โครงการ

การเรียนการสอนเพื่อประสบการณ์

ชื่อโครงการ	บทพิสูจน์ที่รัดกุมของการหาความน่าจะเป็นที่จำนวนนับสองจำนวนเป็นจำนวนเฉพาะสัมพัทธ์กัน A rigorous proof of the probability that two randomly chosen positive integers are relatively prime
ชื่อนิสิต	นายวีรวิชญ์ สวัสดิ์พานิช 5933547723
ภาควิชา	คณิตศาสตร์และวิทยาการคอมพิวเตอร์ สาขาวิชา คณิตศาสตร์
ปีการศึกษา	2562

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

บทพิสูจน์ที่รัดกุมของการหาความน่าจะเป็นที่จำนวนนับสองจำนวนเป็นจำนวนเฉพาะสัมพัทธ์กัน

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โครงงานนี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตร์บัณฑิต

สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2562

A RIGOROUS PROOF OF THE PROBABILITY THAT TWO RANDOMLY CHOSEN POSITIVE INTEGERS ARE RELATIVELY PRIME

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A Project Submitted in Partial Fulfillment of the Requirements
for the Degree of Bachelor of Science Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2019

หัวข้อโครงงาน บทพิสูจน์ที่รัดกุมของการหาความน่าจะเป็นที่จำนวนนับสองจำนวนเป็นจำนวนเฉพาะสัมพัทธ์กัน

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อาจารย์ที่ปรึกษาโครงงานหลัก ผู้ช่วยศาสตราจารย์ ดร.เก่ง วิบูลย์ธัญญ์

ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์ คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย อนุมัติให้นับโครงงานฉบับนี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาบัณฑิต ในรายวิชา 2301499 โครงงานวิทยาศาสตร์

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(ผู้ช่วยศาสตราจารย์ ดร.เก่ง วิบูลย์ธัญญ์)

.... กรรมการ

(ผู้ช่วยศาสตราจารย์ ดร.สำรวม บัวประดิษฐ์)

วีรวิชญ์ สวัสดิ์พานิช: บทพิสูจน์ที่รัดกุมของการหาความน่าจะเป็นที่จำนวนนับสองจำนวนเป็นจำนวนเฉพาะสัมพัทธ์กัน. (A RIGOROUS PROOF OF THE PROBABILITY THAT TWO RANDOMLY CHOSEN POSITIVE INTEGERS ARE RELATIVELY PRIME) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : ผศ. ดร.เก่ง วิบูลย์ธัญญ์, 39 หน้า.

ปัญหาเรื่องการหาความน่าจะเป็นที่จำนวนนับสองจำนวนเป็นจำนวนเฉพาะสัมพัทธ์กัน เป็นหนึ่งในปัญหาที่สำคัญในแขนงทฤษฎีความน่าจะเป็นและทฤษฎีจำนวนเชิงวิเคราะห์ ในโครงงานนี้ เราศึกษาบทพิสูจน์ของปัญหานี้ ทั้งวิธีอย่างง่ายที่พบได้ทั่วไป วิธีการทางทฤษฎีจำนวนเชิงความน่าจะเป็นและทฤษฎีจำนวนเชิงวิเคราะห์ รวมถึงให้บทพิสูจน์ที่รัดกุมมากขึ้น ภายใต้เมเชอร์ความน่าจะเป็นที่ชื่อว่า natural density measure

ภาควิชา	คณิตศาสตร์และวิทยาการคอมพิวเตอร์	ลายมือชื่อนิสิต	....
สาขาวิชา	คณิตศาสตร์	ลายมือชื่อ อ.ที่ปรึกษาหลัก	....
ปีการศึกษา	2562		

5933547723: MAJOR MATHEMATICS

KEYWORDS: COPRIMALITY, NATURAL DENSITY MEASURE, AVERAGE ORDER

WEERAWICH SWASPANICH : A RIGOROUS PROOF OF THE PROBABILITY THAT TWO RANDOMLY CHOSEN POSITIVE INTEGERS ARE RELATIVELY PRIME. ADVISOR : ASST. PROF. KENG WIBOONTON, Ph.D., 39 pp.

The problem of finding probability that two randomly chosen positive integers are relatively prime is one of the most important problem in probability theory and analytic number theory. In this project, we study intuitively folklore proof, probabilistic number theoretical proof and analytic number theoretical proof. Furthermore, we give a rigorous proof under the natural density measure.

Department:	Mathematics and Computer Science	Student's Signature ....
Field of Study:	Mathematics	Advisor's Signature ....
Academic Year:	2019	

Acknowledgements

I would like to express my sincere thanks of gratitude to my advisor, Assistant Professor Keng Wiboonton, Ph.D. for his vital support and valuable support throughout the course of this project. Moreover, I would also like to thank to my project committees, Professor Paisan Nakmahachalasint, Ph.D. and Assistant Professor Samruam Baupradist, Ph.D. for their keen interest and advice which makes this project more complete.

Moreover, I feel very thankful to all of my teachers who taught me all the precious things. I lastly wish to express my thankfulness to my friends and my family for their encouragement.

This project is funded by the Honors Program of Faculty of Science, Chulalongkorn University.

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Chapter I

INTRODUCTION

In 1849, Dirichlet proved in [4] that the probability that two natural numbers, chosen randomly, are relatively prime is equal to $\frac{6}{\pi^2}$.

In 1885, Gegenbauer showed in [5] that the probability that a natural number, chosen randomly, is square-free is $\frac{6}{\pi^2}$.

These show that the probabilities of these two events are coincide. In the literature, there are a number of folklore proofs of these results. We present some of these proofs in this chapter. First let us fix some notations.

Let

$P(k \mid n) = P(\{n \in \mathbb{N} \mid k \mid n\})$ be the probability that a randomly chosen natural number n can be divided by a fixed chosen natural number k,

$P((a,b) = n) = P(\{(a,b) \in \mathbb{N} \times \mathbb{N} \mid (a,b) = n\})$ the probability that the greatest common divisor of randomly chosen natural numbers a, b is equal to a fixed natural number n,

$Sq(a)$ the greatest integer such that its square can divide a,

$P(Sq(a) = n) = P(\{a \in \mathbb{N} \mid Sq(a) = n\})$ the probability that the largest square factor of a is a fixed natural number n.

If $a \in \{a \in \mathbb{N} | \ Sq(x) = 1\}$, then a is called **square-free**. Let $a\mathbb{N}$ be the set of positive multiples of a when $a \in \mathbb{N}$.

Here are some common folklore proofs of the above two results.

First folklore proof of a Dirichlet's theorem. Let α be the probability that a given randomly chosen positive integers a and b are relatively prime.

Let $A_n = \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid (a,b) = n\}$. Note that

$$A_n = \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid n \mid a \ \wedge \ n \mid b \ \wedge \ \left(\frac{a}{n}, \frac{b}{n}\right) = 1\}.$$

Since three events are mutually independent, $P(n \mid a) = \frac{1}{n}$ and the probability that $\frac{a}{n}, \frac{b}{n}$ are coprime is equal to α,

$$P(A_n) = P(n \mid a) \cdot P(n \mid b) \cdot P\left(\left(\frac{a}{n}, \frac{b}{n}\right) = 1\right) = \frac{1}{n} \cdot \frac{1}{n} \cdot \alpha = \frac{\alpha}{n^2}.$$

Since P is a probability measure, the sum of these probabilities is equal to 1. So

$$\alpha = \frac{1}{\sum_{i=1}^{\infty} \frac{1}{n^2}} = \frac{6}{\pi^2}.$$

First folklore proof of a Gegenbauer's theorem. Let β be the probability that a given randomly chosen positive integers b is square-free.

