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ชื่อโครงการ คำตอบใหม่เชิงเฟอร์มิออนิคในทฤษฎีพลศาสตร์ไฟฟ้าแบบเฉิน -ไซม่อนส์

Other Fermionic Solutions of Chern-Simons Electrodynamics

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คำตอบใหม่เชิงเฟอร์มิออนิคในทฤษฎีพลศาสตร์ไฟฟ้าแบบเฉิน -ไซม่อนส์

ปวินท์ทร ทองสาริ

โครงงานนี้ศึกษาคำตอบแบบกระแสวนของระบบเฟอร์มิออนที่ถูกฝังอยู่ในกาลอวกาศ 2+1 มิติผ่านการทบทวนงานของ Cho, Kim และ Park โดยการมีอยู่ของคำตอบเป็นผลอันเนื่องมาจากพลวัตระหว่างสนาม Dirac กับ สนามเกจแบบ Chern-Simons ในขอบเขตระยะไกล จากการทบทวนงานของ Jackiw, Pi เพิ่มเติม ซึ่งเป็นการศึกษาคำตอบแบบกระแสวนที่มีทั้งสนามไฟฟ้าและสนามแม่เหล็กอันเป็นผลมาจากสมการ Schrodinger แบบไม่เชิงเส้นภายใต้สนามเกจแบบ Chern-Simons ทำให้เกิดเป็นความพยายามของโครงงานนี้ที่จะหาคำตอบแบบเดียวกันในระบบเฟอร์มิออนทั้งในขอบเขตที่เป็นสัมพัทธภาพและไม่เป็นสัมพัทธภาพ ผลจากการศึกษาพบว่า คำตอบเชิงสนามแม่เหล็กไฟฟ้าแบบแม่นตรงและแจ่มชัดไม่เกิดขึ้นในระบบเฟอร์มิออนที่เป็นสัมพัทธภาพ จากผลลัพธ์นี้ทำให้เกิดการต่อยอดเป็นการทบทวนงานของ Duval, Horvathy และ Palla ซึ่งเป็นการศึกษาคำตอบแบบกระแสวนของเฟอร์มิออนในระบบที่ไม่เป็นสัมพัทธภาพ เพื่อเปรียบเทียบกับงาน Jackiw, Pi และสืบเสาะหาสาเหตุการมีอยู่ของสนามไฟฟ้าในกระแสวนและตรวจสอบความเป็นไปได้ที่จะเกิดคำตอบแบบเดียวกันในระบบเฟอร์มิออน

คำสำคัญ ระบบเฟอร์มิออน, กาลอวกาศ 2+1 มิติ, สนามเกจแบบ Chern-Simons, คำตอบแบบหมุนวนเชิงสนามแม่เหล็กไฟฟ้า

Other Fermionic Solutions of Chern-Simons Electrodynamics

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Abstract

In this senior project, vortex solutions of the fermionic system constrained in the (2+1)-dimensional spacetime by Cho, Kim and Park (CKP) [1] are reviewed. The existence of solutions comes from dynamics of the Dirac field coupled with pure Chern-Simons gauge field in the long distance limit. Motivated by the study of electromagnetic vortex solutions emerged from the gauged, nonlinear Schrodinger equation by Jackiw and Pi (JP) [2], we attempt to find a new vortex solution in the fermionic and relativistic system that also generates electric field. A well-defined solution has not been found so far, we instead investigate the possibility of a non-vanishing electric field solution in the fermionic system. First, we review the JP solutions and the vortex solutions from CKP construction in the non-relativistic limit provided by Duval, Horváthy and Palla (DHP) [3]. Then, we compare these two solutions and argue the origin of the electric field. Finally, we conclude all the results obtained from these three theories.
Keywords: fermionic system, (2+1)-dimensional space-time, Chern-Simons gauge field, electromagnetic vortex

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Contents

1 Introduction

One of the reasons that makes the gauge theory in 2+1 dimensions interesting to study is the fact that we can introduce another gauge term which is both Lorentz invariant and gauge invariant besides the Maxwell term. That term is called the Chern-Simons term [4]. The Chern-Simons gauge field alone does not have any dynamics whatsoever so the term is useless on its own. However, if we couple it with the Maxwell term, the resulting theory will be able to describe a massive gauge field. Consider the Lagrangian

$$\mathcal{L}_{MCS} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\kappa}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}. \tag{1}$$

The corresponding equations of motion are

$$\partial_\mu F^{\mu\nu} + \frac{\kappa}{2}\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = 0. \tag{2}$$

By rewriting $F^{\mu\nu}$ in term of the pseudovector dual field $\tilde{F}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho}$, the equations of motion can be rewritten as

$$(\partial_\mu\partial^\mu + \kappa^2)\tilde{F}^\nu = 0, \tag{3}$$

which is the relativistic massive wave equation that describes the massive gauge field of mass κ.

We could also add a matter field to the above system and see what additional results we can obtain. The system of the Maxwell and Chern-Simons gauge field coupled to the matter current has the Lagrangian of the form

$$\mathcal{L}_{MMCS} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\kappa}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} - eA_\mu J^\mu. \tag{4}$$

where we use the mostly minus convention $\eta^{\mu\nu} = (+,-,-)$ and the coupling is e.

