

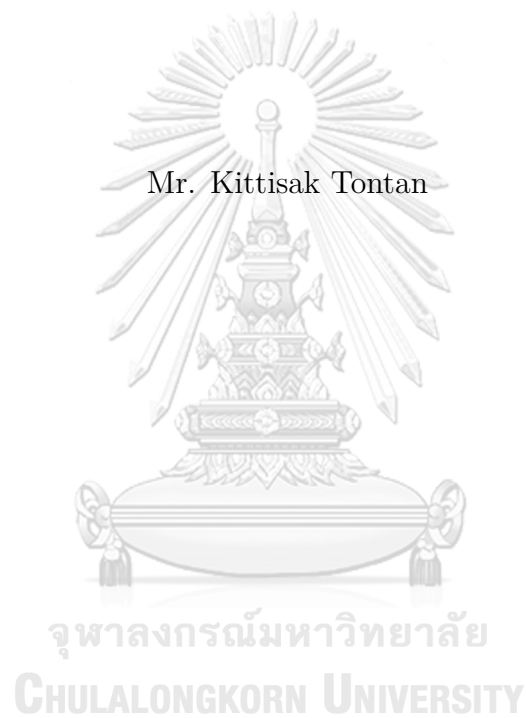
เสถียรภาพโดยแท้ของจุดตรึงและแผนการทำซ้ำ



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STRICT STABILITY OF FIXED POINTS AND ITERATION SCHEMES

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ในวิทยานิพนธ์นี้ เราแนะนำแนวคิดของจุดตรึงเสถียรโดยแท้ของการส่ง และสำรวจความสัมพันธ์ระหว่างเสถียรภาพแบบต่าง ๆ ของจุดตรึง เราพิสูจน์ว่าจุดตรึงทั้งหมดของการส่งกึ่งไม่ขยายนั้นเสถียรโดยแท้ เราขยายแนวความคิดของจุดตรึงเสถียรโดยแท้ของการส่งไปสู่แผนการทำซ้ำ และแนะนำแนวคิดของจุดตรึงเสถียรโดยแท้เชิงนูนของการส่ง และนำไปใช้เพื่อให้ได้เสถียรภาพเสมือนของบางแผนการทำซ้ำที่เป็นที่รู้จักในเซตย่อยนูนของปริภูมิบานาค



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In this thesis, we introduce the concept of a strictly stable fixed point of a selfmap and investigate the relationship among various types of fixed point stability. We prove that all fixed point of a quasi-nonexpansive map is strictly stable and extend the concept of a strictly stable fixed point of a selfmap to an iteration scheme. We introduce the concept of convexly strictly stable fixed points of selfmaps and apply it to obtain virtual stability of some well-known iteration schemes in a convex subset of a Banach space.



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CHAPTER I

INTRODUCTION

There are many notions of fixed point stability in the literatures. The simplest one can be found in the book by C. Adams and R. Fronzosa in 2007 [1]. Later, in 2009 [4], P. Chaoha and W. Atiponrat introduced the notion of virtual stability and used it to define a virtually stable selfmap. As a result, for every virtually stable map f , the fixed point set of f is a retract of the convergence set of f , and hence we can investigate some topological structures of the fixed point set of f via the convergence set. Moreover, in 2014 [7], A.H. Hamza and M. Faris introduced the concepts of strongly stable, c -stable, and strongly c -stable fixed points.

In 2012 [5], P. Chaoha and P. Chanthorn extended the concept virtual stability of a selfmap to an iteration scheme to obtain a retraction from the convergence set of the scheme onto its fixed point set. In particular, the Ishikawa iteration scheme for a quasi-nonexpansive map is virtually stable.

In this thesis, we will define the notion of a strictly stable fixed point of a selfmap. Then we explore the relationship among all mentioned fixed point stabilities. We also define the notion of a strictly stable fixed point of an iteration scheme. Finally, we introduce the concept of a co-strictly stable fixed point of a selfmap in Banach space setting to ensure virtual stability of well-known iteration schemes to guarantee the existence of a retraction from its convergence set onto its fixed point set.

CHAPTER II PRELIMINARIES

In this chapter, we review some definitions and notions used in this work. For more details, see in [1], [8], [10] and [12].

2.1 Topological Spaces

Definition 2.1. Let X be a nonempty set and τ a collection of subset of X such that

1. $\emptyset \in \tau$ and $X \in \tau$.
2. $\bigcup \mathcal{A} \in \tau$, for any subcollection \mathcal{A} of τ .
3. $\bigcap \mathcal{A} \in \tau$, for any finite subcollection \mathcal{A} of τ .

The collection as above τ is called a **topology** on X , an element $U \in \tau$ is called an **open set** in X , and The pair (X, τ) is called a **topological space**. We simply write X when there is no ambiguity.

A subset A of X is called a **closed set** if $A^c = X - A$ is open, for $x \in X$, an open set $U \in \tau$ containing x is called a **neighborhood** of x .

Definition 2.2. A topological space X is said to be

1. **Hausdorff** if any two distinct points in X have disjoint neighborhoods.
2. **regular** if each pair of $x \in X$ and a closed set $B \subseteq X$ disjoint from x , there exist disjoint open sets containing x and B , respectively.

Definition 2.3. Let (X, τ_X) be a topological space and $Y \subseteq X$. The **subspace topology** of Y is defined by

$$\tau_Y = \{U \cap Y \mid U \in \tau_X\}.$$

We say that (Y, τ_Y) is a **subspace** of X .

Definition 2.4. For a nonempty set X and \mathcal{B} a collection of subset of X . we call \mathcal{B} a **basis for a topology on X** if it satisfies the following conditions:

1. for each $x \in X$, there is $B \in \mathcal{B}$ such that $x \in B$.
2. for each $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there is $B_3 \in \mathcal{B}$ such that

$$x \in B_3 \subseteq B_1 \cap B_2.$$

Denote the set $\langle \mathcal{B} \rangle$ by $\{U \subseteq X \mid \forall x \in U \exists B \in \mathcal{B}, x \in B \subseteq U\}$, we have $\langle \mathcal{B} \rangle$ is a topology on X and call it the **topology generated by \mathcal{B}** .

Next, we let X be a nonempty topological space and A a nonempty subset of X .

Definition 2.5. A point $x \in X$ is called a **limit point** of A if for each neighborhood U of x ,

$$U \cap A - \{x\} \neq \emptyset.$$

The set of all limit points of A is called the **derived set** of A , denoted by A' .

Definition 2.6. The **interior** of A , denoted by $\text{Int}_X(A)$ is defined by

$$\text{Int}_X(A) = \bigcup \{U \subseteq X \mid U \subseteq A, \text{ and } U \text{ is open}\}.$$

The **closure** of A , denoted by $\text{Cl}_X(A)$ or \bar{A} is defined by

$$\text{Cl}_X(A) = \bigcap \{U \subseteq X \mid U \supseteq A, \text{ and } U \text{ is closed}\}.$$

Remark 2.7. From the definitions, we have

1. A is open in X if and only if $A = \text{Int}_X(A)$.
2. A is closed in X if and only if $A = \text{Cl}_X(A)$.

Theorem 2.8. Let $A, B \subseteq X$. Then

- (1) $\overline{\bar{A}} = \bar{A}$.
- (2) $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$.

Next, we will recall concepts of continuity. Let X and Y be topological spaces.

Definition 2.9. A map $f : X \rightarrow Y$ is said to be **continuous at $x \in X$** if, for any neighborhood V of $f(x)$, there exists a neighborhood U of x such that

$$f(U) \subseteq V.$$

We say that a map f is **continuous** if it is continuous at each $x \in X$.

