) \\
0 &= -i'\Omega^{-1}E + \lambda_1 i'\Omega^{-1}i + \lambda_3 i'\Omega^{-1}\Omega q \\
0 &= -b + \lambda_1 a + \lambda_3 \\
\lambda_1 &= \frac{b - \lambda_3}{a} .
\end{aligned}
\tag{A24}
$$

Substitute x from (A23) to (A21), we get:

$$
\begin{aligned}
T^2 &= \left(\frac{1}{\lambda_2 + \lambda_3}\right)^2 (-E + \lambda_1 i + \lambda_3 \Omega q)'\Omega^{-1}\Omega\Omega^{-1}(-E + \lambda_1 i + \lambda_3 \Omega q) \\
(\lambda_2 + \lambda_3)^2 T^2 &= E'\Omega^{-1}E - \lambda_1 E'\Omega^{-1}i - \lambda_3 E'\Omega^{-1}\Omega q - \lambda_1 i'\Omega^{-1}E + \lambda_1^2 i'\Omega^{-1}i + \\
&\quad \lambda_1\lambda_3 i'\Omega^{-1}\Omega q - \lambda_3 q'\Omega\Omega^{-1}E + \lambda_1\lambda_3 q'\Omega\Omega^{-1}i + \lambda_3^2 q'\Omega\Omega^{-1}\Omega q \\
(\lambda_2 + \lambda_3)^2 T^2 &= c - \lambda_1 b - \lambda_3 \mu_B - \lambda_1 b + \lambda_1^2 a + \lambda_1\lambda_3 - \lambda_3 \mu_B + \lambda_1\lambda_3 + \lambda_3^2 \sigma_B^2 \\
(\lambda_2 + \lambda_3)^2 T^2 &= c - 2\lambda_1 b - 2\lambda_3 \mu_B + \lambda_1^2 a + 2\lambda_1\lambda_3 + \lambda_3^2 \sigma_B^2 .
\end{aligned}
\tag{A25}
$$

Define $y = \sigma_p^2 - \sigma_B^2 - T^2$. From $x'\Omega x = T^2$ in (A21), we substitute x from (A23) to (A22), we get:

$$
\begin{aligned}
0 &= T^2 - \frac{2}{\lambda_2 + \lambda_3} q'\Omega\Omega^{-1}(-E + \lambda_1 i + \lambda_3 \Omega q) + q'\Omega q - \sigma_p^2 \\
T^2 + \sigma_B^2 - \sigma_p^2 &= \frac{2}{\lambda_2 + \lambda_3} q'\Omega\Omega^{-1}(-E + \lambda_1 i + \lambda_3 \Omega q) \\
-y &= \frac{2}{\lambda_2 + \lambda_3} q'\Omega\Omega^{-1}(-E + \lambda_1 i + \lambda_3 \Omega q) \\
-\frac{(\lambda_2 + \lambda_3)y}{2} &= -q'\Omega\Omega^{-1}E + \lambda_1 q'\Omega\Omega^{-1}i + \lambda_3 q'\Omega\Omega^{-1}\Omega q \\
-\frac{(\lambda_2 + \lambda_3)y}{2} &= -\mu_B + \lambda_1 + \lambda_3 \sigma_B^2 .
\end{aligned}
\tag{A26}
$$

Substitute λ_1 from (A24) to (A26), we get:

$$
\begin{aligned}
-\frac{(\lambda_2 + \lambda_3)y}{2} &= -\mu_B + \frac{b - \lambda_3}{a} + \lambda_3 \sigma_B^2 \\
-\frac{(\lambda_2 + \lambda_3)y}{2} &= \left(-\mu_B + \frac{b}{a}\right) + \lambda_3\left(\sigma_B^2 - \frac{1}{a}\right) \\
-\frac{(\lambda_2 + \lambda_3)y}{2} &= -\Delta_1 + \lambda_3 \Delta_2 \\
\lambda_2 + \lambda_3 &= \frac{2}{y}(\Delta_1 - \lambda_3 \Delta_2) .
\end{aligned}
\tag{A27}
$$