The Chern-Simons term may not look gauge invariant at first. But under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\Lambda$, the Lagrangian changes by the amount

$$\delta\mathcal{L} = \frac{\kappa}{2}\partial_\mu(\Lambda\partial_\nu A_\rho),$$

which vanishes on the boundary so $\mathcal{L}_{MMCS}$ is actually gauge invariant.

By varying the Lagrangian with respect to the gauge field, we obtain the equations of motion

$$\partial_\nu F^{\nu\mu} + \frac{\kappa}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho} = eJ^\mu. \tag{5}$$

Decomposing to each component, we obtain three equations,

$$e\rho = \partial_i E^i - \kappa B, \tag{6a}$$

$$eJ^i = -\partial_t E^i + \epsilon^{ij}\partial_j B - \kappa\epsilon^{ij}E_j. \tag{6b}$$

The original Maxwell equations are now modified by the Chern-Simons term. This leads to some of the fascinating consequences that we cannot expect from the

Maxwell term alone. To see it clearly let us consider the system in the long-distance limit where the Maxwell term can be dropped off. This can be understood by considering the length dimension from each gauge term in (5). We see that both have two gauge potential but the Chern-Simons term has lower derivative than the Maxwell term by one. Thus, κ has inverse length dimension (L^{-1}) and specifies the characteristic length of the system. Consider the length scale L, the Maxwell term is then suppressed by the factor $(1/|\kappa|)/L$ respect to the Chern-Simons term. Therefore, in the long-distance limit where L is large compared to $1/|\kappa|$, the Chern-Simons term dominates the system and the Maxwell term can be neglected. The equations then take the form

$$e\rho = -\kappa B, \tag{7a}$$

$$eJ^i = -\kappa\epsilon^{ij}E_j. \tag{7b}$$

The first equation tells us that the Chern-Simons term binding the matter source with a magnetic (gauge) field. Thus, wherever the particle is, there must be a magnetic flux couple to it as well. The second equation is the typical Hall term. Seeing this, we might assume that the Chern-Simons theory is a suitable field theory to deal with the quantum Hall effect which turns out to be true. In the context of quantum many-body system, many models involving the Chern-Simons terms successfully explain both the integer and fractional quantum Hall effects [5]. However, our purpose here in this senior project does not concern much with these phenomena. We are rather interested in the existence of solutions with certain properties in the Chern-Simons theory in the fermionic system which will be stated in details later.

Because the coupling to a matter field gives nontrivial physical aspects to the Chern-Simons theory, there have been many studies concerned with the consequence after coupling the Chern-Simons terms with scalar and spinor fields in both non-relativistic and relativistic regimes. Interestingly, all admits vortex solutions which are bound states between the matter field and the magnetic flux with some distinct properties depending on the theory. One of the most important work in this field is the study by Jackiw and Pi. They obtained vortex solutions from the gauged, nonlinear Schrodinger equation where the particle is coupled with the magnetic flux through the Chern-Simons interaction. The study is in the long-distance limit which means the Maxwell terms are neglected in this setting just like the CKP construction. Many properties of the solutions are similar to the CKP solutions except that they also have their own electric field. Motivated by this study, we search for a new solution in the CKP theory that also generates a nonzero electric field as well. We managed to find one numerical solution. The magnetic field and electric field are plotted and shown in section 2.

However, because the solution we obtain blows up at a finite distance, the existence of an electromagnetic solution in the fermionic system becomes questionable. It is possible that there might be some kind of mechanisms only occur in the fermionic system that prevent it? Initiating by this idea, We start tackling this problem by reviewing the JP solutions and the vortex solutions solved in the CKP setting in the non-relativistic limit which is provided by Duval, Horváthy and Palla. We then compare these solutions from two different theories and look out for the origin of the electric field. The existence of the electric field is shown to have a connection with the non-linear coupling factor of the system. No evidence support that the electromagnetic vortex could not exist in the fermionic case. We also obtain non-vanishing electric field solutions for the DHP case as a by-product of this speculation. Finally, we conclude all the results we have found thus far from these three theories.

2 Pure Magnetic Vortex CKP Solutions

2.1 Constructing the System

We start with the Lagrangian of the Quantum Electrodynamics (QED) theory in 3+1 dimensions with spinors in the chiral basis defined by

$$\mathcal{L}_{QED(3+1)} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\gamma^{\mu}D_{\mu} - m)\Psi, \tag{8}$$

$$\gamma^{\mu} = \begin{pmatrix} 0 & \bar{\sigma}^{\mu} \\ \sigma^{\mu} & 0 \end{pmatrix}.$$

where we choose the metric convention to be mostly minus $\eta^{\mu\nu} = (+,-,-)$ with charge e. This means $D_{\mu} = \partial_{\mu} + ieA_{\mu}$.

Next, we assume the axial symmetric solutions along the z-axis

$$\Psi = \begin{pmatrix} e^{ip_{+}z}\psi_{+}(t,x,y) \\ e^{ip_{-}z}\psi_{-}(t,x,y) \end{pmatrix}, \tag{9}$$

where the upper and lower components represent the right and left-handed spinor respectively.