Definition 2.10. Let A be a nonempty subset of X . A is said to be **connected** if there exists no pair of subsets U, V of X such that

1. $U \cup V = A$.
2. $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$.
2. $\bar{U} \cap V = \emptyset$ and $U \cap \bar{V} = \emptyset$.

Otherwise, A is said to be **disconnected**.

Example 2.11. The real line \mathbb{R} and the intervals in \mathbb{R} are connected. Let $a < b < c$ be points in an interval I . If we delete the point b , then we obtain a disconnected set $I - \{b\}$. Similarly, $\mathbb{R} - \{b\}$ is also disconnected.

Definition 2.12. Let A be a nonempty subset of X . A is said to be **path connected** if for every pair of $x, y \in A$, there exists a continuous map $p : [0, 1] \rightarrow X$ such that $p(0) = x$ and $p(1) = y$.

Theorem 2.13. *A path connected set is always connected.*

Theorem 2.14. *Let $f : X \rightarrow Y$ be continuous. If A is a (path) connected subspace of X , then $f(A)$ is (path) connected.*

Definition 2.15. Let A be a subset X . A continuous map $R : X \rightarrow A$ is said to be a **retraction** if for any $a \in A$, $R(a) = a$. The set A is said to be a **retract** of X .

Notice that R is surjective since $R(a) = a$.

The next corollary shows that the retract of a (path) connected space is also (path) connected.

Corollary 2.16. *Let $R : X \rightarrow A$ be a retraction. If X is (path) connected, then A is also (path) connected.*

Proof. Let $R : X \rightarrow A$ be a retraction and X a (path) connected space. Since R is surjective, thus $R(X) = A$ is (path) connected by Theorem 2.14. \square

Definition 2.17. A map $f : X \rightarrow Y$ is said to be **open** if for each an open subset U of X , the set $f(U)$ is open Y . A **closed** map is defined in the similar manner.

Definition 2.18. The **sequence** (x_n) in X is defined to be a function

$$(x_n) : \mathbb{N} \rightarrow X.$$

We denote the value $(x_n)(i)$ by x_i .

Definition 2.19. The sequence (x_n) in X is said to **converge** to $x \in X$, we write $x_n \rightarrow x$ or $\lim_{n \rightarrow \infty} x_n = x$, if for each neighborhood U of x , there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. When $x_n \rightarrow x$, we call x a **limit** of (x_n) .

Theorem 2.20. A convergent sequence in a Hausdorff space has a unique limit.

Theorem 2.21. Let $x \in X$. If f is continuous and a sequence (x_n) converges to x , then $(f(x_n))$ converges to $f(x)$.

Remark 2.22. We rewrite the result of the previous theorem as:

$$\lim_{n \rightarrow \infty} f(x_n) = f(x) = f(\lim_{n \rightarrow \infty} x_n).$$

Definition 2.23. A **metric** on X is defined to be a function $d : X \times X \rightarrow \mathbb{R}$ such that

1. $d(x, y) \geq 0$ for all $x, y \in X$.
2. $d(x, y) = 0 \Leftrightarrow x = y$.
3. $d(x, y) = d(y, x)$ for all $x, y \in X$.
4. $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

The pair (X, d) is called a **metric space**.

Given $r > 0$, the set $B_d(x, r) = \{y \in X \mid d(x, y) < r\}$ is called the **open r -ball** centered at x . Let $\tau_d = \{U \subseteq X \mid \forall x \in U \exists r > 0, B_d(x, r) \subseteq U\}$. Then τ_d is a topology on X , it is called the **topology induced by the metric d** . When there is no ambiguity, we will omit d and simply write X .

Example 2.24. The **Euclidean metric** d on \mathbb{R}^n is defined by

$$d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2},$$

for all $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in \mathbb{R}^n .

Definition 2.25. Let (X, d) be a metric space. A sequence (x_n) in X is called a **Cauchy sequence** if for each $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$d(x_m, x_n) < \varepsilon,$$

for all $m, n \geq N$.

Definition 2.26. A metric space (X, d) is said to be **complete** if every Cauchy sequence in X converges in X .

Definition 2.27. Let X be a topological space, (Y, d) a metric space and \mathcal{F} a set of continuous maps from X to Y . For $x_0 \in X$, the set \mathcal{F} is said to be **equicontinuous at x_0** if for each $\varepsilon > 0$, there exists a neighborhood U of x_0 such that

$$d(f(x), f(x_0)) < \varepsilon$$

for every $x \in U$ and $f \in \mathcal{F}$. We simply say that \mathcal{F} is **equicontinuous on X** if it is equicontinuous at each $x_0 \in X$.

Definition 2.28. A **vector space** or **linear space** $\langle V, +, \cdot \rangle$ over \mathbb{R} (or \mathbb{C}) is a nonempty set V with vector addition and scalar multiplication by elements of \mathbb{R} (or \mathbb{C}).

Vector addition is a map $+ : V \times V \rightarrow V$ defined by

$$(x, y) \mapsto x + y,$$

a vector $x + y$ is called **sum** of x and y , and the following conditions hold.

1. $x + y = y + x$ for all $x, y \in V$.
2. $(x + y) + z = x + (y + z)$ for all $x, y, z \in V$.
3. There exists a vector $0 \in V$ is called **identity**, for any $x \in V$,

$$x + 0 = x = 0 + x.$$

4. For any $x \in V$, there exists a vector $-x$ such that

$$x + (-x) = 0 = (-x) + x.$$

Scalar multiplication is a map $\cdot : \mathbb{R}$ (or \mathbb{C}) $\times V \rightarrow V$ defined by

$$(\alpha, x) \mapsto \alpha \cdot x,$$

a vector $\alpha \cdot x$ (also written αx) is called **product** of α and x , and the following conditions hold.

1. $\alpha(\beta x) = (\alpha\beta)x$ for all $x \in V$ and $\alpha, \beta \in \mathbb{R}$ (or \mathbb{C}).
2. $1x = x$ where 1 is multiplicative identity in \mathbb{R} (or \mathbb{C}).

3. $\alpha(x + y) = \alpha x + \alpha y$, for all $x, y \in V$ and $\alpha \in \mathbb{R}$ (or \mathbb{C}).

4. $(\alpha + \beta)x = \alpha x + \beta x$, for all $x \in V$ and $\alpha, \beta \in \mathbb{R}$ (or \mathbb{C}).

Definition 2.29. A **topological vector space** or **linear topological space** $\langle X, +, \cdot \rangle$ over \mathbb{R} (or \mathbb{C}) is a nonempty set X with Hausdorff topology such that vector addition and scalar multiplication by elements of \mathbb{R} (or \mathbb{C}) are continuous.

Definition 2.30. A **norm** on a linear space E is defined to be a real-valued function $\|\cdot\| : V \rightarrow \mathbb{R}$ such that

1. $\|x\| \geq 0$ for all $x \in E$.
2. $\|x\| = 0$ if and only if $x = 0$.
3. $\|ax\| = |a|\|x\|$ for all $x \in E$ and $a \in \mathbb{R}$ (or \mathbb{C}).
4. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in E$.

The **metric induced by a norm** $\|\cdot\|$ is defined by

$$d(x, y) = \|x - y\|.$$

A **normed space** $\langle E, \|\cdot\| \rangle$ is a nonempty set E together with a norm $\|\cdot\|$ defined on it. A **Banach space** is a complete normed space.

Definition 2.31. Let A be a subset of a linear space V . The set A is said to be **convex** if for any $x, y \in A$,

$$(1 - t)x + ty \in A$$

where $t \in [0, 1]$.