Let $B_n = \{b \in \mathbb{N} \mid Sq(b) = n\}$. Note that

$$B_n = \{b \in \mathbb{N} \mid n^2 \mid b \ \wedge \ Sq\left(\frac{b}{n^2}\right) = 1\}.$$

Since two events are mutually independent, $P(n^2 \mid b) = \frac{b}{n^2}$ and the probability that $\frac{b}{n^2}$ is square-free is equal to β,

$$P(B_n) = P(n^2 \mid b) \cdot P\left(Sq\left(\frac{b}{n^2}\right) = 1\right) = \frac{1}{n^2} \cdot \beta = \frac{\beta}{n^2}$$

Since P is a probability measure, the sum of these probabilities is equal to 1. So

$$\beta = \frac{1}{\sum_{i=1}^{\infty} \frac{1}{n^2}} = \frac{6}{\pi^2}.$$

Second folklore proof of a Dirichlet's theorem.

Let a, b be natural numbers such that they are relatively prime. Since a, b are relatively prime, there is no prime number p such that $p \mid a$ and $p \mid b$. Therefore $(a, b) = 1$ if and only if $(2 \nmid a$ or $2 \nmid b)$ and $(3 \nmid a$ or $3 \nmid b)$ and ...). Note that these events are mutually independent. Hence,

$$\begin{aligned} P((a,b) = 1) &= P((2 \nmid a \ \vee \ 2 \nmid b) \ \wedge \ (3 \nmid a \ \vee \ 3 \nmid b) \ \wedge \ \ldots) \\ &= P(2 \nmid a \ \vee \ 2 \nmid b) \ \cdot \ P(3 \nmid a \ \vee \ 3 \nmid b) \ \cdot \ \ldots \\ &= \prod_{\text{p prime}} (p \nmid a \ \vee \ p \nmid b) \ = \ \prod_{\text{p prime}} (1 \ - \ P(p \mid a \ \wedge \ p \mid b)) \\ &= \prod_{p \ prime} \left(1 - \frac{1}{p^2}\right) \end{aligned}$$

By the identity $\prod_{\text{p prime}} \left(1 - \frac{1}{p^n}\right) = \frac{1}{\zeta(n)}$,

$$P((a,b) = 1) = \prod_{\text{p prime}} \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$$

Second folklore proof of a Gegenbauer's theorem

Let n be a natural number such that n is square-free. Since n is square-free, there is no prime p such that $p^2|\ n$. Therefore, $Sq(n) = 1$ if and only if $((2^2 \nmid n)$ and $(3^2 \nmid n)$and...). Note that these events are mutually independent.

$$\begin{aligned} P(Sq(n) = 1) &= P(2^2 \nmid n) \ \wedge \ P(3^2 \nmid n) \ \wedge \ \ldots\ ) \\ &= P((2^2 \nmid n) \ \cdot \ P((3^2 \nmid n) \ \cdot \ \ldots \\ &= \prod_{\text{p prime}} (p^2 \nmid n) \ = \ \prod_{p\ prime} (1 \ - \ P(p^2|\ n)) \\ &= \prod_{\text{p prime}} \left(1 - \frac{1}{p^2}\right) \end{aligned}$$

By the identity $\prod_{\text{p prime}} \left(1 - \frac{1}{p^n}\right) = \frac{1}{\zeta(n)}$,

$$P(Sq(n) = 1) = \prod_{\text{p prime}} \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

In these proofs, we intuitively use that $P(k \mid n) = \dfrac{1}{k}$ when $n \in \mathbb{N}$. The question to ask is that: How one define the probability of a subset of $\mathbb{N}$. That is $P(A) = ?$ for $A \subseteq \mathbb{N}$. If we can define such $P(A)$, then we should ask: Is this P a probability measure on $\mathbb{N}$? It turns out that the reasonable P that we should define will be the so-called the natural density measure on $\mathbb{N}$ and unfortunately this density measure is not a probability measure. We will discuss these in Chapter 3. Now come to the second question. Does there exists a probability measure P on $\mathbb{N}$ such that $P(k \mid n) = \dfrac{1}{k}$ when $n \in \mathbb{N}$? The answer is negative. We will explore this in Chapter 3. In the next chapter, we provide some basic definitions and theorems in number theory, real analysis, probability theory and analytic number theory. After Chapter 3, finally in Chapter 4, we give an analytic number theoretical proof of Dirichlet's result.

Chapter II

PRELIMINARIES

In this chapter, relevant concepts are given in order to prove our main results. These include elementary definitions and theorems in analysis, analytic number theory and probability number theory.

2.1 Some background on Real Analysis

This section covers basic definitions and theorems in real analysis. We state these theorems without proofs. All of these facts and their proofs can be found in [8], [10] and [13].

Definition 2.1.1. A sequence (x_n) converges to a real number x if for all $\epsilon > 0$, then there exists N such that $|x_n - x| < \epsilon$ for all natural number $n \geq N$. The sequence (x_n) is said to **tend to** or **converge to** x, written $x_n \to x$ or $\lim_{n\to\infty} x_n = x$. Symbolically, this is:

$$\lim_{n\to\infty} x_n = x \iff \forall \epsilon > 0 \; \exists n \in \mathbb{N} \; \forall n \geq \mathbb{N}, |x_n - x| < \epsilon.$$

We called a (x_n) **convergent sequence**. Otherwise, (x_n) is called a **divergent sequence**.

Definition 2.1.2. Let (x_n) be a sequence in $\mathbb{R} \cup \{-\infty, \infty\}$. The **limit superior** of x_n, which denoted $\limsup_{n\to\infty} x_n$ or $\overline{\lim_{n\to\infty}} x_n$, is defined by

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup_{k\geq n} x_k \right) = \inf_{n\in\mathbb{N}} \left(\sup_{k\geq n} x_k \right).$$

Similarly, the **limit inferior** of x_n, which is written by $\liminf_{n\to\infty} x_n$ or $\underline{\lim_{n\to\infty}} x_n$, is defined by

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left(\inf_{k\geq n} x_k \right) = \sup_{n\in\mathbb{N}} \left(\sup_{k\geq n} x_k \right).$$

Definition 2.1.3. Let $\emptyset \neq A \subseteq \mathbb{R} \cup \{-\infty, \infty\}$. We called $x \in \mathbb{R} \cup \{-\infty, \infty\}$ a **limit point** or a **cluster point** or an **accumulation point** of A if every neighbourhood of x contains a point in $A \backslash \{x\}$. Contrarily, y is called an **isolation point** if it is not a limit point.

Definition 2.1.4. Let $A \subseteq \mathbb{R}$, let f be a real-valued function such that its domain includes A. Let $a, L \in \mathbb{R} \cup \{-\infty, \infty\}$ where a is a limit point of A. We say that L is a **limit** of f when $x \to a$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in A$ with $0 < |x - a| < \delta$, $|f(x) - L| < \epsilon$. The limit can be written as $\lim_{x \to a} f(x) = L$.