Inserting Ψ into $\mathcal{L}_{3+1}$ and applying the dimensional reduction by integrating out the z component give us the effective QED Lagrangian in 2+1 dimensions

$$
\begin{aligned}
\mathcal{L}_{QED(2+1)} = & -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \bar{\psi}_{+}i\gamma_{+}^{\alpha}D_{\alpha}\psi_{+} + \bar{\psi}_{-}i\gamma_{-}^{\alpha}D_{\alpha}\psi_{-} - p_{+}\bar{\psi}_{+}\psi_{+} - p_{-}\bar{\psi}_{-}\psi_{-} \\
& - m(\psi_{+}^{\dagger}\psi_{-} + \psi_{-}^{\dagger}\psi_{+})
\end{aligned} \tag{10}
$$

where $\gamma_{\pm}^{\alpha}$ ($\alpha = 0,1,2$) are two sets of the gamma matrices in 2+1 dimensions defined as

$$
\gamma_{+}^{\alpha} = (\sigma^{3}, i\sigma^{2}, -i\sigma^{1}) \quad , \quad \gamma_{-}^{\alpha} = (-\sigma^{3}, i\sigma^{2}, -i\sigma^{1}).
$$

Notice that the original 4-spinor Ψ is now separated into two dependent fermionic fields. This Lagrangian becomes two coupled Dirac Lagrangian with different handedness in which the frequency $p_{\pm}$ play the role of the mass terms of fermions on this plane.

Now that our theory lives in 2+1 dimensions, it is possible to induce the Chern-Simons term through the one-loop computation of the fermion effective action [6]. The Lagrangian then describes the Chern-Simons quantum electrodynamics theory with the form

$$
\begin{aligned}
\mathcal{L}_{CSQED(2+1)} = & -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}A_{\alpha}F_{\beta\gamma} + \bar{\psi}_{+}i\gamma_{+}^{\alpha}D_{\alpha}\psi_{+} + \bar{\psi}_{-}i\gamma_{-}^{\alpha}D_{\alpha}\psi_{-} \\
& - p_{+}\bar{\psi}_{+}\psi_{+} - p_{-}\bar{\psi}_{-}\psi_{-} - m(\psi_{+}^{\dagger}\psi_{-} + \psi_{-}^{\dagger}\psi_{+}),
\end{aligned} \tag{11}
$$

where κ is the Chern-Simons coupling and its value is arbitrary.

2.2 Solving for the Solutions

For simplicity, we will consider the system in the long-distance limit where the Chern-Simons term dominates the Maxwell term and we assume we can neglect the Maxwell term overall. To find the solutions, first, we substitute (11) into the Euler-Lagrange equation which gives us the equations of motion

$$
\frac{\kappa}{2}\epsilon^{\alpha\beta\gamma}F_{\beta\gamma} = e\,(\bar{\psi}_{+}\gamma_{+}^{\alpha}\psi_{+} + \bar{\psi}_{-}\gamma_{-}^{\alpha}\psi_{-}), \tag{12a}
$$

$$
(i\gamma_{+}^{\alpha}D_{\alpha} - p_{+})\psi_{+} = m\sigma^{3}\psi_{-}, \tag{12b}
$$

$$
(i\gamma_{-}^{\alpha}D_{\alpha} - p_{-})\psi_{-} = -m\sigma^{3}\psi_{+}. \tag{12c}
$$

Next, we apply the following ansatz to (12a), (12b), (12c)

$$
\begin{aligned}
A_\alpha &= \left\{ \begin{array}{c} 0, \quad \alpha = t, \rho, \\ A(\rho), \alpha = \phi, \end{array} \right. \\
\psi_+ &= e^{iE_+ t} \begin{pmatrix} f_+(\rho) e^{ik_+\phi} \\ i g_+(\rho) e^{il_+\phi} \end{pmatrix}, \\
\psi_- &= e^{iE_- t} \begin{pmatrix} f_-(\rho) e^{ik_-\phi} \\ i g_-(\rho) e^{il_-\phi} \end{pmatrix},
\end{aligned} \tag{13}
$$

to get,

$$k_+ = l_+ - 1 = k_- = l_- - 1, \;\; E_+ = E_- = E, \;\; p_+ = p_- = p, \tag{14a}$$

$$-\frac{\kappa}{\rho}\frac{dA}{d\rho} = e\,(|f_+|^2 + |g_+|^2 + |f_-|^2 + |g_-|^2), \tag{14b}$$

$$0 = f_+ g_+^* - f_- g_-^*, \tag{14c}$$

$$\frac{df_+}{d\rho} = \frac{k_+ + eA}{\rho} f_+ - (E+p) g_+ + m g_-, \tag{14d}$$

$$\frac{dg_+}{d\rho} = -\frac{l_+ + eA}{\rho} g_+ + (E-p) f_+ - m f_-, \tag{14e}$$

$$\frac{df_-}{d\rho} = \frac{k_- + eA}{\rho} f_- + (E-p) g_- - m g_+, \tag{14f}$$

$$\frac{dg_-}{d\rho} = -\frac{l_- + eA}{\rho} g_- - (E+p) f_- + m f_+, \tag{14g}$$

with the on-shell condition

$$E^2 = p^2 + m^2.$$

Now, we are ready to solve the equations.