Definition 2.32. Let A be a subset of a linear space V . The smallest convex set in E containing A is called **convex hull** of A , denoted by $\text{co}\{A\}$.

Proposition 2.33. *Suppose that X is a linear topological space over \mathbb{R} , then the arbitrary intersection of convex sets is convex.*

Proof. Let $\{C_\alpha\}$ be a collection of convex subsets of X , $x, y \in \bigcap_{\alpha \in \Lambda} C_\alpha$ and $t \in [0, 1]$.

Then $x, y \in C_\alpha$ for each $\alpha \in \Lambda$.

By convexity of each C_α , $(1 - t)x + ty \in C_\alpha$ for each $\alpha \in \Lambda$.

Thus, $(1 - t)x + ty \in \bigcap_{\alpha \in \Lambda} C_\alpha$ and hence $\bigcap_{\alpha \in \Lambda} C_\alpha$ is convex. □

Theorem 2.34. [12] Let A be a subset of a linear topological space X . $x \in \text{co}\{A\}$ if and only if there exist $n \in \mathbb{N}$ and points $x_1, x_2, \dots, x_n \in A$ such that

$$x = \sum_{i=1}^n \lambda_i x_i$$

where $\lambda_1, \lambda_2, \dots, \lambda_n \geq 0$ and $\lambda_1 + \lambda_2 + \dots + \lambda_n = 1$.

Theorem 2.35. [12] Let A be a subset of a linear topological space X . If A is an open set, then the convex hull $\text{co}\{A\}$ is open.

Example 2.36. Suppose that X is a normed linear topological space and d is the metric induced by norm $\|\cdot\|$.

For each $x \in X$ and $r > 0$, consider an open ball $B_d(x, r)$.

Let $x, y \in B_d(a, r)$ and $t \in [0, 1]$. Then

$$\|a - ((1-t)x + ty)\| = \|(1-t)(a-x) + t(a-y)\| \leq (1-t)\|a-x\| + t\|a-y\| < r.$$

That is $(1-t)x + ty \in B_d(a, r)$. Hence, an open ball $B_d(x, r)$ is convex.

Example 2.37. Suppose that $\|\cdot\|$ induces the Euclidean metric and $a < b \in \mathbb{R}$.

Let $C = \{(x, y) \mid a < x < b\}$ and $(x_1, y_1), (x_2, y_2) \in C$. Note that $a < x_1, x_2 < b$.

We have

$$a < (1-t)x_1 + tx_2 < b$$

$$\text{and } (1-t)(x_1, y_1) + t(x_2, y_2) = ((1-t)x_1 + tx_2, (1-t)y_1 + ty_2) \in C.$$

Thus C is convex.

Theorem 2.38. A convex subset C of a linear topological space X is path connected and connected.

Proof. For any $x, y \in C$, define $p : [0, 1] \rightarrow X$ by

$$p(t) = (1-t)x + ty.$$

We have $p(0) = x$, $p(1) = y$ and C is path connected. Thus C is connected. \square

2.2 Fixed Point Stability

Let X be a (nonempty) Hausdorff space and $f : X \rightarrow X$ a continuous selfmap. The fixed point set and the convergence set of f is defined to be

$$F(f) = \{x \in X \mid f(x) = x\} \text{ and}$$

$$C(f) = \{x \in X \mid (f^n(x)) \text{ converges}\},$$

respectively. The composition $f_n \circ f_{n-1} \circ \dots \circ f_j$ is denoted by $\prod_{i=j}^n f_i$. In particular,

$\prod_{i=1}^n f$ is the n -th iterate f^n of f . Hence, we naturally get a map $f^\infty : C(f) \rightarrow F(f)$ given by

$$f^\infty(x) = \lim_{n \rightarrow \infty} f^n(x)$$

for each $x \in C(f)$. A map f^∞ may not be continuous in general which is shown in the next example.

Example 2.39. Consider the map $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$. We have $F(f) = \{0, 1\}$ and $C(f) = [-1, 1]$. But $f^\infty : [-1, 1] \rightarrow \{0, 1\}$ given by

$$f^\infty(x) = \begin{cases} 1, & \text{if } x = -1, 1, \\ 0, & \text{otherwise,} \end{cases}$$

is not continuous.

Next, we introduce some terminology and notations of various fixed point stability used throughout our work. In this work, we always assume that $F(f) \neq \emptyset$.

Definition 2.40. [1] A fixed point p of f is said to be **stable** if for each neighborhood U of p , there exists a neighborhood $V \subseteq U$ of p such that

$$f^n(V) \subseteq U$$

for each $n \in \mathbb{N}$.

Definition 2.41. [7] A fixed point p of f is said to be

1. **strongly stable** if p is stable and there exists a neighborhood V of p such that a sequence $(f^n(x))$ converges to p for each $x \in V$,
2. **c-stable** if for each neighborhood U of p , there exists a neighborhood V of p such that $f^n(V) \subseteq \bar{U}$ for each $n \in \mathbb{N}$ and,
3. **strongly c-stable** if p is c-stable and there exists a neighborhood V of p such that a sequence $(f^n(x))$ converges to p for each $x \in V$.

Theorem 2.42. [7] *A stable fixed point is always c-stable.*

Proof. Let $p \in F(f)$ be stable and U a neighborhood of p . There is a neighborhood $V \subseteq U$ of p such that

$$f^n(V) \subseteq U \subseteq \bar{U}.$$

Then p is c-stable. □

Definition 2.43. [4] A fixed point p of f is said to be **virtually f -stable** if for each neighborhood U of p , there exists a neighborhood V of p and a strictly increasing sequence (k_n) in \mathbb{N} such that

$$f^{k_n}(V) \subseteq U$$

for all $n \in \mathbb{N}$. We say that a map f is **virtually stable** if every its fixed point is virtually f -stable.

Definition 2.44. [4] A fixed point p of f is said to be **uniformly virtually f -stable** if there exists a strictly increasing sequence of natural number (k_n) such that for each neighborhood U of p , there exists a neighborhood V of p with

$$f^{k_n}(V) \subseteq U$$

for each $n \in \mathbb{N}$. We simply call f **uniformly virtually stable** if every fixed point is uniformly virtually f -stable with respect to the same (k_n) .

The next theorem shows that a virtually stable fixed point is generally more than a stable fixed point.

Theorem 2.45. *A stable fixed point is uniformly virtually stable with respect to the sequence of all natural number.*

Proof. Let $p \in F(f)$ be stable and U a neighborhood of p . There is a neighborhood $V \subseteq U$ of p such that

$$f^n(V) \subseteq U.$$

We choose $k_n = n$. Then p is uniformly virtually stable with respect to (n) . □

From the previous theorems, we have the following implications of fixed point stability:

$$\begin{array}{ccccc} \text{strongly stable} & \Rightarrow & \text{stable} & \Rightarrow & \text{uniformly virtually } f\text{-stable} \\ \downarrow & & \downarrow & & \downarrow \\ \text{strongly c-stable} & \Rightarrow & \text{c-stable} & & \text{virtually } f\text{-stable.} \end{array}$$

Next, we present some basic maps in a metric space.

Definition 2.46. [3] Let (X, d) be a metric space.

- f is called **nonexpansive** if $d(f(x), f(y)) \leq d(x, y)$ for any $x, y \in X$.
- f is called **quasi-nonexpansive** if $d(f(x), p) \leq d(x, p)$ for any $x \in X$ and $p \in F(f)$.
- f is called **virtually nonexpansive** if $\{f^n \mid n \in \mathbb{N}\}$ is equicontinuous on $F(f)$.

We have following implications:

$$\text{nonexpansive} \Rightarrow \text{quasi-nonexpansive} \Rightarrow \text{virtually nonexpansive}.$$

See more details in [3]. In a metric space, we have the following proposition.

