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PANCYCLICITY AND VERTEX PANCYCLICITY FOR SOME PRODUCTS OF GRAPHS

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กราฟ G อันดับ n เป็นแพนไซคลิก ถ้า G มีวัฏจักรความยาว l เมื่อ $3 \leq l \leq n$ กราฟ G อันดับ n เป็นเวอร์เท็กซ์แพนไซคลิก ถ้าจุดยอดแต่ละจุดของ G อยู่บนวัฏจักรความยาว l สำหรับแต่ละ $3 \leq l \leq n$ ในวิทยานิพนธ์ฉบับนี้ เราพิสูจน์ว่า n-ปริซึมทั่วไปของกราฟกระโปรงใด ๆ เป็นแพนไซคลิก สำหรับ $n \geq 2$ นอกจากนั้นเราได้ศึกษาความเป็นเวอร์เท็กซ์แพนไซคลิกบนผลคูณแบบพจนานุกรมของกราฟบางชนิด เราพบว่า ถ้ากราฟ G_1 เป็นกราฟติดตามได้ที่มีจำนวนจุดยอดเป็นจำนวนคู่ และ G_2 เป็นกราฟที่มีเส้นเชื่อมอย่างน้อยหนึ่งเส้น แล้ว ผลคูณแบบพจนานุกรมของ G_1 และ G_2 (หรือ $G_1 \circ G_2$) เป็นเวอร์เท็กซ์แพนไซคลิก ถ้า G_1 และ G_2 เป็นกราฟติดตามได้ที่มีเส้นเชื่อมอย่างน้อยหนึ่งเส้น แล้ว $G_1 \circ G_2$ เป็นเวอร์เท็กซ์แพนไซคลิก และ ถ้ากราฟ G_1 มีวัฏจักรแฮมิลตันและ G_2 เป็นกราฟที่มีเส้นเชื่อมอย่างน้อยหนึ่งเส้น แล้ว $G_1 \circ G_2$ เป็นเวอร์เท็กซ์แพนไซคลิก

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....

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A graph G of order n is said to be pancyclic if it contains a cycle of each length l for $3 \leq l \leq n$. A graph G of order n is vertex pancyclic if each vertices of G is contained in a cycle of each length l for $3 \leq l \leq n$. In this dissertation, we show that the n-generalized prism over any skirted graph is pancyclic for $n \geq 2$. Furthermore, we study vertex pancyclicity of the lexicographic product of graphs. We obtain that if G_1 is a traceable graph of even order and G_2 is a graph with at least one edge, then the lexicographic product of G_1 and G_2 (or $G_1 \circ G_2$) is vertex pancyclic; if G_1 and G_2 are nontrivial traceable graphs, then $G_1 \circ G_2$ is vertex pancyclic; and if G_1 is Hamiltonian and G_2 is a graph with at least one edge, then $G_1 \circ G_2$ is vertex pancyclic.

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CHAPTER I
INTRODUCTION

In this disseration, we study pancyclicity and vertex pancyclicity of the Cartesian product and the lexicographic product of graphs. We first introduce some basic definitions in graph theory which are used in this dissertation as follows.

1.1 Preliminaries

Every graph that we consider in this dissertation is a finite, undirected and simple graph $G = (V(G), E(G))$ with the vertex set $V(G)$ and the edge set $E(G)$. Most of the basic graph theory terminologies in this research follow from West's textbook [19].

We say that G is a graph of order m if $|V(G)| = m$. The set of all neighbors of a vertex v in G is denoted by $N(v)$ and $d(v)$ is the degree of the vertex v in G, i.e., the number of vertices which are adjacent to v in G. The maximum degree of G is denoted by $\Delta(G)$. The length of a path or a cycle is the number of its edges. A path of length $n-1$ is denoted by P_n. The followings are several terminologies that we use in this dissertation.

Definition 1.1. A graph is called *trivial* if it contains only one vertex and no edges. Otherwise, it is *nontrivial.* An *empty graph* is a graph having no edges.

Definition 1.2. If $S \subseteq V(G)$ and $M \subseteq E(G)$, then we write $G - S$ and $G - M$ for the subgraph obtained by deleting the set of vertices S and the set of edges M, respectively. In particular, if $S = \{v\}$ and $M = \{e\}$ are singleton sets, then we write $G - v$ and $G - e$ instead of $G - \{v\}$ and $G - \{e\}$, respectively.

Definition 1.3. If H and G are graphs such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subgraph* of G. In particular, if $V(H) = V(G)$, then H is a *spanning subgraph* of G.

Definition 1.4. A subgraph H of G is called an *induced subgraph* of G whenever u and v are vertices of H and uv is an edge of G, then uv is an edge of H. If S is a nonempty subset of $V(G)$, the *subgraph of G induced by S* is the induced subgraph with vertex set S and denoted by $G[S]$.

Definition 1.5. In a graph G and its subgraph $H = (V(H), E(H))$, the *contraction of H into a single vertex* is a replacement of H by a single vertex u^* and the edges incident to u^* are all edges formerly incident to some vertices in $V(H)$.

Note that the complete graph of order n is denoted by K_n, the complete bipartite graph with the partite sets X and Y where $|X| = p$ and $|Y| = q$ is denoted by $K_{p,q}$. The notion $P(s,t)$ is referred to an (s,t)-path of a graph G as a path in G from s to t. For paths $P(s,t)$ and $P(t,k)$ of which t is only one common vertex, the union of $P(s,t)$ and $P(t,k)$ is a path from s to k, denoted by $P(s,t)P(t,k)$.

Definition 1.6. A *tree* is a connected graph with no cycles.

Definition 1.7. A *rooted tree* is a tree with one vertex a chosen as its root. For each vertex u of a rooted tree with root a, let $P(u)$ be the unique (a,u)-path. Then,

(i) the *parent* of u is its neighbor on $P(u)$;

(ii) the *children* of u are its other neighbors in the rooted tree;

(iii) the *descendents* of u are the vertices v of the rooted tree such that $P(v)$ contains u;

(iv) the *leaves* are vertices of the rooted tree having no children;

(v) the *internal vertices* are vertices of the rooted tree having children.

Definition 1.8. (i) A graph is called a *planar graph* if it can be drawn in the plane without edges crossing. This drawing is called an *embedding* in the plane or a *planar embedding*.

(ii) A *plane graph* is a planar embedding of a planar graph.

(iii) A *bounded face* of a plane graph is a region bounded by edges. An *unbounded face* of a plane graph is the region with unbounded area.

(iv) An edge e that bounds a face f is said to be *incident* to f. If a vertex v is an endpoint of e, then v is also *incident* to f.

The following definitions are products of two graphs which we consider in this dissertation.

Definition 1.9. (i) Let G and H be two graphs. The *Cartesian product* of graphs G and H, denoted by $G\square H$, is defined as the graph with vertex set $V(G) \times V(H)$ and an edge $\{(u_1, v_1), (u_2, v_2)\}$ is present in the Cartesian product whenever $u_1 = u_2$ and $v_1v_2 \in E(H)$ or symmetrically $v_1 = v_2$ and $u_1u_2 \in E(G)$.

(ii) For $n \geq 2$ and $P_n = v_1v_2v_3 \cdots v_n$, we call the graph $G\square P_n$, the *n-generalized prism over a graph G*. The 2-generalized prism over a graph G is called the *prism over a graph G*. For convenience, the n-generalized prism over a graph G is referred to the family of the n-generalized prisms over a graph G for all $n \geq 2$.

Definition 1.10. Let G and H be two graphs. The *lexicographic product* or *graph composition of G and H*, denoted by $G \circ H$, is defined as a graph with vertex set $V(G) \times V(H)$ and an edge $\{(u_1, v_1), (u_2, v_2)\}$ is present in the lexicographic product whenever $u_1u_2 \in E(G)$ or ($u_1 = u_2$ and $v_1v_2 \in E(H)$). The *double graph* of a graph G is $G \circ P_2$.

Since this dissertation consider hamiltonicity, pancyclicity as well as vertex pancyclicity of a graph, we collect all definitions involved as follows.

Definition 1.11. (i) A path in G is a *Hamiltonian path* or a *spanning path* if it contains all vertices of G.

(ii) A graph G is *traceable* if G contains a Hamiltonian path.

(ii) A cycle of G is a *Hamiltonian cycle* if it contains all vertices of G.

(iv) A graph G is said to be *Hamiltonian* if it contains a Hamiltonian cycle. Otherwise, G is *non-Hamiltonian.*

Definition 1.12. (i) A graph G of order $n \geq 3$ is said to be *pancyclic* if it contains a cycle of each length l for $3 \leq l \leq n$.

(ii) A graph G of order n is *almost pancyclic* [4] if it contains a cycle of each length l for $3 \leq l \leq n$ except possibly for a single even length. We use the term *m-almost pancyclic* for an almost pancyclic graph without a cycle of even length m.

(iii) A vertex of a graph G of order n is *k-vertex pancyclic* if it is contained in a cycle of each length l for $k \leq l \leq n$, and a graph G is *vertex k-pancyclic* if all vertices of G are k-vertex pancyclic. Note that a vertex 3-pancyclic graph is simply called a *vertex pancyclic graph.*

(iv) A graph G of order n is *vertex even pancyclic* if each vertex of G is contained in a cycle of each even length l for $3 < l \leq n$.

1.2 Introduction

The topological structure of an interconnection network or network is usually well-known that it can be represented by a graph. The processors can be regarded by vertices or nodes and the communication links between processors can be expressed by edges connecting two vertices together. The study of structural properties of a network is beneficial for parallel or distributed systems. The problem of finding cycles of various lengths in networks or graphs receives much attention from researchers because this is a key measurement for evaluating the suitability of the network's structure for its applications and more information, see [20].

Pancyclicity in graph theory refers to the problem of finding cycles of all lengths from 3 to its order. It was first investigated in the context of tournaments by Harary and Moser [10], Moon [13] and Alspach [1]. Bondy [3] was the first one who introduced and extended the concept of pancyclicity from directed graphs to undirected graphs. In 1971, Bondy [2] posed a metaconjecture which states that almost any nontrivial condition on a graph which implies that the graph is Hamiltonian also implies that the graph is pancyclic (there may be a simple family of exceptional graphs). There are a number of works that correspond to this metaconjecture. For instances, in 1960, Ore [14] introduced the degree sum condition which states that "for each pair of non-adjacent vertices u, v in G, $d(u) + d(v) \geq n(G)$" and showed that if G is a graph satisfying the degree sum condition, then G is Hamiltonian. Bondy [3] showed that if G is graph satisfying the degree sum condition, then G is pancyclic or $G = K_{n/2,n/2}$. Moreover, in terms of degree sequence of a graph, Chvátal [7] showed that if G is a graph of order $n \geq 3$ with vertex degree sequence $d_1 \leq d_2 \leq d_3 \leq \cdots \leq d_n$ and $d_k \leq k < n/2$ implies $d_{n-k} \geq n - k$, then G is Hamiltonian. Schmeichel and Hakimi [18] showed that if G satisfies such condition introduced by Chvátal [7], then G is either pancyclic or bipartite. Recently, the concept of pancyclicity was also extended to hypergraphs, for example, see [9] and [12].

Meanwhile, for the prism over a graph G, there are some Hamiltonian and pancyclicity results. For example, Paulraja [15] proved in 1993 that if G is a 3-connected 3-regular graph, then the prism $G\Box P_2$ is Hamiltonian. In 2001, Goddard [8] showed that if G is a 3-connected 3-regular graph that contains a triangle, then the prism $G\Box P_2$ is pancyclic. In 2009, Čada et al. [5] showed that if G is a connected almost claw-free graph and $n \geq 4$ is an even integer, then $G\Box P_n$ is Hamiltonian. They also showed that if G is a 1-pendent cactus with $\Delta(G) \leq \frac{1}{2}(n+2)$ and $n \geq 4$ is an even integer, then $G\Box P_n$ is vertex even pancyclic, i.e., each vertex of $G\Box P_n$ is contained in a cycle of each even length.

In this study, we first show that the n-generalized prism over any skirted graph is Hamiltonian. To satisfy the metaconjecture, we investigate the pancyclicity of the n-generalized prism over any skirted graph.

In Chapter II, we first show that the n-generalized prism over any skirted graph is Hamiltonian and show that the n-generalized prism over a skirted graph with three specific types is Hamiltonian by applying the lemma given by Bondy and Lovász [4]. However, this technique cannot be applied to prove the pancyclicity of the n-generalized prism over any skirted graph.

In Chapter III, we prove that the n-generalized prism over any skirted graph is pancyclic. In the final part of this chapter, we discuss the vertex pancyclicity of the n-generalized prism over any skirted graph and we can see that the n-generalized prism over any skirted graph is not always vertex pancyclic. This motivates us to investigate the other product of graphs, that is, the lexicographic product.

In Chapter IV, we study the vertex pancyclicity over the lexicographic product of some graphs. We investigate some sufficient conditions for vertex pancyclicity over the lexicographic product of complete graphs K_n, paths P_n or cycles C_n with a general graph.

In Chapter V, the conclusion for our work is given and the disscussion for our future research as well as some open problems are provided.

CHAPTER II

THE n-GENERALIZED PRISM OVER A SKIRTED GRAPH WITH THREE SPECIFIC TYPES

In this chapter, we study pancyclicity of the n-generalized prism over a skirted graph with three specific types introduced by Bondy and Lovász [4]. We provide some basis definitions in graph theory which are used in Chapter II and Chapter III as follows.

Definition 2.1. Let G be a graph and a path $P_n = v_1v_2v_3 \cdots v_n$. If $u \in V(G)$, then, for convenience, we refer to the vertex u in its i-th copy in $G\Box P_n$ as $u^{(i)}$ instead of (u, v_i).

Definition 2.2. (i) A *Halin graph* [4] is a plane graph $\mathscr{H} = T \cup C$, where T is a planar embedding tree with no vertices of degree two and at least one vertex of degree at least three and C is the cycle connecting the leaves of T in the cyclic order determined by the embedding of T.

(ii) Let x be a vertex of C and a be the neighbor of x in T. Then, the graph $G = \mathscr{H} - x$ is called a *reduced Halin graph with root a*. Clearly, $G = T' \cup P$ where $T' = T - x$ and $P = C - x$. Note that T' has no vertex of degree two except possibly the vertex a.

For technical reasons, Bondy and Lovász [4] regarded that a single vertex is also a reduced Halin graph. Actually, in literatures, a reduced Halin graph which is not a single vertex can be represented by a diagram that is similar to a skirted graph. Hence, in this dissertation, we use the term skirted graph instead of a reduced Halin graph which is not a single vertex.

In this research, we are interested in the pancyclicity of the Cartesian product of a skirted graph G and a path P_n for $n \geq 2$ (the n-generalized prism over a skirted graph G). We can see that the Cartesian product is pancyclic only if the order of G is at least 2. As we mention before, here, we recall that a skirted graph is isomorphic to a reduced Halin graph defined by Bondy and Lovász [4]. However, we exclude the case of a single vertex.

Before giving a definition of a skirted graph, let us introduce a definition of a side skirt as follows.

Definition 2.3. A *side skirt* is a planar embedding rooted tree T, $T \neq P_2$, where the root of T is a vertex of degree at least two and all other vertices, except its leaves, are of degree at least three. In addition, the structure of T is embedded in such a way that the root is at the top.

Definition 2.4. A *skirted graph* is a plane graph $G = T \cup P$, where T is a side skirt and P is the path connecting the leaves of T in the order determined by the embedding of T starting from the vertex on the far left to the vertex on the far right (see Figure 2.1).

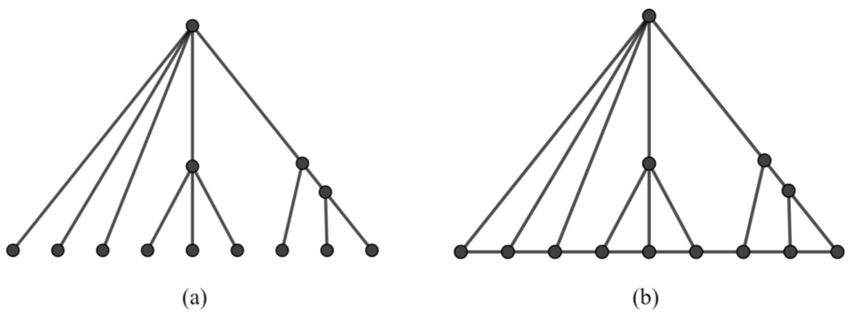

Figure 2.1: (a) A side skirted T and (b) a skirted graph $G = T \cup P$

Let $G = T \cup P$ be a skirted graph, a be the root of T and u_0, u_α be the endpoints of P. Then, the graph G is called a *skirted graph with root* a and is denoted by $G(a, u_0, u_\alpha)$. We notice that if u is an internal vertex of a side skirt T, then u and its descendents induce a skirted subgraph of G with root u.

In the following section, we provide our preliminary results on hamiltonicity and pancyclicity as well as the motivation of the main results of this chapter.

2.1 Preliminary results and motivation

In 1971, Bondy [2] posed a metaconjecture: almost any nontrivial condition on a graph which implies that the graph is Hamiltonian also implies that the graph is pancyclic (there may be a simple family of exceptional graphs). There are a number of works that correspond to this metaconjecture, see [3], [14] and [16] for more examples.

Meanwhile, for the prism over a graph G, there are some Hamiltonian and pancyclicity results. For example, Paulraja [15] proved in 1993 that if G is a 3-connected 3-regular graph, then the prism $G\Box P_2$ is Hamiltonian. In 2001, Goddard [8] showed that if G is a 3-connected 3-regular graph that contains a triangle, then the prism $G\Box P_2$ is pancyclic.

This motivates us to be interested in hamiltonicity and pancyclicity of the n-generalized prism over a skirted graph.

Since our skirted graphs are isomorphic to reduced Halin graphs defined by Bondy and Lovász [4], we obtain the following theorem and lemma from their study.

Theorem 2.5 (Bondy and Lovász [4]). *Any skirted graph is Hamiltonian.*

Definition 2.6. For any skirted graph with root a, $G(a, u_0, u_\alpha)$, we denote the path P of length α by $u_0u_1u_2\cdots u_\alpha$, and the (a, u_α)-path of length β and (a, u_0)-path of length γ in T by $v_0v_1v_2\cdots v_\beta$ and $w_0w_1w_2\cdots w_\gamma$, respectively. Thus, $v_0 = w_0 = a$, $u_0 = w_\gamma$, and $u_\alpha = v_\beta$ (see Figure 2.2).

Lemma 2.7 (Bondy and Lovász [4]). *Let $G = G(a, u_0, u_\alpha)$ be a reduced Halin graph or a skirted graph of order m. Then, G contains:*

(i) an (a, u_α)-path of each length l for $\alpha + \gamma \leq l \leq m - 1$;

(ii) a (u_0, u_α)-path of each length l for $\alpha \leq l \leq m - 1$.

Remark 2.8. We obtain that

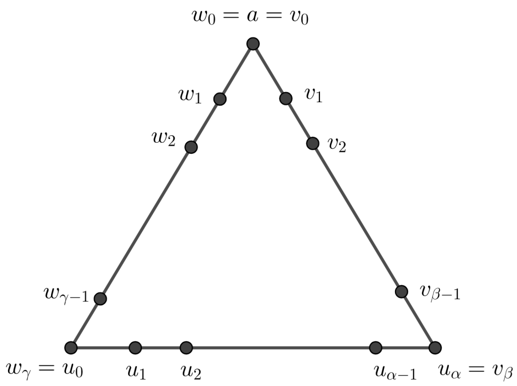

Figure 2.2: The (u_0, u_α)-path, (a, u_α)-path and (a, u_0)-path of $G(a, u_0, u_\alpha)$

(i) Lemma 2.7(i) gives an (a, u_0)-path of each length l for $\alpha + \beta \leq l \leq m - 1$ by the symmetry of $G(a, u_0, u_\alpha)$.

(ii) Since a child of the root a and all of its descendents induce a skirted subgraph of G, we can apply Lemma 2.7(ii) to each of the induced skirted subgraphs of G and obtain that G contains a (u_0, u_α)-path of each length l for $\alpha \leq l \leq m - 2$ (without the root a).

The following theorem is an immediate observation about the existence of a Hamiltonian cycle over the n-generalized prism over any skirted graph.