To write symbolically,

$$\lim_{x \to a} f(x) = L \iff \forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \in A \; 0 < |x - a| < \delta \to |f(x) - L| < \epsilon.$$

Definition 2.1.5. Let f be a real-valued function such that its domain includes A. Let $x \in A$ and $a \in \mathbb{R} \cup \{-\infty, \infty\}$ where a is a limit point of A. For each neighbourhood $B_\epsilon(a)$ when $\epsilon > 0$ define

$$\limsup_{x \to a} f(x) = \lim_{r \to 0^+} \sup\{f(x) \mid x \in A \cap B_\epsilon(r)\} \text{ if } a \in \mathbb{R}.$$

$$\liminf_{x \to a} f(x) = \lim_{r \to 0^+} \inf\{f(x) \mid x \in A \cap B_\epsilon(r)\} \text{ if } a \in \mathbb{R}.$$

$$\limsup_{x \to a} f(x) = \lim_{r \to \infty} \sup\{f(x) \mid x \in A \cap B_\epsilon(r)\} \text{ if } a = \pm\infty.$$

$$\liminf_{x \to a} f(x) = \lim_{r \to \infty} \inf\{f(x) \mid x \in A \cap B_\epsilon(r)\} \text{ if } a = \pm\infty.$$

Clearly, for all $a \in \mathbb{R} \cup \{-\infty, \infty\}$,

$$\liminf_{x \to a} f(x) \leq \limsup_{x \to a} f(x)$$

Theorem 2.1.6. *Let f be a real-valued function such that its domain includes A. Let $x \in A$ and $a \in \mathbb{R} \cup \{-\infty, \infty\}$ where a is a limit point of A. Then, $\lim_{x \to a} f(x)$ exists in $\mathbb{R} \cup \{-\infty, \infty\}$ if and only if*

$$\limsup_{x \to a} f(x) = \lim_{x \to a} f(x) = \liminf_{x \to a} f(x)$$

Definition 2.1.7. Let $(A_n \mid n \in \mathbb{N})$ be a sequence of subsets of a set X.

If $(A_n \mid n \in \mathbb{N})$ is an increasing sequence that is $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, we define

$$\lim_{n \to \infty} A_n = \bigcup_{n \in \mathbb{N}} A_n = \{x \in X \mid x \in A_n \text{ for some } n \in \mathbb{N}\}$$

If $(A_n \mid n \in \mathbb{N})$ is a decreasing sequence that is $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$, we define

$$\lim_{n \to \infty} A_n = \bigcap_{n \in \mathbb{N}} A_n = \{x \in X \mid x \in A_n \text{ for all } n \in \mathbb{N}..\}$$

Definition 2.1.8. The limit inferior and limit superior of a sequence of subsets $(A_n \mid n \in \mathbb{N})$ of a set X are defined by

$$\limsup_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{k\leq\mathbb{N}} A_k$$

$$\liminf_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k\leq\mathbb{N}} A_k$$

Theorem 2.1.9. *Let $(A_n \mid n \in \mathbb{N})$ be a sequence of subsets of a set X.Then*

1. $\liminf_{n\to\infty} A_n = \{x \in X \mid x \in A_n$ for all but finitely many $n \in \mathbb{N}\}$.
2. $\limsup_{n\to\infty} A_n = \{x \in X \mid x \in A_n$ for infinitely many $n \in \mathbb{N}\}$.
3. $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$.

Definition 2.1.10. Let $(A_n \mid n \in \mathbb{N})$ be a sequence of subsets of a set X. We say that the sequence (A_n) **converges** if and only if

$$\limsup_{n\to\infty} A_n = \lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$$

Several theorems on infinite product

Definition 2.1.11. Let (b_n) be a sequence of complex numbers. We say that $\prod_{n=1}^{\infty} b_n$ **converges** if there exists $m \in \mathbb{N}$ such that (b_n) are nonzero for all $n \geq m$ and the limit of the partial products $\prod_{k=m}^{n} b_k$,

$$\lim_{n\to\infty} \prod_{k=m}^{n} b_k = \lim_{n\to\infty} (b_m \cdot b_{m+1} \cdot \ldots \cdot b_n)$$

converges to a nonzero complex number p that is

$$\prod_{n=1}^{\infty} b_n = b_1 \cdot b_2 \cdot \ldots \cdot b_{m-1} \cdot p$$

The infinite product $\prod_{n=1}^{\infty} b_n$ **diverges** if it does not converge, in other word, exactly one of these events occur: there are infinitely many zero, $\lim_{n\to\infty} \prod_{k=m}^{n} b_k$ diverges or the limit converges to zero which is called **diverges to zero**.

Theorem 2.1.12. *An infinite product $\prod_{n=1}^{\infty} a_n$ converges if and only if $a_n \to 0$ and the series $\sum_{k=m+1}^{\infty} \log(1 + a_n) = L$ converges. Moreover, $\prod_{n=1}^{\infty} a_n = (1 + a_1)...(1 + a_m)\ e^L$*

Let $\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}$.

Theorem 2.1.13. *For all* $z \in \mathbb{C}$ with $Re(z) > 1$*,*

$$\zeta(z) = \prod_{p \text{ prime}}^{\infty} \left(1 - \frac{1}{p^z}\right)^{-1} = \prod_{p \text{ prime}}^{\infty} \frac{p^z}{p^z - 1}.$$

In particular,

$$\frac{\pi^2}{6} = \prod_{p \text{ prime}}^{\infty} \left(1 - \frac{1}{p^2}\right)^{-1} = \prod_{p \text{ prime}}^{\infty} \frac{p^2}{p^2 - 1}.$$

Theorem 2.1.14. *For all* $z \in \mathbb{C}$ with $Re(z) > 1$*,*

$$\frac{1}{\zeta(z)} = \prod_{p \text{ prime}}^{\infty} \left(1 - \frac{1}{p^z}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z}.$$

when $\mu(n)$ *is a Möbius function.*

Several theorems on Measure Theory

Definition 2.1.15. Let $\mathfrak{M}$ be a collection of subsets of an arbitrary set X. We will call $\mathfrak{M}$ a **σ-algebra** or **σ-field** if $\mathfrak{M}$ satisfied the following properties

1. $X \in \mathfrak{M}$.

2. If A $\in \mathfrak{M}$, then $A^c \in \mathfrak{M}$.

3. If $A_i \in \mathfrak{M}$ for all $i \in \mathbb{N}$, $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{M}$.

The pair $(X, \mathfrak{M})$ is called a **measurable space**. A set A is **measurable** or $\mathfrak{A}$ **measurable** if $A \in \mathfrak{M}$.

Definition 2.1.16. Let $\mathfrak{M}$ be a collection of subsets of an arbitrary set X. A **measure** μ on $(X, \mathfrak{M})$ is a a nonnegative extended real-valued function if it satisfies the following conditions:

1. $\mu(\emptyset) = 0$.

2. **(Countable Additive Property)** If $A_i \in \mathfrak{M}$ for all $i \in \mathbb{N}$ and $A_i \cap A_j = \emptyset$ for all i $\neq$ j that is they are **pairwise disjoint**, then μ is **finitely additive** that is

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

We will call a triplet $(X, \mathfrak{M}, \mu)$ **measurable space**.

Theorem 2.1.17. *Let μ be a measure on a σ-algebra $\mathfrak{M}$ of subsets of a set X and let $(A_n \mid n \in \mathbb{N})$ be a sequence in $\mathfrak{M}$.*

1. If $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \mu(A_n) = \mu(\lim_{n\to\infty} A_n)$.

2. If $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$ and $\mu(A_1) < \infty$, then $\lim_{n\to\infty} \mu(A_n) = \mu(\lim_{n\to\infty} A_n)$.

Definition 2.1.18. 1. A μ be a measure on a σ-algebra $\mathfrak{M}$ of subsets of a set X is called a **finite measure** if $\mu(X) < \infty$ and $(X,\ \mathfrak{M},\ \mu)$ is called a **finite measure space**.

2. A μ be a measure on a σ-algebra $\mathfrak{M}$ of subsets of a set X is called a **σ-finite measure** if there exists a sequence $(A_n \mid n \in \mathbb{N})$ in with $\bigcup_{n\in\mathbb{N}} A_n = X$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$. In this case, $(X,\ \mathfrak{M},\ \mu)$ is called a **σ-finite measure space**.