Proposition 2.47. [6] Let (X, d) be a metric space. Then f is virtually nonexpansive if and only if f is uniformly virtually stable with respect to (n) .

Theorem 2.48. [4] Suppose X is a regular space. If f is virtually stable, then f^∞ is continuous and hence $F(f)$ is a retract of $C(f)$.

By the previous theorem, we can explore some topological structures of convergence sets and fixed point sets of virtually stable maps, see more details in [4]. Next, we introduce the concept of fixed point iteration schemes.

2.3 Fixed Point Iteration Schemes

For a sequence $\mathcal{S} = (s_n)$ of selfmaps on X ($F(s_n) \neq \emptyset$), the fixed point set and the convergence set of \mathcal{S} are defined to be

$$F(\mathcal{S}) = \bigcap_{f \in \mathcal{S}} F(f) \text{ and}$$

$$C(\mathcal{S}) = \{x \in X \mid (s_n(x)) \text{ converges}\},$$

respectively. Hence, we obtain a map $r : C(\mathcal{S}) \rightarrow X$ given by

$$r(x) = \lim_{n \rightarrow \infty} s_n(x),$$

for each $x \in C(\mathcal{S})$. Clearly, $F(\mathcal{S}) \subseteq r(C(\mathcal{S}))$, but r may not be continuous and $r(C(\mathcal{S})) \not\subseteq F(\mathcal{S})$. So we introduce the notion of an iteration scheme.

Definition 2.49. [5] Let (f_n) be a sequence of selfmaps on X , and $\mathcal{S} = (s_n)$ a sequence where $s_n = \prod_{i=1}^n f_i$. A sequence \mathcal{S} is called a **fixed point iteration scheme** or simply a **scheme** if $\emptyset \neq F(\mathcal{S}) = r(C(\mathcal{S}))$. Moreover, a scheme $\mathcal{S} = (s_n)$ is said to have a **continuous subsequence** if there is a subsequence of \mathcal{S} consisting of continuous maps.

Example 2.50. The iteration scheme such that $f_i = f$ is well-known and usually called the **Picard iteration scheme for f** . We have $\mathcal{S} = (s_n) = (f^n)$. If f is continuous, then

$$r(x) = \lim_{n \rightarrow \infty} f^n(x) \in F(f)$$

for any $x \in C(\mathcal{S})$. Moreover, \mathcal{S} has a continuous subsequence, $F(\mathcal{S}) = F(f)$ and $C(\mathcal{S}) = C(f)$.

Definition 2.51. [5] A fixed point p of the scheme \mathcal{S} is called **virtually stable** if for each neighborhood U of p , there exists a neighborhood V of p and a strictly increasing sequence (k_n) in \mathbb{N} such that

$$\prod_{i=j}^{k_n} f_i(V) \subseteq U$$

for each $n \in \mathbb{N}$ and $j \leq k_n$. We say that an iteration scheme \mathcal{S} is **virtually stable** if every of its fixed point is virtually stable.

Definition 2.52. [5] A fixed point p of a scheme \mathcal{S} is said to be **uniformly virtually stable** if there exists a strictly increasing sequence of natural number (k_n) such that for each neighborhood U of p , there exists a neighborhood V of p with

$$\prod_{i=j}^{k_n} f_i(V) \subseteq U$$

for each $n \in \mathbb{N}$ and $j \leq k_n$. We simply call the scheme S **uniformly virtually stable** if every common fixed point of S is uniformly virtually stable with respect to the same (k_n) .

Example 2.53. Let \mathcal{S} be the Picard iteration scheme for f .

Suppose that $p \in F(\mathcal{S}) = F(f)$ is uniformly virtually f -stable with respect to (n) .

We have

$$\prod_{i=j}^{k_n} f_i = f^{k_n - j + 1}$$

for all $j \leq k_n$. Since $k_n - j + 1$ is a natural number and p is uniformly virtually stable, for each neighborhood U of p , there always exists a neighborhood V of p with

$$f^{k_n-j+1}(V) = \prod_{i=j}^{k_n} f_i(V) \subseteq U.$$

Hence, p is virtually stable.

The next theorem is an extension of Theorem 2.48 from the concept of maps to schemes.

Theorem 2.54. [5] *If X is a regular space and \mathcal{S} is a virtually stable scheme having a continuous subsequence, then the map $r : C(\mathcal{S}) \rightarrow F(\mathcal{S})$ is continuous and hence $F(\mathcal{S})$ is a retract of $C(\mathcal{S})$.*

Next, we introduce some well-known iteration schemes to be used in our work.

Definition 2.55. [11] Let X be a closed convex subset of a Banach space E and $T : X \rightarrow X$ a selfmap. Let $(\alpha_i^{(k)})$ be a sequence in $[0, 1]$ for each $k = 1, \dots, n$. If $\mathcal{S} = (\prod_{i=1}^m f_i)$ is a scheme satisfying:

$$\begin{aligned} g_i^{(1)} &= (1 - \alpha_i^{(1)})I + \alpha_i^{(1)}T \\ g_i^{(2)} &= (1 - \alpha_i^{(2)})I + \alpha_i^{(2)}Tg_i^{(1)} \\ g_i^{(3)} &= (1 - \alpha_i^{(3)})I + \alpha_i^{(3)}Tg_i^{(2)} \\ &\vdots \\ f_i &= g_i^{(n)} = (1 - \alpha_i^{(n)})I + \alpha_i^{(n)}Tg_i^{(n-1)}, \end{aligned}$$

for each $i \in \mathbb{N}$, where I is the identity map, we will call \mathcal{S} the **n -step iteration scheme** for T associated to sequences $(\alpha_i^{(k)})$ where $k = 1, \dots, n$. In particular,

- when $(\alpha_i^{(n)}) = (1)$ and $(\alpha_i^{(k)}) = (0)$ for $k = 1, \dots, n - 1$, we will call \mathcal{S} the **Picard iteration scheme** for T , that is

$$f_i = T,$$

- when $(\alpha_i^{(k)}) = (0)$ for $k = 1, \dots, n - 1$, we will call \mathcal{S} the **Mann iteration scheme** for T associated to $(\alpha_i^{(n)}) = (\alpha_i)$, that is

$$f_i = (1 - \alpha_i)I + \alpha_i T,$$

the Mann iteration scheme associated to a constant sequence is usually called the **Krasnoselskij iteration scheme**, that is

$$f_i = (1 - \lambda)I + \lambda T,$$

- when $(\alpha_i^{(k)}) = (0)$ for $k = 1, \dots, n-2$, we will call \mathcal{S} the **Ishikawa iteration scheme** for T associated to $(\alpha_i^{(n)}) = (\alpha_i)$ and $(\alpha_i^{(n-1)}) = (\beta_i)$, that is

$$f_i = (1 - \alpha_i)I + \alpha_i T((1 - \beta_i)I + \beta_i T),$$

- when $(\alpha_i^{(k)}) = (0)$ for $k = 1, \dots, n-2$, we will call \mathcal{S} the **Noor iteration scheme** for T associated to $(\alpha_i^{(n)}) = (\alpha_i)$, $(\alpha_i^{(n-1)}) = (\beta_i)$ and $(\alpha_i^{(n-2)}) = (\gamma_i)$, that is

$$f_i = (1 - \alpha_i)I + \alpha_i T((1 - \beta_i)I + \beta_i T((1 - \gamma_i)I + \gamma_i T)).$$

Remark 2.56. We always have $F(T) \subseteq F(\mathcal{S})$. If \mathcal{S} is Mann iteration scheme for T with (α_n) where $\alpha_1 \neq 0, 1$, then $F(\mathcal{S}) = F(T)$. But the above definition does not ensure the existence of an n -step iteration scheme unless it is a scheme at the first place, or in a suitable situation (see more details in [2], [5] and [11]).

