# ตัวแบบความเสี่ยงอนุกรมเวลาค่าจำนวนเต็มที่มีการถอนตัวและการลงทุน

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INTEGER-VALUED TIME SERIES RISK MODEL WITH SURRENDER AND INVESTMENT

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In this study, we construct the discrete-time risk models based on integer-valued time series models by incorporating the concepts of surrender and investment. The surrender considered in this study is the situation that the policyholder decides to exit the policy before maturity date. In our study, we provide the probabilistic properties of the model. Moreover, we derive approximation of ruin probabilities of the constructed risk model. Finally, we discuss the trends of the ruin probability and the value at risk of the model by numerical simulations.



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# CHAPTER I

# INTRODUCTION

In actuarial science, the risk model or surplus process describes a measurement of the aggregate amount of premiums and claims corresponding to the insurance company's portfolio. The classical risk model perform as a constant premium rate over time and the aggregate claims process. In recent years, many models have been introduced in the two different ways: (1) the stochastic risk model, and (2) the discrete-time risk model.

The stochastic risk model usually assumes that the number of claims follows a counting process. For example, Huang and Yu (2013) applied a generalized double Poisson-Geometric into insurance risk model. For more precisely, many studies, such as Temnov (2014) and Bao (2006), extended the model by considering the stochastic premiums. Lebbe and Sendova (2009) studied risk models when both premium and claim aggregate processes follow compound Poisson processes. Yu and Huang (2015) introduced the concepts of surrender, where surrender is the situation that the policyholder will get some money from the insurance company if he decides to terminate before maturity date. They introduced surrender and investment into risk models where the processes of claim and surrender follow the thinning process of the premium process.

Beside the family of stochastic risk models, many studies focused on the discrete time risk models where the number of claims follows an integer-valued time series model. The concept of the integer-valued time series models was independently introduced in Al-Osh and Alzaid (1987), McKenzie (1988), and Joe (1997).

Later, the integer-valued time series models were applied in insurance risk models. For example, Cossette (2010) suggested the discrete-time risk model based on Poisson MA(1) and Poisson AR(1) for the number of claim process. Moreover, Hu, Zhang and Sun (2018), Shi and Wang (2014) and Zhang et al. (2011) also considered the discrete-time risk model with different settings.

In this thesis, we will apply the concepts of investment and surrender into the discrete time risk models. The number of premiums, the number of claims and the number of surrenders in the model follow (1) integer-valued moving average model (2) integer-valued autoregressive model. We then derive probabilistic properties of the model. In addition, we study the two risk measures of the model which are ruin probability and value at Risk. We also derive the adjustment coefficient to approximating ruin probability. Finally, we provide numerical examples to discuss the trend of ruin probability and value at Risk comparing with the parameters of premiums, claims and surrenders.

The organization of this thesis is as follows. Chapter 2 gives basic knowledge of probability, definition and properties of integer-valued time series and the concept of ruin probability.



In Chapter 3, we introduce a new risk model based on the first order integer-valued moving average model with surrender and investment. In this model, the numbers of premiums, claims and surrenders follow the first order integer-valued moving average model. In our study, we give its probabilistic properties, derive the adjustment coefficient function to obtain the approximation of ruin probability of the model. Finally, we discuss the trend of ruin probability and value at risk comparing with various parameters, such as premium sizes, claim sizes and surrender values via numerical simulations.

In Chapter 4, we introduce a risk model based on the first order integer-valued autoregressive model. We derive probabilistic properties, derive the adjustment coefficient function. Moreover, numerical studies are also provided to study the trend of ruin probability and value at risk comparing with parameters of the model. Finally, conclusion of this thesis is provided.

# CHAPTER II

# PRELIMINARIES

In this chapter, we review some basic knowledge of probability theory that will be used in this thesis. Moreover, we give the definition and properties of integer-valued time series and a review of the ruin probability.

## 2.1 Random variable and moments of random variable

In this section, we first give some definitions of random variable and some concept of its properties.



**Definition 2.1.1.** The sample space $S$ is the set of all possible outcomes from a random experiment, and the set $\{s \in S | X(s) \in \mathbb{R}\}$ is an event in $S$.

**Definition 2.1.2.** If $S$ is a sample space and $X$ is a real-valued function defined over the elements of $S$, then $X$ is called a random variable.

**Definition 2.1.3.** Let $X$ be a random variable from the sample space $S$. The set $\{x \in \mathbb{R} | x = X(s), s \in S\}$ is the space of the random variable $X$, denoted by $R_X$.

**Definition 2.1.4.** A random variable $X$ is said to be discrete if the space of $X$ is countable.

**Definition 2.1.5.** Let $R_X$ be the space of a discrete random variable $X$. The function $f : R_X \to [0, 1]$ which is defined by

$$f(x) = P(X = x)$$

is called the probability mass function of $X$.

**Definition 2.1.6.** Let $f(\cdot)$ be the probability mass function of $X$. Then the cumulative distribution of $X$, denoted by $F_X(\cdot)$, is defined as

$$F_X(x) = \sum_{t \leq x} f(t) \qquad \text{for } x \in \mathbb{R}$$

**Definition 2.1.7.** Let $R_X$ be the space of the discrete random variable $X$ and $f(\cdot)$ be the probability mass function of $X$. Then

(a) $f(x) \geq 0$ for all $x \in R_X$,

(b) $\sum_{x \in R_X} f(x) = 1$.

**Definition 2.1.8.** Let $X$ be a discrete random variable with space $R_X$, and probability mass function $f(\cdot)$. The expectation or mean of $X$, denoted by $E(X)$, is defined as

$$E(X) = \sum_{x \in R_X} x f(x).$$

**Definition 2.1.9.** The $n^{th}$ moment of the discrete random variable $X$ about the origin, denoted by $E(X^n)$, is defined as

$$E(X^n) = \sum_{x \in R_X} x^n f(x).$$

**Definition 2.1.10.** Let $X$ be a discrete random variable with space $R_X$. The moment generating function of $X$, denoted by $m_X(\cdot)$, is defined by

$$m_X(t) = E(e^{tX}) = \sum_{x \in R_X} e^{tx} f(x),$$

for $t \in \mathbb{R}$ such that $m_X(t)$ exists.

**Definition 2.1.11.** Let $X$ be a discrete random variable with mean $\mu_X$. The variance of $X$, denoted by $\text{Var}(X)$, is defined as

$$\text{Var}(X) = E([X - \mu_X]^2) = E(X^2) - \mu_X^2.$$

**Definition 2.1.12.** Let $X$ and $Y$ be discrete random variables with means $\mu_X$ and $\mu_Y$, respectively. The covariance of $X$ and $Y$, denoted by $\text{Cov}(X, Y)$, is defined as

$$\text{Cov}(X, Y) = E([X - \mu_X][Y - \mu_Y]) = E(XY) - \mu_X\mu_Y.$$

The correlation of $X$ and $Y$, denoted by $Corr(X, Y)$, is defined as

$$Corr(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}}.$$

**Definition 2.1.13.** Let $X$ be a discrete random variable with space $R_X$. The probability generating function of $X$, denoted by $G_X(\cdot)$, is defined as

$$G_X(t) = E(t^X) = \sum_{x \in R_X} t^x f(x),$$

for $t \in \mathbb{R}$ such that $G_X(t)$ exists.



**Lemma 2.1.1.** Let $G_X(\cdot)$ be the probability generating function of a random variable $X$, then the probabilistic properties of $X$ are as follows:

(a) $E(X) = G'_X(1)$,

(b) $E(X(X-1)(X-2)\cdots(X-k+1)) = G_X^{(k)}(1)$, for $k \in \mathbb{N}$ and $G_X^{(k)}$ is $k$th derivative of function $G_X(\cdot)$.

**Definition 2.1.14.** Let $X$ and $Y$ be discrete random variables with the joint density $f(\cdot, \cdot)$ and $f_Y(\cdot)$ is the marginal probability mass of $Y$. Then the function is given by

$$f_X(x|y) = \frac{f(x,y)}{f_Y(y)},$$

for each $x \in R_X$ is called the conditional distribution of $X$ given $Y = y$.

**Definition 2.1.15.** Let $X$ be discrete random variable and $f_X(x|y)$ be the value of the conditional probability distribution of $X$ given $Y = y$. Then the conditional mean of $X$ given $Y = y$ is defined as

$$E(X|Y = y) = \sum_{x \in R_X} x f_X(x|y).$$

**Lemma 2.1.2.** Let $X$ and $Y$ be discrete random variables. Then

(a) $E(X) = E(E(X|Y))$

(b) $\mathrm{Var}(X) = E(\mathrm{Var}(X|Y)) + \mathrm{Var}(E(X|Y))$.

**Definition 2.1.16.** Let $X_1, X_2, ..., X_n$ be any $n$ random variables with probability mass functions $f_{X_1}, ..., f_{X_n}$. They are identically distributed random variables if and only if

$$f_{X_1}(x) = f_{X_2}(x) = \ldots = f_{X_n}(x) \quad \text{for} \quad x \in \mathbb{R}.$$



**Definition 2.1.17.** The random variables $X_1, X_2, ..., X_n$ are said to be independent random variables if and only if, $x_1, x_2, ..., x_n \in \mathbb{R}$

$$f_{X_1,X_2,...,X_n}(x_1, x_2, ..., x_n) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdots f_{X_n}(x_n)$$

**Remark 2.1.1.** The random variables are independent and identically distributed, denoted as i.i.d, if each random variable has the same probability distribution as the others and all are mutually independent.

## 2.2 Compound random variable

Next, we give the definition and properties of compound random variable.

**Definition 2.2.1.** Let $X_1, X_2, ...$ be a sequence of independent and identically distributed (i.i.d.) random variables which are independent of a non-negative integer-valued random variable $N$. Then the random variable $S$ defined as

$$S = \sum_{i=1}^{N} X_i$$

is called a compound random variable.

**Lemma 2.2.1.** The probability properties of the compound random variable $S$ defined in Definition 2.2.1 are as follows

(a) $E(S) = E(N)E(X)$,

(b) $\text{Var}(S) = E(N)\text{Var}(X) + \text{Var}(N)E(X)$,

(c) $\text{Cov}(S, N) = \text{Var}(N)E(X)$.



*Proof.* (a) Since $\{X_1, ..., X_n\}$ are i.i.d. random variables,

$$\begin{aligned} E(S) &= E\left(\sum_{i=1}^{N} X_i\right) \\ &= E\left(E\left(\sum_{i=1}^{N} X_i \middle| N\right)\right) \\ &= E\left(\sum_{i=1}^{N} (EX_i|N)\right) \\ &= E\left(NE\left(X\right)\right) \qquad (2.1) \\ &= E(N)E(X), \end{aligned}$$

where we use the independence of $X$ and $N$ to obtain (2.1)

(b) From Lemma 2.1.2 (b) and the independent of $X$ and $N$, we have

$$\begin{aligned}
\operatorname{Var}(S) &= E\left(\operatorname{Var}\left(\sum_{i=1}^{N} X_i \middle| N\right)\right) + \operatorname{Var}\left(E\left(\sum_{i=1}^{N} X_i \middle| N\right)\right) \\
&= E\left(\sum_{i=1}^{N} \operatorname{Var}\left(X_i|N\right)\right) + \operatorname{Var}\left(\sum_{i=1}^{N} E\left(X_i|N\right)\right) \\
&= E\left(N\operatorname{Var}\left(X\right)\right) + \operatorname{Var}\left(NE\left(X\right)\right) \\
&= E\left(N\right)\operatorname{Var}\left(X\right) + \operatorname{Var}\left(N\right)E^2\left(X\right).
\end{aligned}$$

(c) From (a), we have

$$\begin{aligned}
\operatorname{Cov}(S,N) &= E(SN) - E(S)E(N) \\
&= E\left(E\left(SN|N\right)\right) - E(N)E(X)E(N) \\
&= E\left(E\left(N\sum_{i=1}^{N} X_i \middle| N\right)\right) - E^2(N)E(X) \\
&= E\left(N\sum_{j=1}^{N} E\left(X_i|N\right)\right) - E^2(N)E(X) \\
&= E(N^2E(X)) - E^2(N)E(X) \\
&= (E(N^2) - E^2(N))E(X) \\
&= \operatorname{Var}(N)E(X).
\end{aligned}$$



□

**Lemma 2.2.2.** Let $S_1 = \sum_{j=1}^{N_1} X_{1,j}$ and $S_2 = \sum_{j=1}^{N_2} X_{2,j}$ be compound random variables where $X_1 = \{X_{1,j}\}_{j=1,2,\ldots}$ and $X_2 = \{X_{2,j}\}_{j=1,2,\ldots}$ are sequences of i.i.d. random variables and are independent from $N_1$ and $N_2$, respectively. Then we have

$$\operatorname{Cov}(S_1, S_2) = \operatorname{Cov}(N_1, N_2)E(X_1)E(X_2).$$

*Proof.* Since $X_1$ and $X_2$ are mutually independent. Note that

$$\begin{aligned} E(S_1S_2) &= E\left(\sum_{j=1}^{N_1} X_{1,j} \sum_{j=1}^{N_2} X_{2,j}\right) \\ &= E\left(E\left(\sum_{j=1}^{N_1} X_{1,j} \sum_{j=1}^{N_2} X_{2,j} \middle| N_1, N_2\right)\right) \\ &= E(N_1E(X_1)N_2E(X_2)) \\ &= E(N_1N_2)E(X_1)E(X_2). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{Cov}(S_1, S_2) &= E(S_1S_2) - E(S_1)E(S_2) \\ &= E(N_1N_2)E(X_1)E(X_2) - E(N_1)E(X_1)E(N_2)E(X_2) \\ &= (E(N_1N_2) - E(N_1)E(N_2))E(X_1)E(X_2) \\ &= \mathrm{Cov}(N_1, N_2)E(X_1)E(X_2). \end{aligned}$$

□

## 2.3 Distribution function



**Definition 2.3.1.** A random variable $X$ is said to be the Bernoulli random variable with parameter $p$, denoted by $X \sim Ber(p)$. If its probability mass function of $X$ is in the form of

$$f(x) = p^x(1-p)^{(1-x)} \text{ for } x = 0, 1.$$

**Theorem 2.3.1.** If $X$ is a Bernoulli random variable with parameter $p$. Then its properties are given as follows:

(a) $G_X(t) = (1-p) + pe^t$ for $t \in \mathbb{R}$,

(b) $E(X) = p$,

(c) $\text{Var}(X) = p(1-p)$.

**Definition 2.3.2.** A random variable $X$ is said to be the Poisson random variable with parameter $\lambda$, denoted by $X \sim Poi(\lambda)$. If its probability mass function of $X$ is in the form of

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} \text{ for } x = 1, 2, ... \text{ and } \lambda > 0.$$

**Theorem 2.3.2.** If $X$ is a Poisson random variable with parameter $\lambda$. Then its properties are given as follows:

(a) $G_X(t) = \lambda(e^t - 1)$ for $t \in \mathbb{R}$,

(b) $E(X) = \lambda$,

(c) $\text{Var}(X) = \lambda$.



## 2.4 Binomial thinning operator and integer-valued time series

In this section, we provide the definitions and properties of the binomial thinning operator and integer-valued time series.

**Definition 2.4.1.** Let $X$ be a non-negative integer-valued random variable and $\alpha \in [0,1]$. The binomial thinning operator, denoted by '$\alpha\circ$', is defined by

$$\alpha \circ X = \sum_{i=1}^{X} \delta_i$$

where $\delta_1, \delta_2, ...$ is a sequence of i.i.d. random variables having the Bernoulli distribution with parameter $\alpha$ and is independent of $X$.

**Lemma 2.4.1.** Let $X$ and $Y$ be non-negative integer-valued random variables. Then the following properties hold:

(a) $E(\alpha \circ X) = \alpha E(X)$,

(b) $E((\alpha \circ X)Y) = \alpha E(XY)$,

(c) $G_{\alpha \circ X}(t) = G_X(1 - \alpha + \alpha t)$,

(d) $\mathrm{Var}(\alpha \circ X) = \alpha(1-\alpha)E(X) + \alpha^2 \mathrm{Var}(X)$,

(e) $\mathrm{Cov}(\alpha \circ X, Y) = \alpha \mathrm{Cov}(X, Y)$,

(f) $\mathrm{Cov}(\alpha \circ X, \beta \circ Y) = \alpha\beta \mathrm{Cov}(X, Y)$.



*Proof.* (a) Note that $\alpha \circ X = \sum_{i=1}^{X} \delta_i$ where $\{\delta_i\}_{i=1,2,...}$ is a sequence of i.i.d. Bernoulli random variables with means $\alpha$ and are independent of $X$ and $Y$.

From Lemma 2.2.1 (a),

$$
\begin{aligned}
E(\alpha \circ X) &= E\left(\sum_{i=1}^{X} \delta_i\right) \\
&= E\left(E\left(\sum_{i=1}^{X} \delta_i \middle| X\right)\right) \\
&= E\left(\sum_{i=1}^{X} E\left(\delta_i | X\right)\right) \\
&= E(XE(\delta_i)) \\
&= E(X)E(\delta_i) \\
&= \alpha E(X).
\end{aligned}
$$

(b) Consider

$$
\begin{aligned}
E((\alpha \circ X)Y) &= E\left(Y\sum_{j=1}^{X} \delta_j\right) \\
&= E\left(E\left(Y\sum_{j=1}^{X} \delta_j \middle| X\right)\right) \\
&= E\left(\sum_{j=1}^{X} E\left(Y\delta_j | X\right)\right) \\
&= E(YXE(\delta_j)) \\
&= E(XY)E(\delta_j) \\
&= \alpha E(XY).
\end{aligned}
$$

(c) From Definition 2.4.1,

$$\begin{aligned}
G_{\alpha\circ X}(t) &= E\left(t^{\alpha\circ X}\right)\\
&= E\left(t^{\sum_{i=1}^{X}\delta_i}\right)\\
&= E\left(E\left(t^{\sum_{i=1}^{X}\delta_i}\Big|X\right)\right)\\
&= E\left(\prod_{i=1}^{X}E\left(t^{\delta_i}\big|X\right)\right)\\
&= E\left(\left(E\left(t^{\delta_i}\right)\right)^{X}\right)\\
&= G_X\left(1-\alpha+\alpha t\right).
\end{aligned}$$

(d) From Lemma 2.2.1 (b), we have

$$\begin{aligned}
\mathrm{Var}(\alpha\circ X) &= \mathrm{Var}\left(\sum_{i=1}^{X}\delta_i\right)\\
&= E(X)\mathrm{Var}(\delta)+\mathrm{Var}(X)E^2(\delta)\\
&= \alpha(1-\alpha)E(X)+\alpha^2\mathrm{Var}(X).
\end{aligned}$$

(e) From (a) and (b), we have



$$\begin{aligned}
\mathrm{Cov}(\alpha\circ X,Y) &= \mathrm{Cov}\left(\sum_{i=1}^{X}\delta_i,Y\right)\\
&= E\left(Y\sum_{i=1}^{X}\delta_i\right)-E\left(\sum_{i=1}^{X}\delta_i\right)E(Y)\\
&= \alpha E(XY)-E(X)E(\delta)E(Y)\\
&= \alpha E(XY)-\alpha E(X)E(Y)\\
&= \alpha(E(XY)-E(X)E(Y))\\
&= \alpha\mathrm{Cov}(X,Y).
\end{aligned}$$

(f) Note that $\beta \circ Y = \sum_{i=1}^{Y} \gamma_i$ where $\{\gamma_i\}_{i=1,2,...}$ is a sequence of i.i.d. Bernoulli random variables with mean $\beta$ and are independent of $Y$.
Since $\{\delta_i\}_{i=1,2,...}$ and $\{\gamma_i\}_{i=1,2,...}$ are two mutually independent sequences of Bernoulli random variables with parameters $\alpha$ and $\beta$, respectively. Then

$$\begin{aligned}
\text{Cov}(\alpha \circ X, \beta \circ Y) &= \text{Cov}\left(\sum_{i=1}^{X} \delta_i, \sum_{i=1}^{Y} \gamma_i\right) \\
&= E\left(\sum_{i=1}^{X} \delta_i \sum_{i=1}^{Y} \gamma_i\right) - E\left(\sum_{i=1}^{X} \delta_i\right) E\left(\sum_{i=1}^{Y} \gamma_i\right) \\
&= E\left(E\left(\sum_{i=1}^{X} \delta_i \sum_{i=1}^{Y} \gamma_i \middle| X, Y\right)\right) - E(X)E(\delta)E(Y)E(\gamma) \\
&= E(XE(\delta)YE(\gamma)) - E(X)E(Y)E(\delta)E(\gamma) \\
&= E(XY)E(\delta)E(\gamma) - E(X)E(Y)E(\delta)E(\gamma) \\
&= (E(XY) - E(X)E(Y))E(\delta)E(\gamma) \\
&= \alpha\beta\text{Cov}(X, Y).
\end{aligned}$$

□

Next, we will describe the integer-valued time series that will be used in this thesis. The two integer-valued time series considered in this thesis are the first order integer-valued moving average (INMA(1)) model and the first order integer-valued autoregressive (INAR(1)).