Case 1: $\kappa > 0$

We choose $g_+ = g_- = 0$ to satisfy the constraint and redefine $k_+ = n$. (14) is then reduced to

$$-\frac{\kappa}{\rho}\frac{dA}{d\rho} = e\;\,(|f_+|^2 + |f_-|^2), \tag{15a}$$

$$\frac{df_+}{d\rho} = \frac{n + eA}{\rho} f_+, \tag{15b}$$

$$f_- = \frac{E-p}{m} f_+. \tag{15c}$$

This can be further reduced into a single differential equation with only one unknown variable. The final equation is

$$\frac{1}{\rho}\frac{d^2A}{d\rho} - (n + eA + \frac{1}{2})\frac{2}{\rho^2}\frac{dA}{d\rho} = 0. \tag{16}$$

By solving this nonlinear differential equation above, we obtain the solution

$$\begin{aligned} A(\rho) &= -\frac{2(n+1)}{e}\left(\frac{\rho^{2(eA_0+n+1)}}{\rho^{2(eA_0+n+1)}+\lambda^2}\right) - A_0\left(\frac{\rho^{2(eA_0+n+1)}-\lambda^2}{\rho^{2(eA_0+n+1)}+\lambda^2}\right) \\ f_+ &= \frac{(eA_0+n+1)\lambda}{e}\left[2\mu\left(\frac{E+p}{E}\right)\right]^{1/2}\left(\frac{\rho^{eA_0+n}}{\rho^{2(eA_0+n+1)}+\lambda^2}\right). \end{aligned} \tag{17}$$

Case 2: $\kappa < 0$

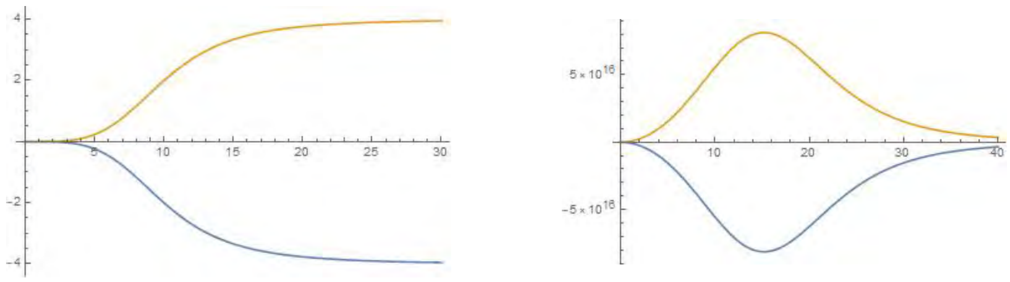
Knowing the solution in the first case, it becomes a trivial task to obtain the solution for this case by observing that setting $\kappa < 0$ is equivalent to the first case but with replacement $e \to -e$. To obtain the equation that has a similar form to (15b), we instead choose $f_+ = f_- = 0$ and redefine $l_- = n$. The set of reduced equations now become almost identical to (15) where $f_+ \to g_-$ and $f_- \to g_+$. Thus, the set of solutions are

$$\begin{aligned} A &= \frac{2(n+1)}{e}\left(\frac{\rho^{2(-eA_0+n+1)}}{\rho^{2(-eA_0+n+1)}+\lambda^2}\right) - A_0\left(\frac{\rho^{2(-eA_0+n+1)}-\lambda^2}{\rho^{2(-eA_0+n+1)}+\lambda^2}\right), \\ g_- &= \frac{(-eA_0+n+1)\lambda}{e}\left[-2\mu\left(\frac{E+p}{E}\right)\right]^{1/2}\left(\frac{\rho^{(-eA_0+n)}}{\rho^{2(-eA_0+n+1)}+\lambda^2}\right), \\ g_+ &= \frac{E-p}{m}g_-. \end{aligned} \tag{18}$$

The solutions in both cases are shown in Figure 1 in which the bottom plot is for $\kappa > 0$ case and the upper plot is for $\kappa < 0$ case respectively.

Let us show that the solutions we obtained above are actually special cases of the Liouville equation. To do that, we will assume a more general ansatz

$$\begin{aligned} A_t = 0, &\quad \partial_t A_i = 0, \\ \psi_+ &= e^{-iE_+t}\begin{pmatrix} F_+(x,y) \\ iG_+(x,y) \end{pmatrix}, \\ \psi_- &= e^{-iE_-t}\begin{pmatrix} F_-(x,y) \\ iG_-(x,y) \end{pmatrix}, \end{aligned} \tag{19}$$



(a) A vs ρ (b) B vs ρ

Figure 1: The vector potential A and its corresponding magnetic field B from the pure magnetic vortex solutions

with

$$\begin{aligned} E_+ = E_- = E, \quad p_+ = p_- = p, \\ E^2 - p^2 = m^2, \quad F_- = \frac{E-p}{m} F_+, \quad G_+ = \frac{E-p}{m} G_-. \end{aligned} \tag{20}$$

(12) then becomes

$$\begin{aligned} \kappa \epsilon^{ij} \partial_i A_j = -\frac{2eE}{E+p}(|F_+|^2 + |G_-|^2), \\ (D_1 + iD_2)F_\pm = 0, \quad (D_1 - iD_2)G_\pm = 0. \end{aligned} \tag{21}$$

By setting either $F_\pm$ or $G_\pm$ to zero, The other will immediately satisfy the Liouville equation and thus, give us an exact solution which can be restored to the form above by imposing the axial symmetry. The reason we emphasize this point here is because in JP case and DHP case, we will encounter with the Liouville equation again which show us how similar they are in terms of mathematical structure.