Theorem 2.9. *The n-generalized prism over any skirted graph is Hamiltonian.*

Proof. Let $G = G(a, u_0, u_\alpha)$ be a skirted graph of order m and P_n be a path of length $n - 1$. We show that $G \square P_n$ is Hamiltonian by finding a cycle of length mn in $G \square P_n$. To show that $G \square P_n$ contains a cycle of length mn, we give the following paths and then link them together with edges joining each copy of G.

- The first and the last copies of G contain paths $P(a^{(1)}, u_\alpha^{(1)})$ and $P(a^{(n)}, u_\alpha^{(n)})$, respectively, of length $m - 1$ by Lemma 2.7(i). Also, a path $P(a^{(n)}, u_0^{(n)})$ of length $m - 1$ of the last copy of G exists by the symmetry of G in Remark 2.8(i) (see Figures 2.3(a) and 2.3(c)).

- The remaining $n - 2$ copies of G contain a path $P(u_0^{(i)}, u_\alpha^{(i)})$ of length $m - 2$ (without the root $a^{(i)}$) for $2 \leq i \leq n - 1$, which exists by Remark 2.8(ii).

- The path $P(a^{(n)}, a^{(1)}) = a^{(n)}a^{(n-1)}a^{(n-2)} \cdots a^{(1)}$ is a path in $G\square P_n$ from the last copy to the first copy of G.

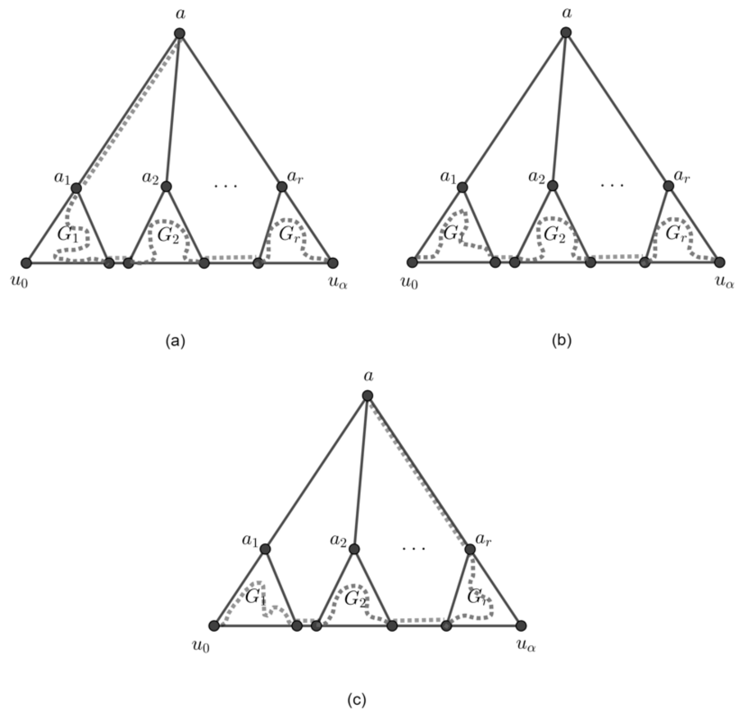

Figure 2.3: (a) (a, u_α)-path, (b) (u_0, u_α)-path and (c) (a, u_0)-path

Now, we link each path by edge $x_i = u_0^{(i)}u_0^{(i+1)}$ when i is even and edge $y_i = u_\alpha^{(i)}u_\alpha^{(i+1)}$ when i is odd. The cycle of length mn is

$$P(a^{(1)}, u_\alpha^{(1)})y_1P(u_\alpha^{(2)}, u_0^{(2)})x_2P(u_0^{(3)}, u_\alpha^{(3)})y_3 \cdots x_{n-1}P(u_0^{(n)}, a^{(n)})P(a^{(n)}, a^{(1)})$$

when n is odd or

$$P(a^{(1)}, u_\alpha^{(1)})y_1P(u_\alpha^{(2)}, u_0^{(2)})x_2P(u_0^{(3)}, u_\alpha^{(3)})y_3 \cdots y_{n-1}P(u_\alpha^{(n)}, a^{(n)})P(a^{(n)}, a^{(1)})$$

when n is even.

This completes the proof. $\square$

By linking paths $P(a^{(1)}, u_\alpha^{(1)})$ and $P(a^{(2)}, u_\alpha^{(2)})$ of length $m-1$ of the first and the second copies of G and edges $u_\alpha^{(1)}u_\alpha^{(2)}$ and $a^{(1)}a^{(2)}$, $G\square P_2$ also contains a Hamiltonian cycle.

We consider a skirted graph of order 7 containing no cycle of length 4 as shown in Figure 2.4.

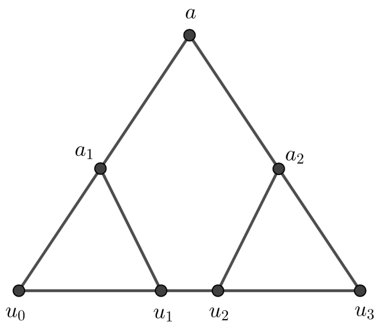

Figure 2.4: A skirted graph of order 7 containing no cycle of length 4

To study the n-generalized prism over a skirted graph, we start by investigating the n-generalized prism over this skirted graph as follows.

Theorem 2.10. *Let $G = G(a, u_0, u_3)$ be the skirted graph shown in Figure 2.4. Then, $G\Box P_n$ is pancyclic for $n \geq 2$.*

Proof. Let $G = G(a, u_0, u_3)$ be a skirted graph of order 7 such that G contains no cycle of length 4 (see Figure 2.4). We show that the n-generalized prism over G is pancyclic by the mathematical induction on n. It is easy to see that $G\Box P_2$ contains a cycle of each length l for $3 \leq l \leq 14$. Thus, $G\Box P_2$ is pancyclic.

For $n = 3$, since $G\Box P_2$ is a subgraph of $G\Box P_3$ and $G\Box P_2$ is pancyclic, $G\Box P_3$ contains a cycle of each length l for $3 \leq l \leq 14$. It suffices to show that $G\Box P_3$ contains a cycle of each length l for $15 \leq l \leq 21$. Two steps are shown. The first one is finding a cycle of each length l for $17 \leq l \leq 21$ and the second one is finding cycles of lengths 15 and 16.

Step 1 : To show that $G\Box P_3$ contains cycles of such lengths, we give the following paths and then link them together with edges joining each copy of G.

- The first copy of G contains a path $P(a^{(1)}, u_3^{(1)})$ of each length l for $5 \leq l \leq 6$ by Lemma 2.7(i). Also, for the last copy of G, a path $P(a^{(3)}, u_0^{(3)})$ of each length l for $5 \leq l \leq 6$ exists by the symmetry of G in Remark 2.8(i).

- The middle copy of G contains a path $P(u_0^{(2)}, u_3^{(2)})$ of each length l for $3 \leq l \leq 5$ (without the root $a^{(2)}$), which exists by Remark 2.8(ii).

- The path $P(a^{(3)}, a^{(1)}) = a^{(3)}a^{(2)}a^{(1)}$ of length 2 is a path in $G\square P_3$ from the last copy to the first copy of G.

Now, we link each path (maybe of different sizes) by edges $e_1 = u_3^{(1)}u_3^{(2)}$ and $e_2 = u_0^{(2)}u_0^{(3)}$. The cycle of length l for $17 \leq l \leq 21$ is

$$P(a^{(1)}, u_3^{(1)})e_1P(u_3^{(2)}, u_0^{(2)})e_2P(u_0^{(3)}, a^{(3)})P(a^{(3)}, a^{(1)}).$$

Step 2 : To show that $G\square P_3$ contains cycles of length 15 and 16, we give the following paths and then link them together with edges joining each copy of G.

- The first copy of G contains $P(a^{(1)}, u_3^{(1)}) = a^{(1)}a_1^{(1)}u_1^{(1)}u_2^{(1)}u_3^{(1)}$ of length 4.

- The middle copy of G contains $P(u_3^{(2)}, u_0^{(2)}) = u_3^{(2)}u_2^{(2)}u_1^{(2)}u_0^{(2)}$ of length 3.

- The last copy of G contains $P^*(u_0^{(3)}, a^{(3)}) = u_0^{(3)}u_1^{(3)}u_2^{(3)}u_3^{(3)}a_2^{(3)}a^{(3)}$ of length 5 and $P(u_0^{(3)}, a^{(3)}) = u_0^{(3)}u_1^{(3)}u_2^{(3)}a_2^{(3)}a^{(3)}$ of length 4.

- The path $P(a^{(3)}, a^{(1)}) = a^{(3)}a^{(2)}a^{(1)}$ of length 2 is a path in $G\square P_3$ from the last copy to the first copy of G.

Now, we link each path by edges $e_1 = u_3^{(1)}u_3^{(2)}$ and $e_2 = u_0^{(2)}u_0^{(3)}$. The cycle of length 16 is $P(a^{(1)}, u_3^{(1)})e_1P(u_3^{(2)}, u_0^{(2)})e_2P^*(u_0^{(3)}, a^{(3)})P(a^{(3)}, a^{(1)})$. The cycle of length 15 is $P(a^{(1)}, u_3^{(1)})e_1P(u_3^{(2)}, u_0^{(2)})e_2P(u_0^{(3)}, a^{(3)})P(a^{(3)}, a^{(1)})$.

Therefore, $G\square P_3$ is pancyclic.

For $n \geq 4$, suppose that $G\square P_{n-1}$ is pancyclic, i.e., $G\square P_{n-1}$ contains a cycle of each length l for $3 \leq l \leq 7(n-1)$. We shall find a cycle of each length l for $7(n-1)+1 \leq l \leq 7n$ in $G\square P_n$.

To show that $G\square P_n$ contains cycles of such lengths, we give the following paths and then link them together with edges joining each copy of G.

- The first copy and the last copy of G contain $P(a^{(1)}, u_3^{(1)})$ and $P(a^{(n)}, u_3^{(n)})$, respectively, of each length l for $5 \le l \le 6$ by Lemma 2.7(i). Also, for the last copy of G a path $P(a^{(n)}, u_0^{(n)})$ of each length l for $5 \le l \le 6$ exists by the symmetry of G in Remark 2.8(i).

- The remaining $n-2$ copies of G contain a path $P(u_0^{(i)}, u_3^{(i)})$ of each length l for $3 \le l \le 5$ (without the root $a^{(i)}$) for $2 \le i \le n-1$, which exists by Remark 2.8(ii).

- The path $P(a^{(n)}, a^{(1)}) = a^{(n)}a^{(n-1)}a^{(n-2)} \cdots a^{(1)}$ of length $n-1$ is a path in $G\square P_n$ from the last copy to the first copy of G.

Now, we link each path (maybe of different sizes) by edge $x_i = u_0^{(i)}u_0^{(i+1)}$ when i is even and edge $y_i = u_3^{(i)}u_3^{(i+1)}$ when i is odd. The cycle of length l for $5n+2 \le l \le 7n$ is

$$P(a^{(1)}, u_3^{(1)})y_1P(u_3^{(2)}, u_0^{(2)})x_2P(u_0^{(3)}, u_3^{(3)})y_3 \cdots x_{n-1}P(u_0^{(n)}, a^{(n)})P(a^{(n)}, a^{(1)})$$

when n is odd or

$$P(a^{(1)}, u_3^{(1)})y_1P(u_3^{(2)}, u_0^{(2)})x_2P(u_0^{(3)}, u_3^{(3)})y_3 \cdots y_{n-1}P(u_3^{(n)}, a^{(n)})P(a^{(n)}, a^{(1)})$$

when n is even.

Since $5n+2 \le 7(n-1)+1$ for $n \ge 4$, $G\square P_n$ contains a cycle of each length l for $7(n-1)+1 \le l \le 7n$.

Therefore, $G\square P_n$ is pancyclic. $\square$

We have that the n-generalized prism over any skirted graph is Hamiltonian. Then, to satisfy the metaconjecture, we are interested to see that *Is the n-generalized prism over any skirted graph pancyclic?* To answer this question, we start by investigating the n-generalized prism over a skirted graph with three specific types. These three types were introduced by Bondy and Lovász [4] in 1985. They studied the pancyclicity for a Halin graph. To show that a Halin graph is almost pancyclic, they restricted the problem into a reduced Halin graph and then showed that a reduced Halin graph is almost pancyclic, i.e., it contains cycles of each length l for $3 \leq l \leq n$, except, possibly, for one even value of l. Moreover, if it contains no cycle of even length m, where $3 < m \leq n$, then it contains a subgraph which is also a skirted graph of order $2m-1$ of type I, II or III (see Figure 2.5).

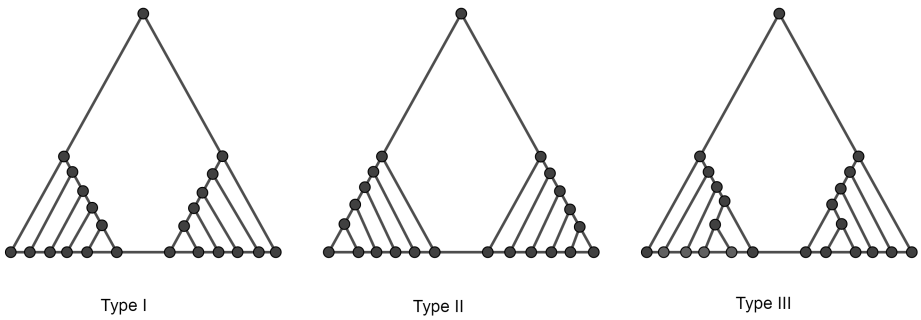

Figure 2.5: Skirted graphs of order $2m-1$ of type I, II and III

From Figure 2.5, we note that types I and III contain $\alpha = m-1, \beta = 2$ and $\gamma = 2$, while $\alpha = m-1, \beta = m/2$ and $\gamma = m/2$ for type II.

Since α, β and γ of types I and III are the same, while the other type has different values of β and γ, we separate the main study of this chapter into two sections. When $m = 4$, we can see that the skirted graphs of these three types are the skirted graph shown in Figure 2.4. Furthermore, we already showed that the n-generalized prism over the skirted graph in Figure 2.4 is pancyclic. Thus, we next consider the case that $m \geq 6$.

In Section 2.2, we prove the pancyclicity results for the n-generalized prism over a skirted graph of type I or III by using Lemma 2.7 and the mathematical induction on n. In Section 2.3, by using a similar idea, we can also prove the pancyclicity of the n-generalized prism over a skirted graph of type II. Finally, conclusion and discussion about this topic are given in Section 2.4.

2.2 The n-generalized prism over a skirted graph of type I or III

We already know that a skirted graph $G = G(a, u_0, u_\alpha)$ of type I or III of order $2m-1$ is m-almost pancyclic, i.e., G contains a cycle of each length l for $3 \le l \le 2m-1$ except for a cycle of even length m. Since G is a subgraph of $G\square P_2$, $G\square P_2$ contains such cycles of length l for $3 \le l \le 2m-1$ except possibly $l = m$. To show that $G\square P_2$ is pancyclic, we first show that $G\square P_2$ contains a cycle of length m.

Lemma 2.11. *Let $G = G(a, u_0, u_\alpha)$ be a skirted graph of order $2m-1$, where m is an even integer such that $m \ge 6$ and G is of type I or III. Then, $G\square P_2$ contains a cycle of each length l where l is an even integer ranging from 4 to $2m+6$.*

Proof. Since G is of type I or III, it contains $m+3$ consecutive vertices which are incident to the unbounded face, called $w_0, w_1, w_2, \ldots, w_{m+2}$, respectively. We define a sequence of $m+2$ cycles in $G\square P_2$ as follows.

$$w_1^{(1)}w_0^{(1)}w_0^{(2)}w_1^{(2)}w_1^{(1)},$$

$$w_2^{(1)}w_1^{(1)}w_0^{(1)}w_0^{(2)}w_1^{(2)}w_2^{(2)}w_2^{(1)},$$

$$w_3^{(1)}w_2^{(1)}w_1^{(1)}w_0^{(1)}w_0^{(2)}w_1^{(2)}w_2^{(2)}w_3^{(2)}w_3^{(1)},$$

$$\cdots,$$

$$w_{m+2}^{(1)}w_{m+1}^{(1)}w_m^{(1)}w_{m-1}^{(1)}\cdots w_1^{(1)}w_0^{(1)}w_0^{(2)}w_1^{(2)}\cdots w_m^{(2)}w_{m+1}^{(2)}w_{m+2}^{(2)}w_{m+2}^{(1)}.$$

The length of each cycle in the sequence increases as an arithmetic sequence with the common difference 2. Then, the last cycle

$$w_{m+2}^{(1)}w_{m+1}^{(1)}w_m^{(1)}w_{m-1}^{(1)}\cdots w_1^{(1)}w_0^{(1)}w_0^{(2)}w_1^{(2)}\cdots w_m^{(2)}w_{m+1}^{(2)}w_{m+2}^{(2)}w_{m+2}^{(1)}$$

of this sequence has length $2m+6$. Since the first cycle $w_1^{(1)}w_0^{(1)}w_0^{(2)}w_1^{(2)}w_1^{(1)}$ is a cycle of length 4, the lengths of the cycles are even integers ranging from 4 to $2m + 6$. $\square$

By Lemma 2.11, we can see that if $G = G(a, u_0, u_\alpha)$ is a skirted graph of order $2m-1$ of type I or III, where $m \geq 6$ is an even integer, then $G \square P_2$ contains a cycle of length m. Next, we need the following lemma to show that the n-generalized prism over a skirted graph of order $2m - 1$ of type I or III is pancyclic.

Lemma 2.12. *Let $G = G(a, u_0, u_\alpha)$ be a skirted graph of order $2m - 1$, where m is an even integer such that $m \geq 6$, and G is of type I or III. Then, $G \square P_2$ is pancyclic.*

Proof. By the result of Bondy and Lovász in [4] that $G = G(a, u_0, u_\alpha)$ is m-almost pancyclic and Lemma 2.11, $G \square P_2$ contains a cycle of each length l for $3 \leq l \leq 2m - 1$. It suffices to show that the prism over G contains a cycle of each length l for $2m \leq l \leq 4m - 2$.

For $1 \leq i \leq 2$, the i-th copy of G contains a path $P(u_0^{(i)}, u_\alpha^{(i)})$ of length l for $m - 1 \leq l \leq 2m - 2$, by Lemma 2.7(ii). We link each path $P(u_0^{(i)}, u_\alpha^{(i)})$ (maybe of different sizes) for $1 \leq i \leq 2$ together with edges $u_0^{(1)} u_0^{(2)}$ and $u_\alpha^{(1)} u_\alpha^{(2)}$. The cycle of each length l for $2m \leq l \leq 4m - 2$ is $P(u_0^{(1)}, u_\alpha^{(1)}) u_\alpha^{(1)} u_\alpha^{(2)} P(u_\alpha^{(2)}, u_0^{(2)}) u_0^{(2)} u_0^{(1)}$.

Therefore, $G \square P_2$ is pancyclic. $\square$

By using Lemma 2.12 as a basis step, we can use the mathematical induction to establish the following result.

Theorem 2.13. *Let $G = G(a, u_0, u_\alpha)$ be a skirted graph of order $2m - 1$, where m is an even integer such that $m \geq 6$, and of type I or III. Then, $G \square P_n$ is pancyclic for $n \geq 2$.*

Proof. We prove by the mathematical induction on the order of P_n. The basis step is already done by Lemma 2.12. For $n \geq 3$, suppose that $G \square P_{n-1}$ is pancyclic, i.e., $G \square P_{n-1}$ contains a cycle of each length l for $3 \leq l \leq (n - 1)(2m - 1)$. We shall find a cycle of each length l for $(n - 1)(2m - 1) + 1 \leq l \leq n(2m - 1)$.

To show that $G\Box P_n$ contains cycles of such length, we give the following paths and then link them together with edges joining each copy of G.

- The first copy and the last copy of G contain $P(a^{(1)}, u_\alpha^{(1)})$ and $P(a^{(n)}, u_\alpha^{(n)})$, respectively, of each length l for $m+1 \leq l \leq 2m-2$ by Lemma 2.7(i). Also, for the last copy of G a path $P(a^{(n)}, u_0^{(n)})$ of each length l for $m+1 \leq l \leq 2m-2$ exists by the symmetry of G in Remark 2.8(i).