Integer-valued time series, such as integer-valued moving average (INMA) and integer-valued autoregressive (INAR), are independently introduced by Al-Osh & Alzaid(1987), Mckenzie(1988) and Joe(1997).

**Definition 2.4.2.** Time series $\{X_t\}_{t=1,2,...}$ is a series of data points indexed in $\{t = 1, 2, ...\}$. If $X_t$ has integer valued, the time series is called the integer-valued time series.

**Definition 2.4.3.** A process $\{X_t\}_{t=1,2,...,n}$ is said to be $n$th-order weakly stationary if all its joint moments up to order $n$ exist and are time variant.

**Definition 2.4.4.** The first order integer-valued moving average (INMA(1)) model for $\{X_t\}_{t>0}$ can be defined as

$$X_t = \alpha \circ \varepsilon_{t-1} + \varepsilon_t$$

where '$\circ$' is binomial thinning operator. The sequence $\varepsilon_1, \varepsilon_2, ...$ is a sequence of independent and identically distributed (i.i.d.) random variables.

**Definition 2.4.5.** The process $\{X_t\}_{t=1,2,...}$ is said to be the first order integer-valued autoregressive (INAR(1)) model if it defined as

$$X_t = \alpha \circ X_{t-1} + \varepsilon_t$$

where '$\circ$' is binomial thinning operator defined in Definition 2.4.1 and $\varepsilon_1, \varepsilon_2, ...$ is a sequence of i.i.d. random variables.

## 2.5 Risk model and ruin probability



In this section, we will provide the basic of discrete time risk model and review the ruin probability.

Let $\{U_n; n \in \mathbb{N}\}$ be the surplus process of insurance company in period $n$. The discrete time risk model can be defined as

$$U_n = u + \pi n - S_n = u + \pi n - \sum_{i=1}^{n} \sum_{j=1}^{N_i} X_{i,j}$$

where $u$ is the initial capital and $\pi$ is the constant premium rate. The process $\{S_n; n \in \mathbb{N}\}$ is the aggregate claims amount in period $n$ and can be written as

$S_n = \sum_{i=1}^{n}\sum_{j=1}^{N_i} X_{i,j}$ where $X_{i,j}$ is the $j$th claim size in period $i$ and $N_i$ is the number of claim in period $i$.

The ruin probability is the one of risk measure for insurance company. To find this measure, this section provides a brief discussion of ruin time, the ruin probability and the adjustment coefficient function.

The first time that the surplus process $\{U_n; n \in \mathbb{N}\}$ changes to be negative which called that the ruin time, denoted by $T$. It can be written as

$$T = \inf_{n\in\mathbb{N}}\{n; U_n < 0\}$$

The probability that ruin time exists, we call that the ruin probability, denoted by $\Psi(u)$ and can be written as

$$\Psi(u) = Pr(T < \infty | U_0 = u)$$

In general, it is difficult to directly obtain the ruin probability. However, there is approximation of ruin probability by Lundberg, proposed in [4], as follows

$$\Psi(u) \simeq e^{-Ru}. \tag{2.2}$$

The main result based on the asymptotic Lundberg type result

$$\lim_{u\to\infty} \frac{-\ln(\Psi(u))}{u} = R,$$

where $R$ is Lundberg adjustment coefficient or adjustment coefficient.

The adjustment coefficient $R$, introduced by Cossette et al. (2010), is the unique positive solution to equation $g(r) = 0$. The function $g(r)$ is called the adjustment coefficient function and is defined as

$$g(r) = \lim_{n\to\infty} \frac{1}{n} g_n(r),$$

where $g_n(r)$ is the cumulative generating function of aggregate loss-profit process $S_n$ defined by

$$g_n(r) = \ln E(e^{-rS_n}). \tag{2.3}$$

# CHAPTER III

# INTEGER-VALUED MOVING AVERAGE RISK MODEL SUBJECT TO INVESTMENT AND SURRENDER

In this chapter, we construct the risk model by incorporating investment and surrender based on the first order integer-valued moving average (INMA(1)) process. Firstly, Section 3.1 introduces the discrete-time risk model and notations used in this chapter. Section 3.2 gives the definition and properties of the INMA(1) risk model. In Section 3.3, we derive the adjustment coefficient function and obtain the adjustment coefficient to calculate ruin probability. Finally, Section 3.4 shows numerical examples of ruin probability and value at risk considering the trend of ruin probability in terms of parameters in the model.

## 3.1 The discrete-time risk model

Let $\{U_n; n \in \mathbb{N}\}$ be the surplus process of insurance company with incorporating investment and surrender at time $n$. For initial capital $u$, the discrete-time risk model can be written as

$$U_n = u + Idn + \sum_{i=1}^{n} A_i - \sum_{i=1}^{n} B_i - \sum_{i=1}^{n} C_i, \tag{3.1}$$

where $I$ is the investment capital for $I < u$, $d$ represents the investment income per unit of time. The sequence $\{A_i; i \in \mathbb{N}\}$ is the sequence of aggregates of premium amounts in period $i$ defined as

$$A_i = \sum_{k=1}^{N_i} X_{i,k}, \tag{3.2}$$

where $\{X_{i,k}; k \in \mathbb{N}\}$ is the sequence of premium sizes in period $i$ assuming to be i.i.d. random variables, and $N_i$ is the number of premiums in period $i$.

The sequence $\{B_i; i \in \mathbb{N}\}$ is the sequence of total of claim sizes in period $i$ and is defined as

$$B_i = \sum_{k=1}^{N_i(p)} Y_{i,k}, \tag{3.3}$$

where the sequence of i.i.d. random variables $\{Y_{i,k}; k \in \mathbb{N}\}$ denotes claim sizes in period $i$, and $N_i(p)$ denotes the number of claims in period $i$. And we say that $N_i(p)$ is the $p$-thinning operator of $N_i$ because the thinning operator of $N_i(p)$ is $\alpha p$ where $0 < p < 1$ which is smaller than the thinning operator $\alpha$ of $N_i$.

The sequence $\{C_i; i \in \mathbb{N}\}$ is the sequence of aggregate of surrender values in period $i$ and is written as

$$C_i = \sum_{k=1}^{N_i(q)} Z_{i,k}, \tag{3.4}$$



where the sequence of i.i.d. random variables $\{Z_{i,k}; k \in \mathbb{N}\}$ represents surrender values in period $i$, and $N_i(q)$ denotes the number of surrenders in period $i$ and is the $q$-thinning operator of $N_i$ for $0 < q < 1$ such that $0 < p + q < 1$.

From (3.1), we will write the model in the form of

$$U_n = u + S_n,$$

where $S_n$ is the loss-profit process defined as

$$S_n = Idn + \sum_{i=1}^{n} A_i - \sum_{i=1}^{n} B_i - \sum_{i=1}^{n} C_i. \tag{3.5}$$

Next, we will give the expectations of aggregate of premium sizes, aggregate of claim sizes and aggregate of surrender values as follows.

**Proposition 3.1.1.** The aggregate of premium amounts $\{A_i; i \in \mathbb{N}\}$, defined in (3.2), is the compound random variable having the expectation as follows.

$$EA_i = EN_iEX.$$

*Proof.* Assume that the sequence of premium sizes $\{X_{i,k}; k \in \mathbb{N}\}$ is a sequence of i.i.d. random variables and is independent of the process $\{N_i; i \in \mathbb{N}\}$, we have

$$\begin{aligned} EA_i &= E\left[\sum_{k=1}^{N_i} X_{i,k}\right] \\ &= E\left[E\left[\sum_{k=1}^{N_i} X_{i,k}\middle| N_i\right]\right] \\ &= E\left[\sum_{k=1}^{N_i} E\left[X_{i,k}\middle| N_i\right]\right] \\ &= E[N_iEX] \\ &= EN_iEX. \end{aligned}$$

□

Similar to the Proposition 3.1.1, we can derive the formulas of the expectations of aggregate of claim sizes and aggregate of surrender values as follows.

**Proposition 3.1.2.** The aggregate of claim sizes $\{B_i; i \in \mathbb{N}\}$ defined in (3.3) is the compound random variable having the expectation

$$EB_i = EN_i(q)EY.$$

**Proposition 3.1.3.** The aggregate of surrender valued $\{C_i; i \in \mathbb{N}\}$ defined in (3.4) is the compound random variable having the expectation

$$EC_i = EN_i(q)EZ.$$

In order to perform a clear profit of insurance company, it is common to assume that the net profit condition that is the expectation of the loss-profit process $S_n$ is greater than 0, written as $E[S_n] > 0$. In the following proposition, we will introduce the factor that satisfies the condition. The factor is called the positive relative safety loading.

**Proposition 3.1.4.** Under the net profit condition, the positive relative safety loading, denoted by $\theta$, can be defined as



$$\theta = \frac{Id + EN_iEX}{EN_i(p)EY + EN_i(q)EZ} - 1 > 0,$$

where

- the processes $\{N_i; i \in \mathbb{N}\}$, $\{N_i(p); i \in \mathbb{N}\}$ and $\{N_i(q); i \in \mathbb{N}\}$ are stationary processes,
- $EX \neq 0$ and $EY \neq 0$.

*Proof.* Under the net profit condition, we have

$$\begin{aligned}
0 &< E[S_n] \\
&= E\left[Idn + \sum_{i=1}^{n} A_i - \sum_{i=1}^{n} B_i - \sum_{i=1}^{n} C_i\right] \\
&= Idn + E\left[\sum_{i=1}^{n} A_i\right] - E\left[\sum_{i=1}^{n} B_i\right] - E\left[\sum_{i=1}^{n} C_i\right] \\
&= Idn + \sum_{i=1}^{n} E\left[A_i\right] - \sum_{i=1}^{n} E\left[B_i\right] - \sum_{i=1}^{n} E\left[C_i\right] \\
&= Idn + \sum_{i=1}^{n} EN_iEX - \sum_{i=1}^{n} EN_i(p)EY - \sum_{i=1}^{n} EN_i(q)EZ,
\end{aligned}$$

where we use Proposition 3.1.1 - 3.1.3 to obtain the last inequality.

Since $N_i$, $N_i(p)$ and $N_i(q)$ are stationary processes for $i \in \mathbb{N}$, then we get,

$$0 < Idn + nEN_iEX - nEN_i(p)EY - nEN_i(q)EZ.$$

For $EX \neq 0$ and $EY \neq 0$, we have

$$\frac{Id + EN_iEX}{EN_i(p)EY + EN_i(q)EZ} - 1 > 0.$$



□

## 3.2 Definition and properties of INMA(1) risk model

In this section, we give definition and properties for the risk model based on the INMA(1) process.

In the INMA(1) risk model considered in this chapter, we assume that each of the processes $\{N_i; i \in \mathbb{N}\}$, $\{N_i(p); i \in \mathbb{N}\}$ and $\{N_i(q); i \in \mathbb{N}\}$ follows an INMA(1) process described as follows.

The process of the number of premiums $\{N_i; i \in \mathbb{N}\}$ can be defined as

$$N_i = \alpha \circ \varepsilon_{i-1} + \varepsilon_i, \tag{3.6}$$

where $\alpha \in [0,1]$, where $\{\varepsilon_i\}_{i=1,2,\ldots}$ is the sequence of i.i.d. random variables following the Possion distribution with mean $\lambda$ and $\alpha \circ \varepsilon_{i-1}$ is the binomial thinning operator defined as

$$\alpha \circ \varepsilon_{i-1} = \sum_{j=1}^{\varepsilon_{i-1}} d_{i-1,j},$$

where $\{d_{i,j}\}_{j=1,2,\ldots}$ is the sequence of i.i.d. Bernoulli random variables with parameter $\alpha$ for all $i$ and is independent of $\varepsilon_i$.

The process of the number of claims $\{N_i(p); i \in \mathbb{N}\}$ can be defined as

$$N_i(p) = (\alpha p) \circ \gamma_{i-1} + \gamma_i, \tag{3.7}$$

where $\{\gamma_i\}_{i=1,2,\ldots}$ is the sequence of i.i.d. random variables following the Possion distribution with mean $\lambda$ and $(\alpha p) \circ \gamma_{i-1}$ is the binomial thinning operator defined as



$$(\alpha p) \circ \gamma_{i-1} = \sum_{j=1}^{\gamma_{i-1}} e_{i-1,j},$$

where $\{e_{i,j}\}_{j=1,2,\ldots}$ is the sequence of i.i.d. Bernoulli random variables with parameter $\alpha p$ for all $i$ and is independent of $\gamma_i$.

The process of the number of surrenders $\{N_i(q); i \in \mathbb{N}\}$ can be defined as

$$N_i(q) = (\alpha q) \circ \mu_{i-1} + \mu_i, \tag{3.8}$$

where $\{\mu_i\}_{i=1,2,\ldots}$ is the sequence of i.i.d. random variables following the Possion distribution with mean $\lambda$ and $(\alpha q) \circ \mu_{i-1}$ is the binomial thinning operator defined as

$$(\alpha q) \circ \mu_{i-1} = \sum_{j=1}^{\mu_{i-1}} f_{i-1,j},$$

where $\{f_{i,j}\}_{j=1,2,\ldots}$ is the sequence of i.i.d. Bernoulli random variables with parameter $\alpha q$ for all $i$ and is independent of $\mu_i$.

Next, we will derive the properties of the number of premiums $(N_i)$, the number of claims $(N_i(p))$ and the number of surrender $(N_i(q))$.

**Proposition 3.2.1.** Let $\{N_i; i \in \mathbb{N}\}$ be defined in (3.6). Then $\{N_i; i \in \mathbb{N}\}$ has the following properties, for $i \in \mathbb{N}$

(1) $G_{N_i}(z) = e^{\lambda(z-1)(\alpha+1)}$ for $z \in \mathbb{R}$,

(2) $N_i$ is stationary process,

(3) $E(N_i) = (1+\alpha)\lambda$,

(4) $\mathrm{Var}(N_i) = (1+\alpha)\lambda$,

(5) $\mathrm{Cov}(N_i, N_{i-k}) = \begin{cases} \alpha\lambda, & \text{if } k = 1 \\ 0, & \text{if } k > 1, \end{cases}$

(6) $Corr(N_i, N_{i-k}) = \begin{cases} \dfrac{\alpha}{1+\alpha}, & \text{if } k = 1 \\ 0, & \text{if } k > 1. \end{cases}$

*Proof.* To prove (1), we will consider the probability generating function of $\{N_i; i \in \mathbb{N}\}$. From Lemma 2.4.1 (c), we note that, for $z \in \mathbb{R}$

$$\begin{aligned} G_{N_i}(z) &= E\left[z^{N_i}\right] \\ &= E\left[z^{\alpha \circ \varepsilon_{i-1} + \varepsilon_i}\right] \\ &= E[z^{\alpha \circ \varepsilon_{i-1}}]E[z^{\varepsilon_i}] \\ &= E[((1-\alpha) + \alpha z)^{\varepsilon_{i-1}}]E[z^{\varepsilon_i}] \\ &= e^{\lambda((1-\alpha)+\alpha z - 1)} e^{\lambda(z-1)} \\ &= e^{\lambda(z-1)(\alpha+1)}, \end{aligned} \tag{3.9}$$

where we use the fact that $\{\varepsilon_i\}_{i=1,2,\ldots}$ is the sequence of Poisson i.i.d. random variables with mean $\lambda$.

To prove (2), from (3.9), we can see that $G_{N_i}(z)$ does not depended on $i$. Therefore, $G_{N_1}(z) = \ldots = G_{N_n}(z)$. Hence, $\{N_i; i \in \mathbb{N}\}$ is a stationary process.

To prove (3), we note from (3.6) that

$$\begin{aligned} E(N_i) &= E(\alpha \circ \varepsilon_{i-1} + \varepsilon_i) \\ &= E(\alpha \circ \varepsilon_{i-1}) + E(\varepsilon_i) \\ &= \alpha E(\varepsilon_{i-1}) + \lambda \\ &= \alpha\lambda + \lambda \\ &= (1+\alpha)\lambda. \end{aligned}$$

For (4), from (3.6)

$$\begin{aligned}
\text{Var}(N_i) &= \text{Var}(\alpha \circ \varepsilon_{i-1} + \varepsilon_i) \\
&= \text{Var}(\alpha \circ \varepsilon_{i-1}) + \text{Var}(\varepsilon_i) \\
&= \alpha(1-\alpha)E(\varepsilon_{i-1}) + \alpha^2\text{Var}(\varepsilon_{i-1}) + \text{Var}(\varepsilon_i) \qquad (3.10) \\
&= \alpha(1-\alpha)\lambda + \alpha^2\lambda + \lambda \\
&= (\alpha(1-\alpha) + \alpha^2 + 1)\lambda \\
&= (1+\alpha)\lambda, \qquad (3.11)
\end{aligned}$$

where we use Lemma 2.4.1 (d) to obtain (3.10) and use the fact that $\{\varepsilon_i\}_{i=1,2,\ldots}$ is a sequence of Poisson i.i.d. random variables with mean $\lambda$ to obtain (3.11), respectively.

To prove (5), we will consider into two cases which are $k = 1$ and $k > 1$.
For $k = 1$, we have

$$\begin{aligned}
\text{Cov}(N_i, N_{i-1}) &= \text{Cov}(\alpha \circ \varepsilon_{i-1} + \varepsilon_i, \alpha \circ \varepsilon_{i-2} + \varepsilon_{i-1}) \\
&= \text{Cov}(\alpha \circ \varepsilon_{i-1}, \alpha \circ \varepsilon_{i-2}) + \text{Cov}(\alpha \circ \varepsilon_{i-1}, \varepsilon_{i-1}) \\
&\quad + \text{Cov}(\varepsilon_i, \alpha \circ \varepsilon_{i-2}) + \text{Cov}(\varepsilon_i, \varepsilon_{i-1}) \\
&= \alpha\text{Cov}(\varepsilon_{i-1}, \varepsilon_{i-1}) \qquad (3.12) \\
&= \alpha\text{Var}(\varepsilon_{i-1}) \\
&= \alpha\lambda, \qquad (3.13)
\end{aligned}$$

where we use Lemma 2.4.1 (e) to obtain (3.12) and the fact that $\{\varepsilon_i\}_{i=1,2,\ldots}$ is a sequence of independent random variables to obtain (3.13).

For $k > 1$. Since $\{\varepsilon_i\}_{i=1,2,\dots}$ is the sequence of independent random variables, we have

$$\begin{aligned}\mathrm{Cov}(N_i, N_{i-k}) &= \mathrm{Cov}(\alpha \circ \varepsilon_{i-1} + \varepsilon_i, \alpha \circ \varepsilon_{i-k-1} + \varepsilon_{i-k}) \\ &= \mathrm{Cov}(\alpha \circ \varepsilon_{i-1}, \alpha \circ \varepsilon_{i-k-1}) + \mathrm{Cov}(\alpha \circ \varepsilon_{i-1}, \varepsilon_{i-k}) \\ &\quad + \mathrm{Cov}(\varepsilon_i, \alpha \circ \varepsilon_{i-k-1}) + \mathrm{Cov}(\varepsilon_i, \varepsilon_{i-k}) \\ &= 0.\end{aligned}$$

To prove (6), from Proposition 3.2.1 (4) and (5), we have

$$\begin{aligned}Corr(N_i, N_{i-k}) &= \frac{\mathrm{Cov}(N_i, N_{i-k})}{\sqrt{\mathrm{Var}(N_i)}\sqrt{\mathrm{Var}(N_{i-k})}} \\ &= \begin{cases} \dfrac{\alpha\lambda}{(1+\alpha)\lambda}, & \text{if } k = 1 \\ 0, & \text{if } k > 1, \end{cases} \\ &= \begin{cases} \dfrac{\alpha}{1+\alpha}, & \text{if } k = 1 \\ 0, & \text{if } k > 1. \end{cases}\end{aligned}$$

□



Similar to Proposition 3.2.1, we can obtain properties of the processes $\{N_i(p); i \in \mathbb{N}\}$ and $\{N_i(p); i \in \mathbb{N}\}$ presented in Proposition 3.2.2 and Proposition 3.2.3, respectively.