Substitute λ_1 from (A24) and $\lambda_2 + \lambda_3$ from (A27) to (A25), we get:

$$\frac{4}{y^2}(\Delta_1 - \lambda_3\Delta_2)^2 T^2 = c - 2b\left(\frac{b-\lambda_3}{a}\right) - 2\lambda_3\mu_B + \left(\frac{b-\lambda_3}{a}\right)^2 a + 2\left(\frac{b-\lambda_3}{a}\right)\lambda_3 + \lambda_3^2\sigma_B^2$$

$$\frac{4T^2}{y^2}(\Delta_1^2 - 2\Delta_1\Delta_2\lambda_3 + \lambda_3^2\Delta_2^2) = c - \frac{2b^2}{a} + \frac{2b\lambda_3}{a} - 2\lambda_3\mu_B + \frac{b^2}{a} - \frac{2b\lambda_3}{a} + \frac{\lambda_3^2}{a} + \frac{2b\lambda_3}{a} - \frac{2\lambda_3^2}{a} + \lambda_3^2\sigma_B^2$$

$$\frac{4T^2}{y^2}(\Delta_1^2 - 2\Delta_1\Delta_2\lambda_3 + \lambda_3^2\Delta_2^2) = c - \frac{b^2}{a} - 2\lambda_3\mu_B + \frac{2b\lambda_3}{a} - \frac{\lambda_3^2}{a} + \lambda_3^2\sigma_B^2$$

$$\frac{4T^2}{y^2}(\Delta_1^2 - 2\Delta_1\Delta_2\lambda_3 + \lambda_3^2\Delta_2^2) = d - 2\lambda_3\left(\mu_B - \frac{b}{a}\right) + \lambda_3^2\left(\sigma_B^2 - \frac{1}{a}\right)$$

$$\frac{4T^2}{y^2}(\Delta_1^2 - 2\Delta_1\Delta_2\lambda_3 + \lambda_3^2\Delta_2^2) = d - 2\lambda_3\Delta_1 + \lambda_3^2\Delta_2$$

$$4T^2\Delta_1^2 - 8T^2\Delta_1\Delta_2\lambda_3 + 4T^2\lambda_3^2\Delta_2^2 = dy^2 - 2\lambda_3\Delta_1 y^2 + \lambda_3^2\Delta_2 y^2$$

$$0 = dy^2 - 2\lambda_3\Delta_1 y^2 + \lambda_3^2\Delta_2 y^2 - 4T^2\Delta_1^2 + 8T^2\Delta_1\Delta_2\lambda_3 - 4T^2\lambda_3^2\Delta_2^2$$

$$0 = (\Delta_2 y^2 - 4T^2\Delta_2^2)\lambda_3^2 - (2\Delta_1 y^2 - 8T^2\Delta_1\Delta_2)\lambda_3 + dy^2 - 4T^2\Delta_1^2$$

$$0 = (\Delta_2 y^2 - 4T^2\Delta_2^2)\lambda_3^2 - \frac{2\Delta_1}{\Delta_2}(\Delta_2 y^2 - 4T^2\Delta_2^2)\lambda_3 + dy^2 - 4T^2\Delta_1^2$$

$$0 = \lambda_3^2 - \frac{2\Delta_1}{\Delta_2}\lambda_3 + \frac{dy^2 - 4T^2\Delta_1^2}{\Delta_2 y^2 - 4T^2\Delta_2^2}. \tag{A28}$$