2.3 Physical Properties and Conserved Quantities of the CKP Solutions

The solutions we obtained from section 2.2 are clearly the vortex solutions (the gauge vector field curled around the origin) where λ determines the size of the vortex because the peak's position of the vortex depends on λ. What is less clear about the solutions is the gauge potential at the origin $A(\rho = 0) = A_0$. We define A as the vector potential in the azimuthal direction which means that if we have a non-vanishing value of A at the origin, it must be a singularity (it points in every direction). The authors argued that this singularity is harmless and physically acceptable because we can think of it as an infinitely thin solenoid pass through the origin along the z-direction. However, it is a complicated task to generate a singular gauge field from a limit of the solenoid so we will assume that $A_0 = 0$ for simplicity.

The first two physical quantities we consider are the magnetic flux Φ and the charge q. The magnetic flux pass through the circle area of radius ρ is

$$\Phi(\rho) = \int_{Area} \vec{B} \cdot d\vec{a} = \int_0^{\rho} \frac{1}{\rho}\frac{dA}{d\rho}\rho d\rho d\phi = 2\pi A(\rho). \tag{22}$$

For the total magnetic flux

$$\Phi(\infty) = 2\pi A(\infty) = \mp\frac{4\pi}{e}(n+1), \tag{23}$$

where the upper sign is for case $\kappa > 0$, lower sign is for case $\kappa < 0$. Notice that there is an additional parameter n contribute to the magnetic flux. Looking back at our ansatz (13), This n is actually the coefficient inside the exponential function of the azimuthal part. For the solutions to be single-valued and thus physical, n must be an integer value with the constraint $n + 1 > 0$ so we could say that n specifies the mode of the vortices.

The total charge can be directly calculated from (6a) where the charge is coupled to the magnetic flux

$$q = \mp\kappa\Phi. \tag{24}$$

The conserved quantities of the system can also be computed from the energy momentum tensor (symmetric and gauge invariant) which is given by

$$T_{\alpha\beta} = \frac{i}{2}[\bar{\psi}_+(\gamma_{+\alpha}D_\beta + \gamma_{+\beta}D_\alpha)\psi_+ + \bar{\psi}_-(\gamma_{-\alpha}D_\beta + \gamma_{-\beta}D_\alpha)\psi_-]. \tag{25}$$

Knowing the energy-momentum tensor, we can calculate the total energy $\mathcal{E}$, and total angular momentum $\mathcal{J}$ of the vortices to be

$$\begin{aligned} \mathcal{E} &= \frac{q}{e}E, \\ \mathcal{J} &= \frac{\kappa}{2e}\Phi = \mp\frac{q}{2e}. \end{aligned} \tag{26}$$

What worth mentioning are that even though the solutions are special cases (axially symmetric) of the solutions that satisfy the Liouville equation. They do have non-vanishing energy unlike the same solutions that was found in Jackiw and Pi work. The total angular momentum $\mathcal{J} = \mp q/2e$ in CKP case is also only half of the value $\mathcal{J} = \mp q/e$ in JP case. This is something to be expected because these solutions describe fermions, unlike them that describe a scalar particle.

3 Electromagnetic Vortex Solution from the CKP Theory

From the work of Jackiw and Pi, they found out that possible solutions for the Chern-Simons gauged, non-linear Schrödinger equations in 2+1 dimension describing a scalar particle are zero mode and have non-vanishing electric field. This becomes our motivation to seek for a vortex solution that also admits a non-vanishing electric field in the Chern-Simons quantum electrodynamics theory.

We initially guess the ansatz of the form

$$\begin{aligned} A_\alpha &= \begin{cases} A_0(\rho), & \alpha = t, \\ 0, & \alpha = \rho, \\ A(\rho), & \alpha = \phi, \end{cases} \\ \psi_+ &= e^{-iE_+t}\begin{pmatrix} f_+(\rho)e^{ik_+\phi} \\ ig_+(\rho)e^{il_+\phi} \end{pmatrix}, \\ \psi_- &= e^{-iE_-t}\begin{pmatrix} f_-(\rho)e^{ik_-\phi} \\ ig_-(\rho)e^{il_-\phi} \end{pmatrix}. \end{aligned} \tag{27}$$

See that our ansatz is almost identical to the one in the pure magnetic vortex case except the non-vanishing gauge field in time component. So we should expect that substituting (27) in (12) will give us almost the same form as (14). Our guess is true in which the only differences are that the energy E is shifted by the electric potential of the value $-eA_0$ and the current equation (14c) is not forced to be zero anymore.

For (12a), we consider each component,

$(12a) \quad \alpha = 1 \;;$

$$\begin{aligned} \kappa\epsilon^{120}F_{20} =& e\left[\begin{pmatrix} f_+^* e^{-ik_+\phi} & -ig_+^* e^{-il_+\phi} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} f_-e^{ik_-\phi} \\ ig_-e^{il_-\phi} \end{pmatrix}\right. \\ &\left. -\begin{pmatrix} f_-^* e^{-ik_-\phi} & -ig_-^* e^{-il_-\phi} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} f_+e^{ik_+\phi} \\ ig_+e^{il_+\phi} \end{pmatrix}\right] \\ -\kappa E_2 =& e\left[if_+^*g_+e^{-i(k_+-l_+)\phi} - ig_+^*f_+e^{i(k_+-l_+)\phi}\right. \\ &\left. -if_-^*g_-e^{-i(k_--l_-)\phi} + ig_-^*f_-e^{i(k_--l_-)\phi}\right] \end{aligned} \tag{28}$$