**Proposition 3.2.2.** The number of claim process $\{N_i(p); i \in \mathbb{N}\}$, defined in (3.7), has properties as follows.

(1) $G_{N_i(p)}(z) = e^{\lambda(z-1)(\alpha p+1)}$ for $z \in \mathbb{R}$,

(2) $N_i(p)$ is stationary process,

(3) $E(N_i(p)) = (1+\alpha p)\lambda$,

(4) $\mathrm{Var}(N_i(p)) = (1+\alpha p)\lambda$,

(5) $$\mathrm{Cov}(N_i(p), N_{i-k}(p)) = \begin{cases} \alpha p\lambda, & \text{if } k = 1 \\ 0, & \text{if } k > 1, \end{cases}$$

(6) $$Corr(N_i(p), N_{i-k}(p)) = \begin{cases} \dfrac{\alpha p}{1+\alpha p}, & \text{if } k = 1 \\ 0, & \text{if } k > 1. \end{cases}$$

**Proposition 3.2.3.** The number of surrenders process $\{N_i(q); i \in \mathbb{N}\}$, defined in (3.8), has properties as follows.

(1) $G_{N_i(q)}(z) = e^{\lambda(z-1)(\alpha q+1)}$ for $z \in \mathbb{R}$,

(2) $N_i(q)$ is stationary process,

(3) $E(N_i(q)) = (1+\alpha q)\lambda$,

(4) $\mathrm{Var}(N_i(q)) = (1+\alpha q)\lambda$,

(5) $$\mathrm{Cov}(N_i(q), N_{i-k}(q)) = \begin{cases} \alpha q\lambda, & \text{if } k = 1 \\ 0, & \text{if } k > 1, \end{cases}$$

(6) $$Corr(N_i(q), N_{i-k}(q)) = \begin{cases} \dfrac{\alpha q}{1+\alpha q}, & \text{if } k = 1 \\ 0, & \text{if } k > 1. \end{cases}$$

The following theorem shows the joint generating function of INMA(1) process.

**Theorem 3.2.1.** Let INMA(1) process $\{N_i; i \in \mathbb{N}\}$ be defined in (3.6). The joint generating function of $\{N_i; i = 1, 2, ..., n\}$ can be written as follows. For $n \in \mathbb{N}$,

$$G_{N_1,...,N_n}(t_1, ..., t_n) = e^{\lambda\alpha(t_1-1)} e^{\lambda[(1-\alpha)t_1+\alpha t_1 t_2 - 1]} e^{\lambda[(1-\alpha)t_2+\alpha t_2 t_3 - 1]} \cdots e^{\lambda[(1-\alpha)t_{n-1}+\alpha t_{n-1} t_n - 1]} \times e^{\lambda(t_n-1)},$$

where $(t_1, ..., t_n) \in \mathbb{R}^n$.

*Proof.* Consider the joint generating function of $\{N_i; i = 1, 2, ..., n\}$ as follows.

$$\begin{aligned} G_{N_1,...,N_n}(t_1, \ldots, t_n) &= E[t_1^{N_1} \ldots t_n^{N_n}] \\ &= E\left[t_1^{\sum_{j=1}^{\varepsilon_0} d_{0j}+\varepsilon_1} t_2^{\sum_{j=1}^{\varepsilon_1} d_{1j}+\varepsilon_2} \cdots t_n^{\sum_{j=1}^{\varepsilon_{n-1}} d_{n-1,j}+\varepsilon_n}\right] \\ &= E\left[t_1^{\sum_{j=1}^{\varepsilon_0} d_{0j}} t_1^{\varepsilon_1} t_2^{\sum_{j=1}^{\varepsilon_1} d_{1j}} t_2^{\varepsilon_2} \cdots t_n^{\sum_{j=1}^{\varepsilon_{n-1}} d_{n-1,j}} t_n^{\varepsilon_n}\right] \\ &= E\left[t_1^{\sum_{j=1}^{\varepsilon_0} d_{0j}}\right] E\left[t_1^{\varepsilon_1} t_2^{\sum_{j=1}^{\varepsilon_1} d_{1j}}\right] \cdots E\left[t_{n-1}^{\varepsilon_{n-1}} t_n^{\sum_{j=1}^{\varepsilon_{n-1}} d_{n-1,j}}\right] E\left[t_n^{\varepsilon_n}\right], \end{aligned} \tag{3.14}$$

where we use the fact that $\{\varepsilon_i\}_{i=1,2,...}$ are independent to obtain the last equation. For the first term of (3.14), we have

$$\begin{aligned} E\left[t_1^{\sum_{j=1}^{\varepsilon_0} d_{1j}}\right] &= G_{\varepsilon_0}(1 - \alpha + \alpha t_1) \\ &= e^{\lambda(1-\alpha+\alpha t_1 - 1)} \\ &= e^{\lambda\alpha(t_1-1)}. \end{aligned} \tag{3.15}$$

For the last term, from (3.14),

$$E[t_n^{\varepsilon_n}] = e^{\lambda(t_n-1)}. \tag{3.16}$$

For the other terms in (3.14), we note that, for $k = 1, 2, ..., n-1$,

$$
\begin{aligned}
E\left[t_{k-1}^{\varepsilon_{k-1}} t_k^{\sum_{j=1}^{\varepsilon_{k-1}} d_{k-1,j}}\right] &= E\left[E\left[t_{k-1}^{\varepsilon_{k-1}} t_k^{\sum_{j=1}^{\varepsilon_{k-1}} d_{k-1,j}} \middle| \varepsilon_{k-1}\right]\right] \\
&= E\left[t_{k-1}^{\varepsilon_{k-1}} E\left[t_k^{\sum_{j=1}^{\varepsilon_{k-1}} d_{kj}} \middle| \varepsilon_{k-1}\right]\right] \\
&= E\left[t_{k-1}^{\varepsilon_{k-1}} (G_d(t_k))^{\varepsilon_{k-1}}\right] \\
&= E\left[(t_{k-1} G_d(t_k))^{\varepsilon_{k-1}}\right] \\
&= E\left[e^{(\ln(t_{k-1} G_d(t_k))) \cdot \varepsilon_{k-1}}\right] \\
&= M_{\varepsilon_{k-1}}(\ln(t_{k-1} G_d(t_k))) \\
&= e^{\lambda(t_{k-1} G_d(t_k) - 1)} \qquad (3.17) \\
&= e^{\lambda(t_{k-1}(1-\alpha+\alpha t_k)-1)} \qquad (3.18) \\
&= e^{\lambda[(1-\alpha)t_{k-1} + \alpha t_{k-1} t_k - 1]}, \qquad (3.19)
\end{aligned}
$$

where we use the fact that $\{\varepsilon_k\}_{k=1,2,...}$ is a sequence of i.i.d. Poisson random variables with mean $\lambda$ to derive (3.17) and $\{d_{i,j}\}_{i,j=1,2,...}$ is the sequence of i.i.d. Bernoulli random variables with parameter $\alpha$ to derive (3.18), respectively.

Substituting (3.15), (3.16) and (3.19) into (3.14), we get



$$
\begin{aligned}
G_{N_1,...,N_n}(t_1, ..., t_n) &= e^{\lambda\alpha(t_1-1)} e^{\lambda[(1-\alpha)t_1+\alpha t_1 t_2 - 1]} e^{\lambda[(1-\alpha)t_2+\alpha t_2 t_3 - 1]} \cdots e^{\lambda[(1-\alpha)t_{n-1}+\alpha t_{n-1} t_n - 1]} \\
&\quad \times e^{\lambda(t_n - 1)}.
\end{aligned}
$$

□

## 3.3 Adjustment coefficient

In this section, we first derive the adjustment coefficient function for the INMA(1) risk model. We then obtain the adjustment coefficient to approximate the ruin probability.

**Theorem 3.3.1.** The adjustment coefficient function of the risk model defined in (3.1) is given by

$$g(r) = -rId + \lambda\big((1-\alpha)M_X(-r) + \alpha M_X^2(-r) + (1-\alpha p)M_Y(r) + \alpha p M_Y^2(r) + (1-\alpha q)M_Z(r) + \alpha q M_Z^2(r) - 3\big).$$

*Proof.* From(2.3) the adjustment coefficient function is defined as

$$g(r) = \lim_{n\to\infty} \frac{1}{n} \ln E(e^{-rS_n}). \tag{3.20}$$

Therefore, we will first derive $E\left[e^{-rS_n}\right]$.

From (3.5) and the fact that $\{A_i; i \in \mathbb{N}\}$, $\{B_i; i \in \mathbb{N}\}$ and $\{C_i; i \in \mathbb{N}\}$ are independent, we have

$$\begin{aligned} E[e^{-rS_n}] &= E\left[e^{-r\left(Idn + \sum_{i=1}^{n} A_i - \sum_{i=1}^{n} B_i - \sum_{i=1}^{n} C_i\right)}\right] \\ &= E\left[e^{-rIdn}\right] \cdot E\left[e^{-r\sum_{i=1}^{n} A_i}\right] \cdot E\left[e^{r\sum_{i=1}^{n} B_i}\right] \cdot E\left[e^{r\sum_{i=1}^{n} C_i}\right]. \end{aligned} \tag{3.21}$$

For the second term of (3.21), note that the aggregate of premium amounts $\{A_i; i \in \mathbb{N}\}$ defined in (3.2). Then

$$
\begin{aligned}
E\left[e^{-r\sum_{i=1}^{n}A_i}\right] &= E\left[e^{-rA_1}\cdots e^{-rA_n}\right] \\
&= E\left[e^{-r\sum_{k=1}^{N_1}X_{1,k}}\cdots e^{-r\sum_{k=1}^{N_n}X_{n,k}}\right] \\
&= E\left[E\left[e^{-r\sum_{k=1}^{N_1}X_{1,k}}\cdots e^{-r\sum_{k=1}^{N_n}X_{n,k}}|N_1,\ldots,N_n\right]\right] \\
&= E[\left(Er^{-rX}\right)^{N_1}\cdots\left(Er^{-rX}\right)^{N_n}|N_1,\ldots,N_n] \\
&= G_{N_1,\ldots,N_n}\left(M_X(-r),\ldots,M_X(-r)\right).
\end{aligned}
$$

From Theorem 3.2.1, we obtain

$$
\begin{aligned}
E\left[e^{-r\sum_{i=1}^{n}A_i}\right] &= e^{\lambda\alpha(M_X(-r)-1)}e^{\lambda(n-1)[(1-\alpha)M_X(-r)+\alpha M_X^2(-r)-1]}e^{\lambda(M_X(-r)-1)} \\
&= \exp\{\lambda(\alpha M_X(-r)-\alpha+(n-1)M_X(-r)-\alpha(n-1)M_X(-r)) \\
&\quad +\lambda(\alpha(n-1)M_X^2(-r)-(n-1)+M_X(-r)-1)\} \\
&= \exp\{\lambda[(2\alpha+n-\alpha n)M_X(-r)-(n+\alpha)+\alpha(n-1)M_X^2(-r)]\} \\
&= \exp\left\{\lambda(n+\alpha)\left[\frac{(n(1-\alpha)+2\alpha)M_X(-r)+\alpha(n-1)M_X^2(-r)}{n+\alpha}-1\right]\right\}.
\end{aligned} \tag{3.22}
$$

By the same technique, the last two terms of (3.21) can be obtained as follows

$$
\begin{aligned}
E\left[e^{r\sum_{i=1}^{n}B_i}\right] &= G_{N_1(p),\ldots,N_n(p)}(M_Y(r),...,M_Y(r)) \\
&= \exp\left\{\lambda(n+\alpha p)\left[\frac{(n(1-\alpha p)+2\alpha p)M_Y(r)+\alpha p(n-1)M_Y^2(r)}{n+\alpha p}-1\right]\right\},
\end{aligned} \tag{3.23}
$$

and

$$
\begin{aligned}
E\left[e^{r\sum_{i=1}^{n}C_i}\right] &= G_{N_1(q),\ldots,N_n(q)}(M_Z(r),...,M_Z(r)) \\
&= \exp\left\{\lambda(n+\alpha q)\left[\frac{(n(1-\alpha q)+2\alpha q)M_Z(r)+\alpha q(n-1)M_Z^2(r)}{n+\alpha q}-1\right]\right\}.
\end{aligned} \tag{3.24}
$$

Substitute (3.22) - (3.24) into (3.21), we have

$$\begin{aligned}
E[e^{-rS_n}] &= E\left[e^{-rIdn}\right]\cdot E\left[e^{-r\sum_{i=1}^n A_i}\right]\cdot E\left[e^{r\sum_{i=1}^n B_i}\right]\cdot E\left[e^{r\sum_{i=1}^n C_i}\right]\\
&= \exp\Bigg\{ -rIdn\\
&\qquad + \lambda(n+\alpha)\left[\frac{(n(1-\alpha)+2\alpha)M_X(-r)+\alpha(n-1)M_X^2(-r)}{n+\alpha}-1\right]\\
&\qquad + \lambda(n+\alpha p)\left[\frac{(n(1-\alpha p)+2\alpha p)M_Y(r)+\alpha p(n-1)M_Y^2(r)}{n+\alpha p}-1\right]\\
&\qquad + \lambda(n+\alpha q)\left[\frac{(n(1-\alpha q)+2\alpha q)M_Z(r)+\alpha q(n-1)M_Z^2(r)}{n+\alpha q}-1\right]\Bigg\}.
\end{aligned} \tag{3.25}$$

Then, we consider the logarithm of the equation (3.25) as follows.

$$\begin{aligned}
\frac{1}{n}\ln E[e^{-rS_n}] &= \frac{1}{n}\ln E\left[\exp\left\{-r\left(Idn+\sum_{i=1}^n A_i-\sum_{i=1}^n B_i-\sum_{i=1}^n C_i\right)\right\}\right]\\
&= \frac{1}{n}\Bigg\{ -rIdn\\
&\qquad + \lambda(n+\alpha)\left[\frac{(n(1-\alpha)+2\alpha)M_X(-r)+\alpha(n-1)M_X^2(-r)}{n+\alpha}-1\right]\\
&\qquad + \lambda(n+\alpha p)\left[\frac{(n(1-\alpha p)+2\alpha p)M_Y(r)+\alpha p(n-1)M_Y^2(r)}{n+\alpha p}-1\right]\\
&\qquad + \lambda(n+\alpha q)\left[\frac{(n(1-\alpha q)+2\alpha q)M_Z(r)+\alpha q(n-1)M_Z^2(r)}{n+\alpha q}-1\right]\Bigg\}\\
&= -rId\\
&\qquad + \lambda(1-\alpha)M_X(-r)+\frac{2\lambda\alpha M_X(-r)}{n}+\frac{\lambda\alpha(n-1)M_X^2(-r)}{n}-\frac{\lambda(n+\alpha)}{n}\\
&\qquad + \lambda(1-\alpha p)M_Y(r)+\frac{2\lambda\alpha pM_Y(r)}{n}+\frac{\lambda\alpha p(n-1)M_Y^2(r)}{n}-\frac{\lambda(n+\alpha p)}{n}\\
&\qquad + \lambda(1-\alpha q)M_Z(r)+\frac{2\lambda\alpha qM_Z(r)}{n}+\frac{\lambda\alpha q(n-1)M_Z^2(r)}{n}-\frac{\lambda(n+\alpha q)}{n}.
\end{aligned}$$

Therefore, we can obtain the equation (3.20) as follows.

$$
\begin{aligned}
g(r) &= \lim_{n\to\infty} \frac{1}{n} \ln E[e^{-rS_n}] \\
&= -rId + \lambda(1-\alpha)M_X(-r) + \lambda\alpha M_X^2(-r) - \lambda + \lambda(1-\alpha p)M_Y(r) \\
&\quad + \lambda\alpha p M_Y^2(r) - \lambda + \lambda(1-\alpha q)M_Z(r) + \lambda\alpha q M_Z^2(r) - \lambda \\
&= -rId + \lambda\big((1-\alpha)M_X(-r) + \alpha M_X^2(-r) + (1-\alpha p)M_Y(r) + \alpha p M_Y^2(r) \\
&\quad + (1-\alpha q)M_Z(r) + \alpha q M_Z^2(r) - 3\big).
\end{aligned}
$$

□

Next, we will show the uniqueness of positive solution of adjustment coefficient equation.

**Proposition 3.3.1.** The adjustment equation $g(r) = 0$ has the unique positive solution $R$ which is called the adjustment coefficent.

*Proof.* To show that $g(r) = 0$ has the unique positive solution, we will show the function $g(\cdot)$ has following properties



(1) $g(0) = 0$,

(2) $g'(0) < 0$,

(3) $g''(r) > 0$ for all $r \in (0, \infty)$,

(4) $\lim_{r\to\infty} g(r) = \infty$.

To prove (1), we can see that

$$g(0) = \lambda(1+1+1) + \lambda\alpha(1-1+p-p+q-q) - 3\lambda = 0.$$

To prove (2), we note that

$$\begin{aligned} g'(r) &= -Id + \lambda(-M'_X(-r) + M'_Y(r) + M'_Z(r)) \\ &\quad + \lambda\alpha[-2M_X(-r)M'_X(-r) + M'_X(-r) + 2pM_Y(r)M'_Y(r) - pM'_Y(r)] \\ &\quad + \lambda\alpha[2qM_Z(r)M'_Z(r) - qM'_Z(r)]. \end{aligned} \tag{3.26}$$

Substituting $r = 0$ into (3.26), we get

$$\begin{aligned} g'(0) &= -Id + \lambda(-EX + EY + EZ) \\ &\quad + \lambda\alpha(-2EX + EX + 2pEY - pEY + 2qEZ - EZ) \\ &= -Id + \lambda(-EX + EY + EZ) + \lambda\alpha(-EX + pEY + qEZ) \\ &= -Id - \lambda(1+\alpha)EX + \lambda(1+\alpha p)EY + \lambda(1+\alpha q)EZ \\ &< 0, \end{aligned}$$

where we use Propositions 3.1.4 and 3.2.1 – 3.2.3 (2) to obtain the last inequality. Hence, $g'(0) < 0$.

To prove (3), we note that



$$\begin{aligned} g''(r) &= \lambda[M''_X(-r) + M''_Y(r) + M''_Z(r)] \\ &\quad + \lambda\alpha[2(M_X(-r)M''_X(-r) + (M'_X(-r))^2) - M''_X(-r)] \\ &\quad + \lambda\alpha[2p(M_Y(r)M''_Y(r) + (M'_Y(r))^2) - pM''_Y(r)] \\ &\quad + \lambda\alpha[2q(M_Z(r)M''_Z(r) + (M'_Z(r))^2) - qM''_Z(r)] \\ &= \lambda[(2\alpha M_X(-r) - \alpha + 1)M''_X(-r) + \alpha(M'_X(-r))^2] \\ &\quad + \lambda[(2\alpha pM_Y(r) - \alpha p + 1)M''_Y(r) + \alpha p(M'_Y(r))^2] \\ &\quad + \lambda[(2\alpha qM_Z(r) - \alpha q + 1)M''_Z(r) + \alpha q(M'_Z(r))^2]. \end{aligned} \tag{3.27}$$

Since the moment generating function is always positive and $0 < \alpha < 1$. So, the term of $2\alpha M_X(-r)-\alpha+1$ is greater than 0. For $0 < p, q < 1$ and $0 < p+q < 1$, we get that the terms of $2\alpha p M_Y(r) - \alpha p + 1$ and $2\alpha q M_Z(r) - \alpha q + 1$ are greater than 0. So, we can conclude that the right hand side of (3.27) is greater than 0. Therefore $g''(r) > 0$.