Definition 2.57. [9] If X is a closed convex subset of a Banach space E and $T : X \rightarrow X$ is a selfmap. Let k be a fixed positive integer, if \mathcal{S} is a Picard iteration scheme for

$$f = \alpha_0 I + \alpha_1 T + \dots + \alpha_k T^k$$

where $\alpha_j \geq 0$, for $j = 0, 1, 2, \dots, k$, $\alpha_1 > 0$ and

$$\alpha_0 + \alpha_1 + \dots + \alpha_k = 1,$$

we will call \mathcal{S} the **Kirk's iteration scheme** for T .

Theorem 2.58. [9] Let \mathcal{S} be a Kirk's scheme for a nonexpansive map T . Then $f(x) = x$ if and only if $T(x) = x$.

Corollary 2.59. If \mathcal{S} be a Kirk's scheme for a nonexpansive map T , then $F(\mathcal{S}) = F(T)$.

Proof. We can consider the Kirk's iteration scheme for T as the Picard iteration scheme for f . Then $F(\mathcal{S}) = F(f) = F(T)$, by the previous theorem. \square

The n -step iteration scheme and Kirk's iteration scheme are generalization of the Picard iteration scheme. The next theorem show that the Ishikawa iteration schemes for a suitable map are virtually stable.

Theorem 2.60. [5] If T is quasi-nonexpansive and \mathcal{S} is the Ishikawa iteration scheme for T associated to sequences (α_n) and (β_n) , then \mathcal{S} is virtually stable with respect to the sequence of all natural numbers.

In the next chapter, we define the notion of strict stability of fixed points which generally captures the fixed points of quasi-nonexpansive maps. Moreover, in Chapter 4, the above theorem is true for an n -step iteration scheme associates to the map whose every of fixed point is co-strictly stable.



CHAPTER III

STRICT STABILITY

In this chapter, we define the notion of strict stability to obtain a retraction of the convergence set onto the fixed point set and explore the relationship among various fixed point stability discussed in Chapter 2.

Let X be a (nonempty) Hausdorff space and $f : X \rightarrow X$ a continuous selfmap with $F(f) \neq \emptyset$.

Definition 3.1. A fixed point p of f is said to be **strictly stable** if for each neighborhood U of p , there exists a neighborhood V of p such that

$$f(V) \subseteq V \subseteq U.$$

Next, we explore the relationship among strict stability, stability, strong stability, c-stability, strong c-stability, virtual stability, uniform virtual stability.

Theorem 3.2. *A strictly stable fixed point is stable.*

Proof. Let $p \in F(f)$ be a strictly stable fixed point and U a neighborhood of p . There exists a neighborhood V of p such that

$$f(V) \subseteq V \subseteq U.$$

Then, for each $n \in \mathbb{N}$, we have

$$f^n(V) \subseteq f^{n-1}(f(V)) \subseteq f^{n-1}(V) \subseteq \cdots \subseteq f(V) \subseteq V \subseteq U.$$

Hence, p is stable. □

The next example show that strict stability does not imply strong stability.

Example 3.3. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $f(x, y) = (-x, y)$. Note that f is the reflection over the y -axis. Then $F(f) = \{(0, p) \mid p \in \mathbb{R}\}$. Consider $(0, p) \in F(f)$ and let $\varepsilon > 0$. For each $(x, y) \in B((0, p), \varepsilon)$, we have

$$\|(0, p) - f(x, y)\| = \|(0, p) - (-x, y)\| = \|(0, p) - (x, y)\| < \varepsilon.$$

That is $f(B((0, p), \varepsilon)) \subseteq B((0, p), \varepsilon)$. Then $(0, p)$ is strictly stable and hence stable. However, $(0, p)$ is not strongly stable because $(f^n(0, p + \varepsilon/2)) = ((0, p + \varepsilon/2))$ does not converge to $(0, p)$.

From the diagram in Chapter 2 and Theorem 3.2, we obtain the following implications:

$$\begin{array}{ccccc}
 & & \text{strictly stable} & & \\
 & & \Downarrow & & \\
 \text{strongly stable} & \Rightarrow & \text{stable} & \Rightarrow & \text{uniformly virtually } f\text{-stable} \\
 \Downarrow & & \Downarrow & & \Downarrow \\
 \text{strongly c-stable} & \Rightarrow & \text{c-stable} & & \text{virtually } f\text{-stable}
 \end{array}$$

Notice that if every fixed point of f is strictly stable, then f is a virtually stable map.

Corollary 3.4. *Let f be an open map and $p \in F(f)$. Then p is strictly stable if and only if p is stable.*

Proof. (\Rightarrow) Theorem 3.2.

(\Leftarrow) Let U be a neighborhood of p . Since p is stable, there exists a neighborhood $V_0 \subseteq U$ of p such that

$$f^n(V_0) \subseteq U,$$

for each $n \in \mathbb{N}$. Set

$$V = \bigcup_{n=0}^{\infty} f^n(V_0).$$

If f is open, $f(V_0)$ is an open set in X . Moreover, $f^n(V_0)$ is open in X for each $n \in \mathbb{N}$.

Thus V is also open in X . Note that $p \in V \subseteq U$ and

$$f(V) = f\left(\bigcup_{n=0}^{\infty} f^n(V_0)\right) = \bigcup_{n=1}^{\infty} f^n(V_0) \subseteq V \subseteq U.$$

Hence, p is strictly stable. □

In a regular space, the next corollary show that f^∞ is always a retraction from $C(f)$ onto $F(f)$ when every of its fixed point is strictly stable.

Corollary 3.5. *Suppose that X is a regular space. If every fixed point of f is strictly stable. Then $f^\infty : C(f) \rightarrow F(f)$ is continuous and hence $F(f)$ is a retract of $C(f)$.*

Proof. Let $p \in F(f)$. Since p is strictly stable, p is virtually stable. Then f is a virtually stable map. By Theorem 2.48, f^∞ is continuous and $F(f)$ is a retract of $C(f)$. □

The next theorem shows that every fixed point of a quasi-nonexpansive map is always strictly stable.

Theorem 3.6. *Let (X, d) be a metric space. If f is quasi-nonexpansive, then every fixed point of f is strictly stable.*

Proof. Let $p \in F(f)$ and U be a neighborhood of p .

There exists an open ball $B(p, \varepsilon) \subseteq U$. Since f is quasi-nonexpansive, we have

$$d(f(x), p) \leq d(x, p) < \varepsilon$$

for each $x \in B(p, \varepsilon)$. Therefore, $f(B(p, \varepsilon)) \subseteq B(p, \varepsilon)$. \square

The next example shows that the converse of the above theorem is not true.

Example 3.7. The map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$f(x, y) = \left(x, \frac{|x| + y}{2} \right).$$

We have $F(f) = \{(x, |x|) \mid x \in \mathbb{R}\}$. Consider $(0, 0) \in F(f)$, we obtain

$$1 = \|(0, 0) - (1, 0)\| < \|(0, 0) - f(1, 0)\| = \|(0, 0) - (1, 1/2)\| = \sqrt{5}/2.$$

Then f is not quasi-nonexpansive.