Solutions for the quadratic equation (A28) is:

$$\lambda_3 = \left[\frac{2\Delta_1}{\Delta_2} \pm \sqrt{\frac{4\Delta_1^2}{\Delta_2^2} - 4\left(\frac{dy^2 - 4T^2\Delta_1^2}{\Delta_2 y^2 - 4T^2\Delta_2^2}\right)}\right]/2$$

$$\lambda_3 = \frac{\Delta_1}{\Delta_2} \pm \sqrt{\frac{\Delta_1^2}{\Delta_2^2} - \frac{dy^2 - 4T^2\Delta_1^2}{\Delta_2 y^2 - 4T^2\Delta_2^2}}$$

$$\lambda_3 = \frac{\Delta_1}{\Delta_2} \pm \sqrt{\frac{\Delta_1^2 y^2 - 4T^2\Delta_1^2\Delta_2}{(y^2 - 4T^2\Delta_2)\Delta_2^2} - \frac{d\Delta_2 y^2 - 4T^2\Delta_1^2\Delta_2}{(y^2 - 4T^2\Delta_2)\Delta_2^2}}$$

$$\lambda_3 = \frac{\Delta_1}{\Delta_2} \pm \sqrt{\frac{y^2(\Delta_1^2 - d\Delta_2)}{(y^2 - 4T^2\Delta_2)\Delta_2^2}}$$

$$\lambda_3 = \frac{\Delta_1}{\Delta_2} \pm \frac{y}{\Delta_2}\sqrt{\frac{\Delta_1^2 - d\Delta_2}{y^2 - 4T^2\Delta_2}}. \tag{A29}$$

Substitute λ_3 from (A29) to (A27), we get:

$$\lambda_2 + \lambda_3 = \frac{2}{y}\left(\Delta_1 - \left(\frac{\Delta_1}{\Delta_2} \pm \frac{y}{\Delta_2}\sqrt{\frac{\Delta_1^2 - d\Delta_2}{y^2 - 4T^2\Delta_2}}\right)\Delta_2\right)$$

$$\lambda_2 + \lambda_3 = \frac{2}{y}\left(\pm(-y)\sqrt{\frac{\Delta_1^2 - d\Delta_2}{y^2 - 4T^2\Delta_2}}\right)$$

$$\lambda_2 + \lambda_3 = \pm(-2)\sqrt{\frac{\Delta_1^2 - d\Delta_2}{y^2 - 4T^2\Delta_2}}. \tag{A30}$$

Define $z = \mu_p - \mu_B$. Substitute λ_1 from (A24), $\lambda_2 + \lambda_3$ from (A30), and λ_3 from (A29) to (A23) and compute $x'E$, we get:

$$
\begin{aligned}
-x'E &= -\left(-\frac{1}{\lambda_2+\lambda_3}\right)(-E + \lambda_1 i + \lambda_3 \Omega q)'\Omega^{-1}E \\
-x'E - q'E + q'E &= \frac{1}{\lambda_2+\lambda_3}\left(-E + \frac{b-\lambda_3}{a}i + \lambda_3\Omega q\right)'\Omega^{-1}E \\
-\mu_p + \mu_B &= \frac{1}{\lambda_2+\lambda_3}\left(-E'\Omega^{-1}E + \frac{b-\lambda_3}{a}i'\Omega^{-1}E + \lambda_3 q'\Omega\Omega^{-1}E\right) \\
-z &= \frac{1}{\lambda_2+\lambda_3}\left(-c + \frac{b-\lambda_3}{a}b + \lambda_3\mu_B\right) \\
-z &= \frac{1}{\lambda_2+\lambda_3}\left(\left(-c + \frac{b^2}{a}\right) + \left(\mu_B - \frac{b}{a}\right)\lambda_3\right) \\
-z &= \frac{1}{\lambda_2+\lambda_3}(-d + \Delta_1\lambda_3) \\
-z &= \pm\left(-\frac{1}{2}\right)\sqrt{\frac{y^2-4T^2\Delta_2}{\Delta_1^2-d\Delta_2}}\left(-d + \Delta_1\left(\frac{\Delta_1}{\Delta_2} \pm \frac{y}{\Delta_2}\sqrt{\frac{\Delta_1^2-d\Delta_2}{y^2-4T^2\Delta_2}}\right)\right) \\
\pm 2z\sqrt{\frac{\Delta_1^2-d\Delta_2}{y^2-4T^2\Delta_2}} &= -d + \frac{\Delta_1^2}{\Delta_2} \pm \frac{y\Delta_1}{\Delta_2}\sqrt{\frac{\Delta_1^2-d\Delta_2}{y^2-4T^2\Delta_2}} \\
\pm 2z\Delta_2\sqrt{\frac{\Delta_1^2-d\Delta_2}{y^2-4T^2\Delta_2}} &= -d\Delta_2 + \Delta_1^2 \pm y\Delta_1\sqrt{\frac{\Delta_1^2-d\Delta_2}{y^2-4T^2\Delta_2}} \\
\pm(2z\Delta_2 - y\Delta_1)\sqrt{\frac{\Delta_1^2-d\Delta_2}{y^2-4T^2\Delta_2}} &= -d\Delta_2 + \Delta_1^2 \\
\pm(2z\Delta_2 - y\Delta_1) &= \sqrt{(d\Delta_2 - \Delta_1^2)(4T^2\Delta_2 - y^2)} \\
\pm(2z\Delta_2 - y\Delta_1) &= \sqrt{(\Delta_1^2 - d\Delta_2)(y^2 - 4T^2\Delta_2)} \\
4\Delta_2^2 z^2 - 4\Delta_1\Delta_2 yz + \Delta_1^2 y^2 &= \Delta_1^2 y^2 - 4T^2\Delta_1^2\Delta_2 - d\Delta_2 y^2 + 4dT^2\Delta_2^2 \\
4\Delta_2^2 z^2 - 4\Delta_1\Delta_2 yz &= 4dT^2\Delta_2^2 - d\Delta_2 y^2 - 4T^2\Delta_1^2\Delta_2 \\
4\Delta_2 z^2 - 4\Delta_1 yz &= 4dT^2\Delta_2 - dy^2 - 4T^2\Delta_1^2 \\
0 &= 4\Delta_2 z^2 - 4\Delta_1 yz - 4dT^2\Delta_2 + dy^2 + 4T^2\Delta_1^2 \\
0 &= dy^2 + 4\Delta_2 z^2 - 4\Delta_1 yz - 4T^2(d\Delta_2 - \Delta_1^2).
\end{aligned}
\tag{A31}
$$

As equation (A15) derived from the maximization problem is the same as equation (A31) which is derived from the minimization problem, we can find portfolio's weight in any portfolios on CTF from MATLAB's numerical search by setting minimize/maximize portfolio's expected return given TE.

REFERENCES

Alexander, G. J., & Baptista, A. M. (2008). Active portfolio management with benchmarking: Adding a value-at-risk constraint. *Journal of Economic Dynamics and Control, 32*(3), 779-820.

Jorion, P. (2003). Portfolio optimization with tracking-error constraints. *Financial Analysts Journal, 59*(5), 70-82.

Luca, R. (2012). Using tracking error volatility to check active management and fee level of investment funds. *Global Business and Economics Review, 14*(3), 139-158.

Markowitz, H. (1952). The utility of wealth. *Journal of Political Economy, 60*(2), 151-158.

Merton, R. C. (1972). An analytic derivation of the efficient portfolio frontier. *Journal of Financial and Quantitative Analysis*, 1851-1872.

Palomba, G. (2008). Multivariate GARCH models and the Black-Litterman approach for tracking error constrained portfolios: an empirical analysis. *Global Business and Economics Review, 10*(4), 379-413.

Palomba, G., & Riccetti, L. (2012). Portfolio frontiers with restrictions to tracking error volatility and value at risk. *Journal of Banking & Finance, 36*(9), 2604-2615.

Riccetti, L. (2016). How risk managers should fix tracking error volatility and value-at-risk constraints in asset management. *Journal of Risk, Forthcoming*.

Zilbering, Y., Jaconetti, C. M., & Kinniry Jr, F. M. (2015). Best practices for portfolio rebalancing. *Valley Forge, Pa.: The Vanguard Group. Vanguard Research PO Box, 2600*, 19482-12600.

VITA

NAME Nuttawoot Ladee

DATE OF BIRTH 1 April 1990

PLACE OF BIRTH Chiang Rai