Definition 2.1.19. Given two measure spaces $(X,\ \mathfrak{M},\ \mu_1)$ and $(Y,\ \mathfrak{N},\ \mu_2)$. Consider the product measurable space $(X \times Y,\ \sigma(\mathfrak{M} \times \mathfrak{N}))$. A measure μ on $\mathfrak{M} \times \mathfrak{N}$ such that

$$\mu(A) = \mu_1(A_1) \cdot \mu_2(A_2)\ for\ A = A_1 \times A_2 \in \mathfrak{M} \times \mathfrak{N}$$

is called a **product measure** of μ_1, μ_2 and it is denoted $\mu_1 \times \mu_2$. The measure space $(X \times Y,\ \sigma(\mathfrak{M} \times \mathfrak{N}),\ \mu_1 \times \mu_2)$ is called a **product measure space** of $(X,\ \mathfrak{M},\ \mu_1)$ and $(Y,\ \mathfrak{N},\ \mu_2)$.

Definition 2.1.20. Given two sets X and Y. Let $A \subseteq X \times Y$ and let f be an extended real-valued function on A.

1. For each $x \in X$, $A(x,\cdot) := \{y \in Y \mid (x,y) \in A\}$ is the **x-section** of A. For each $y \in Y$, $A(\cdot,y) := \{x \in X \mid (x,y) \in A\}$ is the **y-section** of A.

2. For each $x \in X$, $f(x,\cdot)$ on $A(x,\cdot)$ is the **x-section** of f. For each $y \in Y$, $f(\cdot,y)$ on $A(\cdot,y)$ is the **y-section** of f.

Theorem 2.1.21. *Given the product measurable space $(X \times Y, \sigma(\mathfrak{M} \times \mathfrak{N}))$ of the measurable spaces $(X,\ \mathfrak{M})$ and $(Y,\ \mathfrak{N})$ and f is an extended real-valued function on A.*

1. If $A \in \sigma(\mathfrak{M} \times \mathfrak{N})$, then $A(x,\cdot) \in \mathfrak{N}$ for each $x \in X$ and $A(\cdot,y) \in \mathfrak{M}$ for each $y \in Y$.

2. If f is $\sigma(\mathfrak{M} \times \mathfrak{N})$-measurable function on $E \in \sigma(\mathfrak{M} \times \mathfrak{N})$, then $f(x,\cdot)$ is a $\mathfrak{N}$-measurable function on $A(x,\cdot) \in \mathfrak{N}$ for every each $x \in X$ and $f(\cdot,y)$ is a $\mathfrak{M}$-measurable function on $A(\cdot,y) \in \mathfrak{M}$ for every each $y \in Y$.

Theorem 2.1.22. *Given the product measurable space* $(X \times Y,\ \sigma(\mathfrak{M} \times \mathfrak{N}),\ \mu \times \nu)$ *of two* $\sigma-$*finite measurable spaces* $(X,\ \mathfrak{M},\ \mu)$, $(Y,\ \mathfrak{N},\ \nu)$ *and for every* $A \in \sigma(\mathfrak{M} \times \mathfrak{N})$. *Then,*

1. $\nu(A(x, \cdot))$ *is a* $\mathfrak{N}$*-measurable function on* $\in \mathfrak{N}$ *for every each* $x \in X$.

2. $\mu(A(\cdot, y))$ *is a* $\mathfrak{M}$*-measurable function on* $\in \mathfrak{M}$ *for every each* $y \in Y$.

2.2 Some background on Analytic number theory

Basic definitions and relevant theorems are provided in this section. For their proofs and further details, they are available in [1] and [12].

Definition 2.2.1. An **arithmetic function** is a function whose domain is the natural numbers.

Definition 2.2.2. The **Möbius function** μ is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes (k} \geq 0) \\ 0 & \text{if } p^2 | n \text{ for some prime } p. \end{cases}$$

Definition 2.2.3. The **Euler totient function** or **Euler's phi function** $\phi(n)$ is the quantity of natural numbers not exceeding than n which are relatively prime to n that is

$$\phi(n) = \sum_{k=1}^{n} 1$$

Theorem 2.2.4. *If* $n \geq 1$, *we obtain*

$$n = \sum_{d|n} \phi(d)$$

Theorem 2.2.5. *If* $n \geq 1$, *we have*

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$$

Definition 2.2.6. If $g(x) > 0$ for all x $\geq a$, we define

$$f(x) = O(g(x))$$

that $f(x)$ **is big oh of** $g(x)$ is $\dfrac{f(x)}{g(x)}$ if bounded; in other word, there exists $M > 0$ such that

$$|f(x)| \leq Mg(x) \text{ for all } x \geq a$$

Definition 2.2.7. If

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$$

we called that $f(x)$ **is an asymptotic to** $g(x)$. Symbolically,

$$f(x)\sim g(x) \text{ as } x\to\infty$$

Definition 2.2.8. An averaged order of an arithmetic function $f(n)$ is $g(n)$ if

$$\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{k=1}^{n}f(k)}{g(n)}=1.$$

To find the average of an arithmetic function $f(n)$ we can find by considering an arbitrary positive real number x instead which is in the form

$$\sum_{n\le x}f(n)$$

Theorem 2.2.9. *For $x>1$ we have*

$$\sum_{n\le x}\phi(n)=\frac{3}{\pi^2}x^2+O(x\log x).$$

Theorem 2.2.10 (Divergence of the sum of the reciprocals of the primes)**.** *For $x\ge 2$ we have*

$$\sum_{p\ \text{prime}\ \le x}\frac{1}{p}=\log(\log(x))+c+O\left(\frac{1}{\log x}\right).$$

when $c=\gamma-\sum_{\text{p prime}}\left(\log\frac{p}{p-1}-\frac{1}{p}\right)$ and γ is an Euler-Mascheroni constant. If x tends to infinity, the sum of the reciprocals of the primes will diverge.

Theorem 2.2.11 (Mertens' first theorem)**.** *For $x\ge 2$, we have*

$$\sum_{p\ \text{prime}\ \le x}\frac{\log p}{p}=\log x+O(1)$$

Theorem 2.2.12 (Mertens' second theorem)**.** *For $x\ge 2$ and γ is an Euler-Mascheroni constant. Thus,*

$$\prod_{p\ \text{prime}\ \le x}\left(1-\frac{1}{p}\right)=\frac{e^{-\gamma}}{\log x}\left(1+O(\frac{1}{\log x})\right)$$

Corollary 2.2.13 (Mertens' third theorem)**.**

$$\lim_{n\to\infty} \log n \prod_{p \text{ prime } \leq n} \left(1 - \frac{1}{p}\right) = e^{-\gamma}$$

when γ is an Euler-Mascheroni constant.

Theorem 2.2.14 (Prime Number Theorem)**.** *Let $\pi(x)$ be the number of primes not exceeding than x is asymptotic $\frac{x}{\log x}$. Symbolically,*

$$\pi(x) \sim \frac{x}{\log x}$$

Theorem 2.2.15. *Let p_n be the nth primes. Then,*

$$p_n \sim n \log n$$

2.3 Some background on Probability Theory

This section covers basic definitions and theorems whose details and proofs can be found in [2], [11].

Definition 2.3.1. Let $\mathfrak{F}$ be a collection of subsets of a sample space Ω. We will call $\mathfrak{F}$ a **σ-algebra** or **σ-field** if $\mathfrak{F}$ satisfies the following properties

1. $\Omega \in \mathfrak{F}$.

2. If A $\in \mathfrak{F}$, then $A^c \in \mathfrak{F}$.

3. If $A_i \in \mathfrak{F}$ for all $i \in \mathfrak{F}$, $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{F}$.

We will call a member in $\mathfrak{F}$ **event** or a **measurable set**.