For each $(p, |p|) \in F(f)$ and $\varepsilon > 0$, choose

$$V = B((p, |p|), \varepsilon) \cap \{(x, y) \mid \|(p, |p|) - (x, |x|)\| < \varepsilon\}.$$

Let $(x, y) \in V$. Clearly, $f(x, y) \in \{(x, y) \mid \|(p, |p|) - (x, |x|)\| < \varepsilon\}$. Next, consider

$$\begin{aligned} \|(p, |p|) - f(x, y)\|^2 &= \left\| (p, |p|) - \left(x, \frac{|x| + y}{2} \right) \right\|^2 \\ &= (x - p)^2 + \left(|p| - \frac{|x| + y}{2} \right)^2 \\ &= (x - p)^2 + \left(\frac{(|p| - |x|) + (|p| - y)}{2} \right)^2 \\ &\leq (x - p)^2 + \frac{(|p| - |x|)^2}{2} + \frac{(|p| - y)^2}{2} \\ &= \frac{(x - p)^2 + (|p| - |x|)^2}{2} + \frac{(x - p)^2 + (|p| - y)^2}{2} \\ &= \frac{\|(p, |p|) - (x, |x|)\|^2}{2} + \frac{\|(p, |p|) - (x, y)\|^2}{2} < \varepsilon^2 \end{aligned}$$

Then $f(x, y) \in B((p, |p|), \varepsilon)$ and hence, $f(x, y) \in V$.
 Thus $(p, |p|)$ is strictly stable for any $(p, |p|) \in F(f)$.

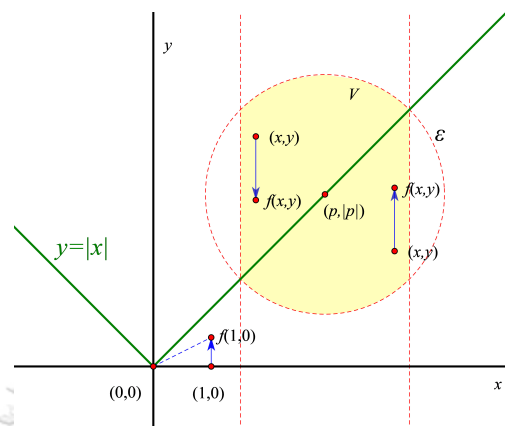


Figure 3.1: The point $f(0, 1)$ and mapping in the neighborhood V .

CHAPTER IV

APPLICATION TO ITERATION SCHEMES

We start this chapter by extending the notion of strictly stable fixed points of selfmaps to fixed points of iteration schemes. We introduce the definition of co-strictly stable fixed points of selfmaps to obtain virtual stability of some well-known iteration schemes in a closed convex subset of a Banach space.

Definition 4.1. A fixed point p of a scheme $\mathcal{S} = \left(\prod_{i=1}^n f_i \right)$ is said to be **strictly stable** if for each neighborhood U of p , there exists a neighborhood V of p such that

$$f_i(V) \subseteq V \subseteq U$$

for each $i \in \mathbb{N}$.

Remark 4.2. The strict stability of a fixed point p of each map in a scheme does not ensure the strict stability of p of this scheme. In Definition 3.1, the neighborhood V depends on U and f , that is $V = V(U, f)$, while the neighborhood V in the Definition 4.1 depends on U and all of f_i , that is $V = V(U, f_1, f_2, \dots)$. Then each neighborhood V for each map in scheme may not be the same.

Next, we show the connection between strict stability and virtual stability.

Theorem 4.3. Let $\mathcal{S} = \left(\prod_{i=1}^n f_i \right)$ be an iteration scheme. If $p \in F(\mathcal{S})$ is strictly stable, then it is uniformly virtually stable with respect to (n) .

Proof. Let $p \in F(\mathcal{S})$ and U a neighborhood of p .

For each $i \in \mathbb{N}$, there exists a neighborhood V of p such that

$$f_i(V) \subseteq V \subseteq U.$$

For each $n \in \mathbb{N}$ and $j \leq n$,

$$\prod_{i=j}^n f_i(V) \subseteq \prod_{i=j+1}^n f_i(f_j(V)) \subseteq \prod_{i=j+1}^n f_i(V) \subseteq \prod_{i=j+2}^n f_i(V) \subseteq \dots \subseteq f_n(V) \subseteq V \subseteq U.$$

Then p is uniformly virtually stable with respect to (n) . □

Corollary 4.4. *Let \mathcal{S} be an iteration scheme such that every of its fixed point is strictly stable. Then \mathcal{S} is uniformly virtually stable with respect to (n) .*

Next, we apply our results to obtain virtual stability of some well-known schemes in a convex subset of a Banach space. However, we require a special version of strict stability to prove it.

Let X be a convex subset of a Banach space E , $T : X \rightarrow X$ a continuous selfmap with $F(T) \neq \emptyset$.

Definition 4.5. A fixed point p of T is said to be **convexly strictly stable** or simply **co-strictly stable** if for each neighborhood U of p , there exists a convex neighborhood V of p such that

$$T(V) \subseteq V \subseteq U.$$

Example 4.6. Let (X, d) be a metric space. If T is quasi-nonexpansive, then every fixed point of T is co-strictly stable because any ball is convex.

Theorem 4.7. *Let \mathcal{S} be the n -step iteration scheme for T associated to sequences $(\alpha_i^{(k)})$ where $k = 1, \dots, n$ such that $F(\mathcal{S}) \subseteq F(T)$. If every fixed point of T is co-strictly stable then every fixed point of \mathcal{S} is strictly stable.*

Proof. Let $p \in F(\mathcal{S}) \subseteq F(T)$ and U a neighborhood of p .

Since $p \in F(T)$ is co-strictly stable, there exists a convex neighborhood V of p such that

$$T(V) \subseteq V \subseteq U.$$

Let $x \in V$ and $i \in \mathbb{N}$. Since V is convex and $T(V) \subseteq V$,

$$g_i^{(1)}(x) = (1 - \alpha_i^{(1)})x + \alpha_i^{(1)}T(x) \in V$$

$$g_i^{(2)}(x) = (1 - \alpha_i^{(2)})x + \alpha_i^{(2)}T(g_i^{(1)}(x)) \in V$$

⋮

$$f_i(x) = g_i^{(n)}(x)$$

$$= (1 - \alpha_i^{(n)})x + \alpha_i^{(n)}T(g_i^{(n-1)}(x)) \in V$$

We have $f_i(V) \subseteq V$ for each $i \in \mathbb{N}$. Therefore, p is strictly stable. □

Corollary 4.8. *Let \mathcal{S} be the Kirk's iteration scheme for T such that $F(\mathcal{S}) \subseteq F(T)$. If every fixed point of T is co-strictly stable then every fixed point of \mathcal{S} is strictly stable.*

Proof. Since \mathcal{S} is the Kirk's iteration scheme for T , it is the Picard iteration scheme for

$$f = \alpha_0 I + \alpha_1 T + \cdots + \alpha_k T^k$$

where k is a fixed integer and $\alpha_j \geq 0$, for $j = 0, 1, 2, \dots, k$, $\alpha_1 > 0$ and

$$\alpha_0 + \alpha_1 + \cdots + \alpha_k = 1.$$

Since \mathcal{S} is the Picard scheme for f , $F(f) = F(\mathcal{S}) \subseteq F(T)$. If every fixed point of f is co-strictly stable, then we complete the proof by Theorem 4.7.

Let $p \in F(f)$ and U a neighborhood of p . Then $p \in F(\mathcal{S}) \subseteq F(T)$.