To prove (4), from Theorem 3.3.1, the adjustment coefficient function is

$$\begin{aligned} g(r) &= -rId + \lambda\big((1-\alpha)M_X(-r) + \alpha M_X^2(-r) + (1-\alpha p)M_Y(r) + \alpha p M_Y^2(r) \\ &\quad + (1-\alpha q)M_Z(r) + \alpha q M_Z^2(r) - 3\big) \\ &= -rId + \lambda\big((1-\alpha+\alpha M_X(-r))M_X(-r) + (1-\alpha p+\alpha p M_Y(r))M_Y(r) \\ &\quad + (1-\alpha q+\alpha q M_Z(r))M_Z(r) - 3\big). \end{aligned} \tag{3.28}$$

From the right hand side of (3.28), we can see that the term of $1-\alpha+\alpha M_X(-r)$, $1-\alpha p+\alpha p M_Y(r)$ and $1-\alpha q+\alpha q M_Z(r)$ are always greater than 0 for $0 < \alpha < 1$. Then, we use the fact that the moment generating function of the right hand side of (3.28), determined by the term of $M_Y(r)$ and $M_Z(r)$, perform as exponential terms and the exponential growth is faster than polynomial growth. Hence, $\lim_{r\to\infty} g(r) = \infty$.

□

## 3.4 Numerical example

In this section, we will apply numerical example to study the effect of the ruin probability and the value at risk comparing with the parameters of premiums, claims and surrenders by using Python and R programming.

In section, we will study the behavior of the ruin probability and the value at risk of the risk model by assuming that the premiums sizes ($X$), the claim sizes ($Y$) and the surrender values ($Z$) follow exponential distributions. The sequence of premium sizes $X = \{X_{i,k}\}_{i,k=1,2,...}$ is a sequence of i.i.d. random variables which are exponentially distributed with mean $\frac{1}{\beta_X}$. The sequence of claim sizes $Y = \{Y_{i,k}\}_{i,k=1,2,...}$ is a sequence of i.i.d. random variables which are exponentially distributed with mean $\frac{1}{\beta_Y}$. The sequence of surrender values $Z = \{Z_{i,k}\}_{i,k=1,2,...}$ is a sequence of i.i.d. random variables which are exponentially distributed with mean $\frac{1}{\beta_Z}$, respectively.

Therefore the moment generating functions of $X$, $Y$ and $Z$ are defined as $M_X(-r) = \frac{\beta_X}{\beta_X + r}$, $M_Y(r) = \frac{\beta_Y}{\beta_Y - r}$ and $M_Z(r) = \frac{\beta_Z}{\beta_Z - r}$, respectively, for $\beta_Y, \beta_Z > r$.

### 3.4.1 Numerical example for the ruin probability

In this section, we study the effect of the ruin probability against the terms of premiums, claims and surrenders.



The approximation ruin probability, defined in (2.2), be written as

$$\Psi(u) \simeq e^{-Ru},$$

where $R$ is the adjustment coefficient.

To approximate ruin probability, we will first calculate the adjustment coefficient from finding the unique positive solution of the adjustment coefficient equation $g(r) = 0$ as follows.

From Theorem 3.3.1, we will consider the function $g(r)$ in the case of the premium size, claim size and surrender values follow the exponential distribution.

$$
\begin{aligned}
0 &= g(r) \\
&= -rId + \lambda\big((1-\alpha)M_X(-r) + \alpha M_X^2(-r) + (1-\alpha p)M_Y(r) + \alpha p M_Y^2(r) \\
&\quad + (1-\alpha q)M_Z(r) + \alpha q M_Z^2(r) - 3\big) \\
&= -rId + \lambda\Bigg((1-\alpha)\left(\frac{\beta_X}{\beta_X + r}\right) + \alpha\left(\frac{\beta_X}{\beta_X + r}\right)^2 \\
&\quad + (1-\alpha p)\left(\frac{\beta_Y}{\beta_Y - r}\right) + \alpha p\left(\frac{\beta_Y}{\beta_Y - r}\right)^2 \\
&\quad + (1-\alpha q)\left(\frac{\beta_Z}{\beta_Z - r}\right) + \alpha q\left(\frac{\beta_Z}{\beta_Z - r}\right)^2 - 3\Bigg),
\end{aligned}
$$

where $0 < p, q < 1$ and $0 < p + q < 1$, where $r < \min\{\beta_Y, \beta_Z\}$.

Next, we will study the trend of ruin probability by varying various parameters of premium size, claim size, surrender values and investment in Section 3.4.1.1. In Section 3.4.1.2, we will study the trend of ruin probability by varying various parameters of probabilities of claims and surrenders.



### 3.4.1.1 Effects from premiums size, claim size and surrender value

In insurance company, income and expenses are quite important to the company financial stability. In our risk model, the income are determined by the premium size and investment and the expenses are determined by the claim size and surrender value. So, this section will study the trends of ruin probability in terms of means of premium size $\left(\frac{1}{\beta_X}\right)$, claim size $\left(\frac{1}{\beta_Y}\right)$ and surrender values $\left(\frac{1}{\beta_Z}\right)$ by varying the parameter $\beta_X$, $\beta_Y$ and $\beta_Z$, respectively. Moreover, we will also study the ruin probability in term of investment by varying parameter $I$.

For this section, we consider the following values of parameters $I = 10, d = 0.2, \lambda = 2, \alpha = 0.25, p = 0.4, q = 0.07$ and the initial surplus $u = 12$.

Scenario 3.1 : The trend of ruin probability in terms of the parameter $\beta_X$ where the parameters $\beta_Y$ and $\beta_Z$ are fixed ($\beta_Y = \beta_Z = 1.5$). In this scenario, we consider different values of $\beta_X$ which are 0.5, 0.75, 1.0, 1.5 and 2.0, respectively. The values of the upper bound of the ruin probability are given in Table 3.1. The corresponding plot is presented in Figure 3.1.

**Table 3.1:** Parameter $\beta_X \in [0.5, 2]$ and their upper bound of ruin probability

| $\beta_X$ | 0.5 | 0.75 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| Upper bound | 0.002942 | 0.006777 | 0.014604 | 0.056045 | 0.171677 |

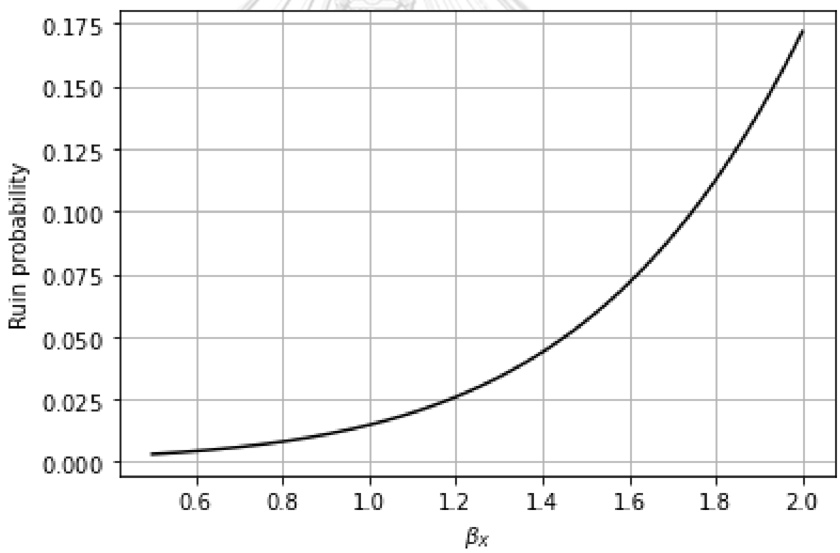


**Figure 3.1:** Trend of the ruin probability when $\beta_X$ increases

From Table 3.1 and Figure 3.1, we can see that the ruin probability increases when $\beta_X$ increases. That is the ruin probability increases when the mean of premium size decreases.

Scenario 3.2 : The trend of ruin probability in terms of the parameter $\beta_Y$ where the parameters $\beta_X$ and $\beta_Z$ are fixed ($\beta_X = 0.5$ and $\beta_Z = 1.5$). In this scenario, we consider different values of $\beta_Y$ which are 0.5, 0.75, 1.0, 1.5 and 2.0, respectively. The values of the upper bound of the ruin probability are given in Table 3.2. The corresponding plot is presented in Figure 3.2.

**Table 3.2:** Parameter $\beta_Y \in [0.5, 2]$ and their upper bound of ruin probability

| $\beta_Y$ | 0.5 | 0.75 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| Upper bound | 0.515049 | 0.118524 | 0.029798 | 0.002830 | 0.000605 |

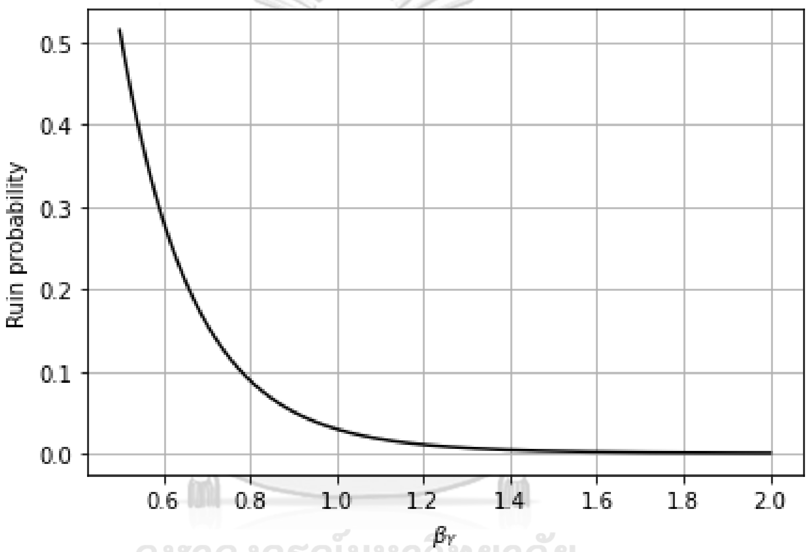



**Figure 3.2:** Trend of the ruin probability when $\beta_Y$ increases

Table 3.2 and Figure 3.2 show that the ruin probability decreases as parameter $\beta_Y$ increases. It means that the ruin probability decreases when the mean of claim size decreases.

Scenario 3.3 : The trend of ruin probability in terms of the parameter $\beta_Z$ where the parameters $\beta_X$ and $\beta_Y$ are fixed ($\beta_X = 0.5$ and $\beta_Y = 1.5$). In this scenario, we consider different values of $\beta_Z$ which are 0.5, 0.75, 1.0, 1.5 and 2.0, respectively. The values of the upper bound of the ruin probability are given in Table 3.3. The corresponding plot is presented in Figure 3.3.

**Table 3.3:** Parameter $\beta_Z \in [0.5, 10]$ and their upper bound of ruin probability

| $\beta_Z$ | 0.5 | 0.75 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| Upper bound | 0.478468 | 0.110136 | 0.028787 | 0.003438 | 0.000897 |

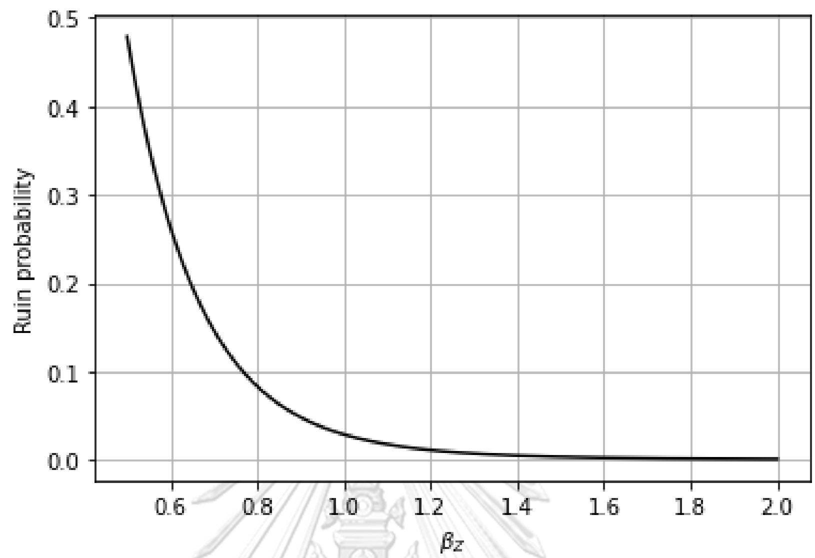


**Figure 3.3:** Trend of the ruin probability when $\beta_Z$ increases

From Table 3.3 and Figure 3.3, we can conclude that the ruin probability decreases when parameter $\beta_Z$ increases. It means that the ruin probability decreases when the mean of surrender value decreases.



Scenario 3.4 : The trend of ruin probability in terms of the investment $I$ where the parameters $\beta_X$, $\beta_Y$ and $\beta_Z$ are fixed ($\beta_X = 0.5$ and $\beta_Y = \beta_Z = 1.5$). In this scenario, we consider different values of investment which are 1.0, 3.0, 5.0, 8.0 and 10.0, respectively. The values of the upper bound of the ruin probability are given in Table 3.4. The corresponding plot is presented in Figure 3.4.

**Table 3.4:** Parameter $I \in [1, 10]$ and their upper bound of ruin probability

| $\beta_Z$ | 1.0 | 3.0 | 5.0 | 8.0 | 10.0 |
|---|---|---|---|---|---|
| Upper bound | 0.059467 | 0.031536 | 0.016207 | 0.005807 | 0.002942 |

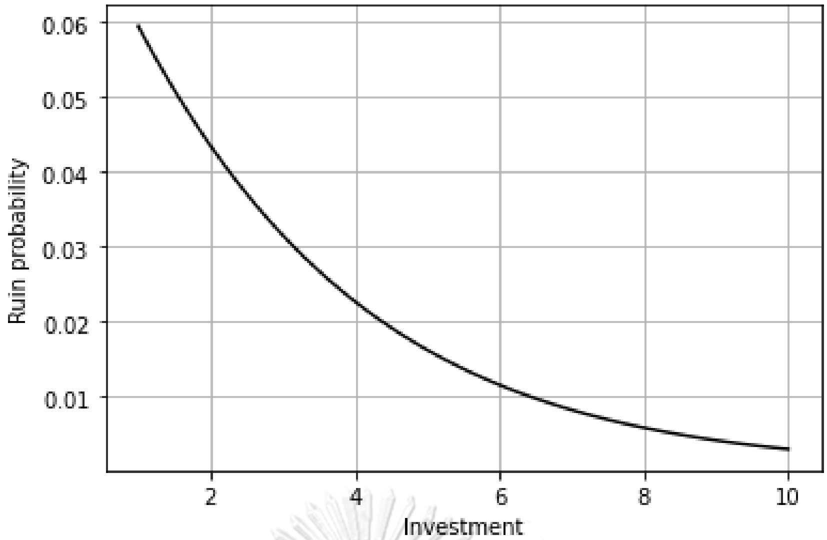


**Figure 3.4:** Trend of the ruin probability when $I$ increases

Table 3.4 and Figure 3.4 show that the ruin probability decreases as the investment increases. It means that the more the insurance company invests in financial markets, the smaller value of ruin probability.

As we know that the closer the ruin probability to 1, the greater the possibility that the insurance company will go bankrupt. From the result in Scenario 3.1 - 3.4, we can see that the ruin probability decreases as the mean of premiums and investment, which are income of the model, increase and the means of claims and surrender, which are expenses of the model, decrease. Therefore, in order to control the risk of the bankrupt, the company should increase the premium values or reduce the payments from claims and surrender.

#### 3.4.1.2 Effects from the probabilities of claims and surrenders

In this section, we will study the trend of ruin probability in terms of the probability of claims, denoted by parameter $p$ and the probability of surrenders, denoted by parameter $q$.

For this section, we set the value $I = 10, d = 0.2, \lambda = 2, \alpha = 0.25, \beta_X = 1, \beta_Y = 2, \beta_Z = 1.5$ and the initial surplus $u = 12$.

Scenario 3.5 : The trend of ruin probability in terms of probability of claims $p$ when $q$ is fixed ($q = 0.04$). In this scenario, we consider different values of $p$ which are 0.001, 0.01, 0.01, 0.5 and 0.9, respectively. The values of the upper bound of the ruin probability are given in Table 3.5. The corresponding plot is presented in Figure 3.5.

**Table 3.5:** Parameter $p \in (0, 1)$ and their upper bound of ruin probability

| $p$ | 0.001 | 0.01 | 0.1 | 0.5 | 0.9 |
|---|---|---|---|---|---|
| Upper bound | 0.001437 | 0.001454 | 0.001635 | 0.002689 | 0.004278 |

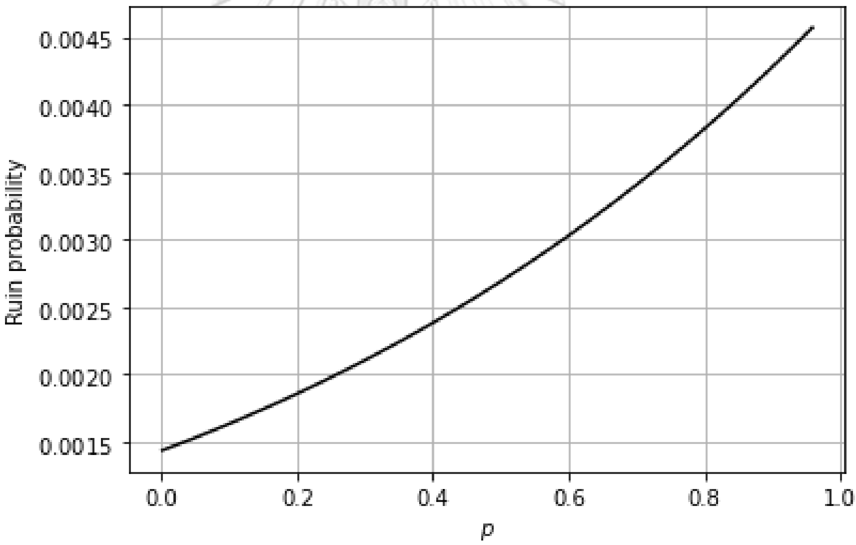


**Figure 3.5:** Trend of the ruin probability when $p$ increases

From Table 3.5 and Figure 3.5, we can see that the ruin probability increases as the probability $p$ increases. It means that the more claims occur, the higher value of ruin probability.

Scenario 3.6 : We consider the trend ruin probability in terms of probability of surrenders $q$ when $p$ is fixed ($p = 0.04$). In this scenario, we consider different values of $q$ which are 0.001, 0.01, 0.01, 0.5 and 0.9, respectively. The values of the upper bound of the ruin probability are given in Table 3.6. The corresponding plot is presented in Figure 3.6.

**Table 3.6:** Parameter $q \in (0, 1)$ and their upper bound of ruin probability

| $q$ | 0.001 | 0.01 | 0.1 | 0.5 | 0.9 |
|---|---|---|---|---|---|
| Upper bound | 0.001383 | 0.001412 | 0.001729 | 0.003813 | 0.007421 |

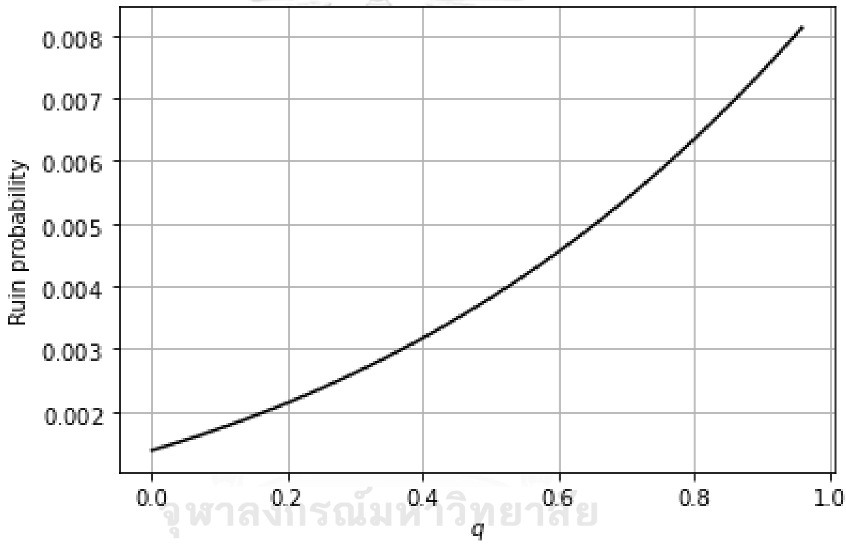




**Figure 3.6:** Trend of the ruin probability when $q$ increases

Table 3.6 and Figure 3.6 show that the ruin probability increases when parameter $q$ increases. It means that the more surrenders occur, the higher value of ruin probability.