Definition 2.3.2. Suppose that $\mathfrak{F}$ be a **σ-algebra** on a sample space Ω. Suppose that P: $\mathfrak{F} \rightarrow$ [0,1] is such that

1. P(Ω) = 1.

2. If $A_i \in \mathfrak{F}$ for all $i \in \mathbb{N}$ and $A_i \cap A_j = \emptyset$ for all i $\neq$ j,

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

We will call P a **probability measure** and the second property is called the **countably additive property**. We will call a triplet (Ω, $\mathfrak{F}$, P) a **probability space**.

Definition 2.3.3. If $A_1, A_2, ..., A_n$ are **mutually independent**, then every $k \geq n$ and every k-element nonempty subsets of events $B_1, A_2, ..., B_n$ of $A_1, A_2, ..., A_n$,

$$P\left(\bigcap_{i=1}^{k} B_i\right) = \prod_{i=1}^{k} P(B_i).$$

Theorem 2.3.4. *If $A_1, A_2, ..., A_n$ are independent, $A_1^C, A_2^C, ..., A_n^C$ are also mutually independent.*

Theorem 2.3.5 (Inclusion-Exclusion Principle)**.** *Consider $A_1, A_2, ..., A_n$. Let $|A|$ denote the number of elements of a set A. Then,*

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} |A_i| - \sum_{1\leq i,j\leq n}\sum |A_i \cap A_j| + \sum_{1\leq i,j,k\leq n}\sum |A_i \cap A_j \cap A_k|$$
$$+ ... + (-1)^{n-1} \sum_{1\leq q_1<q_2<...<q_k\leq n} ... \sum |A_{q_1} \cap A_{q_2} \cap ... \cap A_{q_n}| + (-1)^{k-1}|A_1 \cap A_2 \cap ... \cap A_k|$$

Theorem 2.3.6 (First Borel-Cantelli Lemma)**.** *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of events in a probability space with a probability measure $P(\cdot)$. If $\sum_{n=1}^{\infty} P(A_n)$ converges,*

$$P(\limsup_{n\to\infty}(A_n)) = 0$$

Theorem 2.3.7 (Second Borel-Cantelli Lemma)**.** *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of mutually independent events in a probability space with a probability measure $P(\cdot)$. Then,*

$$P(\limsup_{n\to\infty}(A_n)) = \begin{cases} 0 & \text{if} \sum_{n=1}^{\infty} P(A_n) \text{ converges,} \\ 1 & \text{otherwise.} \end{cases}$$

Chapter III

PROBABILISTIC PROOF

This chapter presents our main result with probabilistic approach. First we define the density measure on $\mathbb{N}$. Then, we show that the density measure is not a probability measure.

3.1 The density measure is not a probability measure.

Before we give a proof that the density measure is not a measure, we give a definition of the density measure which is used intuitively as a probability measure in many folklore proofs.

Definition 3.1.1. Let $\mathbb{N}_n$ be $\{1, 2, ..., n\}$ and A be a subset of $\mathbb{N}$ and $|A|$ denotes its cardinality. If

$$D(A) \;=\; \lim_{n\to\infty} \frac{|A \cap \mathbb{N}_n|}{n} \tag{3.1}$$

exists, then $D(A)$ is called the **density measure**. This quantity represents the possibility of the occurrence of A.

Theorem 3.1.2. $D(k\mathbb{N}) \;= \frac{1}{k}$ *when k is a natural number.*

Proof. Let n be a randomly chosen natural number and k be a natural number. By the Division algorithm,

$$n \;= kq_n \;+\; r$$

when q_n is a quotient of n divided by k and r is a natural number such that $0 \leq \; r < \; k$
Consider occurrences of the event, then we have occurrences are q_n which equal to $\frac{n-r}{k}$ and all possibilities are equal to n.
By the measure, the probability is

$$D(k\mathbb{N}) \;=\; \lim_{n\to\infty} \frac{\frac{n-r}{k}}{n} = \frac{1}{k}$$

as we desired. □

By Theorem 3.1.2, $D(k\mathbb{N}) = \frac{1}{k}$ when k is a natural number. If a and b are coprime natural numbers, then $a\mathbb{N} \cap b\mathbb{N} = ab\mathbb{N}$ and

$$D(a\mathbb{N} \cap b\mathbb{N}) = D(a\mathbb{N}) \cdot D(b\mathbb{N}) \tag{3.2}$$

and by using induction, the Equation 3.2 also holds for any finite collection of $a_1\mathbb{N},\ a_2\mathbb{N}, ...,\ a_k\mathbb{N}$ when $a_1,\ a_2, ...,\ a_k$ are pairwise coprime.

Following the idea in [3], we have the following observation.

Theorem 3.1.3. *If $D(\cdot)$ is a probability measure, then the second Borel-Cantelli Lemma fails. Hence, $D(\cdot)$ is not a measure.*

Proof. Let p_k be the kth prime and $p_k\mathbb{N}$ be the collection of multiples of p_k. Since p_j and p_k are obviously coprime, $p_1,\ p_2, ...,\ p_m, ...,\ p_k$ are pairwise coprime and we can imply that $p_1,\ p_2, ...,\ p_m, ...,\ p_k$ are mutually coprime so $p_1\mathbb{N},\ p_2\mathbb{N}, ...,\ p_k\mathbb{N}$ are mutually independent.
It is obvious that $D(p_k\mathbb{N}) = \frac{1}{p_k}$. By Theorem 2.2.10, $D(p_k\mathbb{N})$ diverges. In the other method, to determine the number of growth asymptotically, we use Theorem 2.2.14(Prime Number Theorem) and Theorem 2.2.15. Then, p_k is approximately $k \log k$.

$$\sum_{k=1}^{\infty} D(p_k\mathbb{N}) \approx \frac{1}{2} + \sum_{k=2}^{\infty} \frac{1}{k \log k}.$$

It is easy to show that $f(x) = \frac{1}{x \log x}$ is continuous, decreasing and positive on $[2, \infty)$.Then, we can apply the Integral Test with the series.
The related improper integral is

$$\int_2^{\infty} \frac{1}{x \log x} dx = \log(\log x)\Big|_2^{\infty} = \infty$$

Since the improper integral diverges, so does the infinite series.
By Theorem 2.3.7 (the second Borel-Cantelli lemma), $D(\limsup_{k\to\infty}(p_k\mathbb{N})) = 1$.
However, $\limsup_{k\to\infty} p_k\mathbb{N} = \cap_{k=1}^{\infty} \cup_{j=k}^{\infty} (p_j\mathbb{N}) = \emptyset$ because $\cap_{k=1}^{\infty} \cup_{j=k}^{\infty} (p_j\mathbb{N})$ consists of integers divisible by infinitely many prime p_k. Since $D(\cdot)$ is a probability measure, $D(\limsup_{k\to\infty}(p_k\mathbb{N})) = 0$ contradicting with the necessary condition of the lemma. ☐

Theorem 3.1.4. *Let $\mathfrak{D}$ be the class of events that have a density under the density measure $D(\cdot)$. Then,*

1. $\mathbb{N} \in \mathfrak{D}$.

2. $\mathfrak{D}$ *is closed under complementation.*

3. $\mathfrak{D}$ *is not closed under countable unions.*

Proof. It is easy to show the first and the second statements by the definition of the measure. Now we need to find the counterexample that if $A_1,\ A_2,\ ...,A_k\ \in \mathfrak{D}$ but $\cup_{j=1}^{\infty} A_j \notin \mathfrak{D}$. In other word, the goal is to find $A_1,\ A_2,\ ...,A_k$ such that there densities under the density measure exist while $\cup_{j=1}^{\infty} A_j \notin \mathfrak{D}$ does not.