Since $p \in F(T)$ is co-strictly stable, there exists a convex neighborhood V of p such that

$$T(V) \subseteq V \subseteq U.$$

Let $x \in V$. Then $T^j(x) \in V$ for $j = 1, 2, \dots, k$. Since V is convex,

$$f(x) = \sum_{j=0}^k \alpha_j T^j(x) \in V.$$

Then $f(V) \subseteq V$. Thus, $p \in F(f)$ is strictly stable. \square

Corollary 4.9. *If every fixed point of T is co-strictly stable, then the n -step iteration scheme for T and the Kirk's iteration scheme for T with $F(\mathcal{S}) \subseteq F(T)$ are uniformly virtually stable with respect to (n) and hence $F(\mathcal{S})$ is a retract of $C(\mathcal{S})$.*

Proof. Since T is continuous, both f_i defined in Definition 2.55 and f defined in Definition 2.57 are continuous. This implies that the n -step iteration scheme for T and the Kirk's iteration scheme for T have continuous subsequences. The result follows directly from Theorem 2.54, Theorem 4.7, Corollary 4.4 and Corollary 4.8. \square

The next example extends Lemma 3.4 in [5].

Example 4.10. The n -step iteration scheme and the Kirk's iteration scheme for a quasi-nonexpansive map T with $F(\mathcal{S}) \subseteq F(T)$ are uniformly virtually stable.

Theorem 4.11. *Let $T : X \rightarrow X$ be a continuous selfmap with $F(T) \neq \emptyset$. Suppose that T satisfies the following condition:*

$$\text{For any } A \subseteq X, \text{ if } T(A) \subseteq A \text{ then } T(\text{co}\{A\}) \subseteq \text{co}\{A\}. \quad (*)$$

Then $p \in F(T)$ is co-strictly stable if and only if p is strictly stable.

Proof. (\Rightarrow) It follows directly from the definitions.

(\Leftarrow) Let $p \in F(T)$ be strictly stable and U a neighborhood of p .

There always exists an open ball $B(p, \varepsilon) \subseteq U$.

Since p is strictly stable, there exists a neighborhood V_0 of p such that

$$T(V_0) \subseteq V_0 \subseteq B(p, \varepsilon).$$

Set $V := \text{co}\{V_0\}$. Since $B(p, \varepsilon)$ is a convex set containing V_0 , $V \subseteq B(p, \varepsilon)$.

Since V_0 is open, V is open by Theorem 2.35, and we have

$$T(V) = T(\text{co}\{V_0\}) \subseteq \text{co}\{V_0\} = V \subseteq B(p, \varepsilon) \subseteq U.$$

Therefore, p is co-strictly stable. □

Finally, we illustrate an example of a map which is not quasi-nonexpansive but all of its fixed points are co-strictly stable.

Example 4.12. The map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$T(x, y) = \left(x, \frac{|x| + y}{2} \right).$$

Notice that T is not quasi-nonexpansive but all of its fixed points are strictly stable and

$$F(T) = \{(x, |x|) \mid x \in \mathbb{R}\},$$

by Example 3.7. Observe that for any $(p, |p|) \in F(T)$,

$$T(V) \subseteq V$$

where

$$V = B((p, |p|), \varepsilon) \cap \{(x, y) \mid \|(p, |p|) - (x, |x|)\| < \varepsilon\}.$$

Next, consider the convexity of V . From Example 2.36, $B((p, |p|), \varepsilon)$ is convex.

If $C := \{(x, y) \mid \|(p, |p|) - (x, |x|)\| < \varepsilon\}$ is convex, by Proposition 2.33, V is the desired open neighborhood of p .

Claim that C can be written as $\{(x, y) \mid a < x < b\}$ for some $a < b \in \mathbb{R}$.

Without loss of generality, suppose that $p \geq 0$.

If $p = 0$, then $\|(x, |x|)\| < \varepsilon$ that is

$$-\frac{\varepsilon}{\sqrt{2}} < x < \frac{\varepsilon}{\sqrt{2}}.$$

If $p \geq \frac{\varepsilon}{\sqrt{2}} > 0$, then $x \geq 0$. Otherwise,

$$\|(p, |p|) - (x, |x|)\| = \|(p, p) - (x, -x)\| = \sqrt{2p^2 + 2x^2} > \sqrt{2}p \geq \varepsilon.$$

We have

$$\|(p, |p|) - (x, |x|)\|^2 = 2(p - x)^2 < \varepsilon^2$$

that is

$$p - \frac{\varepsilon}{\sqrt{2}} < x < p + \frac{\varepsilon}{\sqrt{2}}.$$

If $0 < p < \frac{\varepsilon}{\sqrt{2}}$.

When $x \geq 0$, it is similar to the previous case, we have

$$p - \frac{\varepsilon}{\sqrt{2}} < x < p + \frac{\varepsilon}{\sqrt{2}}.$$

Thus $0 \leq x < p + \frac{\varepsilon}{\sqrt{2}}$, since $p < \frac{\varepsilon}{\sqrt{2}}$.

When $x < 0$,

$$\|(p, |p|) - (x, |x|)\|^2 = \|(p, p) - (x, -x)\|^2 = 2p^2 + 2x^2 < \varepsilon^2.$$

Then

$$x^2 < \frac{\varepsilon^2 - 2p^2}{2}$$

that is $-\sqrt{\frac{\varepsilon^2 - 2p^2}{2}} < x < \sqrt{\frac{\varepsilon^2 - 2p^2}{2}}$.

Since $p < \frac{\varepsilon}{\sqrt{2}}$ and $x < 0$, $-\sqrt{\frac{\varepsilon^2 - 2p^2}{2}} < x < 0$.

Therefore,

$$-\sqrt{\frac{\varepsilon^2 - 2p^2}{2}} < x < p + \frac{\varepsilon}{\sqrt{2}}.$$

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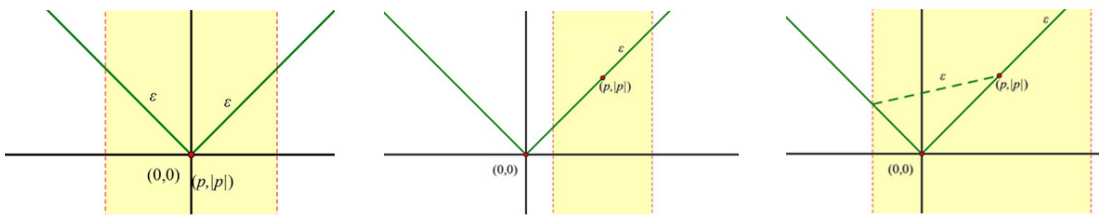


Figure 4.1: The set V when $p = 0$, $p \geq \frac{\varepsilon}{\sqrt{2}} > 0$ and $0 < p < \frac{\varepsilon}{\sqrt{2}}$, respectively.

Thus we obtain the claim, and hence V is convex.

Therefore, all of fixed points of T are not only strictly stable but also co-strictly stable.

However, we have to verify that $\mathcal{S} = \left(\prod_{i=1}^m f_i \right)$ where

$$f_i = g_i^{(n)} = (1 - \alpha_i^{(n)})I + \alpha_i^{(n)}Tg_i^{(n-1)}$$

as defined in Definition 2.55 is an iteration scheme.

Note that $F(T) = \{(x, y) \in \mathbb{R}^2 : y = |x|\}$ and

$$f_i(x, y) = (x, (1 - \lambda_i)y + \lambda_i|x|)$$

where

$$\lambda_i = \frac{\alpha_i^{(n)}}{2} + \frac{\alpha_i^{(n)}\alpha_i^{(n-1)}}{2^2} + \dots + \frac{\alpha_i^{(n)}\alpha_i^{(n-1)}\dots\alpha_i^{(1)}}{2^n}.$$

Then $f_i(x, y) = (x, y)$ if and only if

$$y = (1 - \lambda_i)y + \lambda_i|x|$$

$$\lambda_i y = \lambda_i|x|$$

$$y = |x|.$$

Hence, $F(f_i) = F(T)$ for each i ; that is $F(\mathcal{S}) = F(T)$.