(12a) $\alpha = 2$;

$$\begin{aligned}
\kappa\epsilon^{210}F_{10} =& e\left[\begin{pmatrix} f_+^* e^{-ik_+\phi} & -ig_+^* e^{-il_+\phi}\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix} f_- e^{ik_-\phi}\\ ig_- e^{il_-\phi}\end{pmatrix}\right.\\
&\left.-\begin{pmatrix} f_-^* e^{-ik_-\phi} & -ig_-^* e^{-il_-\phi}\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix} f_+ e^{ik_+\phi}\\ ig_+ e^{il_+\phi}\end{pmatrix}\right] \qquad (29)\\
\kappa E_1 =& e\left[f_+^* g_+ e^{-i(k_+-l_+)\phi} + g_+^* f_+ e^{i(k_+-l_+)\phi}\right.\\
&\left.- f_-^* g_- e^{-i(k_--l_-)\phi} - g_-^* f_- e^{i(k_--l_-)\phi}\right]
\end{aligned}$$

(12a) $\alpha = 0$;

$$-\frac{\kappa}{\rho}\frac{dA}{d\rho} = e(|f_+|^2 + |g_+|^2 + |f_-|^2 + |g_-|^2) \qquad (30)$$

Imposing $k_+ = k_- = l_+ - 1 = l_- - 1 = k$,

(28) ;

$$\begin{aligned}
\kappa\partial_y A_0 =& e[if_+^* g_+ e^{i\phi} - ig_+^* f_+ e^{-i\phi} - if_-^* g_- e^{i\phi} - ig_-^* f_- e^{-i\phi}]\\
=& ie[(f_+^* g_+ - g_+^* f_+)\cos\phi + (f_+^* g_+ + g_+^* f_+) i\sin\phi \qquad (31)\\
&- (f_-^* g_- - g_-^* f_-)\cos\phi - (f_-^* g_- + g_-^* f_-) i\sin\phi]
\end{aligned}$$

(29) ;

$$\begin{aligned}
-\kappa\partial_x A_0 =& e[f_+^* g_+ e^{i\phi} + g_+^* f_+ e^{-i\phi} - f_-^* g_- e^{i\phi} + g_-^* f_- e^{-i\phi}]\\
=& e[(f_+^* g_+ + g_+^* f_+)\cos\phi + (f_+^* g_+ - g_+^* f_+) i\sin\phi \qquad (32)\\
&- (f_-^* g_- + g_-^* f_-)\cos\phi - (f_-^* g_- - g_-^* f_-) i\sin\phi]
\end{aligned}$$

$$\begin{aligned}
;\ \ \partial_y A_0 &= \frac{\partial\rho}{\partial y}\partial_\rho A_0 + \frac{\partial\phi}{\partial y}\partial\phi A_0 = \sin\phi\,\partial_\rho A_0 + \frac{\cos\phi}{\rho}\partial_\phi A_0\\
\partial_x A_0 &= \cos\phi\,\partial_\rho A_0 - \frac{\sin\phi}{\rho}\partial_\phi A_0
\end{aligned}$$

(31) $\sin\phi$ − (32) $\cos\phi$;

$$-\frac{\kappa}{e}\partial_\rho A_0 = f_+^* g_+ + g_+^* f_+ - f_-^* g_- - g_-^* f_- \qquad (33)$$

(31) $\cos\phi$ + (32) $\sin\phi$;

$$\frac{\kappa}{e\rho}\partial_\phi A_0 = -i(f_+^* g_+ - g_+^* f_+ - f_-^* g_- + g_-^* f_-)$$

Input the ansatz $A_0 = A_0(\rho) \to \partial_\phi A_0 = 0$

$$0 = f_+^* g_+ - g_+^* f_+ - f_-^* g_- + g_-^* f_-. \tag{34}$$

To reduce the differential equations and still make the solution satisfies the constraint (34), we choose the choice

$$f_+ = f_-^*, \quad g_+^* = -g_-,$$

$$E_+ = E_- = E, \quad p_+ = -p_- = p,$$

and also assume that $f_\pm$ and $g_\pm$ are real functions. Then,

$$(33) \quad ; \quad 0 = 0$$

$$(34) \quad ; \quad -\frac{\kappa}{e}\partial_\rho A_0 = 4(f_+ g_+) \tag{35}$$

$$(14b) \quad ; \quad -\frac{\kappa}{\rho}\frac{dA}{d\rho} = 2e(f_+^2 + g_+^2) \tag{36}$$

$$(14d) \quad E \to E - eA_0 \quad ; \quad \frac{df_-}{d\rho} - \frac{k + eA}{\rho} f_- - (E - eA_0 + p) g_- = -m g_+ \tag{37}$$

$$(14e) \quad E \to E - eA_0 \quad ; \quad \frac{dg_+}{d\rho} + \frac{k + 1 + eA}{\rho} g_+ - (E - eA_0 - p) f_+ = -m f_- \tag{38}$$

$$(14f) \quad E \to E - eA_0 \quad ; \quad \frac{df_-}{d\rho} - \frac{k + eA}{\rho} f_- - (E - eA_0 - p) g_- = -m g_+ \tag{39}$$

$$(14g) \quad E \to E - eA_0 \quad ; \quad -\frac{dg_+}{d\rho} - \frac{k + 1 + eA}{\rho} g_+ + (E - eA_0 + p) f_+ = m f_- \tag{40}$$