Firstly, construct a sequence of events with density 0 for each $k \in \mathbb{N}$, let

$$A_{2k-1} \ = \ \{m \in \mathbb{N} \mid 2^{2k-2} \ < \ m \ \leq 2^{2k-1}\}$$

$$A_{2k} \ = \ \{2m \in \mathbb{N} \mid 2^{2k-1} \ < \ 2m \ \leq 2^{2k}\}$$

Then, $|A_{2k-1}| = |A_{2k}| = 2^{2k-2}$ and $D(A_k) = 0$ for each $k \in \mathbb{N}$. Let $A = \cup_{k=1}^{\infty} A_k$. Consider the ratios

$$r_{2^{2n-1}} = \frac{|A \cap \mathbb{N}_{2^{2n-1}}|}{2^{2n-1}} \text{ and } r_{2^{2n}} = \frac{|A \cap \mathbb{N}_{2^{2n}}|}{2^{2n}}.$$

For $n \geq 2$,

$$\begin{aligned} |A \cap \mathbb{N}_{2^{2n-1}}| &= |A_1 \cup A_2 \cup ... \cup A_{2n-1}| \\ &= 1 + 1 + 4 + 4 + 2^{2n-4} + 2^{2n-4} + 2^{2n-2} \\ &= \frac{2}{3}(4^{n-1} - 1) + 4^{n-1} \end{aligned}$$

Hence, $\lim_{n\to\infty} r_{2^{2n-1}} = \frac{5}{6}$. By the same method, $\lim_{n\to\infty} r_{2^{2n}} = \frac{2}{3}$. Consequently, $A \notin \mathfrak{D}$ because A does not have a density under $D(\cdot)$ because the sequence of ratio $r_n = \frac{|A \cap \mathbb{N}_n|}{n}$ has two subsequences $r_{2^{2n-1}}$ and $r_{2^{2n}}$ tending to distinct limits. $\square$

Theorem 3.1.5. *Let $\mathfrak{D}$ be the class of events that have a density under the density measure $D(\cdot)$. Then,*

1. $D(\mathbb{N}) = 1$ and $D(\cdot) \geq 0$ on $\mathfrak{D}$

2. $D(\cdot)$ is not countably additive.

Proof. It is explicit to show the first part of the theorem by the definition of the measure. To prove the second, we give two counterexamples. The first is the example we gave on Theorem 3.1.4.3 which show that $D(\cdot)$ cannot give a measure to $A = \cup_{k=1}^{\infty} A_k$. The other is the sequence of mutually exclusive singletons generated from $\mathbb{N}$ which can be verified that each singleton has density zero while the countable union of the sequence is $\mathbb{N}$ having density 1. □

3.2 There is no product probability measure $\mu_{\otimes}$ on $\sigma(\mathfrak{M} \times \mathfrak{M})$ when $\mathfrak{M} \subseteq P(\mathbb{N})$ such that $a\mathbb{N} \times b\mathbb{N} \in \mathfrak{M} \times \mathfrak{M}$ and $\mu_{\otimes}(a\mathbb{N} \times b\mathbb{N}) = \dfrac{1}{ab}$ for all $a, b \in \mathbb{N}$.

Initially, we prove the statement in one-dimension in Theorem 3.2.1 and then we extend this in two-dimension in Theorem 3.2.2.

Theorem 3.2.1. *On a given probability measure space* $(\mathbb{N},\ \mathfrak{M},\ \mu)$*, there is no probability measure* μ *on* $\mathfrak{M} \subseteq P(\mathbb{N})$ *such that* $k\mathbb{N} \in \mathfrak{M}$ *and* $\mu(k\mathbb{N}) = \dfrac{1}{k}$ *for all* $k \in \mathbb{N}$.

Proof. For the sake of contradiction, assume that there is a probability measure μ such that $k\mathbb{N} \in \mathfrak{M} \subseteq P(\mathbb{N})$ and $\mu(k\mathbb{N}) = \dfrac{1}{k}$ for all $k \in \mathbb{N}$. Then, we show that a group of multiples of finite distinct primes are mutually independent.

Let a and b are coprime natural numbers, then $a\mathbb{N} \cap b\mathbb{N} \ = \ ab\mathbb{N}$ and

$$\begin{aligned}\mu(a\mathbb{N} \cap b\mathbb{N}) &= \mu(ab\mathbb{N}) \\ &= \frac{1}{ab} \\ &= \mu(a\mathbb{N}) \cdot \mu(b\mathbb{N})\end{aligned}$$

Hence, $a\mathbb{N}$ and $b\mathbb{N}$ are independent. Next, assume that there are k primes that is not exceeding than fixed n when $n \in \mathbb{N}$. Now we have,

$$\begin{aligned}\bigcap_{\text{p prime} \leq n} p\mathbb{N} &= p_1\mathbb{N} \cap p_2\mathbb{N} \cap \ldots \cap p_{k-1}\mathbb{N} \cap p_k\mathbb{N} \\ &= (p_1\mathbb{N} \cap p_2\mathbb{N}) \cap \ldots \cap (p_{k-1}\mathbb{N} \cap p_k\mathbb{N}) \\ &= p_1p_2\mathbb{N} \cap p_3p_4\mathbb{N} \cap \ldots \cap p_{k-1}p_k\mathbb{N}\end{aligned}$$

By matching a pair of primes inductively, we obtain

$$\bigcap_{p \text{ prime} \leq n} p\mathbb{N} \ = \ \left(\prod_{p \text{ prime} \leq n} p \right) \mathbb{N}$$

Hence, $p_1\mathbb{N}, p_2\mathbb{N}, \ldots, p_{k-1}\mathbb{N}, p_k\mathbb{N}$ are mutually independent.

By Theorem 2.3.4, $(p_1\mathbb{N})^C, (p_2\mathbb{N})^C, ..., (p_{k-1}\mathbb{N})^C, (p_k\mathbb{N})^C$ are also mutually independent. Hence,

$$\mu\left(\bigcap_{p\ \text{prime}\leq n}(p\mathbb{N})^C\right) = \prod_{p\ \text{prime}\leq n}\left(1-\frac{1}{p}\right)$$

$$\limsup_{n\to\infty}\mu\left(\bigcap_{p\ \text{prime}\leq n}(p\mathbb{N})^C\right) = \limsup_{n\to\infty}\prod_{\text{p prime}\leq \text{n}}\left(1-\frac{1}{p}\right)$$

Consider the left hand side, since μ is a probability measure, consequently,

$$\begin{aligned}\limsup_{n\to\infty}\mu\left(\bigcap_{p\ \text{prime}\leq n}(p\mathbb{N})^C\right) &= \mu\left(\left(\bigcup_{p\ \text{prime}\leq n}p\mathbb{N}\right)^C\right)\\ &= 1-\mu\left(\bigcup_{p\ \text{prime}\leq n}p\mathbb{N}\right)\\ &= 1-\sum_{p\ \text{prime}\leq n}^{\infty}\frac{1}{p}\end{aligned}$$

Since $\displaystyle\sum_{p\ \text{prime}\leq n}^{\infty}\frac{1}{p}$ diverges to infinity, by Theorem 2.2.10,

$$\limsup_{n\to\infty}\prod_{p\ \text{prime}\leq n}\left(1-\frac{1}{p}\right) = -\infty$$

That is

$$\lim_{n\to\infty}\prod_{p\ \text{prime}\leq n}\left(1-\frac{1}{p}\right) = -\infty$$

Since $\lim_{n\to\infty}\log(n)=\infty$,

$$\lim_{n\to\infty}\log(n)\prod_{p\ \text{prime}\leq n}\left(1-\frac{1}{p}\right) = -\infty$$