- The remaining $n-2$ copies of G contain a path $P(u_0^{(i)}, u_\alpha^{(i)})$ of each length l for $m-1 \leq l \leq 2m-3$ (without the root $a^{(i)}$) for $2 \leq i \leq n-1$, which exists by Remark 2.8(ii).

- The path $P(a^{(n)}, a^{(1)}) = a^{(n)}a^{(n-1)}a^{(n-2)} \cdots a^{(1)}$ of length $n-1$ is a path in $G\Box P_n$ from the last copy to the first copy of G.

Now, we link each path (maybe of different sizes) by edge $x_i = u_0^{(i)}u_0^{(i+1)}$ when i is even and edge $y_i = u_\alpha^{(i)}u_\alpha^{(i+1)}$ when i is odd. The cycle of length l for $mn+n+2 \leq l \leq n(2m-1)$ is

$$P(a^{(1)}, u_\alpha^{(1)})y_1P(u_\alpha^{(2)}, u_0^{(2)})x_2P(u_0^{(3)}, u_\alpha^{(3)})y_3 \cdots x_{n-1}P(u_0^{(n)}, a^{(n)})P(a^{(n)}, a^{(1)})$$

when n is odd or

$$P(a^{(1)}, u_\alpha^{(1)})y_1P(u_\alpha^{(2)}, u_0^{(2)})x_2P(u_0^{(3)}, u_\alpha^{(3)})y_3 \cdots y_{n-1}P(u_\alpha^{(n)}, a^{(n)})P(a^{(n)}, a^{(1)})$$

when n is even.

We can conclude that $G\Box P_n$ is pancyclic if $mn+n+2 \leq (n-1)(2m-1)+1$, that is, $n \geq 2m/(m-2)$. Since $3 \geq 2m/(m-2)$ for all $m \geq 6$, $n \geq 2m/(m-2)$ for all $n \geq 3$.

Therefore, $G\Box P_n$ is pancyclic. $\square$

2.3 The n-generalized prism over a skirted graph of type II

We already know that a skirted graph $G = G(a, u_0, u_\alpha)$ of type II of order $2m-1$ is m-almost pancyclic, i.e., G contains a cycle of each length l for $3 \leq l \leq 2m-1$ except for a cycle of even length m. Since G is subgraph of $G\Box P_2$, $G\Box P_2$ contains such cycles of length l for $3 \leq l \leq 2m-1$ except possibly $l = m$. To show that $G\Box P_2$ is pancyclic, we first show that $G\Box P_2$ contains a cycle of length m.

Lemma 2.14. *Let $G = G(a, u_0, u_\alpha)$ be a skirted graph of order $2m-1$, where m is an even integer such that $m \geq 6$ and G is of type II. Then, $G\Box P_2$ contains a cycle of each length l where l is an even integer ranging from 4 to $4m-2$.*

Proof. Since G is of type II, it contains $2m-1$ consecutive vertices which are incident to the unbounded face, called $w_0, w_1, w_2, \ldots, w_{2m-2}$, respectively. We define a sequence of $2m-2$ cycles in $G\Box P_2$ as follows.

$$w_1^{(1)}w_0^{(1)}w_0^{(2)}w_1^{(2)}w_1^{(1)},$$

$$w_2^{(1)}w_1^{(1)}w_0^{(1)}w_0^{(2)}w_1^{(2)}w_2^{(2)}w_2^{(1)},$$

$$w_3^{(1)}w_2^{(1)}w_1^{(1)}w_0^{(1)}w_0^{(2)}w_1^{(2)}w_2^{(2)}w_3^{(2)}w_3^{(1)},$$

$$\cdots,$$

$$w_{2m-2}^{(1)}w_{2m-3}^{(1)}w_{2m-4}^{(1)}w_{2m-5}^{(1)}\cdots w_1^{(1)}w_0^{(1)}w_0^{(2)}w_1^{(2)}\cdots w_{2m-4}^{(2)}w_{2m-3}^{(2)}w_{2m-2}^{(2)}w_{2m-2}^{(1)}.$$

The length of each cycle in the sequence increases as an arithmetic sequence with the common difference 2. Then, the last cycle

$$w_{2m-2}^{(1)}w_{2m-3}^{(1)}w_{2m-4}^{(1)}w_{2m-5}^{(1)}\cdots w_1^{(1)}w_0^{(1)}w_0^{(2)}w_1^{(2)}\cdots w_{2m-4}^{(2)}w_{2m-3}^{(2)}w_{2m-2}^{(2)}w_{2m-2}^{(1)}$$

of this sequence has length $4m-2$. Since the first cycle $w_1^{(1)}w_0^{(1)}w_0^{(2)}w_1^{(2)}w_1^{(1)}$ is a cycle of length 4, the lengths of the cycles are even integers ranging from 4 to

$4m-2$. $\square$

By Lemma 2.14, we can see that if $G = G(a, u_0, u_\alpha)$ is a skirted graph of order $2m-1$ of type II, where $m \geq 6$ is an even integer, then $G\square P_2$ contains a cycle of length m. Next, we need the following lemmas to show that the n-generalized prism over a skirted graph of order $2m-1$ of type II is pancyclic.

Lemma 2.15. *Let $G = G(a, u_0, u_\alpha)$ be a skirted graph of order $2m-1$, where m is an even integer such that $m \geq 6$, and G is of type II. Then, $G\square P_2$ is pancyclic.*

Proof. By the result of Bondy and Lovász in [4] that $G = G(a, u_0, u_\alpha)$ is m-almost pancyclic and Lemma 2.14, $G\square P_2$ contains a cycle of each length l, $3 \leq l \leq 2m-1$. It suffices to show that the prism over G contains a cycle of each length l for $2m \leq l \leq 4m-2$.

For $1 \leq i \leq 2$, the i-th copy of G contains a path $P(u_0^{(i)}, u_\alpha^{(i)})$ of length l for $m-1 \leq l \leq 2m-2$, by Lemma 2.7(ii). We link each path $P(u_0^{(i)}, u_\alpha^{(i)})$ (maybe of different sizes) for $1 \leq i \leq 2$ together with edges $u_0^{(1)}u_0^{(2)}$ and $u_\alpha^{(1)}u_\alpha^{(2)}$. The cycle of each length l for $2m \leq l \leq 4m-2$ is $P(u_0^{(1)}, u_\alpha^{(1)})u_\alpha^{(1)}u_\alpha^{(2)}P(u_\alpha^{(2)}, u_0^{(2)})u_0^{(2)}u_0^{(1)}$.

Therefore, $G\square P_2$ is pancyclic. $\square$

Lemma 2.16. *Let $G = G(a, u_0, u_\alpha)$ be a skirted graph of order $2m-1$, where m is an even integer such that $m \geq 6$ and G is of type II. Then, $G\square P_3$ is pancyclic.*

Proof. Let $G = G(a, u_0, u_\alpha) = T \cup P$ be a skirted graph of type II. By Lemma 2.15, $G\square P_3$ contains a cycle of each length l for $3 \leq l \leq 4m-2$. It suffices to show that $G\square P_3$ contains a cycle of each length l for $4m-1 \leq l \leq 6m-3$. Two steps are shown. The first one is finding a cycle of each length l for $4m+1 \leq l \leq 6m-3$ and the second one is finding cycles of length $4m-1$ and $4m$.

Step 1 : To show that $G\square P_3$ contains cycles of such length, we give the following paths and then link them together with edges joining each copy of G.

- The first copy of G contains a path $P(a^{(1)}, u_\alpha^{(1)})$, of each length l for $(3m-2)/2 \leq l \leq 2m-2$ by Lemma 2.7(i). Also, for the last copy of G, a path

$P(a^{(3)}, u_0^{(3)})$ of each length l for $(3m-2)/2 \leq l \leq 2m-2$ exists by the symmetry of G in Remark 2.8(i).

- The middle copy of G contains a path $P(u_0^{(2)}, u_\alpha^{(2)})$ of each length l for $m-1 \leq l \leq 2m-3$ (without the root $a^{(2)}$), which exists by Remark 2.8(ii).

- The path $P(a^{(3)}, a^{(1)}) = a^{(3)}a^{(2)}a^{(1)}$ of length 2 is a path in $G\square P_3$ from the last copy to the first copy of G.

Now, we link each path (maybe of different sizes) by edges $e_1 = u_\alpha^{(1)}u_\alpha^{(2)}$ and $e_2 = u_0^{(2)}u_0^{(3)}$. The cycle of length l for $4m+1 \leq l \leq 6m-3$ is

$$P(a^{(1)}, u_\alpha^{(1)})e_1P(u_\alpha^{(2)}, u_0^{(2)})e_2P(u_0^{(3)}, a^{(3)})P(a^{(3)}, a^{(1)}).$$

Step 2 : To show that $G\square P_3$ contains cycles of lengths $4m-1$ and $4m$, we modify the cycle of length $4m+1$ from Step 1, where $P(a^{(1)}, u_\alpha^{(1)})$ and $P(u_0^{(3)}, a^{(3)})$ have length $(3m-2)/2$ and $P(u_\alpha^{(2)}, u_0^{(2)})$ has length $m-1$. For the first copy of G, let $P(a^{(1)}, u_0^{(1)})$ be the path of length $m/2$ from $a^{(1)}$ to $u_0^{(1)}$ containing all vertices which are incident to the unbounded face of G and $P(u_0^{(1)}, u_\alpha^{(1)})$ be the path of length $m-1$ from $u_0^{(1)}$ to $u_\alpha^{(1)}$ containing all vertices which are incident to the unbounded face of G. Then, $P(a^{(1)}, u_\alpha^{(1)}) = P(a^{(1)}, u_0^{(1)})P(u_0^{(1)}, u_\alpha^{(1)})$ is the path of length $(3m-2)/2$ containing the vertex $u_0^{(1)}$. Similary, for the third copy of G, we have that $P(u_0^{(3)}, a^{(3)}) = P(u_0^{(3)}, u_\alpha^{(3)})P(u_\alpha^{(3)}, a^{(3)})$ is the path of length $(3m-2)/2$ containing the vertex $u_\alpha^{(3)}$. Then, removing vertex $u_0^{(1)}$ (respectively $u_0^{(1)}$ and $u_\alpha^{(3)}$) makes the cycle of length $4m+1$ to become a cycle of length $4m$ (respectively a cycle of length $4m-1$).

Therefore, $G\square P_3$ is pancyclic. $\square$

We see that, in the proof of Lemma 2.16, the Cartesian product of $G = G(a, u_0, u_\alpha)$ and a path of order 3, we have to consider the special case as shown in Step 2. However, there is no special case when we show that $G\square P_n$ is pancyclic for $n \geq 4$.

By using Lemmas 2.15 and 2.16 as a basis step, we can use the mathematical induction to establish the following result.

Theorem 2.17. *Let $G = G(a, u_0, u_\alpha)$ be a skirted graph of order $2m-1$, where m is an even integer such that $m \geq 6$ and G is of type II. Then, $G \square P_n$ is pancyclic for $n \geq 2$.*

Proof. We prove by the mathematical induction on the order of P_n. The basis step is already done by Lemmas 2.15 and 2.16 for $n = 2$ and $n = 3$, respectively. For $n \geq 4$, suppose that $G \square P_{n-1}$ is pancyclic, i.e., $G \square P_{n-1}$ contains a cycle of each length l for $3 \leq l \leq (n-1)(2m-1)$. We shall find a cycle of each length l for $(n-1)(2m-1)+1 \leq l \leq n(2m-1)$.

To show that $G \square P_n$ contains cycles of such lengths, we give the following paths and then link them together with edges joining each copy of G.

- The first copy and the last copy of G contain $P(a^{(1)}, u_\alpha^{(1)})$ and $P(a^{(n)}, u_\alpha^{(n)})$, respectively, of each length l for $(3m-2)/2 \leq l \leq 2m-2$ by Lemma 2.7(i). Also, for the last copy of G a path $P(a^{(n)}, u_0^{(n)})$ of each length l for $(3m-2)/2 \leq l \leq 2m-2$ exists by the symmetry of G in Remark 2.8(i).

- The remaining $n-2$ copies of G contain a path $P(u_0^{(i)}, u_\alpha^{(i)})$ of each length l for $m-1 \leq l \leq 2m-3$ (without the root $a^{(i)}$) for $2 \leq i \leq n-1$, which exists by Remark 2.8(ii).

- The path $P(a^{(n)}, a^{(1)}) = a^{(n)}a^{(n-1)}a^{(n-2)} \cdots a^{(1)}$ of length $n-1$ is a path in $G \square P_n$ from the last copy to the first copy of G.

Now, we link each path (maybe of different sizes) by edge $x_i = u_0^{(i)}u_0^{(i+1)}$ when i is even and edge $y_i = u_\alpha^{(i)}u_\alpha^{(i+1)}$ when i is odd. The cycle of length l for $mn + m + n - 2 \leq l \leq n(2m-1)$ is

$$P(a^{(1)}, u_\alpha^{(1)})y_1P(u_\alpha^{(2)}, u_0^{(2)})x_2P(u_0^{(3)}, u_\alpha^{(3)})y_3 \cdots x_{n-1}P(u_0^{(n)}, a^{(n)})P(a^{(n)}, a^{(1)})$$

when n is odd or

$$P(a^{(1)}, u_\alpha^{(1)})y_1P(u_\alpha^{(2)}, u_0^{(2)})x_2P(u_0^{(3)}, u_\alpha^{(3)})y_3 \cdots y_{n-1}P(u_\alpha^{(n)}, a^{(n)})P(a^{(n)}, a^{(1)})$$

when n is even.

We can conclude that $G\square P_n$ is pancyclic if $mn+m+n-2 \leq (n-1)(2m-1)+1$, that is, $n \geq (3m-4)/(m-2)$. Since $4 > (3m-4)/(m-2)$ for all $m \geq 6$, $n \geq (3m-4)/(m-2)$ for all $n \geq 4$.

Therefore, $G\square P_n$ is pancyclic. $\square$

2.4 Conclusion and discussion

In this chapter, we prove that the n-generalized prism over a skirted graph of type I, II or III are pancyclic by applying the lemma given by Bondy and Lovász [4]. To apply the lemma, we have to know the exact number of vertices which are incident to the unbounded face of each skirted graph. This constraint is the reason why the technique in this chapter cannot be directly applied to the n-generalized prism of any skirted graphs. Thus, we will develop a technique to overcome this difficulty in the next chapter.

CHAPTER III

THE n-GENERALIZED PRISM OVER A SKIRTED GRAPH

In this chapter, we study pancyclicity of the n-generalized prism over a skirted graph. We first provide our preliminary results on hamiltonicity and pancyclicity as well as the motivation of the main results of this chapter as follows.

3.1 Preliminary results and motivation

In 1971, Bondy [2] posed a metaconjecture: almost any nontrivial condition on a graph which implies that the graph is Hamiltonian also implies that the graph is pancyclic (there may be a simple family of exceptional graphs). From Chapter II, we have proved that the n-generalized prism over any skirted graph is Hamiltonian. This metaconjecture motivates us to investigate the pancyclicity of the n-generalized prism over any skirted graph. However, the technique that we use in Chapter II cannot be directly applied to any skirted graphs other than those three types since we do not know the exact configuration of their vertices and edges. Thus, we develop a technique to overcome this difficulty in this chapter.

From Chapter II, we have proved the following theorem.

Theorem 3.1. *The n-generalized prism over any skirted graph is Hamiltonian.*

Now, we notice that a skirted graph $G = T \cup P$ contains a cycle of length 3 where one of the edges of such cycle belongs to the path P as follows.

Lemma 3.2. *A skirted graph $G = T \cup P$ contains a cycle of length* 3 *with exactly one edge of the cycle belongs to the path P.*

Proof. To prove this statement, we let $P = u_0u_1u_2\cdots u_\alpha$. Let T'' be a rooted tree obtained by deleting all leaves of T. If T'' is a singleton, then it means that all children of the root a of T are leaves of T. Since a has at least two children, $G = T \cup P$ contains a cycle of length 3 with exactly one edge of the cycle belongs to the path P. Otherwise, T'' contains a vertex u of degree one. This implies that u is an internal vertex of T such that all of its children are leaves of T. Since u has at least two children. Let U be the set of all children of u. Thus, $U \subseteq V(P)$ and $|U| \geq 2$. Let $u_i \in U$ and i be the minimum index of vertices in U. Since u has at least two children and P is obtained by connecting the leaves of T in the order determined by the embedding of T, $u_{i+1} \in U$. Thus, $\{u, u_i, u_{i+1}\}$ induces a cycle of length 3 in G. Moreover, this cycle has one edge u_iu_{i+1} belongs to the path P. □

In general, a triangle in graph theory usually means a cycle of length 3. However, in this research, we define a triangle as follows.

Definition 3.3. Let $G(a, u_0, u_\alpha) = T \cup P$ be a skirted graph with $P = u_0u_1u_2\cdots u_\alpha$. For $i, j \in \{0, 1, 2, \ldots, \alpha\}$ and $i < j$, an induced subgraph $C(u, u_i, u_j)$ of $G(a, u_0, u_\alpha)$ is said to be a *triangle* in $G(a, u_0, u_\alpha)$ if

- u is an internal vertex of T such that all children of u are leaves of T and;
- u_i is the first vertex and u_j is the last vertex in P in which u_i and u_j are children of u.

Moreover, since P is obtained by connecting the leaves of T in the order determined by the embedding of T, vertices between u_i and u_j in the path P, $u_{i+1}, u_{i+2}, u_{i+3}, \ldots, u_{j-1}$, are all children of u (see Figure 3.1).

Observation 3.4. *From Definition 3.3, a triangle $C(u, u_i, u_j)$ of $G(a, u_0, u_\alpha)$ is also a skirted graph $T' \cup P'$ containing the side skirt T' with root u and the path $P' = u_iu_{i+1}u_{i+2}\cdots u_j$. Note that u has degree at least two because $i < j$.*

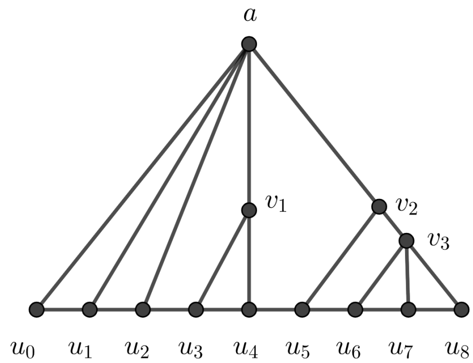

Figure 3.1: $C(v_1, u_3, u_4)$ and $C(v_3, u_6, u_8)$ are triangles in $G(a, u_0, u_8)$, while $C(a, u_0, u_2)$ is not a triangle

We obtain from the proof of Lemma 3.2 that a skirted graph $G = T \cup P$ contains a cycle of length 3 induced by $\{u, u_i, u_{i+1}\}$ in G. Since all children of u are leaves of T, we can extend such cycle into a triangle. Therefore, a skirted graph contains a triangle.

Lemma 3.5. *Let $G(a, u_0, u_\alpha) = T \cup P$ be a skirted graph with $P = u_0u_1u_2 \cdots u_\alpha$. If G' is a simple graph obtained from a skirted graph $G(a, u_0, u_\alpha)$ by contracting a triangle $C(u, u_i, u_j)$ of $G(a, u_0, u_\alpha)$ where $u \neq a$. Then, G' is a skirted graph.*

Proof. Let $G(a, u_0, u_\alpha) = T \cup P$ be a skirted graph and $C(u, u_i, u_j)$ be a triangle in $G(a, u_0, u_\alpha)$ for some $0 \leq i \leq \alpha - 1$ and $i < j$. Let G' be a simple graph obtained from $G(a, u_0, u_\alpha)$ by contracting $C(u, u_i, u_j)$ and u^* be the vertex of G' representing the triangle $C(u, u_i, u_j)$, i.e., all vertices $u, u_i, u_{i+1}, u_{i+2}, \ldots, u_j$ are contracted into one vertex u^*. Since $u \neq a$, G' is not a trivial graph.