From the result of Scenario 3.5 and 3.6, we can see that when either parameters $p$ or $q$ increases, the upper bound of ruin probability also increases. This suggests that the insurance company will be in the high risk when either the probability of claims or surrenders increases.

### 3.4.2 Numerical example for the value at risk

In this section, we will study value at risk (VaR) which is a risk measure measuring the risk of loss-profit process in our risk model.

The value at risk at the confidence level $\omega$ for the INMA(1) risk model, denoted by $\text{VaR}_\omega(S_n)$, is the $\omega$-quantile of the distribution of the loss-profit $S_n$ of the risk model. The $\text{VaR}_\omega(S_n)$ can be written as

$$\text{VaR}_\omega(S_n) = \inf\{k \in \mathbb{R} | F_{S_n}(k) > \omega\}, \tag{3.29}$$

where $F_{S_n}(k)$ be the cumulative distribution function of $S_n$. For our model, the loss-profit process $S_n$, define in (3.5), can be express as

$$\begin{aligned} S_n &= Idn + \sum_{i=1}^{n} A_i - \sum_{i=1}^{n} B_i - \sum_{i=1}^{n} C_i \\ &= Idn + \sum_{i=1}^{n} \sum_{k=1}^{N_i} X_{i,k} - \sum_{i=1}^{n} \sum_{k=1}^{N_i(p)} Y_{i,k} - \sum_{i=1}^{n} \sum_{k=1}^{N_i(q)} Z_{i,k}, \end{aligned}$$

where $\{N_i; i \in \mathbb{N}\}$, $\{N_i(p); i \in \mathbb{N}\}$ and $\{N_i(q); i \in \mathbb{N}\}$ follow INMA(1) model, defined in (3.6) - (3.8), respectively.



From (3.29), we can see that we need to know the distribution of $S_n$ in order to obtain the value at risk. However, it is difficult to obtain the distribution of $S_n$. Therefore, we will apply the Fast Fourier Transform (FFT) algorithm, proposed by Gray and Pitts (2012), to approximate the distribution of $S_n$.

The characteristic function of $S_n$, denoted by $\phi_{S_n}(r)$, can be written as follow, Similar to provide $E[e^{-rS_n}]$ in the proof of Theorem 3.3.1, we have

$$\begin{aligned}\phi_{S_n}(r) &= E[e^{irS_n}] \\ &= E[e^{irIdn}] \cdot G_{N_1,...,N_n}(\phi_X(r), ..., \phi_X(r)) \cdot G_{N_1(p),...,N_n(p)}(\phi_Y(-r), ..., \phi_X(-r)) \\ &\quad \cdot G_{N_1(q),...,N_n(q)}(\phi_Z(-r), ..., \phi_Z(-r)),\end{aligned}$$

where

(1) $G_{N_1,...,N_n}(\phi_X(r), ..., \phi_X(r))$
$$= \exp\left\{\lambda\left[(n(1-\alpha)+2\alpha)\phi_X(r)+\alpha(n-1)\phi_X^2(r)-n-\alpha\right]\right\},$$

(2) $G_{N_1(p),...,N_n(p)}(\phi_Y(-r), ..., \phi_X(-r))$
$$= \exp\left\{\lambda\left[(n(1-\alpha p)+2\alpha p)\phi_Y(-r)+\alpha p(n-1)\phi_Y^2(-r)-n-\alpha p\right]\right\},$$

and

(3) $G_{N_1(q),...,N_n(q)}(\phi_Z(-r), ..., \phi_Z(-r))$
$$= \exp\left\{\lambda\left[(n(1-\alpha q)+2\alpha q)\phi_Z(-r)+\alpha q(n-1)\phi_Z^2(-r)-n-\alpha q\right]\right\}.$$



Next, we will study the trend of the value at risk by varying various parameters of claim sizes and surrender values.

#### 3.4.2.1 Effects from claim size and surrender values

Since the value at risk focuses on the financial losses. Then we see the behavior of the value at risk against the loss parameters, which are the claim size and surrender value. In this section, we study the trends of the value at risk comparing with means of claims $\left(\frac{1}{\beta_Y}\right)$ and surrenders $\left(\frac{1}{\beta_Z}\right)$ by varying the parameters $\beta_Y$ and $\beta_Z$, respectively.

For this section, we set the values of parameters $I = 10, d = 0.2, \lambda = 2, \alpha = 0.25, \beta_X = 0.5, p = 0.4, q = 0.07$ and $n = 12$.

Scenario 3.7 : The trend of value at risk in the terms of parameter $\beta_Y$ where the parameter $\beta_Z$ is fixed ($\beta_Z = 1.5$). The confidence level considered in this scenario is $\omega = 0.95$. In this scenario, we consider different values of $\beta_Y$ which are 0.5, 0.75, 1.0, 1.5 and 2.0, respectively. The Values at Risk are given in Table 3.7.

**Table 3.7:** Parameter $\beta_Y \in [0.5, 2]$ and $\text{VaR}_{0.95}$ $(S_{12})$

| $\beta_Y$ | 0.5 | 0.75 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| $\text{VaR}_{0.95}$ $(S_{12})$ | 118.56 | 118.41 | 117.24 | 113.43 | 110.13 |

Table 3.7 show that the value at risk decreases as parameter $\beta_Y$ increases. That is the value at risk decrease when the mean of claim size decreases.

Scenario 3.8 : The trend of value at risk in terms of parameter $\beta_Z$ where the parameter $\beta_Y$ is fixed ($\beta_Y = 1.5$). The confidence level considered in this scenario is $\omega = 0.95$. In this scenario, we consider different values of $\beta_Z$ which are 0.5, 0.75, 1.0, 1.5 and 2.0, respectively. The Values at Risk are given in Table 3.8.



**Table 3.8:** Parameter $\beta_Z \in [0.5, 2]$ and $\text{VaR}_{0.95}$ $(S_{12})$

| $\beta_Z$ | 0.5 | 0.75 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| $\text{VaR}_{0.95}$ $(S_{12})$ | 118.56 | 118.32 | 117.09 | 113.43 | 110.37 |

Table 3.8 show that the value at risk decreases as parameter $\beta_Z$ increases. That is the value at risk decrease when the mean of surrender value decreases.

The result from Scenario 3.7 and Scenario 3.8 show that the value at risk decrease when either the mean of claim sizes or the mean of surrender values, perform as the loss of the model, decreases. Therefore, it is reasonable that if the claim sizes and surrender values decrease, the maximum loss of the company also decrease.

# CHAPTER IV

# INTEGER-VALUED AUTOREGRESSIVE RISK MODEL WITH INVESTMENT AND SURRENDER

In Chapter 3, we constructed INMA(1) risk model. For this chapter, we consider the first order integer-valued autoregressive (INAR(1)) process into the risk model. Because the forecasted data in INAR(1) process have correlation with all other previous data. Section 4.1 introduces the model and describes notations used in this chapter. In Section 4.2, we provide definition and some properties of INAR(1) risk model. We then obtain the adjustment coefficient function to approximate the ruin probability in Section 4.3. Moreover, we also give numerical examples to study the trend of ruin probability and value at risk in terms of model parameters in Section 4.4.

## 4.1 The discrete-time risk model



Let $\{U_n; n \in \mathbb{N}\}$ be the surplus process of insurance company with incorporating investment and surrender at time $n$. For initial capital $u$, the discrete-time risk model can be written as

$$U_n = u + Idn + \sum_{i=1}^{n} A_i - \sum_{i=1}^{n} B_i - \sum_{i=1}^{n} C_i, \tag{4.1}$$

where $I$ is the investment capital for $I < u$, $d$ represents the investment income per unit of time. The sequence of aggregates of premium amounts in period $i$, denoted by $\{A_i; i \in \mathbb{N}\}$, is defined as

$$A_i = \sum_{k=1}^{N_i} X_{i,k}, \tag{4.2}$$

where $\{X_{i,k}; k \in \mathbb{N}\}$ is the sequence of premium sizes in period $i$ assuming to be i.i.d. random variables and $N_i$ is the number of premiums in period $i$.

The sequence of aggregate of claim sizes in period $i$, denoted by $\{B_i; i \in \mathbb{N}\}$, is defined as, for $0 < p < 1$,

$$B_i = \sum_{k=1}^{N_i(p)} Y_{i,k},$$

where the sequence of i.i.d. random variables $\{Y_{i,k}; k \in \mathbb{N}\}$ denotes claim sizes in period $i$, and $N_i(p)$ is the $p$-thinning process of $N_i$ denoting the number of claims in period $i$.

The sequence of aggregate of surrender values in period $i$, denoted by $\{C_i; i \in \mathbb{N}\}$, is defined as

$$C_i = \sum_{k=1}^{N_i(q)} Z_{i,k},$$



where the sequence of i.i.d. random variables $\{Z_{i,k}; k \in \mathbb{N}\}$ represents surrender values in period $i$, and $N_i(q)$ is the $q$-thinning process of $N_i$ denotes the number of surrenders in period $i$ for $0 < q < 1$ such that $0 < p + q < 1$.

## 4.2 Definition and properties of INAR(1) risk model

In this section, we provide definition and some properties of the INAR(1) risk model.

For the INAR(1) risk model considered in this chapter, we suppose that the processes $\{N_i; i \in \mathbb{N}\}$, $\{N_i(p); i \in \mathbb{N}\}$ and $\{N_i(q); i \in \mathbb{N}\}$ follow INAR(1) processes. The processes are defined under the condition that $N_1$, $N_1(p)$ and $N_1(q)$ follow Poisson distribution with means $\dfrac{\lambda}{1-\alpha}$, $\dfrac{\lambda}{1-\alpha p}$ and $\dfrac{\lambda}{1-\alpha q}$, respectively.
The structures of $\{N_i; i \in \mathbb{N}\}$, $\{N_i(p); i \in \mathbb{N}\}$ and $\{N_i(q); i \in \mathbb{N}\}$ are described as follows.
The process of the number of premiums $\{N_i; i \in \mathbb{N}\}$ can be defined as

$$N_i = \alpha \circ N_{i-1} + \varepsilon_i, \tag{4.3}$$

where $\alpha \in [0, 1]$, $\{\varepsilon_i\}_{i=1,2,\ldots}$ is a sequence of i.i.d. random variables following the Poisson distribution with mean $\lambda$ and $\alpha \circ N_{i-1}$ is the binomial thinning operator defined as

$$\alpha \circ N_{i-1} = \sum_{j=1}^{N_{i-1}} d_{i-1,j},$$

where $\{d_{i,j}\}_{j=1,2,\ldots}$ is the sequence of i.i.d. Bernoulli random variables with parameter $\alpha$ for all $i$.
The process of the number of claims $\{N_i(p); i \in \mathbb{N}\}$ can be defined as



$$N_i(p) = (\alpha p) \circ N_{i-1}(p) + \gamma_i, \tag{4.4}$$

where $\{\gamma_i\}_{i=1,2,\ldots}$ is the sequence of i.i.d. random variables following the Poisson distribution with mean $\lambda$ and $(\alpha p) \circ N_{i-1}(p)$ is the binomial thinning operator defined as

$$(\alpha p) \circ N_{i-1}(p) = \sum_{j=1}^{N_{i-1}(p)} e_{i-1,j},$$

where $\{e_{i,j}\}_{j=1,2,\ldots}$ is the sequence of i.i.d. Bernoulli random variables with parameter $\alpha p$.

The process of the number of surrenders $\{N_i(q); i \in \mathbb{N}\}$ can be defined as

$$N_i(q) = (\alpha q) \circ N_{i-1}(q) + \mu_i, \tag{4.5}$$

where $\{\mu_i\}_{i=1,2,\ldots}$ is the sequence of i.i.d. random variables following the Poisson distribution with mean $\lambda$ and $(\alpha q) \circ N_{i-1}$ is the binomial thinning operator defined as

$$(\alpha q) \circ N_{i-1}(q) = \sum_{j=1}^{N_{i-1}(q)} f_{i-1,j},$$

where $\{f_{i,j}\}_{j=1,2,\ldots}$ is the sequence of i.i.d. Bernoulli random variables with parameter $\alpha q$.

Next, we give the properties of the number of premiums, the number of claims and the number of surrenders, denoted by $\{N_i; i \in \mathbb{N}\}$, $\{N_i(p); i \in \mathbb{N}\}$ and $\{N_i(q); i \in \mathbb{N}\}$, respectively.

**Proposition 4.2.1.** Let $\{N_i; i \in \mathbb{N}\}$ be defined in (4.3). Then $\{N_i; i \in \mathbb{N}\}$ has the following properties, for all $i \in \mathbb{N}$



(1) $G_{N_i}(z) = e^{\frac{\lambda}{1-\alpha}(z-1)}$ for $z \in \mathbb{R}$,

(2) $N_i$ is stationary process,

(3) $E(N_i) = \dfrac{\lambda}{1-\alpha}$,

(4) $\text{Var}(N_i) = \dfrac{\lambda}{1-\alpha}$,

(5) $\text{Cov}(N_i, N_{i-k}) = \dfrac{\lambda \alpha^k}{1-\alpha}$ for $k \geq 1$,

(6) $Corr(N_i, N_{i-k}) = \alpha^k$ for $k \geq 1$.

*Proof.* To prove (1) and (2), we will show that $N_i$ follow the Poisson distribution with mean $\dfrac{\lambda}{1-\alpha}$ for all $i$.

Note from the assumption that $N_1$ has Poisson distribution with mean $\dfrac{\lambda}{1-\alpha}$.

Next, we will prove that $\{N_i; i \in \mathbb{N}\}$ is stationary process with mean $\dfrac{\lambda}{1-\alpha}$ by the mathematical induction as follows.

For the inductive step of the mathematical induction, we assume that $G_{N_i}(z) = e^{\frac{\lambda}{1-\alpha}(z-1)}$. Then we have,

$$
\begin{aligned}
G_{N_{i+1}}(z) &= E[z^{N_{i+1}}] \\
&= E[z^{\alpha \circ N_i + \varepsilon_{i+1}}] \\
&= E[z^{\alpha \circ N_i}]E[z^{\varepsilon_{i+1}}] \qquad (4.6) \\
&= E[((1-\alpha)+\alpha z)^{N_i}]E[z^{\varepsilon_{i+1}}] \\
&= e^{\frac{\lambda}{1-\alpha}((1-\alpha)+\alpha z-1)} \qquad (4.7) \\
&= e^{\frac{\lambda}{1-\alpha}(z-1)},
\end{aligned}
$$

where we use the fact that the process $\{N_i; i \in \mathbb{N}\}$ is independent of $\{\varepsilon_i\}_{i=1,2,\ldots}$ to obtain (4.6) and use the assumption that $G_{N_i}(z) = e^{\frac{\lambda}{1-\alpha}(z-1)}$ and the fact that $\{\varepsilon_i\}_{i=1,2,\ldots}$ is a sequence of i.i.d. random variables following the Poisson distribution with mean $\lambda$ to obtain (4.7).



Therefore, we can conclude that $G_{N_i}(z) = e^{\frac{\lambda}{1-\alpha}(z-1)}$ for all $i \in \mathbb{N}$ and $\{N_i; i \in \mathbb{N}\}$ is a stationary process.

To prove (3), note that

$$
\begin{aligned}
E(N_i) &= E(\alpha \circ N_{i-1} + \varepsilon_i) \\
&= E(\alpha \circ N_{i-1}) + E(\varepsilon_i) \\
&= \alpha E(N_{i-1}) + E(\varepsilon_i)
\end{aligned}
$$

From Proposition 4.2.1 (2), $E(N_i) = E(N_{i-1})$. Therefore,

$$\begin{aligned} E(N_i) &= \frac{E(\varepsilon_i)}{1-\alpha} \\ &= \frac{\lambda}{1-\alpha}, \end{aligned} \tag{4.8}$$

where we use the fact that $\{\varepsilon_i\}_{i=1,2,\ldots}$ has the Poisson distribution mean $\lambda$ to obtain (4.8).

To prove (4), from Lemma 2.1.1 (b), note that

$$G_{N_i}^{(2)}(1) = E(N_i(N_i - 1)) = E(N_i^2) - E(N_i)$$

Then, from Proposition 4.2.1 (1),

$$\begin{aligned} E(N_i^2) &= G_{N_i}^{(2)}(1) + E(N_i) \\ &= \left(\frac{\lambda}{1-\alpha}\right)^2 e^{\frac{\lambda}{1-\alpha}(z-1)}\Big|_{z=1} + \frac{\lambda}{1-\alpha} \\ &= \left(\frac{\lambda}{1-\alpha}\right)^2 + \frac{\lambda}{1-\alpha}. \end{aligned}$$



Therefore, we can obtain the variance of $\{N_i; i \in \mathbb{N}\}$ as follows.

$$\begin{aligned} \operatorname{Var}(N_i) &= E(N_i^2) - E^2(N_i) \\ &= \left(\frac{\lambda}{1-\alpha}\right)^2 + \frac{\lambda}{1-\alpha} - \left(\frac{\lambda}{1-\alpha}\right)^2 \\ &= \frac{\lambda}{1-\alpha}. \end{aligned}$$

To prove (5), for $k \geq 1$, we consider

$$\begin{aligned}
\mathrm{Cov}(N_i, N_{i-k}) &= \mathrm{Cov}(\alpha \circ N_{i-1} + \varepsilon_i, N_{i-k}) \\
&= \mathrm{Cov}(\alpha \circ N_{i-1}, N_{i-k}) + \mathrm{Cov}(\varepsilon_i, N_{i-k}) \\
&= \alpha \mathrm{Cov}(N_{i-1}, N_{i-k}) && (4.9) \\
&= \alpha \mathrm{Cov}(\alpha \circ N_{i-2} + \varepsilon_{i-1}, N_{i-k}) \\
&= \alpha^2 \mathrm{Cov}(N_{i-2}, N_{i-k}) && (4.10) \\
&= \alpha^2 \mathrm{Cov}(\alpha \circ N_{i-k} + \varepsilon_{i-2}, N_{i-3}) \\
&= \alpha^3 \mathrm{Cov}(N_{i-3}, N_{i-k}), && (4.11)
\end{aligned}$$

where we use the fact that the process $\{N_i; i \in \mathbb{N}\}$ be independent of $\{\varepsilon_i\}_{i=1,2,\ldots}$ to obtain (4.9) – (4.11).

By recursively, we have

$$\begin{aligned}
\mathrm{Cov}(N_i, N_{i-k}) &= \alpha^k \mathrm{Cov}(N_{i-k}, N_{i-k}) \\
&= \alpha^k \mathrm{Var}(N_{i-k}) \\
&= \frac{\alpha^k \lambda}{1-\alpha}.
\end{aligned}$$

To prove (6), from Proposition 4.2.1 (4) and (5), we have

$$\begin{aligned}
Corr(N_i, N_{i-k}) &= \frac{\mathrm{Cov}(N_i, N_{i-k})}{\sqrt{\mathrm{Var}(N_i)}\sqrt{\mathrm{Var}(N_{i-k})}} \\
&= \left(\frac{\alpha^k \lambda}{1-\alpha}\right)\left(\frac{1-\alpha}{\lambda}\right) \\
&= \alpha^k.
\end{aligned}$$

□

Similar to Proposition 4.2.1, we can provide the properties of the processes $\{N_i(p); i \in \mathbb{N}\}$ and $\{N_i(q); i \in \mathbb{N}\}$ presented in Proposition 4.2.2 and Proposition 4.2.3, respectively.