Meanwhile, by the Mertens' third theorem,

$$\lim_{n\to\infty}\log(n)\prod_{\text{p prime}\leq \text{n}}\left(1-\frac{1}{p}\right) = e^{-\gamma}$$

where γ is the Euler-Mascheroni constant. This result yield the desired contradiction. □

Theorem 3.2.2. *On a given product measure space* $(\mathbb{N}\times\mathbb{N},\ \sigma(\mathfrak{M}\times\mathfrak{M}),\ \mu_{\otimes})$*, there is no product probability measure* $\mu_{\otimes}$ *on* $\mathfrak{M}\subseteq P(\mathbb{N})$ *such that* $a\mathbb{N}\times b\mathbb{N}\in\mathfrak{M}\times\mathfrak{M}$ *and* $\mu_{\otimes}(a\mathbb{N}\times b\mathbb{N})=\dfrac{1}{ab}$ *for all* $a,b\in\mathbb{N}$*.*

Proof. Let us give a proof by contradiction. Suppose that there is a product probability measure $\mu_{\otimes}$ on a given product measure space $(\mathbb{N}\times\mathbb{N},\ \sigma(\mathfrak{M}\times\mathfrak{M}),\ \mu_{\otimes})$ when $\mathfrak{M}\subseteq P(\mathbb{N})$ such that $a\mathbb{N}\times b\mathbb{N}\in\mathfrak{M}\times\mathfrak{M}$ and $\mu(a\mathbb{N}\times b\mathbb{N})=\dfrac{1}{ab}$ for all $a,b\in\mathbb{N}$ and let $A\in\sigma(\mathfrak{M}\times\mathfrak{M}))$.
Then, $\mu_{\otimes}(a\mathbb{N}\times\mathbb{N})=\dfrac{1}{a}$.
Since there exists a sequence $(n\mathbb{N}\mid n\in\mathbb{N})$ in $P(\mathbb{N})$ such that $\bigcup_{n\in\mathbb{N}}(n\mathbb{N})=\mathbb{N}$ and $\mu(n\mathbb{N})<\infty$ for every $n\in\mathbb{N}$, $(\mathbb{N},\ \mathfrak{M},\ \mu)$ is a σ-finite measure space and $a\mathbb{N}\times\mathbb{N}\in\mathfrak{M}\subseteq P(\mathbb{N}\times\mathbb{N})$, by Theorem 2.1.22 and the definition a product measure,

$$\mu(a\mathbb{N})=\mu(a\mathbb{N})\cdot\mu(\mathbb{N})=\mu_{\otimes}(a\mathbb{N},n)$$

is measurable on $\mathbb{N}$ when $n\in\mathbb{N}$.
Since $\mathrm{dom}(\mu)=\mathrm{dom}(\mu_{\otimes})$ and $\mu(a\mathbb{N})=\mu_{\otimes}(a\mathbb{N},n)$, $\mu(\cdot)=\mu_{\otimes}(A(\cdot,n))$, in addition, $\mu_{\otimes}(A(\cdot,n))$ is a probability measure on $\mathfrak{M}\subseteq P(\mathbb{N})$ contradicting Theorem 3.2.1 as we desired. $\square$

3.3 Main proof of the statement

In this section, we can conclude that we cannot define a probability measure satisfying the conditions so we need to prove the statement without properties of a probability measure to complete the objective.

Theorem 3.3.1. *The probability that two natural numbers, chosen randomly, are relatively prime is equal to* $\frac{6}{\pi^2}$.

Proof. Let N be a fixed natural number and there are by k distinct primes $p_1, p_2, ..., p_k$ which less than or equal to N. Let a, b be randomly chosen natural numbers not exceeding than N and n be a fixed natural number. We have done that $D(aA) = \frac{1}{a}$ when $a \in \mathbb{N}$. Consider an event $(a, b) > 1$. The event $(a, b) > 1$ if and only if there is a prime p such that $p \mid a$ and $p \mid b$;in other word, there are at least m distinct primes $p_1, p_2, ..., p_m$ such that $p_1 p_2 ... p_m \mid (a, b)$ when $1 \leq m \leq k$. For each prime p, the event that $p \mid a$ and $p \mid b$ has probability $\frac{1}{p^2}$ and let S_m denote the probabilities of events that at least m distinct primes can divide (a, b). Then,

$$S_m = \sum_{p_1 < p_2 < ... < p_m} \cdots \sum \frac{1}{(p_1 p_2 ... p_m)^2}$$

By Theorem 2.3.5 (Inclusion-Exclusion Principle), we obtain

$$\begin{aligned} P((a,b) > 1) &= P((a,b) \text{ can be divided by at least 1 prime}) - P((a,b) \text{ can be divided by at} \\ &\quad \text{least 2 primes}) + ... + (-1)^{k-1} P((a,b) \text{ can be divided by at least } k \text{ primes}) \\ &= \sum_{i=1}^{k} \frac{N}{(p_i)^2} - \sum_{1 \leq i,j \leq k} \sum \frac{N}{(p_i p_j)^2} + ... + (-1)^{k-1} \sum_{p_1 < p_2 < ... < p_k} \cdots \sum \frac{N}{(p_1 p_2 ... p_k)^2} \\ &= NS_1 - NS_2 + ... + (-1)^{k-1} NS_k \end{aligned}$$

Therefore,

$$\begin{aligned}
D((a,b)=1) &= \lim_{N\to\infty} \frac{|(p_1^2 \nmid N) \ \wedge \ P((p_2^2 \nmid N) \ \wedge \ \ldots \ \wedge \ P((p_k^2 \nmid N)|}{N} \\
&= 1 - \lim_{N\to\infty} \frac{|Sq(N) > 1|}{N} \\
&= \lim_{N\to\infty} \frac{N - |(p_1^2 \nmid N) \ \vee \ P((p_2^2 \nmid N) \ \vee P((p_k^2 \nmid N)|}{N} \\
&= \lim_{N\to\infty} \frac{N \ - \ NS_1 \ + \ NS_2 \ + \ \ldots \ + (-1)^{k-1} NS_k}{N} \\
&= \lim_{N\to\infty} 1 \ - \ S_1 \ + \ S_2 \ + \ \ldots \ + (-1)^{k-1} S_k.
\end{aligned}$$

By Theorem 2.1.14, if we denote in this proof that $\mu(n)$ is a Möbius function, thus,

$$D((a,b)=1) \ = \ \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\mu(n)}{n^2} \ = \ \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$$

Hence, the density of the event, in other word the probability of the event, is $\frac{6}{\pi^2}$. □

Similarly, we obtain Corollary 3.3.2 and we can show that the probabilities of the events in Theorem 3.3.1 and Corollary 3.3.2 are coincide.

Corollary 3.3.2. The probability that a natural number, chosen randomly, is square-free is $\frac{6}{\pi^2}$

Chapter IV

ANALYTIC NUMBER THEORETICAL PROOF

In this chapter, an analytic number theoretical approach for a proof of the statement will be presented. Properties of arithmetic functions and average order are crucial topics for our proof. In this chapter, we follow the argument in [1].

4.1 Main proof by analytic number theoretical approach

Definition 4.1.1. A lattice point is a point in a Cartesian coordinate system such that both every coordinates of the point is an integer.

Remark. In this chapter, we are interested in lattice points which all coordinates are positive integers only.

Definition 4.1.2. Two lattice points A and B are called to be **mutually visible** if there are no lattice points excluding the endpoints A and B containing in the line segment which join two endpoints.

Example 4.1.3. (1,2) and (0,0) are mutually visible but (1,1) and (3,3) are not mutually invisible.