Consider the side skirt T of $G(a, u_0, u_\alpha)$. It can be regarded that we obtain T' from T by deleting all children of u and then turn the internal vertex u to be a leaf u^* of T'. The contraction does not affect the degree of other vertices in $G(a, u_0, u_\alpha)$. Thus, T' is a side skirt. Now, we consider the path P of $G(a, u_0, u_\alpha)$. The contraction turns the path $P = u_0u_1u_2 \ldots u_\alpha$ into the path $P' = u_0u_1 \ldots u_{i-1}u^*u_{j+1} \ldots u_\alpha$ in G'. Since the contraction does not affect the degree of other vertices outside the triangle, all leaves of T except $u_i, u_{i+1}, u_{i+2}, \ldots, u_j$ are still the leaves of T'. Thus, all vertices of P' are all leaves of T'. Since G' is a union $T' \cup P'$, G' is a skirted graph. $\square$

Note that $G' = G'(a, u_0, u_\alpha)$ if $i, j \notin \{0, \alpha\}$, $G' = G'(a, u^*, u_\alpha)$ if $i = 0$ (in this case, $j \neq \alpha$) and $G' = G'(a, u_0, u^*)$ if $j = \alpha$ (in this case, $i \neq 0$). However, to prove Lemma 3.5, we do not care about the endpoints of the path P' in G'. Thus, we just wrote G'.

The following figure shows skirted graphs $G'(a, u_0, u_8)$ and $G'(a, u_0, u^*)$ obtained from skirted graph $G(a, u_0, u_8)$ by contracting triangles $C(v_1, u_3, u_4)$ and $C(v_3, u_6, u_8)$, respectively

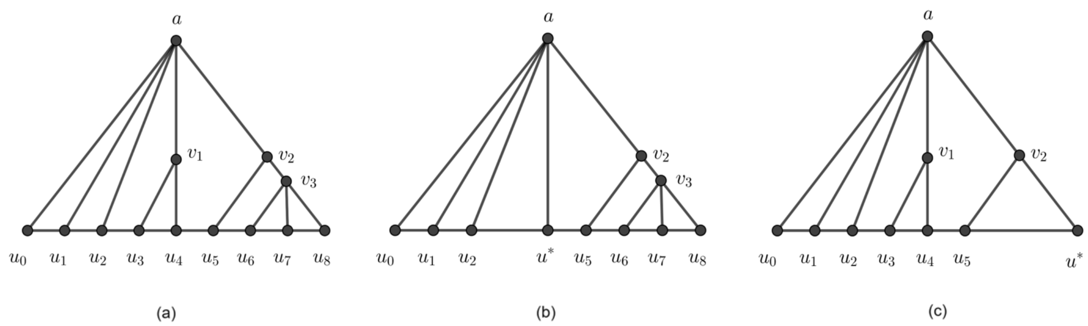

Figure 3.2: (a) a skirted graph $G(a, u_0, u_8)$, (b) and (c) skirted graphs obtained from $G(a, u_0, u_8)$ by contracting triangles $C(v_1, u_3, u_4)$ and $C(v_3, u_6, u_8)$, respectively

From Lemma 3.5, we already know that if G' is a simple graph obtained from a skirted graph $G(a, u_0, u_\alpha)$ by contracting a triangle $C(u, u_i, u_j)$ of $G(a, u_0, u_\alpha)$ where $u \neq a$. Then, G' is a skirted graph. Next, we investigate the case that $u = a$. By the definition of a triangle, we obtain that $i = 0$ and $j = \alpha$. Thus, in this case, the skirted graph $G(a, u_0, u_\alpha)$ is a triangle. In the next section, we prove the pancyclicity results for the n-generalized prism over a triangle.

3.2 Pancyclicity of the n-generalized prism over a triangle

To show that the n-generalized prism over a triangle is pancyclic, we need the following lemmas.

Lemma 3.6. *Let $C = C(u, u_0, u_\alpha)$ be a triangle of order $\alpha+2$. Then, C contains:*

(i) a (u, u_α)-path of each length l for $1 \leq l \leq \alpha + 1$;

(ii) a (u_0, u_α)-path of lengths α and $\alpha + 1$.

Proof. Let $C = C(u, u_0, u_\alpha) = T \cup P$ be a triangle of order $\alpha + 2$ and $P = u_0u_1u_2 \cdots u_\alpha$. We prove this statement by the mathematical induction on α. If $\alpha = 1$, then C is a cycle of length 3. It contains (i) a (u, u_1)-path of lengths 1 and 2 and (ii) a (u_0, u_1)-path of lengths 1 and 2. Now, we suppose that the statement holds for all triangles of order less than $\alpha + 2$ where $\alpha > 1$.

Let $C' = (T - u_\alpha) \cup (P - u_\alpha)$. Then, $C' = C(u, u_0, u_{\alpha-1})$ is a triangle subgraph of C. By the induction hypothesis, we obtain that $C(u, u_0, u_{\alpha-1})$ contains (i) a $(u, u_{\alpha-1})$-path of each length l for $1 \leq l \leq \alpha$ and (ii) a $(u_0, u_{\alpha-1})$-path of lengths $\alpha - 1$ and α.

Since u_α is adjacent to u in C, C contains a (u, u_α)-path of length 1. Since u_α is adjacent to $u_{\alpha-1}$ in C, we can extend a $(u, u_{\alpha-1})$-path of length l to a (u, u_α)-path of length $l+1$. Thus, C contains (i) a (u, u_α)-path of each length l for $1 \leq l \leq \alpha+1$ and (ii) a (u_0, u_α)-path of lengths α and $\alpha + 1$. □

Remark 3.7. We obtain that

(i) Lemma 3.6(i) gives a (u, u_0)-path of each length l for $1 \leq l \leq \alpha + 1$ by the symmetry of $C(u, u_0, u_\alpha)$.

(ii) $P = u_0u_1u_2 \ldots u_\alpha$ is a (u_0, u_α)-path of length α (without the vertex u) in $C(u, u_0, u_\alpha)$.

The following lemma is an immediate observation about the pancyclicity of the prism over a triangle.

Lemma 3.8. *The prism over a triangle is pancyclic.*

Proof. Let $\alpha \geq 1$ and $C = C(u, u_0, u_\alpha)$ be a triangle of length $\alpha+2$. For $1 \leq s \leq 2$, the s-th copy of C contains a $(u^{(s)}, u_\alpha^{(s)})$-path of each length l for $1 \leq l \leq \alpha + 1$

by Lemma 3.6(i). We link each $(u^{(1)}, u_\alpha^{(1)})$-path and $(u^{(2)}, u_\alpha^{(2)})$-path (maybe of different sizes) together with edges $u^{(1)}u^{(2)}$ and $u_\alpha^{(1)}u_\alpha^{(2)}$. We obtain a cycle of each length l for $4 \le l \le 2\alpha + 4$. Since C contains a cycle of length 3, $C \square P_2$ is pancyclic. $\square$

By using Lemma 3.8 as a basic step, we can use the mathematical induction to establish the following result.

Theorem 3.9. *The n-generalized prism over a triangle is pancyclic.*

Proof. Let $\alpha \ge 1$ and $C = C(u, u_0, u_\alpha)$ be a triangle of order $\alpha + 2$ and P_n be a path of order $n \ge 2$. We prove that $C \square P_n$ is pancyclic by the mathematical induction on n. The basic step is already done by Lemma 3.8. For $n \ge 3$, suppose that $C \square P_{n-1}$ is pancyclic. Since $C \square P_{n-1}$ is a subgraph of $C \square P_n$, $C \square P_n$ contains a cycle of each length l for $3 \le l \le (\alpha + 2)(n - 1)$. We shall find a cycle of each length l for $(\alpha + 2)(n - 1) + 1 \le l \le (\alpha + 2)n$.

To show that $C \square P_n$ contains a cycle of such lengths, we give the following paths and link them together with edges joining each copy of C.

- The first copy and the last copy of C contain $P(u^{(1)}, u_\alpha^{(1)})$ and $P(u^{(n)}, u_\alpha^{(n)})$, respectively, of each length l for $1 \le l \le \alpha + 1$ by Lemma 3.6(i). Also, for the last copy of C, a path $P(u^{(n)}, u_0^{(n)})$ of each length l for $1 \le l \le \alpha + 1$ exists by the symmetry of C in Remark 3.7(i).

- The remaining $n - 2$ copies of G contain the path $P(u_0^{(s)}, u_\alpha^{(s)})$ of length α (without the root $u^{(s)}$) for $2 \le s \le n - 1$, which exists by Remark 3.7(ii).

- The path $P(u^{(n)}, u^{(1)}) = u^{(n)}u^{(n-1)}u^{(n-2)} \cdots u^{(1)}$ of length $n - 1$ is a path in $C \square P_n$ from the last copy to the first copy of C.

Now, we link each path (maybe of different sizes) by edge $u_\alpha^{(s)}u_\alpha^{(s+1)}$ when s is odd and by edge $u_0^{(s)}u_0^{(s+1)}$ when s is even. We obtain a cycle of each length l for $(\alpha + 2)n - 2\alpha \le l \le (\alpha + 2)n$. Since $(\alpha + 2)n - 2\alpha \le (\alpha + 2)(n - 1) + 1$ for all $\alpha \ge 1$, $C \square P_n$ contains a cycle of each length l for $(\alpha+2)(n-1)+1 \le l \le (\alpha+2)n$. Therefore, $C \square P_n$ is pancyclic. $\square$

3.3 Pancyclicity of the n-generalized prism over a skirted graph

To show that the n-generalized prism over a skirted graph is pancyclic, we first establish the preliminary results of even cycles in the n-generalized prism over a skirted graph. Note that since a skirted graph is traceable, we investigate the n-generalized prism over a path instead of the n-generalized prism over a skirted graph as follows.

3.3.1 Even cycles in the n-generalized prism over a path

Let $n \geq 2$ be an even integer and $m \geq 2$, we need the following lemma to prove that $P_m \Box P_n$ contains a cycle of each even length l where l is an even integer ranging from 4 to mn.

Lemma 3.10. *Suppose that $m \geq 2$. Then, the prism over P_m contains a cycle of each length l where l is an even integer ranging from 4 to $2m$. Moreover, if $P_m = v_1v_2v_3 \cdots v_m$, then the edges $v_1^{(1)}v_2^{(1)}$ and $v_1^{(2)}v_2^{(2)}$ of the first copy and the second copy of $P_m \Box P_2$, respectively, are contained in a cycle of each even length l for $4 \leq l \leq 2m$.*

Proof. Let $P_m = v_1v_2v_3 \cdots v_m$. We define a sequence of $m-1$ cycles in $P_m \Box P_2$ as follows.

$$v_2^{(1)}v_1^{(1)}v_1^{(2)}v_2^{(2)}v_2^{(1)},$$

$$v_3^{(1)}v_2^{(1)}v_1^{(1)}v_1^{(2)}v_2^{(2)}v_3^{(2)}v_3^{(1)},$$

$$v_4^{(1)}v_3^{(1)}v_2^{(1)}v_1^{(1)}v_1^{(2)}v_2^{(2)}v_3^{(2)}v_4^{(2)}v_4^{(1)},$$

$$\cdots,$$

$$v_m^{(1)}v_{m-1}^{(1)}v_{m-2}^{(1)}v_{m-3}^{(1)} \cdots v_2^{(1)}v_1^{(1)}v_1^{(2)}v_2^{(2)} \cdots v_{m-2}^{(2)}v_{m-1}^{(2)}v_m^{(2)}v_m^{(1)}.$$

The length of each cycle in the sequence increases as an arithmetic sequence with the common difference 2. Then, the last cycle

$$v_m^{(1)}v_{m-1}^{(1)}v_{m-2}^{(1)}v_{m-3}^{(1)}\cdots v_2^{(1)}v_1^{(1)}v_1^{(2)}v_2^{(2)}\cdots v_{m-2}^{(2)}v_{m-1}^{(2)}v_m^{(2)}v_m^{(1)}$$

of this sequence has length $2m$. Since the first cycle $v_2^{(1)}v_1^{(1)}v_1^{(2)}v_2^{(2)}v_2^{(1)}$ is a cycle of length 4, the lengths of the cycles are even integers ranging from 4 to $2m$. Moreover, $v_1^{(1)}v_2^{(1)}$ and $v_1^{(2)}v_2^{(2)}$ are edges contained in all even cycles. $\square$

Observation 3.11. *For $n \geq 2$ is an even integer and $m \geq 2$, if $P_m = v_1v_2v_3\cdots v_m$, then the edges $v_1^{(1)}v_2^{(1)}$ and $v_1^{(n)}v_2^{(n)}$ of the first copy and the last copy of $P_m \square P_n$, respectively, are contained in a cycle of length mn (see Figure 3.3).*

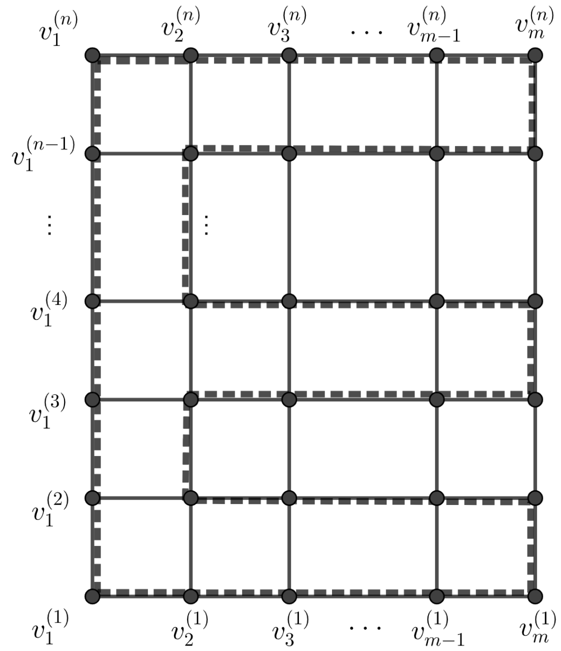

Figure 3.3: The dashed line represents a spanning cycle of length mn containing edges $v_1^{(1)}v_2^{(1)}$ and $v_1^{(n)}v_2^{(n)}$

By using Lemma 3.10 as a basic step, we can use the mathematical induction to establish the following result.

Lemma 3.12. *Suppose that $n \geq 2$ is an even integer and $m \geq 2$. Then, the n-generalized prism over P_m contains a cycle of each length l where l is an even* *integer ranging from 4 to mn. Moreover, if $P_m = v_1v_2v_3\cdots v_m$, then the edge $v_1^{(1)}v_2^{(1)}$ of the first copy of $P_m\Box P_n$ is contained in a cycle of each even length l for $4 \leq l \leq mn$.*

Proof. Let $P_m = v_1v_2v_3\cdots v_m$ where $m \geq 2$ and $n = 2k$ for some positive integer k. We prove by the mathematical induction on k. The basic step is already done by Lemma 3.10. For $k \geq 2$, suppose that $P_m\Box P_{2(k-1)}$ contains a cycle of each even length l where l is an even integer ranging from 4 to $2m(k-1)$. We shall find an even cycle of each length l for $2m(k-1)+2 \leq l \leq 2mk$.

Here, let us regard $P_m\Box P_{2(k-1)}$ as a subgraph of $P_m\Box P_{2k}$ induced by the set of all vertices of the first $2(k-1)$ copies of P_m. By Observation 3.11, there is a cycle C^* of length $2m(k-1)$ in $P_m\Box P_{2(k-1)}$ containing the edges $v_1^{(1)}v_2^{(1)}$ and $v_1^{(2k-2)}v_2^{(2k-2)}$.

Now, we consider the last two copies of P_m. The vertices of these two copies induce a subgraph $P_m\Box P_2$ of $P_m\Box P_{2k}$. By Lemma 3.10, an edge $v_1^{(2k-1)}v_2^{(2k-1)}$ is contained in a cycle of each even length l for $4 \leq l \leq 2m$ in $P_m\Box P_{2k}$. Since $v_1^{(2k-2)}v_1^{(2k-1)}$ and $v_2^{(2k-2)}v_2^{(2k-1)}$ are edges of $P_m\Box P_{2k}$, we delete edges $v_1^{(2k-2)}v_2^{(2k-2)}$ and $v_1^{(2k-1)}v_2^{(2k-1)}$ and then join $v_1^{(2k-1)}$ to $v_1^{(2k-2)}$ and $v_2^{(2k-1)}$ to $v_2^{(2k-2)}$, respectively. Then, C^* can be extended to a cycle of each even length l for $2m(k-1)+4 \leq l \leq 2mk$. Next, we extend C^* to be a cycle of even length $2m(k-1)+2$ by replacing the edge $v_1^{(2k-2)}v_2^{(2k-2)}$ with the path $v_1^{(2k-2)}v_1^{(2k-1)}v_2^{(2k-1)}v_2^{(2k-2)}$.

Moreover, since the cycle C^* contains edge $v_1^{(1)}v_2^{(1)}$ and the extension of C^* does not affect the edge $v_1^{(1)}v_2^{(1)}$, it is contained in a cycle of each even length l for $4 \leq l \leq mn$. □

By Lemma 3.12, $P_m\Box P_n$ contains an even cycle of each length l for $4 \leq l \leq mn$ when n is even. Next, to investigate the case that n is odd, we will only examine the case that $n = 3$ as follows.

Lemma 3.13. *Suppose that $m \geq 2$. Then, the 3-generalized prism over P_m contains a cycle of each length l where l is an even integer ranging from 4 to $3m$. Moreover, if $P_m = v_1v_2v_3\cdots v_m$, then the edge $v_1^{(1)}v_2^{(1)}$ of the first copy of $P_m\Box P_3$*

is contained in:

(i) a cycle of each even length l for $4 \leq l \leq 3m$ if m is even;

(ii) a cycle of each even length l for $4 \leq l \leq 3m-1$ if m is odd.

Proof. Let $m \geq 2$ and $P_m = v_1v_2v_3\cdots v_m$. Here, let us regard $P_m\square P_2$ as a subgraph of $P_m\square P_3$ induced by vertices of the first two copies of P_m. By Lemma 3.10 and $P_m\square P_2$ is a subgraph of $P_m\square P_3$, $P_m\square P_3$ contains a cycle of each length l where l is an even integer ranging from 4 to $2m$ and the edge $v_1^{(1)}v_2^{(1)}$ of the first copy of $P_m\square P_n$ is contained in a cycle of each length l where l is an even integer ranging from 4 to $2m$. We shall find an even cycles of each length l for $2m+2 \leq l \leq 3m$. By Lemma 3.10, $P_m\square P_2$ contains a cycle

$$C^* = v_m^{(1)}v_{m-1}^{(1)}v_{m-2}^{(1)}v_{m-3}^{(1)}\cdots v_2^{(1)}v_1^{(1)}v_1^{(2)}v_2^{(2)}\cdots v_{m-2}^{(2)}v_{m-1}^{(2)}v_m^{(2)}v_m^{(1)}$$

of length $2m$ in which it contains $v_1^{(1)}v_2^{(1)}$.

Now, we consider the second and the third copies of P_m. For an odd integer j such that $1 \leq j \leq m-1$, there is a path $P_j = v_j^{(2)}v_j^{(3)}v_{j+1}^{(3)}v_{j+1}^{(2)}$ of length 3 in $P_m\square P_3$.

Since $v_j^{(3)}$ and $v_{j+1}^{(3)}$ have not been contained in C^* for all odd integers j, we replace each edge $v_j^{(2)}v_{j+1}^{(2)}$ with each path P_j. Then, C^* can be extended to a cycle of each even length l for $2m+2 \leq l \leq 3m$. Since this extension does not change anything in the first copy of P_m, the extended cycle still contains the edge $v_1^{(1)}v_2^{(1)}$.

Moreover, we can see that (i) if m is even, then $v_1^{(1)}v_2^{(1)}$ is contained in a cycle of each even length l for $4 \leq l \leq 3m$ ($3m$ is even); (ii) if m is odd, then $v_1^{(1)}v_2^{(1)}$ is contained in a cycle of each even length l for $4 \leq l \leq 3m-1$ ($3m$ is odd). □

Figure 3.4 shows examples of cycles of length 18 and 20 in $P_6\square P_3$ and $P_7\square P_3$, respectively.

Remark 3.14. From the proof of Lemma 3.13, we obtain the cycles of length $3m$ when m is even and $3m-1$ when m is odd. We notice that, apart from edge $v_1^{(1)}v_2^{(1)}$, these two cycles also contain an edge $v_1^{(3)}v_2^{(3)}$ when $m \geq 3$.