**Proposition 4.2.2.** Let $\{N_i(p); i \in \mathbb{N}\}$ be defined in (4.4). Then $\{N_i(p); i \in \mathbb{N}\}$ has the following properties, for all $i \in \mathbb{N}$.

(1) $G_{N_i(p)}(z) = e^{\frac{\lambda}{1-\alpha p}(z-1)}$ for $z \in \mathbb{R}$,

(2) $N_i(p)$ is stationary process,

(3) $E(N_i(p)) = \dfrac{\lambda}{1-\alpha p}$,

(4) $\mathrm{Var}(N_i(p)) = \dfrac{\lambda}{1-\alpha p}$,

(5) $\mathrm{Cov}(N_i(p), N_{i-k}(p)) = \dfrac{\lambda(\alpha p)^k}{1-\alpha p}$ for $k \geq 1$,

(6) $Corr(N_i(p), N_{i-k}(p)) = (\alpha p)^k$ for $k \geq 1$.

**Proposition 4.2.3.** Let $\{N_i(q); i \in \mathbb{N}\}$ be defined in (4.5). Then $\{N_i(q); i \in \mathbb{N}\}$ has the following properties, for all $i \in \mathbb{N}$.



(1) $G_{N_i(q)}(z) = e^{\frac{\lambda}{1-\alpha q}(z-1)}$ for $z \in \mathbb{R}$,

(2) $N_i(q)$ is stationary process,

(3) $E(N_i(q)) = \dfrac{\lambda}{1-\alpha q}$,

(4) $\mathrm{Var}(N_i(q)) = \dfrac{\lambda}{1-\alpha q}$,

(5) $\mathrm{Cov}(N_i(q), N_{i-k}(q)) = \dfrac{\lambda(\alpha q)^k}{1-\alpha q}$ for $k \geq 1$,

(6) $Corr(N_i(q), N_{i-k}(q)) = (\alpha q)^k$ for $k \geq 1$.

Next, following Joe (1997), we will give the dependence structure of the process $\{N_i; i \in \mathbb{N}\}$ by writting $N_{i+1}$ in the terms of $N_1$ for $i \in \mathbb{N}$.

**Theorem 4.2.1** ([8], p. 263)**.** The dependence structure of the Poisson INAR(1) process can be defined by,

$$N_{i+1} = \sum_{j=1}^{N_1} d_{21j} d_{31j} \ldots d_{i+1,1j} + \sum_{k=2}^{i} \sum_{j=1}^{\varepsilon_k} d_{k+1,kj} \ldots d_{i+1,kj} + \varepsilon_{i+1}$$

where $\{d_{i,j}\}_{j=1,2,\ldots}$ is the sequence of i.i.d. Bernoulli random variables for all $i$.

The following theorem obtains the generating function of the sum of INAR(1) process.

**Theorem 4.2.2.** Let INAR(1) process $\{N_i; i = 1, 2, ...\}$ be defined in (4.3). The generating function of the sum of $\{N_i; i = 1, 2, .., n\}$ process can be written as follows.

$$G_{P_n}(t) = \exp\left\{\lambda\left(2t\sum_{k=0}^{n-2}(\alpha t)^k + (1-\alpha)t\sum_{k=0}^{n-2}(n-k-2)(\alpha t)^k - (n-1)\right) + \frac{\lambda}{1-\alpha}\left(\alpha^{n-1}t^n - 1\right)\right\},$$



where $P_n = N_1 + \cdots + N_n$ for $n = 2, 3, ...$

*Proof.* From Theorem 4.2.1, we have

$$\begin{aligned} G_{P_n}(t) &= E\left[t^{N_1+\cdots+N_n}\right] \\ &= E\left[t^{N_1} t^{\sum_{i=1}^{n-1} N_{i+1}}\right] \\ &= E\left[t^{N_1} t^{\sum_{i=1}^{n-1}\sum_{j=1}^{N_1} d_{21j}d_{31j}\cdots d_{i+1,1j} + \sum_{i=1}^{n-1}\sum_{k=2}^{i}\sum_{j=1}^{\varepsilon_k} d_{k+1,kj}\cdots d_{i,kj} + \sum_{i=1}^{n-1}\varepsilon_{i+1}}\right]. \end{aligned} \tag{4.12}$$

From Theorem 4.2.1 and the fact that $N_1$ follows the Poisson distribution with mean $\dfrac{\lambda}{1-\alpha}$, we will provide the generating function $G_{P_n}(t)$ in the case of $n = 2, 3$.
For $n = 2$, from (4.12), we have

$$\begin{aligned}
G_{P_2}(t) &= E[t^{N_1+N_2}] \\
&= E\left[t^{N_1} t^{\sum_{j=1}^{N_1} d_{21j}} t^{\varepsilon_2}\right] \\
&= E\left[t^{N_1+\sum_{j=1}^{N_1} d_{21j}}\right] E\left(t^{\varepsilon_2}\right) \\
&= E\left[t^{N_1} E\left[t^{\sum_{j=1}^{N_1} d_{21j}} \middle| N_1\right]\right] E\left(t^{\varepsilon_2}\right) && (4.13) \\
&= E\left[t^{N_1} E\left[\prod_{j=1}^{N_1} t^{d_{21j}} \middle| N_1\right]\right] E\left(t^{\varepsilon_2}\right) \\
&= E\left[t^{N_1} \prod_{j=1}^{N_1} (1-\alpha+\alpha t)\right] E\left(t^{\varepsilon_2}\right) && (4.14) \\
&= E\left[t^{N_1}(1-\alpha+\alpha t)^{N_1}\right] E\left(t^{\varepsilon_2}\right) \\
&= E\left[((1-\alpha)t+\alpha t^2)^{N_1}\right] E\left(t^{\varepsilon_2}\right) \\
&= e^{\frac{\lambda}{1-\alpha}[(1-\alpha)t+\alpha t^2-1]} e^{\lambda(t-1)}, && (4.15)
\end{aligned}$$

where we use the fact that the random variable $N_1$ is independent of $\varepsilon_2$ to obtain (4.13), use the fact that $\{d_{i,j}\}_{j=1,2,\ldots}$ be the sequence of i.i.d. Bernoulli random variables with parameter $\alpha$ to obtain (4.14) and we use the fact that the random variables $N_1$ and $\varepsilon_2$ be the Poisson distribution with means $\dfrac{\lambda}{1-\alpha}$ and $\lambda$ to obtain (4.15).

For $n = 3$, from (4.12), we get

$$\begin{aligned}
G_{P_3}(t) &= E[t^{N_1+N_2+N_3}] \\
&= E\left[t^{N_1+\sum_{j=1}^{N_1} d_{21j}+\sum_{j=1}^{N_1} d_{21j}d_{31j}} t^{\sum_{j=1}^{\varepsilon_2} d_{32j}} t^{\varepsilon_2+\varepsilon_3}\right] \\
&= E\left[t^{N_1+\sum_{j=1}^{N_1} d_{21j}+\sum_{j=1}^{N_1} d_{21j}d_{31j}}\right] E\left[t^{\varepsilon_2+\sum_{j=1}^{\varepsilon_2} d_{32j}}\right] E\left(t^{\varepsilon_3}\right) && (4.16) \\
&= E\left[t^{N_1+\sum_{j=1}^{N_1} d_{21j}+\sum_{j=1}^{N_1} d_{21j}d_{31j}}\right] e^{\lambda[(1-\alpha)t+\alpha t^2-1]} e^{\lambda(t-1)}, && (4.17)
\end{aligned}$$

where we use the fact that the random variable $N_1$ is independent of $\{\varepsilon_i\}_{i=1,2,...}$ to obtain (4.16) and the fact that $\{\varepsilon_i\}_{i=1,2,...}$ be the Poisson distribution with means $\lambda$ to obtain (4.17).

Next, we will separately consider the first term of (4.17). Then

$$\begin{aligned}
&E\left[t^{N_1+\sum_{j=1}^{N_1} d_{21j}+\sum_{j=1}^{N_1} d_{21j}d_{31j}}\right] \\
&= E\left[t^{N_1}E\left[t^{\sum_{j=1}^{N_1} d_{21j}}E\left[t^{\sum_{j=1}^{N_1} d_{21j}d_{31j}}\middle|N_1, d_{21j}\right]\middle|N_1\right]\right] \\
&= E\left[t^{N_1}E\left[t^{\sum_{j=1}^{N_1} d_{21j}}E\left[\prod_{j=1}^{N_1} t^{d_{21j}d_{31j}}\middle|N_1, d_{21j}\right]\middle|N_1\right]\right] \\
&= E\left[t^{N_1}E\left[\prod_{j=1}^{N_1} t^{d_{21j}}\prod_{j=1}^{N_1}(1-\alpha+\alpha t^{d_{21j}})\middle|N_1\right]\right] && (4.18) \\
&= E\left[t^{N_1}E\left[\prod_{j=1}^{N_1}(1-\alpha)t^{d_{21j}}+\alpha t^{2d_{21j}}\middle|N_1\right]\right] \\
&= E\left[t^{N_1}\prod_{j=1}^{N_1}(1-\alpha)(1-\alpha+\alpha t)+\alpha(1-\alpha+\alpha t^2)\right] && (4.19) \\
&= E\left[t^{N_1}((1-\alpha)^2+\alpha(1-\alpha)t+\alpha(1-\alpha)+\alpha^2t^2)^{N_1}\right] \\
&= E\left[((1-\alpha)^2t+\alpha(1-\alpha)t^2+\alpha(1-\alpha)t+\alpha^2t^3)^{N_1}\right] \\
&= e^{\frac{\lambda}{1-\alpha}[(1-\alpha)t+\alpha(1-\alpha)t^2+\alpha^2t^3-1]}, && (4.20)
\end{aligned}$$

where we use the fact that $\{d_{i,j}\}_{j=1,2,...}$ is the sequence of i.i.d. Bernoulli random variables with parameter $\alpha$ to obtain (4.18) and (4.19) and we use the fact that $N_1$ be the Poisson distribution with mean $\dfrac{\lambda}{1-\alpha}$ to obtain (4.20).

Substitute (4.20) into (4.17), we obtain

$$G_{P_3}(t) = e^{\frac{\lambda}{1-\alpha}[(1-\alpha)t+\alpha(1-\alpha)t^2+\alpha^2t^3-1]}e^{\lambda[(1-\alpha)t+\alpha t^2-1]}e^{\lambda(t-1)}. \quad (4.21)$$

By the same technique in obtaining (4.15) and (4.21), we can write in the general form of $G_{P_n}(t)$. For $n = 2, 3, ...$, we have

$$
\begin{aligned}
G_{P_n}(t) &= E\Big[t^{N_1+\sum_{i=1}^{n-1}\sum_{j=1}^{N_1} d_{21j}d_{31j}\cdots d_{i+1,1j}} t^{\varepsilon_2+\sum_{j=1}^{\varepsilon_2}\prod_{i=3}^{n} d_{i,2j}} t^{\varepsilon_3+\sum_{j=1}^{\varepsilon_3}\prod_{i=4}^{n} d_{i,3j}} \\
&\qquad \ldots t^{\varepsilon_{n-2}+\sum_{j=1}^{\varepsilon_{n-2}}\prod_{i=n-1}^{n} d_{i,n-2j}} t^{\varepsilon_{n-1}+\sum_{j=1}^{\varepsilon_{n-1}} d_{n,n-1j}} t^{\varepsilon_n}\Big] \\
&= E\left[t^{N_1+\sum_{i=1}^{n-1}\sum_{j=1}^{N_1} d_{21j}d_{31j}\cdots d_{i+1,1j}}\right] E\left[t^{\varepsilon_2+\sum_{j=1}^{\varepsilon_2}\prod_{i=3}^{n} d_{i,2j}}\right] E\left[t^{\varepsilon_3+\sum_{j=1}^{\varepsilon_3}\prod_{i=4}^{n} d_{i,3j}}\right] \\
&\quad \cdots E\left[t^{\varepsilon_{n-2}+\sum_{j=1}^{\varepsilon_{n-2}}\prod_{i=n-1}^{n} d_{i,n-2j}}\right] E\left[t^{\varepsilon_{n-1}+\sum_{j=1}^{\varepsilon_{n-1}} d_{n,n-1j}}\right] E\left(t^{\varepsilon_n}\right) \\
&= e^{\frac{\lambda}{1-\alpha}[(1-\alpha)t+\alpha(1-\alpha)t^2+\alpha^2(1-\alpha)t^3+\cdots+\alpha^{n-2}(1-\alpha)t^{n-1}+\alpha^{n-1}t^n-1]} \\
&\quad \cdot e^{\lambda[(1-\alpha)t+\alpha(1-\alpha)t^2+\alpha^2t^3+\cdots+\alpha^{n-3}(1-\alpha)t^{n-2}+\alpha^{n-2}t^{n-1}-1]} \cdots e^{\lambda[(1-\alpha)t+\alpha t^2-1]} e^{\lambda(t-1)} \\
&= e^{\lambda[t+\alpha t^2+\alpha^2t^3+\cdots+\alpha^{n-2}t^{n-1}]} e^{\frac{\lambda}{1-\alpha}[\alpha^{n-1}t^n-1]} \\
&\quad \cdot e^{\lambda[(1-\alpha)t+\alpha(1-\alpha)t^2+\alpha^2t^3+\cdots+\alpha^{n-3}(1-\alpha)t^{n-2}+\alpha^{n-2}t^{n-1}-1]} \cdots e^{\lambda[(1-\alpha)t+\alpha t^2-1]} e^{\lambda(t-1)} \\
&= \exp\left\{\lambda\left(2t\sum_{k=0}^{n-2}(\alpha t)^k + (1-\alpha)t\sum_{k=0}^{n-2}(n-k-2)(\alpha t)^k - (n-1)\right)\right. \\
&\qquad \left. + \frac{\lambda}{1-\alpha}\left(\alpha^{n-1}t^n - 1\right)\right\}.
\end{aligned}
$$

□

## 4.3 Adjustment coefficient

In this section, we derive the adjustment coefficient function for the INAR(1) risk model and adjustment coefficient to obtain the approximation of ruin probability.

**Theorem 4.3.1.** The adjustment coefficient function of the risk model defined in (4.1) is given by

$$
\begin{aligned}
c(r) = -rId + \lambda(1-\alpha)M_X(-r)\left(\frac{1}{1-\alpha M_X(-r)}\right) + \lambda(1-\alpha p)M_Y(r)\left(\frac{1}{1-\alpha p M_Y(r)}\right) \\
+ \lambda(1-\alpha q)M_Z(r)\left(\frac{1}{1-\alpha q M_Z(r)}\right) - 3\lambda,
\end{aligned}
$$

for all $r$ such that $M_Y(r) < \dfrac{1}{\alpha p}$ and $M_Z(r) < \dfrac{1}{\alpha q}$.

*Proof.* From (2.3), the adjustment coefficient function is defined as,

$$c(r) = \lim_{n\to\infty} \frac{1}{n} \ln E(e^{-rS_n}). \tag{4.22}$$

Therefore, We will first derive $E[e^{-rS_n}]$.

From (3.5) and the fact that $\{A_i; i \in \mathbb{N}\}$, $\{B_i; i \in \mathbb{N}\}$ and $\{C_i; i \in \mathbb{N}\}$ are independent, we have

$$\begin{aligned} E[e^{-rS_n}] &= E\left[ e^{-r\left( Idn + \sum\limits_{i=1}^{n} A_i - \sum\limits_{i=1}^{n} B_i - \sum\limits_{i=1}^{n} C_i \right)} \right] \\ &= E\left[e^{-rIdn}\right] \cdot E\left[e^{-r\sum_{i=1}^{n} A_i}\right] \cdot E\left[e^{r\sum_{i=1}^{n} B_i}\right] \cdot E\left[e^{r\sum_{i=1}^{n} C_i}\right]. \end{aligned} \tag{4.23}$$

For the second term of (4.23), note that the aggregate of premium amounts $\{A_i; i \in \mathbb{N}\}$ defined in (4.2). Then

$$\begin{aligned} &E\left[e^{-r\sum_{i=1}^{n} A_i}\right] \\ &= E[e^{-rA_1} \cdots e^{-rA_n}] \\ &= E[e^{-r\sum_{k=1}^{N_1} X_{1,k}} \cdots e^{-r\sum_{k=1}^{N_n} X_{n,k}}] \\ &= E[E[e^{-r\sum_{k=1}^{N_1} X_{1,k}} \cdots e^{-r\sum_{k=1}^{N_n} X_{n,k}} | N_1, \ldots, N_n]] \\ &= E\left[(M_X(-r))^{N_1} \cdots (M_X(-r))^{N_n}\right] \\ &= G_{P_n}(M_X(-r)), \end{aligned}$$

where $P_n = N_1 + \cdots + N_n$.

From Theorem 4.2.2, we obtain

$$
\begin{aligned}
&E\left[e^{-r\sum_{i=1}^{n}A_i}\right] \\
&= \exp\Bigg\{\lambda\Bigg(2M_X(-r)\sum_{k=0}^{n-2}(\alpha M_X(-r))^k + (1-\alpha)M_X(-r)\sum_{k=0}^{n-2}(n-k-2)(\alpha M_X(-r))^k\Bigg) \\
&\qquad + \frac{\lambda}{1-\alpha}\big(\alpha^{n-1}M_X^n(-r)-1\big) - \lambda(n-1)\Bigg\} \\
&= \exp\Bigg\{2\lambda M_X(-r)\sum_{k=0}^{n-2}(\alpha M_X(-r))^k + \lambda(1-\alpha)M_X(-r)(n-2)\sum_{k=0}^{n-2}(\alpha M_X(-r))^k \\
&\qquad - \lambda(1-\alpha)M_X(-r)\sum_{k=0}^{n-2}k(\alpha M_X(-r))^k \\
&\qquad + \frac{\lambda}{1-\alpha}\big(\alpha^{n-1}M_X^n(-r)-1\big) - \lambda(n-1)\Bigg\} \\
&= \exp\Bigg\{2\lambda M_X(-r)\left(\frac{1-(\alpha M_X(-r))^{n-1}}{1-\alpha M_X(-r)}\right) \\
&\qquad + \lambda(1-\alpha)M_X(-r)(n-2)\left(\frac{1-(\alpha M_X(-r))^{n-1}}{1-\alpha M_X(-r)}\right) \\
&\qquad - \lambda(1-\alpha)M_X(-r)\left(\frac{(n-2)(\alpha M_X(-r))^{n-1}}{1-\alpha M_X(-r)} + \frac{\alpha M_X(-r)-(\alpha M_X(-r))^{n-1}}{(1-\alpha M_X(-r))^2}\right) \\
&\qquad + \frac{\lambda}{1-\alpha}\big(\alpha^{n-1}M_X^n(-r)-1\big) - \lambda(n-1)\Bigg\} \\
&= \exp\Bigg\{2\lambda M_X(-r)\left(\frac{1-(\alpha M_X(-r))^{n-1}}{1-\alpha M_X(-r)}\right) \\
&\qquad + \lambda(1-\alpha)M_X(-r)\left(\frac{n-2}{1-\alpha M_X(-r)} - \frac{\alpha M_X(-r)-(\alpha M_X(-r))^{n-1}}{(1-\alpha M_X(-r))^2}\right) \\
&\qquad + \frac{\lambda}{1-\alpha}\big(\alpha^{n-1}M_X^n(-r)-1\big) - \lambda(n-1)\Bigg\}. \qquad (4.24)
\end{aligned}
$$

By the same technique, we can obtain the last two terms of (4.23) as the following,

$$
\begin{aligned}
E[e^{r\sum_{i=1}^{n}B_i}] &= G_{P_{n(p)}}(M_Y(r)) \\
&= \exp\Bigg\{2\lambda M_Y(r)\left(\frac{1-(\alpha p M_Y(r))^{n-1}}{1-\alpha p M_Y(r)}\right) \\
&\qquad + \lambda(1-\alpha p)M_Y(r)\left(\frac{n-2}{1-\alpha p M_Y(r)} - \frac{\alpha p M_Y(r)-(\alpha p M_Y(r))^{n-1}}{(1-\alpha p M_Y(r))^2}\right) \\
&\qquad + \frac{\lambda}{1-\alpha p}\big((\alpha p)^{n-1}M_Y^n(r)-1\big) - \lambda(n-1)\Bigg\}, \qquad (4.25)
\end{aligned}
$$

and

$$
\begin{aligned}
E[e^{r\sum_{i=1}^{n}C_i}] &= G_{P_{n(q)}}\left(M_Z(r)\right) \\
&= \exp\bigg\{2\lambda M_Z(r)\bigg(\frac{1-(\alpha q M_Z(r))^{n-1}}{1-\alpha q M_Z(r)}\bigg) \\
&\qquad + \lambda(1-\alpha q)M_Z(r)\bigg(\frac{n-2}{1-\alpha q M_Z(r)} - \frac{\alpha q M_Z(r)-(\alpha q M_Z(r))^{n-1}}{(1-\alpha q M_Z(r))^2}\bigg) \\
&\qquad + \frac{\lambda}{1-\alpha q}\big((\alpha q)^{n-1}M_Z^n(r)-1\big) - \lambda(n-1)\bigg\}, \qquad (4.26)
\end{aligned}
$$

where $P_n(p) = N_1(p) + \cdots + N_n(p)$ and $P_n(q) = N_1(q) + \cdots + N_n(q)$.