Theorem 4.1.4. *Two lattice points (a,b) and (c,d) are mutually visible if, and only if, $a-c$ and $b-d$ are relatively prime.*

Proof. Obviously, (a,b) and (c,d) are mutually visible if and only if $(a-c,b-d)$ is visible from the origin. Thus, it is sufficient to prove that (a,b) is visible from the origin if and only if a and b are relatively prime. Suppose that (a,b) is visible from the origin we must show that $(a,b)=1$ so we proof by contradiction by assuming that $(a,b)=d$ where $d>1$.

Then, $a = da', \ b = db'$ when a', b' are natural numbers. Hence, the line segment connecting the origin and (a, b) through (a', b'). This contradicts the assumption that the lattice point (a, b) is visible from the origin. Conversely, suppose that $(a, b) = 1$. If there is a lattice point a', b' on the line segment connecting the origin and (a,b), then

$$a' = ta, \qquad b' = tb \qquad \text{where } 0 < t < 1.$$

We obtain that t is rational that is $t = \frac{r}{s}$ when r, s are natural numbers and r, s are relatively prime. Hence,

$$sa' = ar, \qquad sb' = br.$$

so $s|ar$ and $s|br$. Since $(r, s) = 1$, $s|a$ and $s|b$ so $s = 1$ that is $t = 1$ because r is a natural number and $(a, b) = 1$. This contradicts the condition $0 < t < 1$.

Consequently, the lattice point (a, b) is visible from the origin. □

To proof our statement, by Theorem 4.1.4, we need to find the quantity of lattice points distributed in the plane. Consider a square region with extremely large x- and y- coordinates r in the xy-plane with the condition

$$1 \leq x \leq r, \qquad 1 \leq y \leq r.$$

Let $N(r)$ denote the quantity of all lattice points in the square and let $N'(r)$ denote of the quantity of lattice points which are visible from the origin. The quotient $\frac{N'(r)}{N(r)}$ tells the ratio of lattice points in the square which are visible from the origin. The next theorem shows that the ratio tends to a limit as r tends to infinity which is called the **density** of the lattice points which are visible from the origin.

Theorem 4.1.5. *The density of the set of lattice points which are visible from the origin is* $\frac{6}{\pi^2}$.

Proof. We will show that

$$\lim_{r\to\infty} \frac{N'(r)}{N(r)} = \frac{6}{\pi^2}$$

Firstly, consider the quantity of $N'(r)$. There is one lattice point nearest the origin is all visible from the origin. By the symmetry (see Figure 4.1), $N'(r)$ is equal to 1 plus 2 times the number of visible points in the region

$$\{(x,y)|\ 2 \le x \le r, \quad 1 \le y \le x,\}$$

(the shaded region in Figure 4.1), Hence,

$$\begin{aligned} N'(r) \ &= \ 1 \ + \sum_{2\le n\le r}\sum_{1\le m<n} 1 \\ &= 2\sum_{1\le m<n} \phi(n) - 1. \end{aligned}$$

By Theorem 2.2.9, we obtain

$$\begin{aligned} N'(r) &= \frac{6}{\pi^2}r^2 + O(1) + O(r\log r) \\ &= \frac{6}{\pi^2}r^2 + O(r\log r). \end{aligned}$$

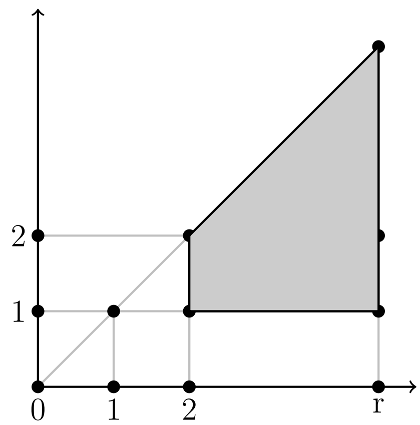

Figure 4.1

Meanwhile, the total quantity of lattice points in the square is

$$N(r) = ([r])^2 = (r + O(1))^2 = r^2 + 2rO(1) + (O(1))^2$$

$$= r^2 + O(r).$$

Hence,

$$\lim_{r\to\infty} \frac{N'(r)}{N(r)} = \lim_{r\to\infty} \frac{\frac{6}{\pi^2}r^2 + O(r\log r)}{r^2 + O(r)} = \frac{6}{\pi^2}$$

□

Now we can conclude that if two natural numbers a and b are chosen randomly, the probability of the event is $\frac{6}{\pi^2}$. Furthermore, by the same method, the probability of the event that two randomly chosen integers are relatively prime is equal to $\frac{6}{\pi^2}$ as in Corollary 4.1.6.

Corollary 4.1.6. If two integers a and b are chosen randomly, the probability of the event is $\frac{6}{\pi^2}$.

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กลุ่มที่ 23

เอกสารนี้ได้รับการอนุมัติจากอาจารย์ที่ปรึกษาโครงงานแล้ว

ลงชื่อ ....

(วันที่ ....)

The Project Proposal of Course 2301399 Project Proposal Academic Year 2019

Project Title (Thai) บทพิสูจน์ที่รัดกุมของการหาความน่าจะเป็นที่จำนวนนับสองจำนวนเป็นจำนวนเฉพาะสัมพัทธ์กัน

Project Title (English) A rigorous proof of the probability that two randomly chosen positive integers are relatively prime

Project Advisor Asst. Prof. Keng Wiboonton

By Weerawich Swaspanich ID 5933547723

Mathematics, Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University

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Background and Rationale

In 1849, Peter Gustav Lejeune Dirichlet prove that a proof that probability that two integers, chosen randomly, are relatively prime is equal to $\frac{6}{\pi^2}$ in [1].

This result of Dirichlet also mentioned in the classic book [3] on the theory of numbers by Hardy and Wright and in the books of [5],[6] and in the article [4]. The question we should ask is "What is the interpretation of the phrase 'What is the probability that two positive integers, chosen randomly, are relatively prime' ?".

When we talked about the probability of an event that a randomly chosen natural number n is divisible by a natural number k, that is we need to know the sample space and the probability measure assigned to the set of all events. In fact, it turns out that the probability considered in the Dirichlet's result is not (countably additive) measure on $\mathbb{N} \times \mathbb{N}$ (see in [1]).

Hence, we should give a precise meaning of the Dirichlet's result.

Objectives

Give a precise meaning of the statement "the probability that two positive integers, chosen randomly, are relatively prime". Also give two rigorous proofs of this statement using the probabilistic method and using analytic-number-theory method.

Scope

We study the so-called density measure on the set of all natural numbers and prove that the density measure of the event that two positive integers, chosen randomly, are relatively prime is equal to $\frac{6}{\pi^2}$.

Project Activities

1. Study relevant researches from [2], [4] and other related researches.
2. Review the related knowledge for our project.
3. Give a reason why we cannot construct a probability measure on $\mathbb{N} \times \mathbb{N}$ satisfying that the probability that two randomly chosen natural numbers (a,b) is divisible by a natural number k is $\dfrac{1}{k^2}$.
4. Prove the theorem.
5. Write a report and present a project.

Scheduled operations

Procedures	August 2019 - April 2020								
	Aug	Sep	Oct	Nov	Dec	Jan	Feb	Mar	Apr
1. Study relevant researches from [2], [4] and other related researches.									
2. Review the related knowledge for our project.									
3. Give a reason why we cannot construct a measure probability on $\mathbb{N} \times \mathbb{N}$ satisfying that a probability that two randomly chosen natural numbers a,b is divisible by a natural number k is $\frac{1}{k^2}$.									
4. Prove the theorem.									
5. Write a report and present a project.									

Benefits

The result might lead to understand the Dirichlet's result and the concept of the natural density measure more .

Equipment

1. Computer
2. Paper
3. Printer
4. Stationery
5. Word processing software

Budget

1. Textbook 5000 Baht

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Biography

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