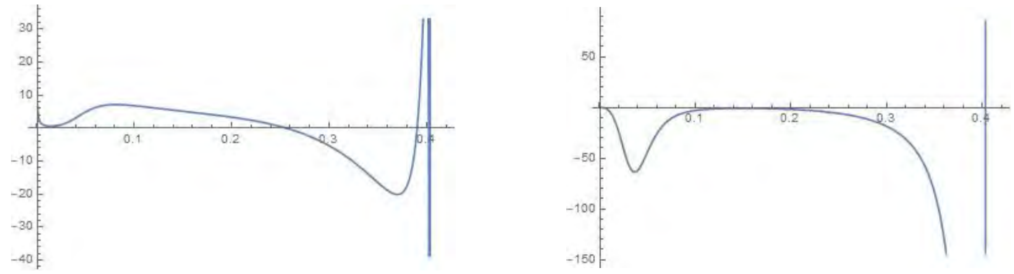
We see that (37)=(39) and (38)=(40) when $p = m = 0$ so we will set it like this. This also forces $E = 0$ from the on-shell condition. Now, the four equations below are reduced to two and we are left with total four coupled differential equations with four unknown variables

$$(35) \quad ; \quad -\frac{\kappa}{e}\frac{dA_0}{d\rho} = 4(f_+ g_+) \tag{41}$$

$$(36) \quad ; \quad -\frac{\kappa}{\rho}\frac{dA}{d\rho} = e(f_+^2 + g_+^2) \tag{42}$$

$$(37) \quad ; \quad \frac{df_+}{d\rho} - \frac{k + eA}{\rho} f_+ - eA_0 g_+ = 0 \tag{43}$$

$$(38) \quad ; \quad \frac{dg_+}{d\rho} + \frac{k + 1 + eA}{\rho} g_+ + eA_0 f_+ = 0 \tag{44}$$



(a) E vs ρ

(b) B vs ρ

Figure 2: The electric field and magnetic field of the electromagnetic vortex solution

There is no known exact solutions for this set of equations up until now. The best we can do is to solve it numerically by setting appropriate boundary conditions for each function.

One solution that we found so far is from the boundary conditions

$$A(0.001) = 0.001, \;\; A_0(0.001) = 0.001, \;\; f(0.001) = 1, \;\; g(0.001) = 1,$$

with parameters $\mu = 1, e = 1.6, n = 1$. The results are shown in Figure 2.

Even though we could find a solution, it does have a crucial problem. Both electric field and magnetic field blow up at a finite radial distance which should not be possible in nature. we suspect either that the boundary condition in this case is still inappropriate or the numerical method we use is still not efficient. But put that aside, if we look far beyond the problem of boundary conditions and ask its physics instead, an open question arise. Is it possible for the fermionic system to admit non-vanishing electric field from Chern-Simons dynamics in the first place? The motivation we got come from the system of scalar particles so it is possible that there might be some mechanisms that prevent this electric property in ferminonic particles. An attempt to answer this question is shown in the next section.

4 Possibility of Electromagnetic Vortex Solutions in Fermionic System

4.1 Vortex Solutions by Jackiw and Pi

To answer whether the fermionic system under the influence of the Chern-Simons gauge field do admit an electromagnetic vortex solution or not, we first investigate where our motivation arises. Jackiw and Pi showed us that the gauged, nonlinear Schrödinger equation naturally gives zero mode vortex solutions if the scalar field couple to the Chern-Simons gauge field. They start with the Lagrangian

$$\mathcal{L}_{JP} = \frac{\kappa}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} + i\hbar\Psi^* \left(\partial_t + \frac{ie}{\hbar}A^0\right)\Psi - \frac{\hbar^2}{2m}\left|\left(\nabla - \frac{ie}{\hbar c}\vec{A}\right)\Psi\right|^2 + \frac{g}{2}(\Psi^*\Psi)^2. \tag{45}$$

where the convention of the metric they use is mostly minus $\eta^{\mu\nu} = (+,-,-)$ and the charge is e. For the importance of each term, the Chern-Simons terms give us dynamics of gauge fields. The second and third term are the Schrödinger Lagrangian. And the last term represents nonlinear behavior of the system which is the interaction of the particle with itself. This term may not look familiar yet but after we specify the nonlinear coupling g later, the physical meaning of the last term will reveal itself.

The gauge equations are identical to the ones we show in section 1 after we apply the long-distance limit. They are

$$e\rho = -\kappa B, \tag{46a}$$
$$eJ^i = -\kappa\epsilon^{ij}E_j. \tag{46b}$$

Knowing these gauge field equations, we can express the gauge field A_μ in terms of the matter source and current as

$$\vec{A}(t,\vec{r}) = \frac{1}{\kappa}\int d^2\vec{r}'\vec{G}(\vec{r}-\vec{r}')\rho(t,\vec{r}'), \tag{47a}$$
$$A_0(t,\vec{r}) = \frac{1}{\kappa}\int d^2\vec{r}'\vec{G}(\vec{r}-\vec{r}')\cdot\vec{j}(t,\vec{r}'), \tag{47b}$$

where $\vec{G}$ is the Green's function

$$G^i(\vec{r}) = \frac{1}{2\pi}\epsilon^{ij}\partial_j \ln r, \tag{48}$$

that satisfies

$$\nabla\times\vec{G}(\vec{r}) = -\delta^2(\vec{r}). \tag{49}$$

Because $\vec{A}$ and A_0 are dependent on Ψ^* and Ψ, substituting $\mathcal{L}_{JP}$ into the Euler-Lagrange equation to obtain the matter field Ψ equation will make the task complicated. Instead, we will use the Hamiltonian formalism to deal with the problem. Legendre transformation from L_{JP} give us the Hamiltonian

$$H_{JP} = \int d^2\vec{r} \, \left\{ \frac{\hbar^2}{2m} |\vec{D}\Psi|^2 - \frac{g}{2}(\Psi^*\Psi)^2 \right\}. \tag{50}$$

where $\vec{D} \equiv \left(\nabla - \frac{ie}{\hbar c}\vec{A}\right)$.