Substitute (4.24) – (4.26) into (4.23), we have

$$
\begin{aligned}
E[e^{-rS_n}] &= E\left[e^{-rIdn}\right]\cdot E\left[e^{-r\sum_{i=1}^{n}A_i}\right]\cdot E\left[e^{r\sum_{i=1}^{n}B_i}\right]\cdot E\left[e^{r\sum_{i=1}^{n}C_i}\right] \\
&= \exp\bigg\{-rIdn + 2\lambda M_X(-r)\bigg(\frac{1-(\alpha M_X(-r))^{n-1}}{1-\alpha M_X(-r)}\bigg) \\
&\qquad + \lambda(1-\alpha)M_X(-r)\bigg(\frac{n-2}{1-\alpha M_X(-r)} - \frac{\alpha M_X(-r)-(\alpha M_X(-r))^{n}}{(1-\alpha M_X(-r))^2}\bigg) \\
&\qquad + \frac{\lambda}{1-\alpha}\big(\alpha^{n-1}M_X^n(-r)-1\big) - \lambda(n-1) \\
&\qquad + 2\lambda M_Y(r)\bigg(\frac{1-(\alpha p M_Y(r))^{n-1}}{1-\alpha p M_Y(r)}\bigg) \\
&\qquad + \lambda(1-\alpha p)M_Y(r)\bigg(\frac{n-2}{1-\alpha p M_Y(r)} - \frac{\alpha p M_Y(r)-(\alpha p M_Y(r))^{n}}{(1-\alpha p M_Y(r))^2}\bigg) \\
&\qquad + \frac{\lambda}{1-\alpha p}\big((\alpha p)^{n-1}M_Y^n(r)-1\big) - \lambda(n-1) \\
&\qquad + 2\lambda M_Z(r)\bigg(\frac{1-(\alpha q M_Z(r))^{n-1}}{1-\alpha q M_Z(r)}\bigg) \\
&\qquad + \lambda(1-\alpha q)M_Z(r)\bigg(\frac{n-2}{1-\alpha q M_Z(r)} - \frac{\alpha q M_Z(r)-(\alpha q M_Z(r))^{n}}{(1-\alpha q M_Z(r))^2}\bigg) \\
&\qquad + \frac{\lambda}{1-\alpha q}\big((\alpha q)^{n-1}M_Z^n(r)-1\big) - \lambda(n-1)\bigg\}.
\end{aligned}
$$

Then, we consider the logarithm of the last equation as follows.

$$\begin{aligned}
\frac{1}{n}\ln E(e^{-rS_n}) &= \frac{1}{n}\ln E\Big[\exp\Big\{-r\Big(Idn+\sum_{i=1}^{n}A_i-\sum_{i=1}^{n}B_i-\sum_{i=1}^{n}C_i\Big)\Big\}\Big] \\
&= \frac{1}{n}\Big\{-rIdn+\frac{(n-2)\lambda(1-\alpha)M_X(-r)}{1-\alpha M_X(-r)} \\
&\quad +\left(\frac{\lambda M_X(-r)\,(-2+\alpha(3-\alpha)M_X(-r))}{(1-\alpha p M_X(-r))^2}\right)\big((\alpha M_X(-r))^{n-1}-1\big) \\
&\quad +\frac{\lambda\alpha}{1-\alpha}\,((\alpha M_X(-r))^{n}-1)+\frac{(n-2)\lambda(1-\alpha p)M_Y(r)}{1-\alpha p M_Y(-r)} \\
&\quad +\left(\frac{\lambda M_Y(r)\,(-2+\alpha p(3-\alpha p)M_Y(r))}{(1-\alpha p M_Y(r))^2}\right)\big((\alpha p M_Y(r))^{n-1}-1\big) \\
&\quad +\frac{\lambda\alpha p}{1-\alpha p}\,((\alpha p M_Y(r))^{n}-1)+\frac{(n-2)\lambda(1-\alpha q)M_Z(r)}{1-\alpha q M_Z(r)} \\
&\quad +\left(\frac{\lambda M_Z(r)\,(-2+\alpha q(3-\alpha q)M_Y(r))}{(1-\alpha p M_Y(r))^2}\right)\big((\alpha q M_Z(r))^{n-1}-1\big) \\
&\quad +\frac{\lambda\alpha q}{1-\alpha q}\,((\alpha q M_Z(r))^{n}-1)-3\lambda(n-1)\Big\} \qquad (4.27)
\end{aligned}$$

Since $0<\alpha M_X(-r)<1$, then the limit of the term $\frac{(\alpha M_X(-r))^n}{n}$ as $n$ approaches infinity is equal to zero. From the assumption that $\alpha p M_Y(r)<1$ and $\alpha q M_Z(r)<1$, then the terms of $\frac{(\alpha p M_Y(r))^n}{n}$ and $\frac{(\alpha q M_Z(r))^n}{n}$ go to zero as $n$ go to infinity. Then we will take the limit into (4.27) as $n$ approaches infinity, we have



$$\begin{aligned}
c(r) &= \lim_{n\to\infty}\frac{1}{n}\ln E(e^{-rS_n}) \\
&= -rId+\lambda(1-\alpha)M_X(-r)\left(\frac{1}{1-\alpha M_X(-r)}\right) \\
&\quad +\lambda(1-\alpha p)M_Y(r)\left(\frac{1}{1-\alpha p M_Y(r)}\right) \\
&\quad +\lambda(1-\alpha q)M_Z(r)\left(\frac{1}{1-\alpha q M_Z(r)}\right)-3\lambda.
\end{aligned}$$

□

Next, we will show that the solution of adjustment equation is the unique positive solution.

**Proposition 4.3.1.** The adjustment equation $c(r) = 0$ has the unique positive solution $R$ which is called the adjustment coefficient.

*Proof.* To show that $c(r) = 0$ has the unique positive solution, we will show the function $c(\cdot)$ has following properties

(1) $c(0) = 0$,

(2) $c'(0) < 0$,

(3) $c''(r) > 0 \ \forall r \in (0, \infty)$,

(4) $\lim_{r \to \infty} c(r) = \infty$.

To prove (1), we will substitute $r = 0$ into the adjustment coefficient $c(r)$ defined in Theorem 4.3.1, then we have

$$\begin{aligned} c(0) &= \lambda(1-\alpha)\left(\frac{1}{1-\alpha}\right) + \lambda(1-\alpha p)\left(\frac{1}{1-\alpha p}\right) + \lambda(1-\alpha q)\left(\frac{1}{1-\alpha q}\right) - 3\lambda \\ &= 0. \end{aligned}$$



To prove (2), note that

$$\begin{aligned} c'(r) = -Id &+ \lambda(1-\alpha)\left(\frac{-(1-\alpha M_X(-r))M'_X(-r) - \alpha M_X(-r)M'_X(-r)}{(1-\alpha M_X(-r))^2}\right) \\ &+ \lambda(1-\alpha p)\left(\frac{(1-\alpha p M_Y(r))M'_Y(r) + \alpha p M_Y(r)M'_Y(r)}{(1-\alpha p M_Y(r))^2}\right) \\ &+ \lambda(1-\alpha q)\left(\frac{(1-\alpha q M_Z(r))M'_Z(r) + \alpha q M_Z(r)M'_Z(r)}{(1-\alpha q M_Z(r))^2}\right) \\ = -Id &+ \lambda(1-\alpha)\left(\frac{-M'_X(-r)}{(1-\alpha M_X(-r))^2}\right) + \lambda(1-\alpha p)\left(\frac{M'_Y(-r)}{(1-\alpha p M_Y(r))^2}\right) \\ &+ \lambda(1-\alpha q)\left(\frac{M'_Z(-r)}{(1-\alpha q M_Z(r))^2}\right). \end{aligned} \tag{4.28}$$

Then we substitute $r = 0$ into (4.28) and get,

$$
\begin{aligned}
c'(0) &= -Id + \lambda(1-\alpha)\frac{-EX}{(1-\alpha)^2} + \lambda(1-\alpha p)\frac{EY}{(1-\alpha p)^2} \\
&\quad + \lambda(1-\alpha q)\frac{EZ}{(1-\alpha q)^2} \\
&= -Id - \frac{\lambda}{1-\alpha}EX + \frac{\lambda}{1-\alpha p}EY + \frac{\lambda}{1-\alpha q}EZ \\
&< 0,
\end{aligned}
$$

where we use Proposition 3.1.4 and Propositions 4.2.1 – 4.2.3 (2) to obtain the last inequality.

Hence, $c'(0) < 0$.

To prove (3), consider

$$
\begin{aligned}
c''(r) &= \lambda(1-\alpha)\left(\frac{(M''_X(-r)(1-\alpha M_X(-r))^2 + 2\alpha(M'_X(-r))^2(1-\alpha M_X(-r))}{(1-\alpha M_X(-r))^4}\right) \\
&\quad + \lambda(1-\alpha p)\left(\frac{M''_Y(r)(1-\alpha p M_Y(r))^2 + 2\alpha p(M'_Y(r))^2(1-\alpha p M_Y(r))}{(1-\alpha p M_Y(r))^4}\right) \\
&\quad + \lambda(1-\alpha q)\left(\frac{M''_Z(r)(1-\alpha q M_Z(r))^2 + 2\alpha q(M'_Z(r))^2(1-\alpha q M_Z(r))}{(1-\alpha q M_Z(r))^4}\right) \\
&= \lambda(1-\alpha)\left(\frac{M''_X(-r)}{(1-\alpha M_X(-r))^2} + \frac{2\alpha(M'_X(-r))^2}{(1-\alpha M_X(-r))^3}\right) \\
&\quad + \lambda(1-\alpha p)\left(\frac{M''_Y(r)}{(1-\alpha p M_Y(r))^2} + \frac{2\alpha p(M'_Y(r))^2}{(1-\alpha p M_Y(r))^3}\right) \\
&\quad + \lambda(1-\alpha q)\left(\frac{M''_Z(r)}{(1-\alpha q M_Z(r))^2} + \frac{2\alpha q(M'_Z(r))^2}{(1-\alpha q M_Z(r))^3}\right).
\end{aligned}
$$

As we know that moment generating function is always positive. From the fact that $0 < M_X(-r) < 1$ and $0 < \alpha < 1$. So, the term of $1-\alpha M_X(-r)$ is greater than 0. Since $M_Y(r) < \frac{1}{\alpha p}$ and $M_Z(r) < \frac{1}{\alpha q}$, we can get that the terms of $1-\alpha p M_Y(r)$ and $1-\alpha q M_Z(r)$ are greater than 0 for $0 < p, q < 1$ and $0 < p+q < 1$. Hence, we can conclude that the right hand side of the last equation is positive.

Therefore, $c''(r) > 0$.

To prove (4), From Theorem 4.3.1, the adjustment coefficient can be express as

$$c(r) = -rId + \lambda(1-\alpha)M_X(-r)\left(\frac{1}{1-\alpha M_X(-r)}\right) + \lambda(1-\alpha p)M_Y(r)\left(\frac{1}{1-\alpha p M_Y(r)}\right)$$
$$+ \lambda(1-\alpha q)M_Z(r)\left(\frac{1}{1-\alpha q M_Z(r)}\right) - 3\lambda.$$

From the right hand side of the equation, we can see that the terms of moment generating functions $M_Y(r)$ and $M_Z(r)$ grow faster than the polynomial term, determined by the term of $-rId$. Moreover, from Proposition 4.3.1 (3), we have $1-\alpha M_X(-r)$, $1-\alpha p M_Y(r)$ and $1-\alpha q M_Z(r)$ are positive for $0 < p, q < 1$ and $0 < p+q < 1$. Hence, $\lim_{r\to\infty} c(r) = \infty$. □

## 4.4 Numerical example

In this section, we study the effect of ruin probability and value at risk comparing with the parameters of premiums, claims and surrenders via numerical example by using Python and R programming.

Our examples are performed for a special case where we assume that the sequence of premium sizes $X = \{X_{i,k}\}_{i,k=1,2,...}$ is a sequence of i.i.d. random variables which are exponentially distributed with mean $\frac{1}{\beta_X}$, the sequence of claim sizes $Y = \{Y_{i,k}\}_{i,k=1,2,...}$ is a sequence of i.i.d. random variables which are exponentially distributed with mean $\frac{1}{\beta_Y}$ and the sequence of surrender values $Z = \{Z_{i,k}\}_{i,k=1,2,...}$ is a sequence of i.i.d. random variables which are exponentially distributed with mean $\frac{1}{\beta_Z}$, respectively.

Therefore, the moment generating functions of $X$, $Y$ and $Z$ are defined by $M_X(-r) = \dfrac{\beta_X}{\beta_X + r}$, $M_Y(r) = \dfrac{\beta_Y}{\beta_Y - r}$ and $M_Z(r) = \dfrac{\beta_Z}{\beta_Z - r}$, respectively, for $r < \min\{\beta_Y, \beta_Z\}$.

### 4.4.1 Numerical example for the ruin probability

In this section, we study the effect of the ruin probability in the term of premiums, claims and surrenders.

The approximation ruin probability, defined in (2.2), be written as

$$\Psi(u) \simeq e^{-Ru}$$

where $R$ is the adjustment coefficient.

Firstly, we will calculate the unique positive solution of the adjustment coefficient equation as follows.
From theorem 4.3.1, we have,

$$\begin{aligned}
0 &= c(r) \\
&= -rId + \lambda(1-\alpha)M_X(-r)\left(\frac{1}{1-\alpha M_X(-r)}\right) + \lambda(1-\alpha p)M_Y(r)\left(\frac{1}{1-\alpha p M_Y(r)}\right) \\
&\quad + \lambda(1-\alpha q)M_Z(r)\left(\frac{1}{1-\alpha q M_Z(r)}\right) - 3\lambda \\
&= -rId + \lambda(1-\alpha)\left(\frac{\beta_X}{(1-\alpha)\beta_X + r}\right) + \lambda(1-\alpha p)\left(\frac{\beta_Y}{(1-\alpha p)\beta_Y - r}\right) \\
&\quad + \lambda(1-\alpha q)\left(\frac{\beta_Z}{(1-\alpha q)\beta_Z - r}\right),
\end{aligned}$$

where $0 < p, q < 1$, $0 < p + q < 1$ and $r < \min\{(1-\alpha p)\beta_Y, (1-\alpha q)\beta_Z\}$.

Next, we will study the effect of ruin probability by changing the parameters of premium size, claim size, surrender values and investment in Section 4.4.1.1. In Section 4.4.1.2, we will consider the effect of ruin probability in terms of probabilities of claims and surrenders.

#### 4.4.1.1 Effects from premiums size, claim size and surrender value

In this section, we will discuss the trend of ruin probability comparing with income and expenses of insurance company where the income are determined by the premium sizes and investments and the expenses are determined by the claim sizes and surrender values, respectively. The parameters of the model considered in this section are the mean of premiums $\left(\frac{1}{\beta_X}\right)$, the mean of claims $\left(\frac{1}{\beta_Y}\right)$ and the mean of surrenders $\left(\frac{1}{\beta_Z}\right)$ and investment $(I)$.

For this section, we set the values of $I = 10, d = 0.2, \lambda = 2, \alpha = 0.25, p = 0.4, q = 0.07$ and the initial surplus $u = 12$.

Scenario 4.1 : The trend of ruin probability in terms of the parameter $\beta_X$ where the parameters $\beta_Y$ and $\beta_Z$ are fixed ($\beta_Y = \beta_Z = 1.5$). In this scenario, we consider different values of $\beta_X$ which are 0.5, 0.75, 1.0, 1.5 and 2.0, respectively. The values of the upper bound of the ruin probability are given in Table 4.1. The corresponding plot is presented in Figure 4.1.

**Table 4.1:** Parameter $\beta_X \in [0.5, 2]$ and their upper bound of ruin probability

| $\beta_X$ | 0.5 | 0.75 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| Upper bound | 0.003013 | 0.006638 | 0.013735 | 0.049486 | 0.145657 |

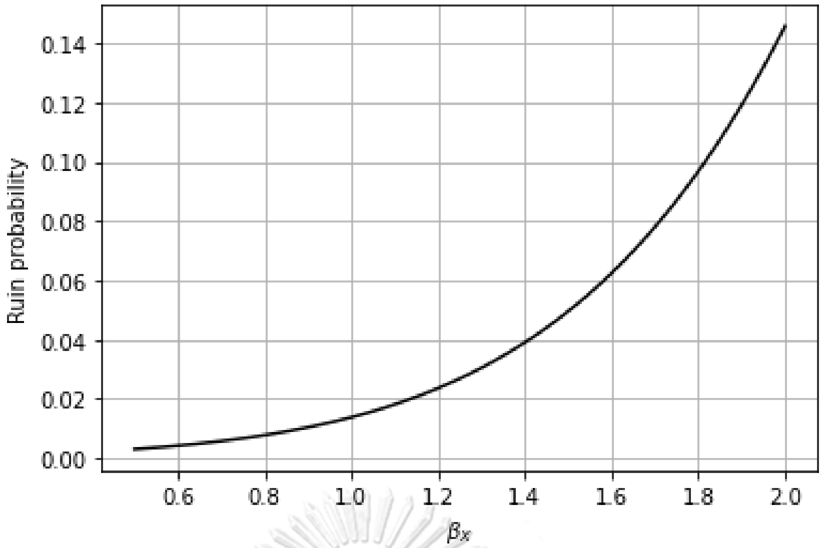


**Figure 4.1:** Trend of the ruin probability when $\beta_X$ increases for INAR(1) risk model

From Table 4.1 and Figure 4.1, we can see that the ruin probability increases when $\beta_X$ increases. That is the ruin probability increases when the mean of premium size decreases.

Scenario 4.2 : The trend of ruin probability in terms of the parameter $\beta_Y$ where the parameters $\beta_X$ and $\beta_Z$ are fixed ($\beta_X = 0.5$ and $\beta_Z = 1.5$). In this scenario, we consider different values of $\beta_Y$ which are 0.5, 0.75, 1.0, 1.5 and 2.0, respectively. The values of the upper bound of the ruin probability are given in Table 4.2. The corresponding plot is presented in Figure 4.2.