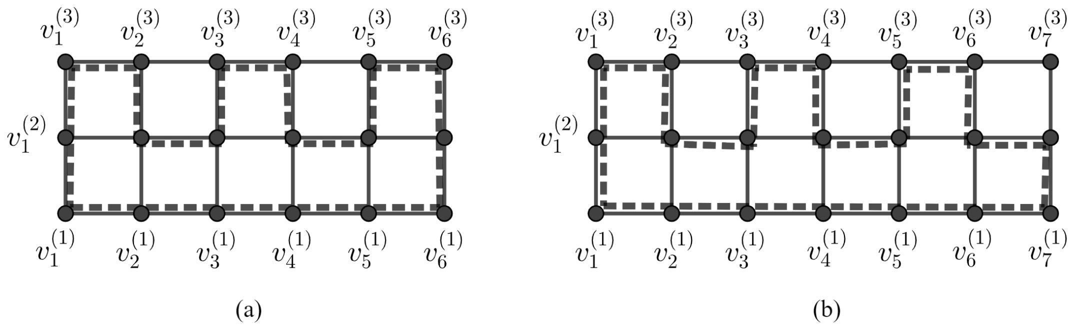

Figure 3.4: (a) The dashed line represents a cycle of length 18 in $P_6 \square P_3$ and (b) The dashed line represents a cycle of length 20 in $P_7 \square P_3$

3.3.2 Main results

To show that the n-generalized prism over any skirted graph is pancyclic, we start by providing some observations and investigating the pancyclicity of the prism over a skirted graph; and the pancyclicity of the 3-generalized prism over a skirted graph as follows.

Observation 3.15. *Let $m \geq 3$, $\alpha \geq 2$ and $G(a, u_0, u_\alpha) = T \cup P$ be a skirted graph of order m with $P = u_0 u_1 u_2 \cdots u_\alpha$ and $C = C(u, u_i, u_j)$ be a triangle of order t in $G(a, u_0, u_\alpha)$ such that $u \neq a$. Then, $m - t > 1$. Let G' be a skirted graph of order $m - (t-1)$ obtained from a skirted graph $G(a, u_0, u_\alpha)$ by contracting the triangle C and u^* be the vertex of G' representing the triangle C. By Theorem 2.5, G' is Hamiltonian. Let $C' = u^* v_1 v_2 v_3 \cdots v_{m-t} u^*$ be a spanning cycle in G'. Then, there is a spanning path $P' = u^* v_1 v_2 v_3 \cdots v_{m-t}$ in G'.*

Since u^* is the vertex of G' representing the triangle C and v_1 is adjacent to u^*, v_1 is adjacent to either u, u_i or u_j in $G(a, u_0, u_\alpha)$. Let $G = G(a, u_0, u_\alpha)$.

- *If $v_1 u_j \in E(G)$, then $P(u_i, v_{m-t}) = u_i u_{i+1} u_{i+2} \cdots u_j v_1 v_2 \cdots v_{m-t}$ is a path of length $m-2$ (without the vertex u) in G.*

- *If $v_1 u_i \in E(G)$, then $P(u_j, v_{m-t}) = u_j u_{j-1} u_{j-2} \cdots u_i v_1 v_2 \cdots v_{m-t}$ is a path of length $m-2$ (without the vertex u) in G.*

- *If $v_1u \in E(G)$, then v_{m-t} is adjacent to either u_i or u_j in $G(a, u_0, u_\alpha)$. Note that $v_1 \neq v_{m-t}$ since $m - t > 1$.*
  - *If $v_{m-t}u_j \in E(G)$, then $P(u_i, v_1) = u_iu_{i+1}u_{i+2}\cdots u_jv_{m-t}v_{m-t-1}v_{m-t-2}\cdots v_1$ is a path of length $m - 2$ (without the vertex u) in G.*
  - *If $v_{m-t}u_i \in E(G)$, then $P(u_j, v_1) = u_ju_{j-1}u_{j-2}\cdots u_iv_{m-t}v_{m-t-1}v_{m-t-2}\cdots v_1$ is a path of length $m - 2$ (without the vertex u) in G.*

We notice that the vertex u is not contained in each of these four paths and the vertex u is adjacent to the first two vertices of such paths. This note is used in the proof of the following theorems.

Theorem 3.16. *The prism over any skirted graph is pancyclic.*

Proof. First, we consider a single skirted graph. Let $G = G(a, u_0, u_\alpha) = T \cup P$ be a skirted graph of order m with $P = u_0u_1u_2\cdots u_\alpha$. Let $C = C(u, u_i, u_j)$ be a triangle of order t in $G(a, u_0, u_\alpha)$, where $t \leq m$. If $u = a$, then G itself is a triangle. By Theorem 3.9, the prism over G is pancyclic. Now, we assume that $u \neq a$.

Let G' be a skirted graph of order $m - (t - 1)$ obtained from a skirted graph G by contracting the triangle C and u^* be the vertex of G' representing the triangle C. By Theorem 2.5, G' is Hamiltonian. Let $C' = u^*v_1v_2v_3\cdots v_{m-t}u^*$ be a spanning cycle in G'. Then, $P' = u^*v_1v_2v_3\cdots v_{m-t}$ is a spanning path in G'.

Since u^* is the vertex of G' representing the triangle C and v_1 is adjacent to u^*, v_1 is adjacent to either u, u_i or u_j. By Observation 3.15, without loss of generality, let v_1 be adjacent to u_j. Then, $P(u_i, v_{m-t}) = u_iu_{i+1}u_{i+2}\cdots u_jv_1v_2\cdots v_{m-t}$ is a path of length $m - 2$ (without the vertex u) in G.

Now, consider prism over a skirted graph which contains the first and the second copies of the same skirted graph. By Lemma 3.10, $P(u_i, v_{m-t})\square P_2$ contains a cycle C^* of each even length l for $4 \leq l \leq 2(m-1)$ in which it contains the edge $u_i^{(1)}u_{i+1}^{(1)}$. Since $P(u_i, v_{m-t})\square P_2$ is a subgraph of $G\square P_2$, the prism over G contains a cycle of each even length l for $4 \leq l \leq 2(m-1)$.

We shall find a cycle of each odd length l for $5 \leq l \leq 2m-1$. Since $P = u_i^{(1)} u^{(1)} u_{i+1}^{(1)}$ is a path of length 2 in $G \square P_2$ and $u^{(1)}$ is not contained in C^*, we replace edge $u_i^{(1)} u_{i+1}^{(1)}$ with the path P. Then, C^* can be extended to a cycle of length $l+1$. Since $4 \leq l \leq 2(m-1)$, we obtain a cycle of each odd length l for $5 \leq l \leq 2m-1$.

Since G contains a cycle of length 3, the prism over G also contains a cycle of length 3. By Theorem 3.1, the prism over G is Hamiltonian, i.e., it contains a cycle of length $2m$. Therefore, the prism over G is pancyclic. $\square$

Remark 3.17. From the proof of Theorem 3.16, the edge $v_{m-t-1}^{(2)} v_{m-t}^{(2)}$ of the second copy of $G \square P_2$ is contained in the odd cycle of length $2m-1$ (see Figure 3.5).

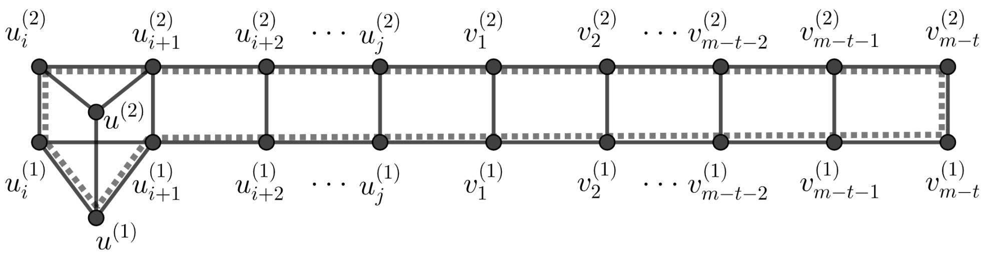

Figure 3.5: The dashed line represents a cycle of length $2m-1$ in $G \square P_2$ containing edge $v_{m-t-1}^{(2)} v_{m-t}^{(2)}$ where G is a skirted graph in Theorem 3.16

Next, we consider the pancyclicity of the 3-generalized prism over a skirted graph.

Theorem 3.18. *The 3-generalized prism over a skirted graph is pancyclic.*

Proof. First, we consider a single skirted graph. Let $G = G(a, u_0, u_\alpha) = T \cup P$ be a skirted graph of order m with $P = u_0 u_1 u_2 \cdots u_\alpha$. Let $C = C(u, u_i, u_j)$ be a triangle of order t in $G(a, u_0, u_\alpha)$, where $t \leq m$. If $u = a$, then G itself is a triangle. By Theorem 3.9, $G \square P_3$ is pancyclic. Now, we assume that $u \neq a$.

Let G' be a skirted graph of order $m-(t-1)$ obtained from a skirted graph G by contracting the triangle C and u^* be the vertex of G' representing the triangle

C. By Theorem 2.5, G' is Hamiltonian. Let $C' = u^*v_1v_2v_3\cdots v_{m-t}u^*$ be a spanning cycle in G'. Then, we let $P' = u^*v_1v_2v_3\cdots v_{m-t}$ be a spanning path in G'.

Since u^* is the vertex of G' representing the triangle C and v_1 is adjacent to u^*, v_1 is adjacent to either u, u_i or u_j. By Observation 3.15, without loss of generality, let v_1 be adjacent to u_j. Then, $P(u_i, v_{m-t}) = u_iu_{i+1}u_{i+2}\cdots u_jv_1v_2\cdots v_{m-t}$ is a path of length $m-2$ (without the vertex u) in G.

Now, consider the 3-generalized prism over a skirted graph which contains three copies of the same skirted graph. Since $P(u_i, v_{m-t})\Box P_3$ is a subgraph of $G\Box P_3$, we show that $G\Box P_3$ is pancyclic by applying Lemma 3.13. Then, we consider two cases as follows.

Case 1. $m-1$ is even. By Lemma 3.13(i), $P(u_i, v_{m-t})\Box P_3$ contains a cycle of each even length l for $4 \le l \le 3(m-1)$ in which it contains the edge $u_i^{(1)}u_{i+1}^{(1)}$. Note that, for all $1 \le s \le 3$, vertex $u^{(s)}$ has not been contained in $P(u_i, v_{m-t})\Box P_3$. To find an odd cycle, we replace $u_i^{(1)}u_{i+1}^{(1)}$ of such cycles with a path $u_i^{(1)}u^{(1)}u_{i+1}^{(1)}$ and then obtain a cycle of each odd length l for $5 \le l \le 3(m-1)+1 = 3m-2$. Let C' be the cycle of length $3m-2$ without the vertex $u^{(3)}$ (see Figure 3.6 (a)). By Remark 3.14, C' contains the edge $u_i^{(3)}u_{i+1}^{(3)}$. Then, we replace $u_i^{(3)}u_{i+1}^{(3)}$ of C' with a path $u_i^{(3)}u^{(3)}u_{i+1}^{(3)}$ and then obtain a cycle of length $3m-1$. Thus, we obtain that $G\Box P_3$ contains a cycle of each length l for all $4 \le l \le 3m-1$.

Case 2. $m-1$ is odd. By Lemma 3.13(ii), $P(u_i, v_{m-t})\Box P_3$ contains a cycle of each even length l for $4 \le l \le 3(m-1)-1$ in which it contains edge $u_i^{(1)}u_{i+1}^{(1)}$. Note that, for all $1 \le s \le 3$, vertex $u^{(s)}$ has not been contained in $P(u_i, v_{m-t})\Box P_3$. To find an odd cycle, we replace $u_i^{(1)}u_{i+1}^{(1)}$ of such cycles with a path $u_i^{(1)}u^{(1)}u_{i+1}^{(1)}$ and then obtain a cycle of each odd length l for $5 \le l \le 3(m-1) = 3m-3$. Let C' be the cycle of length $3m-3$ without vertex $u^{(3)}$ (see Figure 3.6 (b)). By Remark 3.14, C' contains edge $u_i^{(3)}u_{i+1}^{(3)}$. Thus, we replace $u_i^{(3)}u_{i+1}^{(3)}$ of C' with a path $u_i^{(3)}u^{(3)}u_{i+1}^{(3)}$ and then obtain a cycle of length $3m-2$. Therefore, $G\Box P_3$ contains a cycle of each length l for all $4 \le l \le 3m-2$.

We shall find a cycle of length $3m-1$ in $G\Box P_3$. Recall that $C = C(u, u_i, u_j)$ is a triangle of order t in $G = G(a, u_0, u_\alpha)$ such that $u \ne a$. To show that $G\Box P_3$

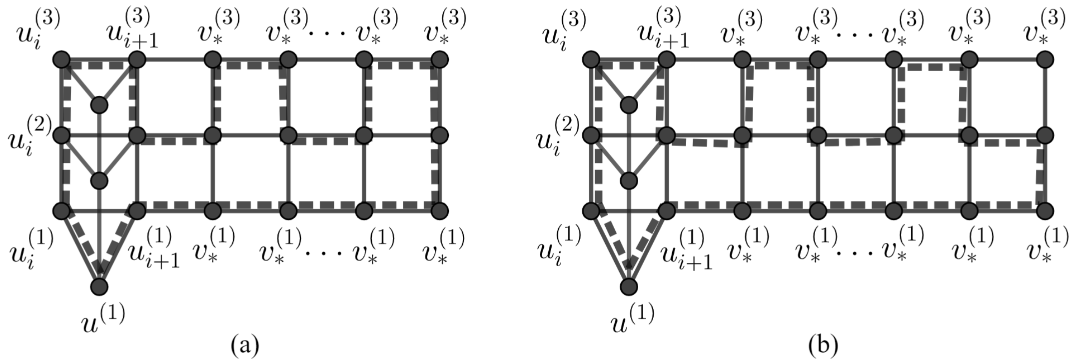

Figure 3.6: (a) The dashed line represents a cycle of length $3m-2$ in $G\square P_3$ when $m-1$ is even and (b) The dashed line represents a cycle of length $3m-3$ in $G\square P_3$ when $m-1$ is odd

contains a cycle of length $3m-1$, we give the following paths and link them together with edges joining each copy of G.

- For the first copy of G, we consider subgraph G'.

  In the first case, let $u_j = u_\alpha$. Since $u \neq a$, we have $u_j \neq u_\alpha$ or $u_i \neq u_0$. Thus, in this case, $u_i \neq u_0$ and $C(u, u_i, u_j) = C(u, u_i, u_\alpha)$. Then, $G' = G'(a, u_0, u^*)$. Since G' is a skirted graph, by Lemma 2.7, G' contains an (a, u^*)-path $P_{G'}(a, u^*)$ of length $m-t$. Suppose that v' is adjacent to u^* in $P_{G'}(a, u^*)$. Then, v' is adjacent to either u or u_i in G. We consider two cases as follows.

  - If v' is adjacent to u, then $P(v', u_j) = v'uu_{i+1}u_{i+2}\cdots u_j$ is a path of length $t-1$ (without the vertex u_i).
  - If v' is adjacent to u_i, then $P(v', u_j) = v'u_iu_{i+1}u_{i+2}\cdots u_j$ is a path of length $t-1$ (without the vertex u).

  Therefore, we can extend the path $P_{G'}(a, u^*)$ of length $m-t$ in G' to be a path $P(a, u_\alpha)$ of length $m-2$ in G by replacing the edge $v'u^*$ of G' with the path $P(v', u_j)$.

  Now, let $u_j \neq u_\alpha$. Then, $G' = G'(a, w, u_\alpha)$. Note that $w = u^*$ if $u_i = u_0$.

Otherwise, $w = u_0$. Since $G'(a, w, u_\alpha)$ is a skirted graph, by Lemma 2.7, G' contains an (a, u_α)-path $P_{G'}(a, u_\alpha)$ of length $m - t$. Since $P_{G'}(a, u_\alpha)$ is a spanning path in G', $P_{G'}(a, u_\alpha)$ contains the vertex u^*. Suppose that v' and v'' are adjacent to u^* in $P_{G'}(a, u_\alpha)$. Then, each of v' and v'' is adjacent to either u, u_i or u_j in G. We consider three cases as follows.

– If $v'u_i, u_jv'' \in E(G)$, then $P(v', v'') = v'u_iu_{i+1}u_{i+2} \cdots u_jv''$ is a path of length t (without the vertex u).

– If $v'u, u_jv'' \in E(G)$, then $P(v', v'') = v'uu_{i+1}u_{i+2} \cdots u_jv''$ is a path of length t (without the vertex u_i).

– If $v'u, u_iv'' \in E(G)$, then $P(v', v'') = v'uu_{j-1}u_{j-2} \cdots u_{i+1}u_iv''$ is a path of length t (without the vertex u_j).

Therefore, we can extend the path $P_{G'}(a, u_\alpha)$ of length $m - t$ in G' to be a path $P(a, u_\alpha)$ of length $m-2$ in G by replacing the path $v'u^*v''$ in $P_{G'}(a, u_\alpha)$ with the path $P(v', v'')$. Thus, the first copy of G contains a path $P(a^{(1)}, u_\alpha^{(1)})$ of length $m - 2$.

- By Remark 2.8(ii), the second copy of G contains a $(u_0^{(2)}, u_\alpha^{(2)})$-path $P(u_0^{(2)}, u_\alpha^{(2)})$ of length $m - 2$ (without the root $a^{(2)}$).

- By Remark 2.8(i), the last copy of G contains an $(a^{(3)}, u_0^{(3)})$-path $P(a^{(3)}, u_0^{(3)})$ of length $m - 1$.

- The path $P^* = a^{(3)}a^{(2)}a^{(1)}$ of length 2 is a path in $G\Box P_3$ from the last copy to the first copy of G.

Now, we link each path by edges $u_\alpha^{(1)}u_\alpha^{(2)}$ and $u_0^{(2)}u_0^{(1)}$. The cycle of length $3m-1$ is

$$P(a^{(1)}, u_\alpha^{(1)})P(u_\alpha^{(2)}, u_0^{(2)})P(u_0^{(3)}, a^{(3)})P^*.$$

Therefore, $G\Box P_3$ contains a cycle of length $3m - 1$.

From these two cases, we obtain that $G\Box P_3$ contains a cycle of each length l for all $4 \le l \le 3m - 1$. Since G is a skirted graph, by Lemma 3.2, G contains a cycle of length 3. By Theorem 3.1, $G\Box P_3$ is Hamiltonian, i.e., it contains a cycle of length $3m$. Therefore, $G\Box P_3$ is pancyclic. $\square$

By the proof of Theorem 3.18, the pancyclicity of the 3-generalized prism over a skirted graph, we need to consider the special case using the technique that we have used in Chapter II. However, there is no special case when we show that $G\Box P_n$ is pancyclic for $n \geq 4$. Therefore, we prove the following theorem by considering $n \geq 4$.

Theorem 3.19. *The n-generalized prism over any skirted graph is pancyclic.*

Proof. First, we consider a single skirted graph. Let $G = G(a, u_0, u_\alpha) = T \cup P$ be a skirted graph of order m with $P = u_0u_1u_2 \cdots u_\alpha$. Let P_n be a path of order $n \geq 2$. If $n = 2$ or 3, then we respectively obtain from Theorems 3.16 and 3.18 that $G\Box P_n$ is pancyclic. Suppose now that $n \geq 4$.

Let $C = C(u, u_i, u_j)$ be a triangle of order t in $G(a, u_0, u_\alpha)$, where $t \leq m$. If $u = a$, then G itself is a triangle. By Theorem 3.9, the n-generalized prism over G is pancyclic. Now, we assume that $u \neq a$.

Let G' be a skirted graph of order $m-(t-1)$ obtained from a skirted graph G by contracting the triangle C and u^* be the vertex of G' representing the triangle C. By Theorem 2.5, G' is Hamiltonian. Let $C' = u^*v_1v_2v_3 \cdots v_{m-t}u^*$ be a spanning cycle in G'. Then, $P' = u^*v_1v_2v_3 \cdots v_{m-t}$ is a spanning path in G'.

