CHAPTER I

INTRODUCTION

In the classical case, an **integer-valued function** on the ring of integers has been defined for a long time. It is the function $f(t)$ that sends the set of non-negative integers ($\mathbb{N}_0$) to the set of integers. It is obvious that the ring of polynomials over $\mathbb{Z}$ is a subset of the set of integer-valued functions. Moreover, for each $m, n \in \mathbb{N}_0$, the binomial coefficients $\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}$, $n \neq 0$ and $\binom{m}{0} = 1$, are integers, so the function from $\mathbb{N}_0$ defined by $\binom{t}{n} = \frac{t(t-1)\cdots(t-n+1)}{n!}$, $n \neq 0$ and $\binom{t}{0} = 1$, is an integer-valued function. This implies that the linear combination over $\mathbb{Z}$ of the binomial functions, $\sum_{i=1}^{n} b_i \binom{t}{i}$, where $b_i \in \mathbb{Z}$ is an integer-valued function. Indeed, each integer-valued function $f(t)$ can be uniquely represented by an interpolation series

$$f(t) = b_0 + \sum_{i=1}^{\infty} b_i \binom{t}{i},$$

where $b_i \in \mathbb{Z}$. This series is well-defined because for each $m \in \mathbb{N}_0$, $\binom{m}{i} = 0$ if $m < i$, so $\sum_{i=1}^{\infty} b_i \binom{m}{i}$ becomes a finite sum and the representation is interpreted as yielding the same value of $f(m)$.

Based on the definition of an integer-valued function, de Bruijn [3] defined a universal function by adding the congruence condition as follows. An integer-valued function $f(t)$ is called a **universal function** if it satisfies

$$f(t+m) - f(t) \equiv 0 \pmod{m}$$

for all $t \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Note that every polynomial over $\mathbb{Z}$ is a universal function. From the definition, a universal function is also an integer-valued function. By using the previous interpolation series with some additional condition on coefficients, de Bruijn obtained the explicit shapes of universal functions states that, for each integer-valued function $f(t)$, it is a universal function if and only if it can be written in the form

$$f(t) = c_0 + \sum_{i=1}^{\infty} c_i s_i \binom{t}{i},$$

where $c_i \in \mathbb{Z}$ and $s_i = \operatorname{lcm}(1, 2, \ldots, i)$ for all positive integers i. If $f(t)$ is given, then the c_i's are uniquely determined.

De Bruijn also defined a modular function as follows. A **modular function** $f(t)$ is a function from the set of all integers to itself satisfying

$$f(t+m) - f(t) \equiv 0 \pmod{m}$$

for all $t \in \mathbb{Z}$ and $m \in \mathbb{Z}\backslash\{0\}$. Since the domain of the universal function is extended, the previous interpolation series is not available. However, de Bruijn obtained the certain interpolation series that represented a modular function. He showed that, for each $f : \mathbb{Z} \to \mathbb{Z}$, $f(t)$ is a modular function if and only if it has the form

$$f(t) = c_0 + \sum_{i=1}^{\infty} c_i s_i \binom{t + \lfloor \frac{i}{2} \rfloor}{i},$$

where $c_i \in \mathbb{Z}$ and $s_i = \operatorname{lcm}(1, 2, \ldots, i)$. If $f(t)$ is given, then the c_i's are uniquely determined.

Hall [4] worked on universal functions but referred to them as **pseudo-polynomials**, and proved some result of de Bruijn. In addition, he also studied some algebraic structures of the set of all pseudo-polynomials. He used an asymptotic notation to characterize polynomials in this set. Moreover, he showed the set of all pseudo-polynomials is an integral domain but it is not a unique factorization domain.

For the case of function fields, let $\mathbb{F}_q$ be the finite field of q elements, $\mathbb{F}_q[x]$ the ring of polynomials over $\mathbb{F}_q$, and $\mathbb{F}_q(x)$ its field of quotients. Carlitz [1] started his work on function fields by defining polynomials

$$\psi_0(t) = t \quad \text{and} \quad \psi_k(t) = \prod_{\deg M < k} (t - M)$$

for $k \in \mathbb{N}$ which play the role analogous to the binomial expansion and derived their properties. Later, Wagner [6], making use of Carlitz's preparatory work, defined a **linear pseudo-polynomial over** $\mathbb{F}_q[x]$ as a function $f : \mathbb{F}_q[x] \to \mathbb{F}_q[x]$ satisfying the congruence equation

$$f(M + K) \equiv f(M) \pmod{K}$$

for all $M \in \mathbb{F}_q[x]$ and $K \in \mathbb{F}_q[x]\backslash\{0\}$ and the linear properties

$$f(cM) = cf(M) \quad \text{and} \quad f(M + K) = f(M) + f(K)$$

for all $c \in \mathbb{F}_q$ and $M, K \in \mathbb{F}_q[x]$. By linear properties on the $\mathbb{F}_q$ - vector space $\mathbb{F}_q[x]$, the previous congruence equation can be reduced to

$$f(K) \equiv 0 \pmod{K}.$$

Wagner also presented some properties of linear pseudo-polynomials over $\mathbb{F}_q[x]$ resembling the results in the classical case of de Bruijn and Hall. He showed that any linear pseudo-polynomial over $\mathbb{F}_q[x]$ can be uniquely represented into the interpolation series by using Carlitz's polynomials $\psi_k(t)$ as follows. For any linear function $f(t)$ over $\mathbb{F}_q[x]$, it is a pseudo-polynomial over $\mathbb{F}_q[x]$ if and only if it can be represented by the interpolation series

$$f(t) = \sum_{i=0}^{\infty} A_i L_i \frac{\psi_i(t)}{F_i}.$$

where F_i is a product of all monic polynomials over $\mathbb{F}_q[x]$ of degree i, and L_i is the least common multiple of all polynomials over $\mathbb{F}_q[x]$ of degree i. For the algebraic structures, he showed that the set of all linear pseudo-polynomials is a non-commutative ring under addition and composition and has no zero divisor.

In our work, we extend among other things the result of Wagner generalizing the pseudo-polynomials.

Chapter II consists of some notation, definitions and related theorems, mainly without proofs, from [1], [2], [5], [6] and [7], that will be used throughout this thesis.

In Chapter III, we obtain some arithmetic properties of the set of all pseudo-polynomials over $\mathbb{F}_q[x]$ including the representation by the certain interpolation series, the factorization and algebraic properties of pseudo-polynomials over $\mathbb{F}_q[x]$. Finally, the sets of the difference and higher order differences of pseudo-polynomials over $\mathbb{F}_q[x]$ generalized the sets introduced by Wagner [7] are studied. The interpolation series representing their elements are established.