**Table 4.2:** Parameter $\beta_Y \in [0.5, 2]$ and their upper bound of ruin probability

| $\beta_Y$ | 0.5 | 0.75 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| Upper bound | 0.656398 | 0.189402 | 0.060250 | 0.008994 | 0.002348 |

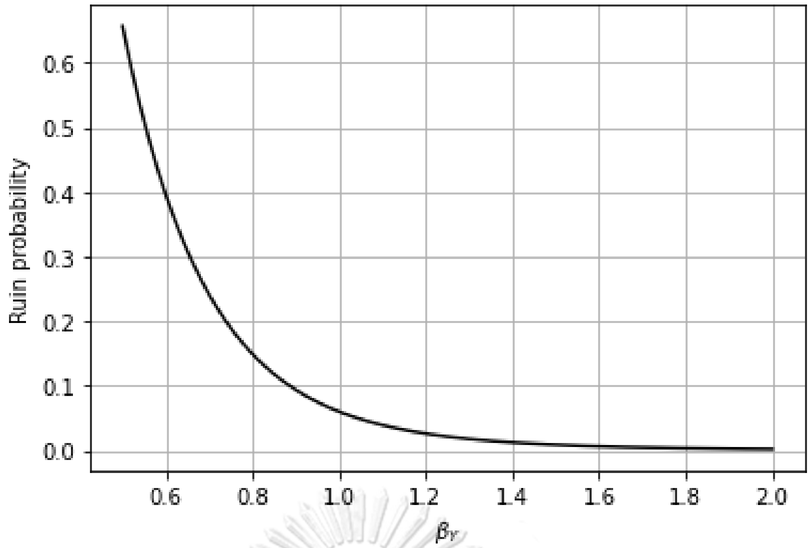


**Figure 4.2:** Trend of the ruin probability when $\beta_Y$ increases for INAR(1) risk model

Table 4.2 and Figure 4.2 show that the ruin probability decreases and reaches to 0 as parameter $\beta_Y$ increases. It means that the ruin probability decreases when the mean of claim size decreases.

Scenario 4.3 : The trend of ruin probability in terms of the parameter $\beta_Z$ where the parameters $\beta_X$ and $\beta_Y$ are fixed ($\beta_X = 0.5$ and $\beta_Y = 1.5$). In this scenario, we consider different values of $\beta_Z$ which are 0.5, 0.75, 1.0, 1.5 and 2.0, respectively. The values of the upper bound of the ruin probability are given in Table 4.3. The corresponding plot is presented in Figure 4.3.

**Table 4.3:** Parameter $\beta_Z \in [0.5, 2]$ and their upper bound of ruin probability

| $\beta_Z$ | 0.5 | 0.75 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| Upper bound | 0.558193 | 0.156177 | 0.050661 | 0.008994 | 0.002994 |

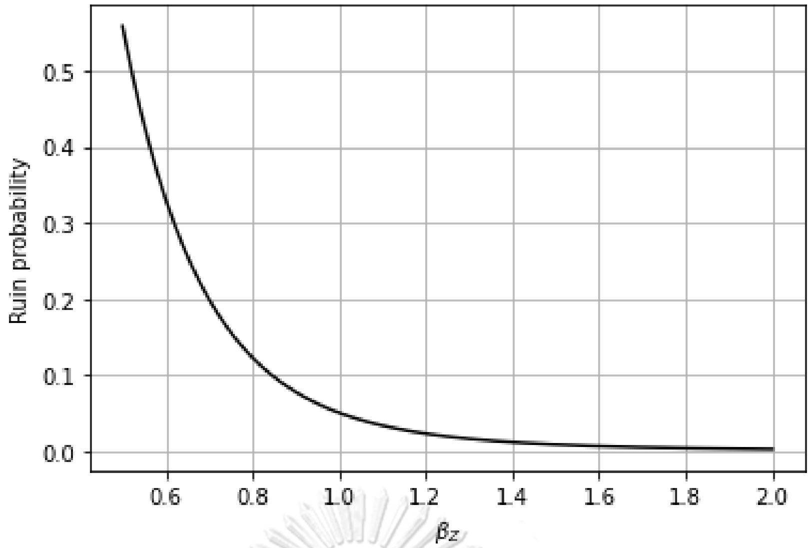


**Figure 4.3:** Trend of the ruin probability when $\beta_Z$ increases for INAR(1) risk model

From Table 4.3 and Figure 4.3, we can see that the ruin probability decreases and reaches to 0 when parameter $\beta_Z$ increases. It means that the ruin probability decreases when the mean of surrender value decreases.

Scenario 4.4 : The trend of ruin probability in terms of the investment $I$ where the parameters $\beta_X$, $\beta_Y$ and $\beta_Z$ are fixed ($\beta_X = 0.5$ and $\beta_Y = \beta_Z = 1.5$). In this scenario, we consider different values of investment which are 1.0, 3.0, 5.0, 8.0 and 10.0, respectively. The values of the upper bound of the ruin probability are given in Table 4.4. The corresponding plot is presented in Figure 4.4.



**Table 4.4:** Parameter $I \in [1, 10]$ and their upper bound of ruin probability

| Investment | 1.0 | 3.0 | 5.0 | 8.0 | 10.0 |
|---|---|---|---|---|---|
| Upper bound | 0.053791 | 0.029349 | 0.015519 | 0.005795 | 0.003013 |

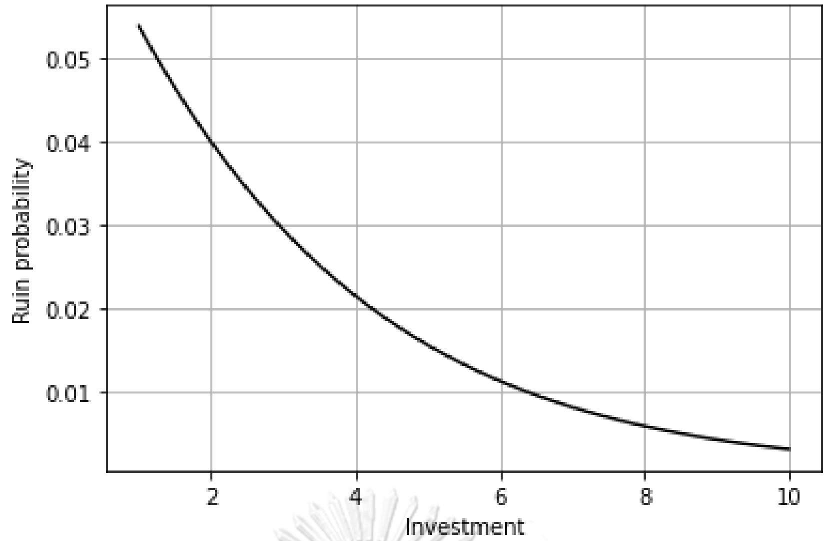


**Figure 4.4:** Trend of the ruin probability when $I$ increases for INAR(1) risk model

Table 4.4 and Figure 4.4 show that the ruin probability decreases when the investment increase. It means that the more amount the insurance company invests in financial markets, the smaller value of ruin probability.

From the result in Scenario 4.1 - 4.4, we can see that the ruin probability increases as the income of the model, determined by the mean of premiums and investment, decrease. Whereas the ruin probability decreases as the expenses of the model, determined by means of claims and surrender, decrease.

#### 4.4.1.2 Effects from the probabilities of claims and surrenders

In this section, we will study the change of the ruin probability in terms of the probability of claims ($p$) and the probability of surrenders ($q$).

For this section, we set the values of $I = 10, d = 0.2, \lambda = 2, \alpha = 0.25, \beta_X = 1, \beta_Y = 2, \beta_Z = 1.5$ and the initial surplus $u = 12$.

Scenario 4.5 : The trend of ruin probability in terms of probability of claims $p$ when $q$ is fixed ($q = 0.04$). In this scenario, we consider different values of $p$ which are 0.001, 0.01, 0.01, 0.5 and 0.9, respectively. The values of the upper bound of the ruin probability are given in Table 4.5. The corresponding plot is presented in Figure 4.5.

**Table 4.5:** Parameter $p \in (0, 0.9]$ and their upper bound of ruin probability

| $p$ | 0.001 | 0.01 | 0.1 | 0.5 | 0.9 |
|---|---|---|---|---|---|
| Upper bound | 0.00129 | 0.00127 | 0.001471 | 0.002683 | 0.005553 |

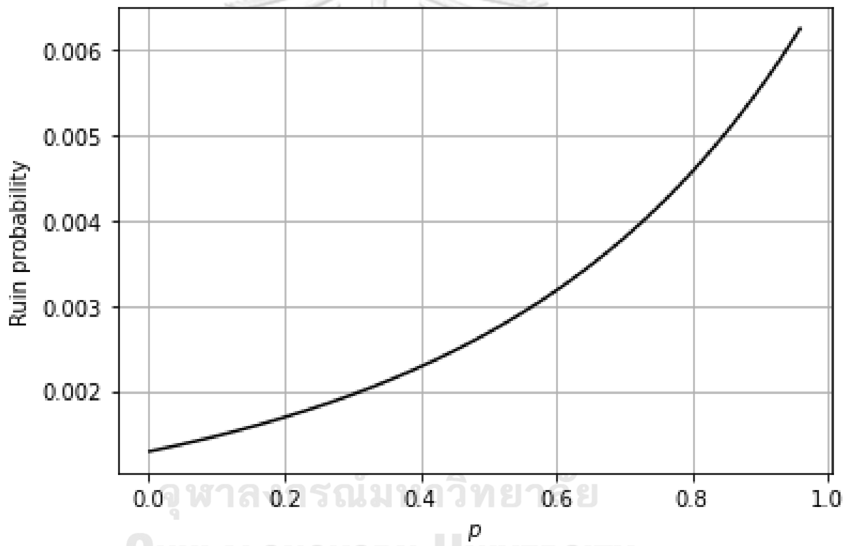




**Figure 4.5:** Trend of the ruin probability when $p$ increases for INAR(1) risk model

From Table 4.5 and Figure 4.5, we can see that the ruin probability increases as the probability $p$ increases. It means that the more claims occur, the higher value of ruin probability.

Scenario 4.6 : The trend of ruin probability in terms of probability of claims $q$ when $p$ is fixed ($p = 0.04$). In this scenario, we consider different values of $q$ which are 0.001, 0.01, 0.01, 0.5 and 0.9, respectively. The values of the upper bound of the ruin probability are given in Table 4.6. The corresponding plot is presented in Figure 4.6.

**Table 4.6:** Parameter $q \in (0, 0.9]$ and their upper bound of ruin probability

| $q$ | 0.001 | 0.01 | 0.1 | 0.5 | 0.9 |
|---|---|---|---|---|---|
| Upper bound | 0.001241 | 0.001267 | 0.001561 | 0.004093 | 0.011302 |

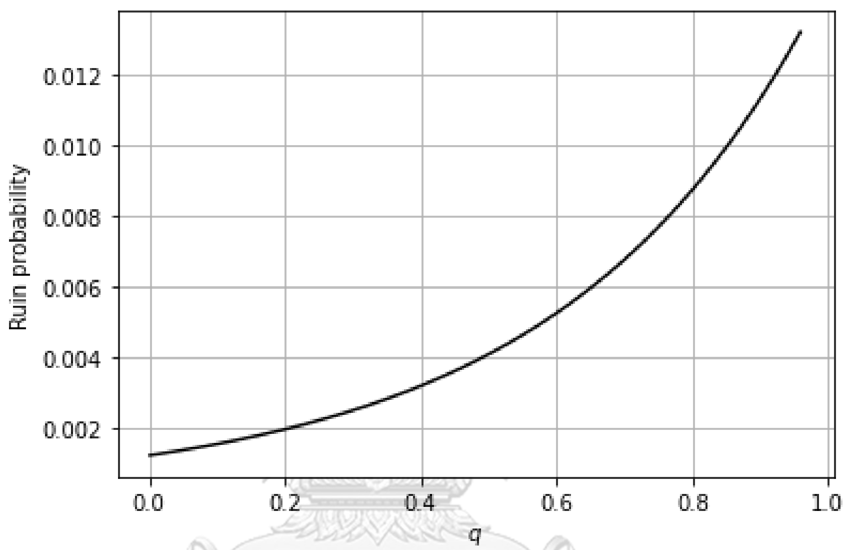


**Figure 4.6:** Trend of the ruin probability when $q$ increases for INAR(1) risk model



Table 4.6 and Figure 4.6 show that the ruin probability increases when parameter $q$ increases. It means that the more surrenders occur, the higher value of ruin probability.

The result of Scenario 4.5 and 4.6 show that the increased in the probability of claims ($p$) and the probability of surrenders ($q$) make more the value of ruin probability.

### 4.4.2 Numerical Example for the value at risk

In this section, we will study the value at risk (VaR) which is a risk measure measuring the risk of loss-profit process in our risk model.

The value at risk at the confidence level $\omega$ for the INAR(1) risk model, denoted by $\mathrm{VaR}_\omega(S_n)$, is the $\omega$-quantile of the distribution of the loss-profit $S_n$ of the risk model. The $\mathrm{VaR}_\omega(S_n)$ can be written as

$$\mathrm{VaR}_\omega(S_n) = \inf\{k \in \mathbb{R} | F_{S_n}(k) > \omega\}, \tag{4.29}$$

where $F_{S_n}(k)$ be the cumulative distribution function of $S_n$.
For our model, the loss-profit process $S_n$, define in (3.5), can be expressed as

$$\begin{aligned} S_n &= Idn + \sum_{i=1}^{n} A_i - \sum_{i=1}^{n} B_i - \sum_{i=1}^{n} C_i \\ &= Idn + \sum_{i=1}^{n}\sum_{k=1}^{N_i} X_{i,k} - \sum_{i=1}^{n}\sum_{k=1}^{N_i(p)} Y_{i,k} - \sum_{i=1}^{n}\sum_{k=1}^{N_i(q)} Z_{i,k}, \end{aligned}$$

where $\{N_i; i \in \mathbb{N}\}$, $\{N_i(p); i \in \mathbb{N}\}$ and $\{N_i(q); i \in \mathbb{N}\}$ follow INAR(1) model, defined in (4.3) - (4.5), respectively.



From (4.29), we can see that we need to have the distribution of $S_n$ in order to obtain the value at risk. However, it is difficult to obtain the distribution of $S_n$. Therefore, we will apply the Fast Fourier Transform (FFT) algorithm, proposed in [5], to approximate the distribution of $S_n$.

The characteristic function of $S_n$, denoted by $\phi_{S_n}(r)$, can be written as follows. Similar to provide $E[e^{-rS_n}]$ in the proof of Theorem 4.3.1, we have

$$\begin{aligned}\phi_{S_n}(r) &= E[e^{irS_n}]\\ &= E[e^{irIdn}] \cdot G_{P_n}(\phi_X(r)) \cdot G_{P_n(p)}(\phi_Y(-r)) \cdot G_{P_n(q)}(\phi_Z(-r)),\end{aligned}$$

where $P_n = N_1 + \cdots + N_n$, $P_n(p) = N_1(p) + \cdots + N_n(p)$ and $P_n(q) = N_1(q) + \cdots + N_n(q)$, where

(1) $G_{P_n}(\phi_X(r))$

$$\begin{aligned}&= \exp\left\{2\lambda\phi_X(r)\left(\frac{1-(\alpha\phi_X(r))^{n-1}}{1-\alpha\phi_X(r)}\right)\right.\\ &\quad + \lambda(1-\alpha)\phi_X(r)\left(\frac{n-2}{1-\alpha\phi_X(r)} - \frac{\alpha\phi_X(r)-(\alpha\phi_X(r))^n}{(1-\alpha\phi_X(r))^2}\right)\\ &\quad \left.+ \frac{\lambda}{1-\alpha}\big(\alpha^{n-1}\phi_X^n(r)-1\big) - \lambda(n-1)\right\},\end{aligned}$$

(2) $G_{P_n(p)}(\phi_Y(-r))$

$$\begin{aligned}&= \exp\left\{2\lambda\phi_Y(-r)\left(\frac{1-(\alpha p\phi_Y(-r))^{n-1}}{1-\alpha p\phi_Y(-r)}\right)\right.\\ &\quad + \lambda(1-\alpha p)\phi_Y(-r)\left(\frac{n-2}{1-\alpha p\phi_Y(-r)} - \frac{\alpha p\phi_Y(-r)-(\alpha p\phi_Y(-r))^n}{(1-\alpha p\phi_Y(-r))^2}\right)\\ &\quad \left.+ \frac{\lambda}{1-\alpha p}\big((\alpha p)^{n-1}\phi_Y^n(-r)-1\big) - \lambda(n-1)\right\},\end{aligned}$$

and

(3) $G_{P_n(q)}(\phi_Z(-r))$

$$\begin{aligned}&= \exp\left\{2\lambda\phi_Z(-r)\left(\frac{1-(\alpha q\phi_Z(-r))^{n-1}}{1-\alpha q\phi_Z(-r)}\right)\right.\\ &\quad + \lambda(1-\alpha q)\phi_Z(-r)\left(\frac{n-2}{1-\alpha q\phi_Z(-r)} - \frac{\alpha q\phi_Z(-r)-(\alpha q\phi_Z(-r))^n}{(1-\alpha q\phi_Z(-r))^2}\right)\\ &\quad \left.+ \frac{\lambda}{1-\alpha q}\big((\alpha q)^{n-1}\phi_Z^n(-r)-1\big) - \lambda(n-1)\right\}.\end{aligned}$$

Next, we will study the trend of the value at risk by varying various parameter of claim size and surrender values.

#### 4.4.2.1 Effects from claim size and surrender values

Since the value at risk focuses on the financial losses. Then we see the behavior of the value at risk against the loss parameters, which are the claim size and surrender value. In this section, we study the trend of the value at risk in terms of mean of claims $\left(\frac{1}{\beta_Y}\right)$ and the mean of surrenders $\left(\frac{1}{\beta_Z}\right)$ by varying the parameters $\beta_Y$ and $\beta_Z$, respectively.

For this section, we set the values of parameters $I = 10, d = 0.2, \lambda = 2, \alpha = 0.25, \beta_X = 0.5, p = 0.4, q = 0.07$ and $n = 12$.

Scenario 4.7 : The trend of value at risk in the terms of parameter $\beta_Y$ where the parameter $\beta_Z$ is fixed ($\beta_Z = 1.5$). The confidence level considered in this scenario is $\omega = 0.95$. In this scenario, we consider different values of $\beta_Y$ which are 0.5, 0.75, 1.0, 1.5 and 2.0, respectively. The Values at Risk are given in Table 4.7.



**Table 4.7:** Parameter $\beta_Y \in [0.5, 2]$ and $\mathrm{VaR}_{0.95}$ $(S_{12})$

| $\beta_Y$ | 0.5 | 0.75 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| $\mathrm{VaR}_{0.95}$ $(S_{12})$ | 118.47 | 117.96 | 116.40 | 112.44 | 110.22 |

Table 4.7 shows that the value at risk decreases when parameter $\beta_Y$ increases. That is the value at risk decreases when the mean of claim size decreases.

Scenario 4.8 : The trend of value at risk in terms of parameter $\beta_Z$ where the parameter $\beta_Y$ is fixed ($\beta_Y = 1.5$). The confidence level considered in this scenario is $\omega = 0.95$. In this scenario, we consider different values of $\beta_Z$ which are 0.5, 0.75, 1.0, 1.5 and 2.0, respectively. The Values at Risk are given in Table 4.8.

**Table 4.8:** Parameter $\beta_Z \in [0.5, 2]$ and $\text{VaR}_{0.95}$ $(S_{12})$

| $\beta_Z$ | 0.5 | 0.75 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| $\text{VaR}_{0.95}$ $(S_{12})$ | 118.50 | 117.81 | 116.19 | 112.44 | 110.37 |

Table 4.8 shows that the value at risk decreases as parameter $\beta_Z$ increases. That is the value at risk decreases when the mean of surrender value decreases.

From Scenario 4.7 and Scenario 4.8, we can conclude that the value at risk, described as the maximum loss of the company, decreases when the either claim sizes or surrender values decreases.

# CHAPTER V

# CONCLUSION AND FUTURE WORK

In this thesis, we introduce two discrete-time risk models by incorporating the concepts of investment and surrender where the number of premiums, claims and surrenders follow integer-valued time series.

In chapter 3, we construct the first order integer-valued moving average risk model with investment and surrender. We also provide some properties of the model. Moreover, we study the risk measures for this model which are the approximation of ruin probability and the value at risk. Finally, we discuss the risk measures of the model by numerical simulations.

In chapter 4, we construct the first order integer-valued autoregressive risk model with investment and surrender and derive some of its properties. Then, we provide the approximation of ruin probability of the model. We derive the adjustment coefficient of this model and prove that it has a unique positive solution. We discuss the trend of ruin probability and value at risk against the model parameters by numerical simulations in the last section of this chapter.



The research can be extended in many direction. For example, we can consider the concepts of investment and surrender to the higher orders of integer-valued moving average and integer-valued autoregressive processes of our data. We can also consider other general time series models such as autoregressive moving average process.

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