Since u^* is the vertex of G' representing the triangle C and v_1 is adjacent to u^*, v_1 is adjacent to either u, u_i or u_j. By Observation 3.15, without loss of generality, let v_1 be adjacent to u_j. Then, $P(u_i, v_{m-t}) = u_iu_{i+1}u_{i+2} \cdots u_jv_1v_2 \cdots v_{m-t}$ is a path of length $m-2$ (without the vertex u) in G.

Now, consider the n-generalized prism over a skirted graph which contains n copies of the same skirted graph. Since $u_iu, uu_{i+1} \in E(G)$, there is a path $P'_m = u_iuu_{i+1}u_{i+2} \cdots u_jv_1 \cdots v_{m-t}$ of length $m-1$ in G, i.e., P'_m is a spanning path in G. We can see that $P'_m\Box P_n$ is a subgraph of $G\Box P_n$.

To show that $G\Box P_n$ is pancyclic, we consider two cases as follows.

Case 1. n is even. By Lemma 3.12, $P'_m \square P_n$ contains a cycle of each even length l for $4 \leq l \leq mn$. Since $P'_m \square P_n$ is a subgraph of $G \square P_n$, $G \square P_n$ contains a cycle of each even length l for $4 \leq l \leq mn$. We shall find a cycle of each odd length in $G \square P_n$ by considering two disjoint induced subgraphs $G \square P_2$ and $G \square P_{n-2}$ of $G \square P_n$, where $G \square P_2$ is induced by the first two copies of G and $G \square P_{n-2}$ is induced by the last $n-2$ copies of G.

First, we consider $G \square P_2$. By Theorem 3.16, $G \square P_2$ contains a cycle of each length l for $3 \leq l \leq 2m$. Since $G \square P_2$ is a subgraph of $G \square P_n$, we obtain that $G \square P_n$ contains a cycle of each length l for $3 \leq l \leq 2m$. Let C^* be the cycle of length $2m-1$ in $G \square P_n$ containing edge $v^{(2)}_{m-t-1}v^{(2)}_{m-t}$, which exists by Remark 3.17.

Next, we consider subgraph $G \square P_{n-2}$ induced by the last $n-2$ copies of G, in order to show that $G \square P_n$ contains a cycle of each odd length l for $2m+1 \leq l \leq mn-1$. Since $P'_m \square P_{n-2}$ is a subgraph of $G \square P_{n-2}$, we can consider cycles in $P'_m \square P_{n-2}$ instead of $G \square P_{n-2}$. Since $n-2$ is even, by Lemma 3.12 and the reverse of the path P'_m, the edge $v^{(3)}_{m-t-1}v^{(3)}_{m-t}$ is contained in a cycle of each length l where l is an even integer ranging from 4 to $m(n-2)$ in $P'_m \square P_{n-2}$. Since $v^{(2)}_{m-t-1}v^{(3)}_{m-t-1}, v^{(2)}_{m-t}v^{(3)}_{m-t}, v^{(3)}_{m-t-1}v^{(3)}_{m-t} \in E(G \square P_n)$, we delete the edge $v^{(2)}_{m-t-1}v^{(2)}_{m-t}$ of C^* and then join $v^{(2)}_{m-t-1}$ to $v^{(3)}_{m-t-1}$ and $v^{(2)}_{m-t}$ to $v^{(3)}_{m-t}$. Then, we can extend C^* to be a cycle of length $2m+1$. In addition, we delete the edge $v^{(3)}_{m-t-1}v^{(3)}_{m-t}$ of each cycle of each length l in $P'_m \square P_{n-2}$ and then join $v^{(2)}_{m-t-1}$ to $v^{(3)}_{m-t-1}$ and $v^{(2)}_{m-t}$ to $v^{(3)}_{m-t}$. Then, we can extend C^* to be a cycle of each length l for $2m+3 \leq l \leq mn-1$. Therefore, $G \square P_n$ is pancyclic.

Case 2. n is odd. Since $n-3 \geq 2$ is even, by Case 1, $G \square P_{n-3}$ contains a cycle of each length l for $3 \leq l \leq m(n-3)$. Thus, we consider two disjoint induced subgraphs $G \square P_{n-3}$ and $G \square P_3$ of $G \square P_n$, where $G \square P_{n-3}$ is induced by the first $n-3$ copies of G and $G \square P_3$ is induced by the last three copies of G.

We shall find a cycle of each remaining length l for $m(n-3)+1 \leq l \leq mn$. Recall that G is a skirted graph of order m and $P'_m = u_i u u_{i+1} u_{i+2} \cdots u_j v_1 \cdots v_{m-t}$ is a spanning path in G. Then, $P'_m \square P_n$ is a subgraph of $G \square P_n$. Let C_{odd} be the cycle of odd length $m(n-3)-1$ in $P'_m \square P_{n-3}$ containing the edge $v^{(n-3)}_{m-t}v^{(n-3)}_{m-t-1}$

(see Figure 3.7(a)) and C_{even} be the cycle of even length $m(n-3)$ in $P'_m \Box P_{n-3}$ containing the edge $v_{m-t}^{(n-3)} v_{m-t-1}^{(n-3)}$ (see Figure 3.7(b)).

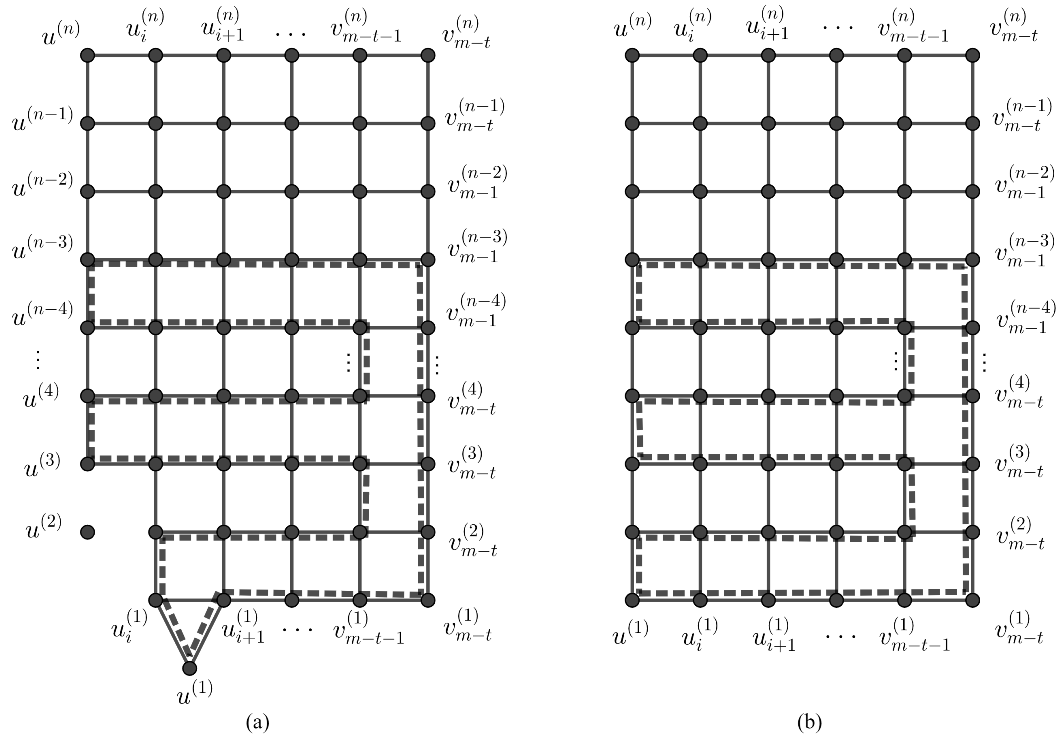

Figure 3.7: (a) The dashed line represents C_{odd} of length $m(n-3)-1$ and (b) The dashed line represents C_{even} of length $m(n-3)$

Consider $G\Box P_3$. By Lemma 3.13(i) and the reverse of the path P'_m, $G\Box P_3$ contains a cycle of each even length l for $4 \le l \le 3m$ containing edge $v_{m-t}^{(n-2)} v_{m-t-1}^{(n-2)}$.

First of all, we replace the edge $v_{m-t-1}^{(n-3)} v_{m-t}^{(n-3)}$ of the cycle C_{odd} with the path $v_{m-t-1}^{(n-3)} v_{m-t-1}^{(n-2)} v_{m-t}^{(n-2)} v_{m-t}^{(n-3)}$ and then obtain a cycle of odd length $m(n-3)+1$. Next, we delete the edge $v_{m-t-1}^{(n-3)} v_{m-t}^{(n-3)}$ of C_{odd} and the edge $v_{m-t-1}^{(n-2)} v_{m-t}^{(n-2)}$ of each cycle of each even length l in $G\Box P_3$ and then join $v_{m-t-1}^{(n-3)}$ to $v_{m-t-1}^{(n-2)}$ and $v_{m-t}^{(n-3)}$ to $v_{m-t}^{(n-2)}$. Hence, we can extend C_{odd} of length $m(n-3)-1$ to be a cycle of each odd length l for $m(n-3)+3 \le l \le mn-1$ when m is even and extend C_{odd} to be a cycle of each odd length l for $m(n-3)+3 \le l \le mn-2$ when m is odd. Thus, $G\Box P_n$ contains a cycle of each odd length l for $m(n-3)+1 \le l \le mn-1$ when m is even and a cycle of each odd length l for $m(n-3)+1 \le l \le mn-2$ when m is odd.

For cycles of even length, in a similar way, we extend C_{even} of length $m(n-3)$ to be a cycle of each even length l for $m(n-3)+2 \leq l \leq mn$ when m is even and extend C_{even} to be a cycle of each even length l for $m(n-3)+2 \leq l \leq mn-1$ when m is odd. Since $G\square P_n$ is Hamiltonian, it contains a cycle of length mn.

Thus, $G\square P_n$ contains a cycle of each length l for $m(n-3)+1 \leq l \leq mn$. Therefore, $G\square P_n$ is pancyclic. $\square$

3.4 Conclusion and discussion

In this chapter, we prove that the n-generalized prism over a skirted graph is pancyclic. The result holds for any skirted graph, even though we have not known the exact configuration of this family of graphs. Moreover, since the Cartesian product of a graph G and a path P_n (or $G\square P_n$) is a subgraph of $G\square C_n$ and $G\square K_n$, the results can be concluded in a similar way when P_n is replaced by C_n or K_n for $n \geq 3$.

For the vertex pancyclicity of the n-generalized prism over a skirted graph G, we notice that there are vertices of G in which it is not contained in any cycle of length 3 in G. Moreover, the Cartesian product of G and a path does not generate a cycle of length 3. Thus, the n-generalized prism over a skirted graph is not vertex pancyclic. This motivates us to investigate the other product of graphs in the next chapter.

CHAPTER IV
THE LEXICOGRAPHIC PRODUCTS OF SOME GRAPHS

To study vertex pancyclicity over lexicographic products of some graphs, we first provide the preliminary results and motivation of the main results of this chapter as follows.

4.1 Preliminary results and motivation

Apart from pancyclicity, there are a number of works showing that several nontrivial sufficient conditions on a graph which implies that the graph is Hamiltonian also implies that the graph is vertex k-pancyclic for some k. For instance, in 1960, Ore [14] introduced the degree sum condition which was stated that "for each pair of non-adjacent vertices u, v in G, $d(u) + d(v) \geq n(G)$" and showed that if G is a graph satisfying the degree sum condition, then G is Hamiltonian. Bondy [3] showed that if G is graph satisfying the degree sum condition, then G is pancyclic or $G = K_{n/2,n/2}$. In 1984, Cai [6] considered the degree sum condition and proved that a graph G satisfying this condition is vertex 4-pancyclic or $G = K_{n/2,n/2}$, see [16] for more examples.

For Cartesian product of graphs, there also are a bunch of works relating to the metaconjecture, i.e., almost any nontrivial condition on the Cartesian product of graphs which implies that the Cartesian product of graphs is Hamiltonian also implies that the Cartesian product of graphs is pancyclic (there may be a simple family of exceptional graphs). The following theorems are some conditions concerning hamiltonicity of the Cartesian product of graphs which imply pancyclicity.

Theorem 4.1. *The conditions concerning hamiltonicity are provided as follows.*

(i) *[15] If G is a 3-connected cubic graph, then $G\Box P_2$ is Hamiltonian.*

(ii) *[15] If G is an even 3-cactus, then $G\Box P_2$ is Hamiltonian.*

(iii) *[17] If G is a connected graph, then $G\Box K_n$ is Hamiltonian for $\Delta(G) \leq n$.*

(iv) *[5] If G is a connected graph, then $G\Box C_n$ is Hamiltonian for $\Delta(G) \leq n$.*

(v) *[5] Let G be a connected almost claw-free graph and $n \geq 4$ be an even integer. Then, $G\Box P_n$ is Hamiltonian.*

A *cactus* is a connected graph in which every block is a K_2 or a cycle, where a *block* is a maximal 2-connected subgraph. A 3-*cactus* is a cactus with maximum degree 3. An *even* 3-*cactus* is a 3-cuctus in which all of its cycles are of even length.

However, such conditions only imply that the Cartesian product of graphs is vertex even pancyclic as follows.

Theorem 4.2. *The conditions concerning vertex even pancyclicity are provided as follows.*

(i) *[8] If G is a 3-connected cubic graph, then $G\Box P_2$ is vertex even pancyclic.*

(ii) *[8] If G is an even 3-cactus, then $G\Box P_2$ is vertex even pancyclic.*

(iii) *[5] Let n be even and $n \geq 4$. If G is a 1-pendent cactus with $\Delta(G) \leq \frac{1}{2}(n+2)$, then $G\Box P_n$ is vertex even pancyclic.*

A *claw* is a $K_{1,3}$. The vertex of degree 3 is its *center*. For a set $B \subseteq V(G)$, B is a *dominating set* if every vertex of G is in B or has a neighbor in B. A graph G is 2-*dominated* if the size of a minimum dominating set of G is at most 2. A graph G is called an *almost claw-free graph* if the set of center vertices of induced claws in G is independent and the neighborhood of each center vertex induces a 2-dominated subgraph. For a graph G, a vertex of degree 1 in G is called *pendent* if its neighbor is a vertex of degree at least 3 in G. A 1-*pendent cactus* is a cactus in which every vertex v has at most 1 pendent neighbor (v can have other non pendent neighbors).

Here, we notice that vertex pancyclicity over the Cartesian product of graphs is affected by the number of edges between each copy of a graph. This motivates us to consider the lexicographic product of graphs that contains more edges.

Observation 4.3. *From the definitions of the Cartesian product of graphs and the lexicographic product of graphs G and H given in Chapter I, we can see that $V(G\Box H) = V(G\circ H)$ and $E(G\Box H) \subset E(G\circ H)$. Therefore, the vertex pancyclicity over $G\Box H$ implies the vertex pancyclicity over $G\circ H$. Here, we only consider vertex pancyclicity over $G \circ H$ on the conditions that do not imply vertex pancyclic over $G\Box H$.*

For the pancyclicity of the lexicographic product of graphs, there are a few results. In 2006, Kaiser and Kriesell [11] investigated toughness conditions on a graph G that the lexicographic product of G and a graph is Hamiltonian and also pancyclic in which states that if G is 4-tough and H contains at least one edge, then $G \circ H$ is pancyclic. In addition, they proved the following theorem.

Theorem 4.4. *[11] If G and H are graphs with at least one edge each, then $G \circ H$ either has no cycles, or it contains cycles of all lengths between the length of the shortest cycle and the length of the longest cycle.*

The following theorem on vertex pancyclic will be used in this chapter.

Theorem 4.5. *[6] Let G be a graph of order $n \geq 4$ with $d(u) + d(v) \geq n$ for distinct nonadjacent vertices u, v in G. Then, G is vertex 4-pancyclic unless n is even and $G = K_{n/2,n/2}$.*

We know that a vertex pancyclic graph is Hamiltonian. Then, a non-Hamiltonian graph is not vertex pancyclic. Here, we provide a necessary condition for a graph to be Hamiltonian as follows.

Theorem 4.6. *[19] If G has a Hamiltonian cycle, then for each nonempty set $S \subseteq V$, the graph $G - S$ has at most $|S|$ components.*

To study vertex pancyclicity over the lexicographic products of graphs, we start by investigating the lexicographic product of K_n and a graph G in Subsection 2.1. By Theorem 4.5, we obtain that $K_n \circ G$ is vertex pancyclic for $n \geq 3$. In Subsection 2.2, we show that $G_1 \circ G_2$ is vertex pancyclic if G_1 is a traceable graph of even order and G_2 is a graph with at least one edge. Since G_1 is traceble, we consider the lexicographic product of a path and G_2 instead of the lexicographic product of G_1 and G_2. Furthermore, we directly show that if G_1 and G_2 are nontrivial traceable graphs, then $G_1 \circ G_2$ is vertex pancyclic. In Subsection 2.3, we show that if G_1 is Hamiltonian and G_2 is a graph with at least one edge, then $G_1 \circ G_2$ is vertex pancyclic. Since G_1 is Hamiltonian, we consider the lexicographic product of a cycle and G_2 instead of the lexicographic product of G_1 and G_2.

4.2 Vertex pancyclicity of some lexicographic products

4.2.1 Complete Graphs

First of all, we investigate the lexicographic product of a complete graph and a general graph. Let K_n be a complete graph of order n and A_k be an empty graph of order k. Theorem 4.5 gives us the following theorem.

Theorem 4.7. *$K_n \circ A_k$ is vertex pancyclic for $n \geq 3$ and $k \geq 1$.*

Proof. Let (x, y) be any vertex of $K_n \circ A_k$. Then, $N((x, y)) = \bigcup_{x' \in V(K_n)-\{x\}} \{(x', y) | y \in V(A_k)\}$. Since $n \geq 3$, there are $x_i, x_j \in V(K_n) - \{x\}$ such that $x_i \neq x_j$. Then, $(x, y)(x_i, y)(x_j, y)(x, y)$ forms a cycle of length 3 containing (x, y).

Next, we can see that the order of $K_n \circ A_k$ is nk and $|N((x, y))| = (n-1)k$. Let $u, v \in V(K_n \circ A_k)$ such that $uv \notin E(K_n \circ A_k)$. Then, $d(u) + d(v) = 2(n-1)k = 2nk - 2k \geq nk$. Since $K_n \circ A_k$ is not isomorphic to a balance complete bipartite graph $K_{\frac{nk}{2}, \frac{nk}{2}}$, by Theorem 4.5, we obtain that $K_n \circ A_k$ is vertex 4-pancyclic. Thus, (x, y) is contained in a cycle of each length l for $4 \leq l \leq nk$. Therefore, $K_n \circ A_k$ is vertex pancyclic. $\square$

Since A_k is a spanning subgraph of all graphs of order k, we obtain the following corollary.

Corollary 4.8. *Let $n \geq 3$ and G be a graph. Then, $K_n \circ G$ is vertex pancyclic.*

By Corollary 4.8, since C_3 is a complete graph of order 3, we obtain the following corollary.

Corollary 4.9. *Let G be a graph. Then, $C_3 \circ G$ is vertex pancyclic.*

4.2.2 Paths

We start this section by considering the lexicographic product of a path P_2 and any graph as follows.

Let $P_2 = x_1x_2$ and A_k be an empty graph of order k. Then, $P_2 \circ A_k$ is isomorphic to a balanced complete bipartite graph $K_{k,k}$ with two partite sets, V_1 and V_2, where $V_1 = \{(x_1, y) | y \in A_k\}$ and $V_2 = \{(x_2, y) | y \in A_k\}$.

Since a balanced complete bipartite graph $K_{k,k}$ is Hamiltonian and also vertex even pancyclic for $k \geq 2$ (to prove that $K_{k,k}$ is vertex even pancyclic, we can use the result that it is Hamiltonian), we obtain that $P_2 \circ A_k$ is vertex even pancyclic for $k \geq 2$. Since A_k is a spanning subgraph of any graph of order k, we obtain the following remark.

Remark 4.10. Let G be a nontrivial graph. Then, $P_2 \circ G$ is vertex even pancyclic.

Now, we investigate the lexicographic product of P_2 and a graph G as follows.