If $\alpha_i^{(n)} = 0$, then $f_i = I$. This is not interesting, so we assume that $\alpha_i^{(n)} > 0$ for each $i \in \mathbb{N}$. Since $0 \leq \alpha_i^{(k)} \leq 1$ for $k \in \{1, 2, \dots, n\}$, $0 < \lambda_i \leq 1$; that is $0 \leq 1 - \lambda_i < 1$ and we have

$$\lim_{m \rightarrow \infty} (1 - \lambda_1)(1 - \lambda_2)\dots(1 - \lambda_m) = 0.$$

Next, we will show that $\mathcal{S} = \left(\prod_{i=1}^m f_i \right)$ is an iteration scheme.

Notice that for any $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned} r(x, y) &= \lim_{m \rightarrow \infty} s_m(x) \\ &= \lim_{m \rightarrow \infty} \prod_{i=1}^m f_i(x, y) \\ &= \lim_{m \rightarrow \infty} \prod_{i=2}^m f_i(f_1(x, y)) \\ &= \lim_{m \rightarrow \infty} \prod_{i=2}^m f_i(x, (1 - \lambda_1)y + \lambda_1|x|) \\ &= \lim_{m \rightarrow \infty} \prod_{i=3}^m f_i(f_2(x, (1 - \lambda_1)y + \lambda_1|x|)) \\ &= \lim_{m \rightarrow \infty} \prod_{i=3}^m f_i(x, (1 - \lambda_1)(1 - \lambda_2)y + (1 - (1 - \lambda_1)(1 - \lambda_2))|x|) \end{aligned}$$

$$\begin{aligned}
&= \lim_{m \rightarrow \infty} \prod_{i=4}^m f_i(x, (1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3)y + (1 - (1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3))|x|) \\
&\quad \vdots \\
&= \lim_{m \rightarrow \infty} (x, (1 - \lambda_1) \dots (1 - \lambda_m)y + (1 - (1 - \lambda_1) \dots (1 - \lambda_m))|x|) \\
&= (x, |x|) \in F(\mathcal{S})
\end{aligned}$$

Thus $C(\mathcal{S}) = \mathbb{R}^2$ and \mathcal{S} is the n -step iteration scheme for T .

Since every fixed point of T is co-strictly stable and $F(\mathcal{S}) = F(T)$, \mathcal{S} is uniformly virtually stable with respect to (n) , by Corollary 4.9.

Hence, the map $r : C(\mathcal{S}) \rightarrow F(\mathcal{S})$ is a retraction, by Theorem 2.54.

To use the above theorem, we have to show that T satisfies the condition (*). Let $A \subseteq \mathbb{R}^2$ such that $T(A) \subseteq A$ and $(x, y) \in T(\text{co}\{A\})$. We want to prove that $T(\text{co}\{A\}) \subseteq \text{co}\{A\}$. First, we will show that T is injective.

Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ such that

$$T(x_1, y_1) = T(x_2, y_2).$$

Then

$$\left(x_1, \frac{|x_1| + y_1}{2}\right) = \left(x_2, \frac{|x_2| + y_2}{2}\right)$$

that implies $(x_1, y_1) = (x_2, y_2)$. Hence, T is injective.

Since $(x, y) \in T(\text{co}\{A\})$, there exists $(x', y') \in (\text{co}\{A\})$ such that $(x, y) = T(x', y')$.

Since $T(x, 2y - |x|) = (x, y)$ and T is injective, we have

$$(x', y') = (x, 2y - |x|) \in \text{co}\{A\}.$$

Suppose that $(x, y) \notin \text{co}\{A\}$. Then $(x, |x|) \notin \text{co}\{A\}$, otherwise,

$$\frac{1}{2}(x, |x|) + \frac{1}{2}(x, 2y - |x|) = (x, y) \in \text{co}\{A\}.$$

Consider the projection onto 2nd-coordinate $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$\pi(x, y) = y.$$

Note π is continuous. Since $\text{co}\{A\}$ is convex, by Theorem 2.38, it is connected and $\pi(\text{co}\{A\})$ is connected. Since $(x, y) \notin \text{co}\{A\}$, $y \notin \pi(\text{co}\{A\})$. By connectedness of $\pi(\text{co}\{A\})$, either $\pi(\text{co}\{A\}) \subseteq (y, \infty)$ or $\pi(\text{co}\{A\}) \subseteq (-\infty, y)$.

Without loss of generality, suppose that $x \geq 0$ and $y \leq |x|$.

Since $2y - |x| \in \pi(\text{co}\{A\})$ and $2y - |x| < y$, $\pi(\text{co}\{A\}) \subseteq (-\infty, y)$, see Figure 4.2.

Since $(x, 2y - |x|) \in \text{co}\{A\}$, by Theorem 2.34, there exist $n \in \mathbb{N}$ and $(x_i, y_i) \in A$ for each $i = 1, \dots, n$ such that

$$(x, 2y - |x|) = \sum_{i=1}^n \lambda_i (x_i, y_i)$$

where $\lambda_1, \lambda_2, \dots, \lambda_n \geq 0$ and $\lambda_1 + \lambda_2 + \dots + \lambda_n = 1$.

We can assume that $x_1 = \max\{x_1, x_2, \dots, x_n\}$, then

$$x = \sum_{i=1}^n \lambda_i x_i \leq \sum_{i=1}^n \lambda_i x_1 = x_1.$$

Since $(x_1, y_1) \in A$, $T^k(x_1, y_1) \in A$ for all $k \in \mathbb{N}$.

Observe that $T^k(x_1, y_1) = (x_1, (1 - 2^{-k})|x_1| + 2^{-k}y_1)$ and $T^\infty(x_1, y_1) = (x_1, |x_1|)$.

We have

$$(1 - t)(x_1, y_1) + t(x_1, |x_1|) = (x_1, (1 - t)y_1 + t|x_1|) \in \text{co}\{A\} \text{ for each } t \in [0, 1].$$

Since $y_1 \in \pi(A) \subseteq \pi(\text{co}\{A\}) \subseteq (-\infty, y)$ and $y \leq |x| \leq |x_1|$,

$$y_1 < y \leq |x_1|.$$

There exists $t \in [0, 1]$ such that $y = (1 - t)y_1 + t|x_1|$.

If $t \in [0, 1)$, $(x_1, y) \in \text{co}\{A\}$, i.e. $y \in \pi(\text{co}\{A\})$, this contradicts to $(x, y) \notin \text{co}\{A\}$.

If $t = 1$, then $|x| \geq y = |x_1| \geq |x|$, i.e., $y = |x|$, and $(x, y) = (x, 2y - |x|) \in \text{co}\{A\}$.

This is a contradiction.

Therefore, $(x, y) \in \text{co}\{A\}$ and hence $T(\text{co}\{A\}) \subseteq \text{co}\{A\}$.

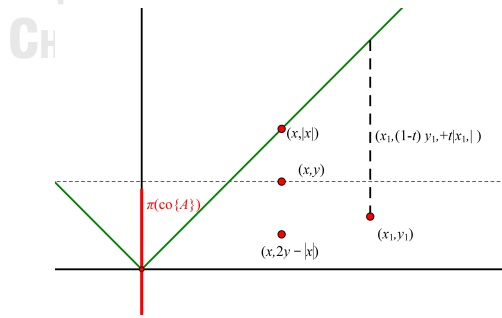


Figure 4.2: $\pi(\text{co}\{A\}) \subseteq (-\infty, y)$ and $y_1 < y$.

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