Theorem 4.11. *Let G be a nontrivial graph with at least one edge. Then, $P_2 \circ G$ is vertex pancyclic.*

Proof. Let $P_2 = x_1x_2$ and $V(G) = \{y_1, y_2, y_3, \ldots, y_k\}$ for $k \geq 2$. Since G contains at least one edge, assume that $y_1y_2 \in E(G)$. Then, $(x_1, y_1)(x_1, y_2)$ and $(x_2, y_1)(x_2, y_2)$ are edges of $P_2 \circ G$. Let $(x, y) \in V(P_2 \circ G)$. If $x = x_1$, then (x, y) is adjacent to both vertices (x_2, y_1) and (x_2, y_2). Thus, $(x, y)(x_2, y_1)(x_2, y_2)(x, y)$ is a cycle of length 3 containing (x, y). If $x = x_2$, then (x, y) is adjacent to both vertices (x_1, y_1) and

(x_1, y_2). Thus, $(x, y)(x_1, y_1)(x_1, y_2)(x, y)$ is a cycle of length 3 containing (x, y). Thus, each vertex of $P_2 \circ G$ is contained in a cycle of length 3.

Since G contains a cycle of length 3, $P_2 \circ G$ is not isomorphic to any complete bipartite graph. Since $P_2 \circ G$ is of order $2k \geq 4$ with $d(u) + d(v) \geq 2k$ for any pair of distinct nonadjacent vertices u and v in $P_2 \circ G$, by Theorem 4.5, G is vertex 4-pancyclic.

Therefore, $P_2 \circ G$ is vertex pancyclic. $\square$

Now, we consider the lexicographic product of a path P_n for $n \geq 2$ and a graph G as follows.

Remark 4.12. For any k and $n \geq 3$, $P_n \circ A_k$ is non-Hamiltonian.

Let $P_n = x_1 x_2 x_3 \cdots x_n$ and $V(A_k) = \{y_1, y_2, y_3, \ldots, y_k\}$. Choose $S = \{(x_2, y) | y \in V(A_k)\}$. Then, $|S| = k$. Let H denote the graph $(P_n \circ A_k) - S$. Then, H has at least $k+1$ components, namely, $H[(x_1, y_1)]$, $H[(x_1, y_2)]$, $H[(x_1, y_3)]$, $\ldots$, $H[(x_1, y_k)]$ and $H[\{(x_i, y) | i \in \{3, 4, 5, \ldots, n\}, y \in V(G)\}]$. By Theorem 4.6, $P_n \circ A_k$ is non-Hamiltonian.

From Remark 4.12, we can see that the lexicographic product of P_n and an empty graph is non-Hamiltonian and not vertex pancyclic. We invertigate the condition of a graph G for $P_n \circ G$ to be vertex pancyclic and show that $P_n \circ G$ is vertex pancyclic when n is even and G contains at least one edge. We start with the following lemmas.

Lemma 4.13. *Let $k \geq 2$. If u and v are on different partite sets of a complete bipartite graph $K_{k,k}$, then there is a path $P(u, v)$ in $K_{k,k}$ of each odd length l for $1 \leq l \leq 2k - 1$.*

Proof. Let $K_{k,k}$ be a complete bipartite graph for $k \geq 2$ with partite sets V_1 and V_2. Assume that $u \in V_1$ and $v \in V_2$. For $k = 2$, let $u' \in V_1 - \{u\}$ and $v' \in V_2 - \{v\}$. We obtain that uv and $uv'u'v$ are paths $P(u, v)$ of length 1 and 3, respectively.

For $k \geq 3$, let $V_1^* = V_1 - \{u\}$ and $V_2^* = V_2 - \{v\}$. We can see that $K_{k-1,k-1}$ is a subgraph of $K_{k,k}$ induced by $V_1^* \cup V_2^*$. Since a balanced complete bipartite graph is vertex even pancyclic, $K_{k-1,k-1}$ contains a cycle of each even length l for $4 \leq l \leq 2(k-1)$. Let $C = v_1v_2v_3\cdots v_lv_1$ be a cycle in $K_{k-1,k-1}$ of even length l for some $4 \leq l \leq 2(k-1)$. Then, any two consecutive vertices of C contain in the different partite sets. Without loss of generality, let $v_1 \in V_1^*$ and $v_2 \in V_2^*$. We see that $v_i \in V_1^*$ if i is odd and $v_i \in V_2^*$ if i is even and $v_1v, v_2u \in E(K_{k,k})$. Then, $uv_2v_3\cdots v_lv_1v$ is a path $P(u,v)$ in $K_{k,k}$ of length $l+1$. Note that $l+1$ is an odd number. Since l is an arbitrary even number between 4 and $2(k-1)$, there exists a path $P(u,v)$ in $K_{k,k}$ of each odd length l for $5 \leq l \leq 2k-1$. In addition, uv and uv_2v_1v are paths from u to v in $K_{k,k}$ of length 1 and 3, respectively.

Therefore, there exists a path $P(u,v)$ in $K_{k,k}$ of each odd length l for $1 \leq l \leq 2k-1$. □

Lemma 4.14. *Let $n \geq 2$ be even and G be a nontrivial graph of order k. If $P_n = x_1x_2x_3\cdots x_n$ is a path and $y_1y_2 \in E(G)$, then $P_n \circ G$ contains a path $P((x_1,y_1),(x_1,y_2))$ of each length l for $1 \leq l \leq nk-1$.*

Proof. Let $P_n = x_1x_2x_3\cdots x_n$ and $V(G) = \{y_1,y_2,y_3,\ldots,y_k\}$ for $k \geq 2$. Since $y_1y_2 \in E(G)$, vertices $(x_1,y_1),(x_1,y_2),(x_2,y_1)$ and (x_2,y_2) form a clique of order 4. Then, there are paths $P((x_1,y_1),(x_1,y_2))$ of length l for $1 \leq l \leq 3$.

We prove by the mathematical induction on n. For $n=2$, let $V_1^* = \{(x_1,y)|y \in V(G)-\{y_1\}\}$ and $V_2^* = \{(x_2,y)|y \in V(G)-\{y_1\}\}$. We can see that $K_{k-1,k-1}$ of which its vertex set is $V_1^* \cup V_2^*$ is a subgraph of $P_n \circ G$. Since $(x_1,y_2) \in V_1^*$ and $(x_2,y_2) \in V_2^*$, by Lemma 4.13, there exists a path $P((x_1,y_2)(x_2,y_2))$ in $K_{k-1,k-1}$ of each odd length l for $1 \leq l \leq 2(k-1)-1$. To show that there exists a path $P((x_1,y_1),(x_1,y_2))$ of each length l for $1 \leq l \leq 2k-1$, we extend the path $P((x_1,y_2),(x_2,y_2))$ of each length l for $1 \leq l \leq 2k-3$ as follows.

(a) Join the vertex (x_1,y_1) with the vertex (x_2,y_2) of $P((x_1,y_2),(x_2,y_2))$ (see Figure 4.1(a)).

(b) Join the vertex (x_2,y_1) of the edge $(x_1,y_1)(x_2,y_1)$ with the vertex (x_2,y_2) of $P((x_1,y_2),(x_2,y_2))$ (see Figure 4.1(b)).

Then, a path $P((x_1,y_2),(x_2,y_2))$ of each odd length l for $1 \leq l \leq 2k-3$ can

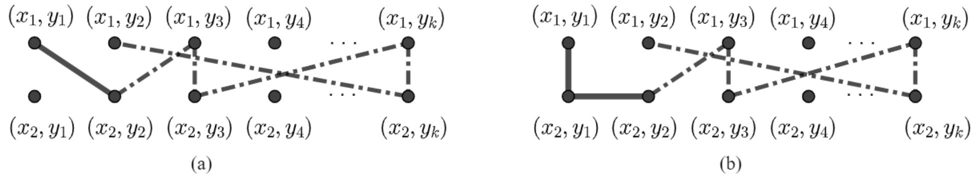

Figure 4.1: (a) Joining vertex (x_1, y_1) to a path $P((x_1, y_2), (x_2, y_2))$ and (b) Joining the edge $(x_1, y_1)(x_2, y_1)$ to a path $P((x_1, y_2), (x_2, y_2))$

be extended to a path $P((x_1, y_1), (x_1, y_2))$ of each even length l for $2 \leq l \leq 2k - 2$ by (a), and of each odd length l for $3 \leq l \leq 2k - 1$ by (b). Thus, we obtain that there exists a path $P((x_1, y_1), (x_1, y_2))$ of each length l for $1 \leq l \leq 2k - 1$.

For the induction step, let $t \in \mathbb{N}$ and suppose that the statement holds for all even n, $n \leq 2t$. We show that the statement still holds for $n = 2t + 2$. Let $V_i = \{(x_i, y) | y \in V(G)\}$ for $i \in \{1, 2, 3, \ldots, 2t + 2\}$. The set $\bigcup_{i=1}^{2t} V_i$ induces a subgraph $P_{2t} \circ G$ of $P_{2t+2} \circ G$. By the induction hypothesis, $P_{2t+2} \circ G$ contains paths $P((x_1, y_1), (x_1, y_2))$ of each length l for $1 \leq l \leq 2tk - 1$. In order to show that there exists a path $P((x_1, y_1), (x_1, y_2))$ of each length l for $2tk \leq l \leq (2t + 2)k - 1$, we perform the following three steps.

(i) We show that there is a path $P((x_{2t}, y_2), (x_1, y_2))$ of length $2t(k - 1) - 1$ (without vertices (x_i, y_1) for all i). Let $V_i^* = \{(x_i, y) | y \in V(G) - \{y_1\}\}$ for all $i \in \{1, 2, 3, \ldots, 2t\}$. Consider each pair of vertex set V_{2j-1}^* and V_{2j}^* for all $j \in \{1, 2, 3, \ldots, t\}$. We can see that the set $V_{2j-1}^* \cup V_{2j}^*$ induces a subgraph $K_{k-1,k-1}$ of $P_n \circ G$. By Lemma 4.13, there is a path $P((x_{2j-1}, y_2), (x_{2j}, y_2))$ of length $2k - 3$. We connect such t paths, $P((x_{2j-1}, y_2), (x_{2j}, y_2))$ for all $j \in \{1, 2, 3, \ldots, t\}$, together to obtain path $P((x_1, y_2), (x_{2t}, y_2))$ of length $2t(k - 1) - 1$. By reversing path $P((x_1, y_2), (x_{2t}, y_2))$, there is a path $P((x_{2t}, y_2), (x_1, y_2))$ of length $2t(k-1)-1$ (see Figure 4.2(a)).

(ii) We show that there is a path $P((x_1, y_1), (x_1, y_2))$ of length $2tk$. From (i), we get $P((x_{2t}, y_2), (x_1, y_2))$ of length $2t(k - 1) - 1$ and the path $P((x_1, y_1), (x_{2t}, y_2)) = (x_1, y_1)(x_2, y_1)(x_3, y_1) \cdots (x_{2t}, y_1)(x_{2t+1}, y_1)(x_{2t}, y_2)$ is a path of length $2t + 1$ (see Figure 4.2(b)). Connecting $P((x_{2t}, y_2), (x_1, y_2))$ to $P((x_1, y_1), (x_{2t}, y_2))$ yields a path $P((x_1, y_1), (x_1, y_2))$ of length $2tk$.

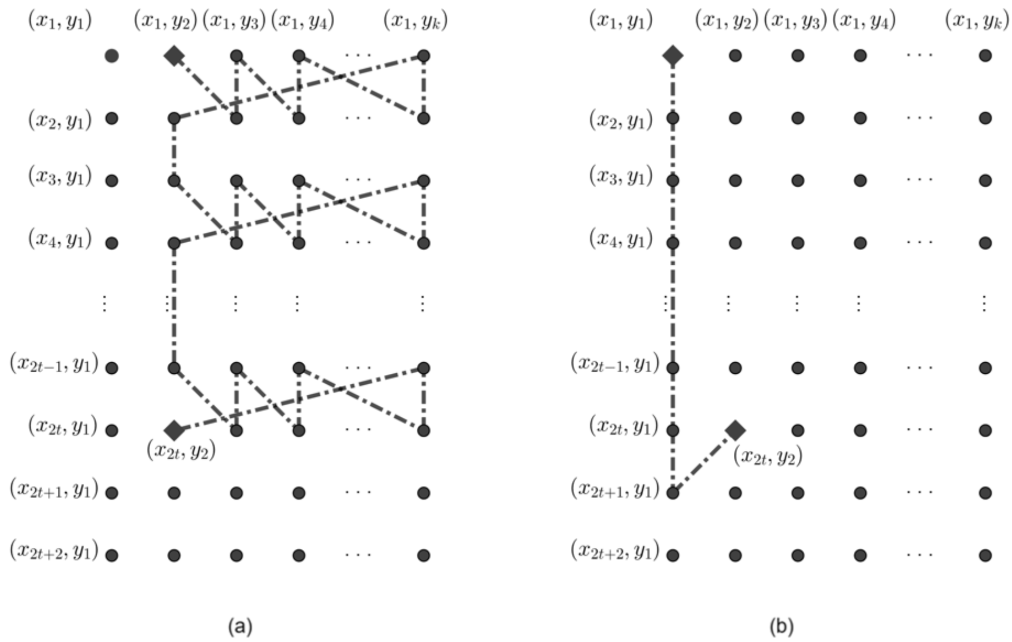

Figure 4.2: (a) A path $P((x_{2t}, y_2), (x_1, y_2))$ and (b) A path $P((x_1, y_1), (x_{2t}, y_2))$ of length $2t + 1$

(iii) We show that there is a path $P((x_1, y_1), (x_1, y_2))$ of each length l for $2tk + 1 \leq l \leq (2t + 2)k - 1$. Let $P((x_1, y_1), (x_{2t}, y_1)) = (x_1, y_1)(x_2, y_1)(x_3, y_1) \cdots (x_{2t}, y_1)$ be a path of length $2t - 1$. By connecting $P((x_1, y_1), (x_{2t}, y_1))$ with the path $P((x_{2t}, y_2), (x_1, y_2))$ of length $2t(k-1) - 1$ from (i), we obtain $P^*((x_1, y_1), (x_1, y_2))$ of length $2tk - 1$ (see Figure 4.3(a)). Consider the set $V_{2t+1} \cup V_{2t+2}$. The set $V_{2t+1} \cup V_{2t+2}$ induces a subgraph $P_2 \circ G$ of $P_{2t+2} \circ G$. Then, $P_{2t+2} \circ G$ contains a path $P((x_{2t+1}, y_1), (x_{2t+1}, y_2))$ of each length l for $1 \leq l \leq 2k-1$ where each vertex of $P((x_{2t+1}, y_1), (x_{2t+1}, y_2))$ contains in the set $V_{2t+1} \cup V_{2t+2}$ (see Figure 4.3(b)). Since (x_{2t+1}, y_1) and (x_{2t+1}, y_2) are adjacent to vertices (x_{2t}, y_1) and (x_{2t}, y_2), we replace the edge $(x_{2t}, y_1)(x_{2t}, y_2)$ of $P^*((x_1, y_1), (x_1, y_2))$ by $P((x_{2t+1}, y_1), (x_{2t+1}, y_2))$ of each length l for $1 \leq l \leq 2k - 1$ and obtain a path $P((x_1, y_1), (x_1, y_2))$ of each length l for $2tk + 1 \leq l \leq (2t + 2)k - 1$.

Therefore, there exist paths $P((x_1, y_1), (x_1, y_2))$ of each length l for $1 \leq l \leq$

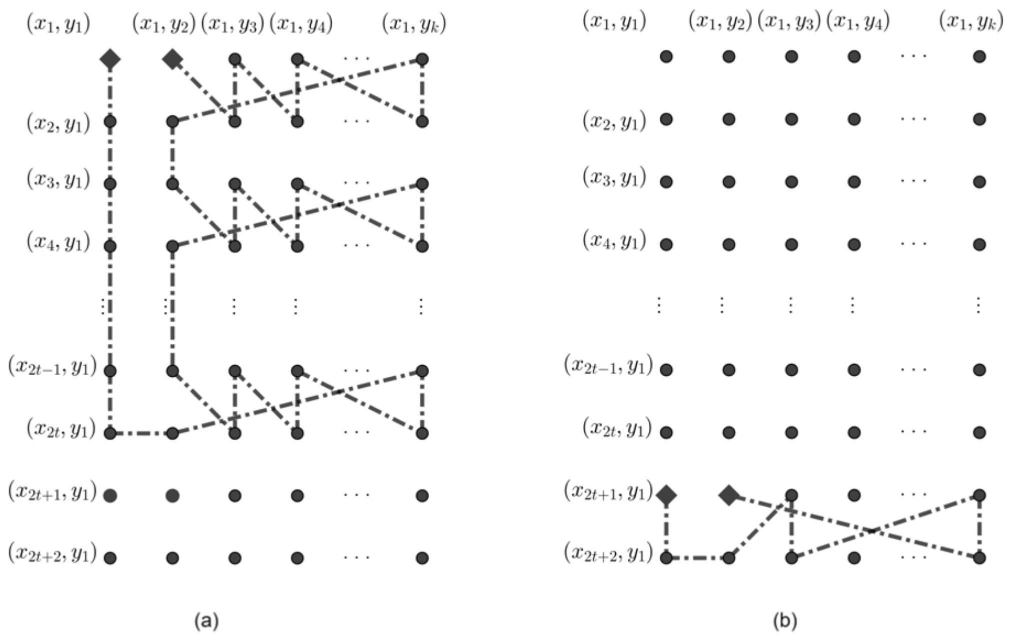

Figure 4.3: (a) A path $P^*((x_1, y_1), (x_1, y_2))$ of length $2tk - 1$ and (b) A path $P((x_{2t+1}, y_1), (x_{2t+1}, y_2))$

$nk - 1$ for n is an even number $n \geq 2$. $\square$

By reversing path $P_n = x_1x_2x_3 \cdots x_n$ into $x_nx_{n-1}x_{n-2} \ldots x_1$, we also obtain that $P_n \circ G$ contains path $P((x_n, y_1), (x_n, y_2))$ of each length l for $1 \leq l \leq nk - 1$ when n is even.

Theorem 4.15. *Let $n \geq 2$ be even. If G is a graph with at least one edge, then $P_n \circ G$ is vertex pancyclic.*

Proof. Let $P_n = x_1x_2x_3 \cdots x_n$ and $V(G) = \{y_1, y_2, y_3, \ldots, y_k\}$ for $k \geq 2$. Since G contains at least one edge, without loss of generality, we assume that $y_1y_2 \in E(G)$.

We prove by the mathematical induction on n. For $n = 2$, Theorem 4.11 yileds that $P_2 \circ G$ is vertex pancyclic.

For the induction step, let $t \in \mathbb{N}$ and suppose that the statement holds for all even n, where $n \leq 2t$. We show that the statement still holds for $n = 2t + 2$. Let $V_i = \{(x_i, y) | y \in V(G)\}$ for $i \in \{1, 2, 3, \ldots, 2t + 2\}$. Then, each $\bigcup_{i=1}^{2t} V_i$ and $\bigcup_{i=2}^{2t+2} V_i$ induces a subgraph $P_{2t} \circ G$ of $P_{2t+2} \circ G$. By the induction hypothesis, a vertex in the induced subgraph $P_{2t} \circ G$ is contained in a cycle of each length l for $3 \leq l \leq 2tk$. Then, each vertex of $P_{2t+2} \circ G$ is contained in a cycle of each length l for $3 \leq l \leq 2tk$. In order to show that $P_{2t+2} \circ G$ is vertex pancyclic, we show that each vertex of $P_{2t+2} \circ G$ is contained in a cycle of each length l for $2tk + 1 \leq l \leq (2t + 2)k$.

Let (x, y) be a vertex of $P_n \circ G$. Without loss of generality, we assume that $(x, y) \in \bigcup_{i=1}^{2t} V_i$. We perform two steps as follows.

(i) We show that there is a cycle of length $2tk+1$ containing (x, y). By Lemma 4.14 and the reversing path, there is a path $P((x_{2t}, y_1), (x_{2t}, y_2))$ of length $2tk-1$ in the subgraph of $P_{2t+2} \circ G$ induced by $\bigcup_{i=1}^{2t} V_i$. Moreover, $P((x_{2t}, y_1), (x_{2t}, y_2))$ contains (x, y). Since (x_{2t+1}, y_1) is adjacent to two end verties of $P((x_{2t}, y_1), (x_{2t}, y_2))$, we connect (x_{2t+1}, y_1) to each end vertex of $P((x_{2t}, y_1), (x_{2t}, y_2))$. Then, a cycle of length $2tk + 1$ containing (x, y) is obtained.

(ii) We show that there is a cycle of each length l for $2tk + 2 \leq l \leq (2t + 2)k$ containing (x, y). We can see that $P_2 \circ G$ is the subgraph of $P_{2t+2} \circ G$ induced by $V_{2t+1} \cup V_{2t+2}$. By Lemma 4.14, there is a path $P((x_{2t+1}, y_1), (x_{2t+1}, y_2))$ in $P_2 \circ G$ of each length l for $1 \leq l \leq 2k - 1$. For the subgraph of $P_{2t} \circ G$ of $P_{2t+2} \circ G$ induced by $\bigcup_{i=1}^{2t} V_i$, we obtain (from Lemma 4.14 and the reversing path) a path $P((x_{2t}, y_1), (x_{2t}, y_2))$ of length $2tk - 1$ containing vertex (x, y). Since (x_{2t+1}, y_1) and (x_{2t+1}, y_2) are adjacent to (x_{2t}, y_1) and (x_{2t}, y_2), respectively, we connect each end vertex of $P((x_{2t+1}, y_1), (x_{2t+1}, y_2))$ to each end vertex of $P((x_{2t}, y_1), (x_{2t}, y_2))$ together. Then, (x, y) is contained in a cycle of each length l for $2tk + 2 \leq l \leq (2t + 2)k$.

Therefore, $P_n \circ G$ is vertex pancyclic for even n. □

By Theorem 4.15, we obtain that $P_n \circ G$ is vertex pancyclic if n is even and G is a graph with at least one edge. Since a path P_n is a subgraph of traceable graphs of order n, we obtain the following corollary.

Corollary 4.16. *If G_1 is a traceable graph of even order and G_2 is a graph with at least one edge, then $G_1 \circ G_2$ is vertex pancyclic.*

Example 4.17. The Petersen graph is a graph of order 10 containing a Hamiltonian path. By Corollary 4.16, the lexicographic product of the Petersen graph and a graph of at least one edge is vertex pancyclic.

Next, we investigate the lexicographic product of odd paths and a graph.

Theorem 4.18. *Let $n > 2$ be odd. If G is a graph of order $k > \frac{n+1}{2}$ with exactly one edge, then $P_n \circ G$ is not vertex pancyclic.*

Proof. Let $P_n = x_1x_2x_3 \cdots x_n$ and $V(G) = \{y_1, y_2, y_3, \ldots, y_k\}$ where $k > \frac{n+1}{2}$. Assume that $E(G) = \{y_1y_2\}$. Choose $S = \bigcup_{i\in\{2,4,6,\ldots,n-1\}}\{(x_i, y)|y \in V(G)\}$. Then, $|S| = k(\frac{n-1}{2})$. Let H denote the graph $(P_n \circ G) - S$. Then, H has $(k-1)(\frac{n+1}{2})$ components, namely, $H[\{(x_i, y_1), (x_i, y_2)\}]$, $H[(x_i, y_3)]$, $H[(x_i, y_4)]$, $\ldots$, $H[(x_i, y_k)]$ for all $i \in \{1, 3, 5, \ldots, n\}$. Since $k > \frac{n+1}{2}$, $(k-1)(\frac{n+1}{2}) > k(\frac{n-1}{2})$. By Theorem 4.6, $P_n \circ G$ is non-Hamiltonian. Therefore, $P_n \circ G$ is not vertex pancyclic. □

Therefore, if n is odd and G is a graph with the same condition as in Theorem 4.15, i.e., G is a graph with at least one edge, then we cannot conclude anything about vertex pancyclic of $P_n \circ G$.

Now, we investigate the condition that provide vertex pancyclic over the lexicographic product of graphs. We consider nontrivial traceable graphs G_1 and G_2 as follows.

Theorem 4.19. *If G_1 and G_2 are nontrivial traceable graphs, then $G_1 \circ G_2$ is vertex pancyclic.*

Proof. Let G_1 and G_2 be traceable graphs of order n and m, respectively, for $n, m \geq 2$. Let $P_n = x_1x_2x_3 \cdots x_n$ and $P_m = y_1y_2y_3 \cdots y_m$ be spanning paths in G_1 and G_2, respectively.

If n is even, by Corollary 4.16, $G_1 \circ G_2$ is vertex pancyclic. Assume that n is odd. Let $P_{n-1} = x_1x_2x_3 \cdots x_{n-1}$ and $P^*_{n-1} = x_2x_3x_4 \cdots x_n$ be subgraphs of P_n. We can see that $P_{n-1} \circ G_2$ and $P^*_{n-1} \circ G_2$ are subgraphs of $G_1 \circ G_2$. By Theorem 4.15, $P_{n-1} \circ G_2$ and $P^*_{n-1} \circ G_2$ are vertex pancyclic. Then, each vertex of $G_1 \circ G_2$ is contained in a cycle of each length l for $3 \leq l \leq k(n-1)$.

We show that each vertex of $G_1 \circ G_2$ is contained in a cycle of each length l for $(n-1)k+1 \leq l \leq nk$. Let (x_i, y_j) be a vertex of $G_1 \circ G_2$ for some $i \in \{1, 2, 3, \ldots, n\}$ and $j \in \{1, 2, 3, \ldots m\}$.

By the symmetry of $G_1 \circ G_2$, the idea of proof for the vertex (x_n, y_j) is similar to the proof of the vertex (x_1, y_j). Then, without loss of generality, let $i \in \{1, 2, 3, \ldots, n-1\}$. Now, we consider the subgraph $P_{n-1} \circ G_2$. Similar to the prove of Theorem 4.15, by reversing a path P_{n-1} of Lemma 4.14, there is a path $P((x_{n-1}, y_1), (x_{n-1}, y_2))$ of length $(n-1)k-1$ containing vertex (x_i, y_j). Consider subgraph $\{x_n\} \circ G_2$ of $G_1 \circ G_2$. This subgraph contains a path $P((x_n, y_1), (x_n, y_j)) = (x_n, y_1)(x_n, y_2)(x_n, y_3) \cdots (x_n, y_j)$ where $j \in \{1, 2, 3, \ldots, k\}$. Since each vertex of $P((x_n, y_1), (x_n, y_j))$ is adjacent to vertices (x_{n-1}, y_1) and (x_{n-1}, y_2), we connect $P((x_{n-1}, y_1), (x_{n-1}, y_2))$ with each end vertex of $P((x_n, y_1), (x_n, y_j))$, respectively, for all $j \in \{1, 2, 3, \ldots, k\}$. Then, (x_i, y_j) is contained in a cycle of length l for $(n-1)k+1 \leq l \leq nk$.

Therefore, $G_1 \circ G_2$ is vertex pancyclic. □

By Theorem 4.19, we obtain that $P_n \circ P_2$ is vertex pancyclic for all $n \geq 2$ even though n is an odd number, the following corollary is proved.

Corollary 4.20. *If G is a nontrivial traceable graph, then the double graph of G is vertex pancyclic.*

4.2.3 Cycles

Theorem 4.21. *Let $n \geq 3$, $k \geq 1$ and A_k be an empty graph of order k. Then, $C_n \circ A_k$ is Hamiltonian.*

Proof. We see that $C_n \circ A_1$ is C_n which is Hamiltonian. Assume that $k > 1$. Let $C_n = x_1 x_2 x_3 \cdots x_n x_1$ and $V(A_k) = \{y_1, y_2, y_3, \ldots, y_k\}$. We can see that the path $x_1 x_2 x_3 \cdots x_n$ in C_n forms the path $P_i = (x_1, y_i)(x_2, y_i)(x_3, y_i) \cdots (x_n, y_i)$ in $C_n \circ A_k$ for each $i \in \{1, 2, 3, \ldots, k\}$. Let $e_i = (x_n, y_i)(x_1, y_{i+1})$ for $i \in \{1, 2, 3, \ldots, k-1\}$ and $e_k = (x_n, y_k)(x_1, y_1)$. For $i \in \{1, 2, 3, \ldots, k-1\}$, each pair of paths P_i and P_{i+1}

is connected by the edge e_i and the paths P_k and P_1 are connected by the edge e_k. A Hamiltonian cycle in $C_n \circ A_k$ is

$$P_1 e_1 P_2 e_2 P_3 e_3 \cdots e_{k-1} P_k e_k.$$

$\square$

Since $C_n \circ A_k$ is a subgraph of $C_n \circ G$ for any graph G of order k, we obtain the following corollaries.

Corollary 4.22. *If $n \geq 3$ and G is a graph, then $C_n \circ G$ is Hamiltonian.*

Corollary 4.23. *If G_1 is Hamiltonian and G_2 is a graph, then $G_1 \circ G_2$ is Hamiltonian.*

Corollary 4.23 does not hold for the Cartesian product $G_1 \square G_2$. For counter example, let G_2 be disconnected. Then, $G_1 \square G_2$ is disconnected (and of course non-Hamiltonian) although G_1 is Hamiltonian.

By Corollary 4.9, $C_3 \circ A_k$ is vertex pancyclic for $k \geq 1$. Unfortunately, the lexicographic product of cycle C_n for $n \geq 4$ and empty graph A_k for $k \geq 1$ is not always vertex pancyclic. For instance, $C_7 \circ A_2$ contains no cycle of length 5. Now, we investigate the condition of G that allows the product $C_n \circ G$ to be vertex pancyclic.

Theorem 4.24. *Let $n \geq 3$. If G is a graph with exactly one edge, then $C_n \circ G$ is vertex pancyclic.*

Proof. Let $C_n = x_1 x_2 x_3 \cdots x_n x_1$ and $V(G) = \{y_1, y_2, y_3, \ldots, y_k\}$ for $k \geq 2$. Since G contains exactly one edge, assume that $y_1 y_2 \in E(G)$. We can see that $P_n \circ G$ is a spanning subgraph of $C_n \circ G$ where $P_n = x_1 x_2 x_3 \cdots x_n$. By Theorem 4.15, $C_n \circ G$ is vertex pancyclic if n is even.

Assume that n is odd. Let $P_{n-1} = x_1 x_2 x_3 \cdots x_{n-1}$ and $P^*_{n-1} = x_2 x_3 x_4 \cdots x_n$. We can see that $P_{n-1} \circ G$ and $P^*_{n-1} \circ G$ are subgraphs of $C_n \circ G$ induced by $V((P_n - x_n) \circ G)$ and $V((P_n - x_1) \circ G)$, respectively. By Theorem 4.15, $P_{n-1} \circ G$

and $P^*_{n-1} \circ G$ are vertex pancyclic. Then, each vertex of $C_n \circ G$ is contained in a cycle of each length l such that $3 \leq l \leq (n-1)k$.

By Theorem 4.4 and Corollary 4.22, $C_n \circ G$ contains a cycle of each length l for $3 \leq l \leq nk$. Now, we show that each vertex is contained in a cycle of each length l for $(n-1)k+1 \leq l \leq nk$. For $(n-1)k+1 \leq l \leq nk$, let $C_l = (x_{i_1}, y_{j_1})(x_{i_2}, y_{j_2})(x_{i_3}, y_{j_3}) \cdots (x_{i_l}, y_{j_l})(x_{i_1}, y_{j_1})$ be a cycle in $C_n \circ G$ of length l. We consider two cases as follows.

Case 1. y_1y_2 does not induce an edge in C_l. Then, C_l is a cycle in $C_n \circ A_k$. Let (x_s, y_t) be a vertex of $C_n \circ G$ where $s \in \{1, 2, 3, \ldots, n\}$ and $t \in \{1, 2, 3, \ldots, k\}$. We consider two subcases as follows.

Subcase 1.1. If $x_s = x_{i_\beta}$ for some $\beta \in \{1, 2, 3, \ldots l\}$, then $(x_s, y_{j_\beta}) = (x_{i_\beta}, y_{j_\beta}) \in C_l$. Since C_l is in $C_n \circ A_k$, $x_{i_\alpha} \neq x_{i_{\alpha+1}}$ for any $\alpha \in \{1, 2, 3, \ldots l-1\}$ and $x_{i_l} \neq x_{i_1}$. This implies that $x_{i_{\beta-1}}x_{i_\beta}, x_{i_\beta}x_{i_{\beta+1}} \in E(C_n)$. Since $x_s = x_{i_\beta}$, $(x_{i_{\beta-1}}, y_{j_{\beta-1}})(x_s, y_t)$ and $(x_s, y_t)(x_{i_{\beta+1}}, y_{j_{\beta+1}})$ are edges in $C_n \circ G$. Thus, we can replace $(x_{i_\beta}, y_{j_\beta})$ in C_l by (x_s, y_t). Therefore, (x_s, y_t) is contained in a cycle of length l.

Subcase 1.2. If $x_s \neq x_{i_\alpha}$ for all $\alpha \in \{1, 2, 3, \ldots l\}$, we translate cycle C_l to be C^*_l by defining an injective function. Let $i_w = \max\{i_\alpha | (x_{i_\alpha}, y_{j_\alpha}) \in C_l\}$. We define an injective function $\varphi : \{1, 2, 3, \ldots, n\} \to \mathbb{Z}_n$ by $\varphi(i_\alpha) = (i_\alpha + s - i_w)(\mathrm{mod}\, n)$. This function translates indices in each vertex $(x_{i_\alpha}, y_{j_\alpha})$ of the cycle C_l. The vertices with new indices are vertices of cycle C^*_l. From this function, vertex (x_{i_w}, y_{j_w}) is translate into vertex (x_s, y_{j_w}). If $y_t = y_{j_w}$, then $(x_s, y_t) = (x_s, y_{j_w})$ is contained in C^*_l. Assume that $y_t \neq y_{j_w}$. We can replace vertex (x_s, y_{j_w}) by vertex (x_s, y_t) as shown in Subcase 1.1. Hence, (x_s, y_t) is contained in a cycle of length l.

Case 2. y_1y_2 induces an edge in C_l. Let S be a subgraph of G induced by the set $\{y_1, y_2\}$. Then, S is a path y_1y_2. If $k = 2$, then $C_n \circ G = C_n \circ P_2$. By Theorem 4.20, $C_n \circ G$ is vertex pancyclic. Now, we assume that $k > 2$. Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be subgraphs of $C_n \circ G$ induced by $C_n \circ S$ and $C_n \circ (G - S)$, respectively. Then, $V(\mathbb{S}_1) = \{(x_i, y_j) | i \in \{1, 2, 3, \ldots n\} \text{ and } j \in \{1, 2\}\}$ and $V(\mathbb{S}_2) = \{(x_i, y_j) | i \in \{1, 2, 3, \ldots n\} \text{ and } j \in \{3, 4, 5, \ldots, k\}\}$. We can see that $V(C_n \circ G) = V(\mathbb{S}_1) \cup V(\mathbb{S}_2)$.

We first show that all vertices of $\mathbb{S}_1$ are contained in a cycle of length l. Since y_1y_2 forms an edge in C_l, C_l contains an edge of $\mathbb{S}_1$. Then, there are vertices (x_i, y_1) and (x_i, y_2) contained in C_l as consecutive vertices for some $i \in \{1, 2, 3, \ldots, n\}$. We translate cycle C_l into C_l^*, as shown in Subcase 1.2, and obtain that all vertices in $\mathbb{S}_1$ are contained in a cycle of length l.

Next, we show that each vertex of $\mathbb{S}_2$ is contained in a cycle of length l. Consider a cycle of maximum length in $\mathbb{S}_1$. The length of such cycles is at most $2n$. Since the length of C_l is at least $(n-1)k+1$ and $(n-1)k+1 > 2n$ for $k > 2$, the cycle C_l contains a vertex of $\mathbb{S}_2$. Let (x_s, y_t) be any vertex in $C_n \circ \mathbb{S}_2$. If $x_s = x_{i_\beta}$ for some $i_\beta \in \{i_\alpha | (x_{i_\alpha}, y_{j_\alpha}) \in \mathbb{S}_2\}$, then $(x_{i_\beta}, y_{j_\beta}) \in C_l$. Similar to Subcase 1.1, we can replace vertex $(x_{i_\beta}, y_{j_\beta})$ by (x_s, y_t). Thus, (x_s, y_t) is in a cycle of length l. If $x_s \neq x_{i_\beta}$ for all $i_\beta \in \{i_\alpha | (x_{i_\alpha}, y_{j_\alpha}) \in \mathbb{S}_2\}$, then let $i_w = \max\{i_\alpha | (x_{i_\alpha}, y_{j_\alpha}) \in \mathbb{S}_2\}$. Similar to Subcase 1.2, we can translate cycle C_l into C_l^*. Then, vertex (x_{i_w}, y_{j_w}) is translated into (x_s, y_{j_w}). If $y_t = y_{j_w}$, then (x_s, y_t) is contained in C_l^*. Otherwise, we can replace vertex (x_s, y_{j_w}) by (x_s, y_t) as shown in Subcase 1.1.

From these two cases, we conclude that each vertex is contained in a cycle of each length l for $(n-1)k+1 \leq l \leq nk$. Therefore, $C_n \circ G$ is vertex pancyclic. $\square$

From Theorem 4.24, we can see that adding more edges into the graph G does not affect vertex pancyclic property. Thus, we obtain the following corollary.

Corollary 4.25. *Let $n \geq 3$. If G is a graph with at least one edge, then $C_n \circ G$ is vertex pancyclic.*

If G_1 is Hamiltonian containing a spanning cycle C_n, then C_n is a subgraph of G_1. We can extend Corollary 4.25 as follows.

Corollary 4.26. *If G_1 is Hamiltonian and G_2 is a graph with at least one edge, then $G_1 \circ G_2$ is vertex pancyclic.*

4.3 Conclusion and discussion

This chapter obtains that $C_n \circ G$ is vertex pancyclic provided that $|E(G)| \geq 1$ and $n \geq 3$ and $K_n \circ G$ is vertex pancyclic for all positive integers n. However, the vertex pancyclicity of $P_n \circ G$ can be obtained only for $n \geq 2$ is an even integer. If $n = 1$, then $P_1 \circ G = G$. Thus, the vertex pancyclicity of $P_1 \circ G$ depends on G. If $n \geq 3$ is an odd integer, then we can see from Theorem 4.18 that the vertex pancyclicity of $P_n \circ G$ may depend on some conditions on n and k. Therefore, our future research will try to find the conditions which imply the vertex pancyclicity of the $P_n \circ G$ when $n \geq 3$ is odd integer.

CHAPTER V

CONCLUSIONS

The present research was conducted to investigate the pancyclicity of the n-generalized prism over any skirted graph and the vertex pancyclicity of the lexicographic product of some graphs. It was found that

(i) the n-generalized prism over any skirted graph is Hamiltonian (see Theorem 2.9);

(ii) the n-generalized prism over a skirted graph with three specific types given by Bondy and Lovász [4] is pancyclic (see Theorems 2.13 and 2.17);

(iii) the n-generalized prism over any skirted graph is pancyclic (see Theorem 3.19);

(iv) if G_1 is a traceable graph of even order and G_2 is a graph with at least one edge, then $G_1 \circ G_2$ is vertex pancyclic (see Theorem 4.15 and Corollary 4.16);

(v) if G_1 and G_2 are nontrivial traceable graphs, then $G_1 \circ G_2$ is vertex pancyclic (see Theorem 4.19);

(vi) if G_1 is Hamiltonian and G_2 is a graph with at least one edge, then $G_1 \circ G_2$ is vertex pancyclic (see Theorem 4.24).

Although the third result implies the second result, the cycles obtained from the proof of the second result is more elective than the cycles from the proof of the third result. Thus, we still provide the proof of the second result.

For the lexicographic product of graphs, since a skirted graph is Hamiltonian, the sixth result implies that the lexicographic product of a skirted graph and a graph with at least one edge is vertex pancyclic. In particular, the lexicographic product of a skirted graph and a path is vertex pancyclic.

However, we have not investigated the vertex k-pancyclicity for some k of the n-generalized prism over any skirted graph. Therefore, it is recommended that further studies investigating more details about the vertex k-pancyclicity for some k of the n-generalized prism over any skirted graph should be conducted.

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