Then, Ψ equation can be obtained from the Hamiltonian field equation

$$i\hbar \, \partial_t \Psi(t, \vec{r}) = \frac{\delta H}{\delta \Psi^*(t, \vec{r})}. \tag{51}$$

Supplemented by (47), we obtain the equation

$$i\hbar \, \partial_t \Psi = \left[-\frac{\hbar^2}{2m}\vec{D}^2 + eA^0 - g(\Psi^*\Psi) \right] \Psi. \tag{52}$$

With the form of the Schrödinger equation, we could say that Ψ describes quantum particles moving in the vicinity of the nonlinear potential with strength $-g$. Thus, the matter current J^μ follow directly from the usual Schrödinger equation.

$$J^\mu = (\rho c, \vec{J}) = \left(c\Psi^*\Psi, \frac{\hbar}{2mi}[\Psi^*(\vec{D}\Psi) - \Psi(\vec{D}\Psi)^*] \right). \tag{53}$$

Now, to solve the equation, we assume a static system in which the zero mode solutions satisfy the self-dual ansatz

$$(D_1 \pm iD_2)\Psi = 0. \tag{54}$$

We can make the equation looks more solvable by assuming one more ansatz

$$\Psi = \exp\left(i\frac{e}{\hbar c}\omega\right) \rho^{1/2}. \tag{55}$$

Substituting Ψ into the self-dual equation gives us the gauge field

$$\vec{A} = \nabla\omega \pm \frac{\hbar c}{2e} \nabla \times \ln \rho. \tag{56}$$

We see that ω is just an arbitrary gauge which can be set to zero for simplicity.

To solve for J^μ we substitute $\vec{A}$ into the gauge equation (46a)

$$\begin{aligned} B &= -\frac{e}{\kappa}\rho \\ \partial_1\partial_2\omega - \partial_2\partial_1\omega \pm \frac{\hbar c}{2e}(\epsilon^{21}\partial_1^2\, ln\rho - \epsilon^{12}\partial_2^2\, ln\rho) &= -\frac{e}{\kappa}\rho \\ \nabla^2 \,\ln\rho &= \pm\frac{2e^2}{\hbar c\kappa}\rho. \end{aligned}$$

This is the Liouville equation which admits an exact solution only when the coefficient on the right hand side has a negative value. The sign $\pm$ we choose must be opposite to the sign of κ. Therefore, we could write the equation as

$$\nabla^2 \,\ln\rho = -\frac{2e^2}{\hbar c|\kappa|}\rho, \tag{57}$$

and in order to avoid sign confusion, here and henceforth we will replace $\pm$ with $-\kappa/|\kappa|$.

The corresponding solution is of the form

$$\rho = \frac{4}{\alpha}\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}, \tag{58}$$

where z is a complex variable, $\alpha = e^2/(\hbar c|\kappa|)$ and f(z) is an arbitrary function.

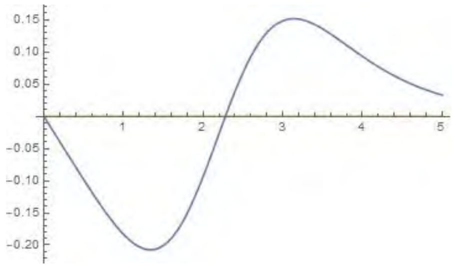
We can construct a general axially symmetric solution out of the Liouville equation by setting $f(z) = \left(\frac{r_0}{r}\right)^n$. The solution becomes

$$\rho = \frac{4n^2}{\alpha r^2}\left[\left(\frac{r_0}{r}\right)^n + \left(\frac{r}{r_0}\right)^n\right]^{-2}. \tag{59}$$

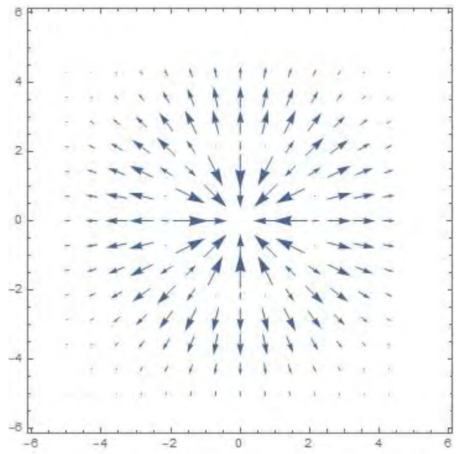
Notice the similarity between this solution and the one we solve in the CKP theory, this is because both are solutions of the same Liouville equation and assume the same axially symmetric property. B also relates to ρ in the same way thus they are solutions that describe a vortex.

Next, we can relate $\vec{J}$ to ρ using the definition (53). Substituting $\vec{A}$ from (56) to get

$$\begin{aligned} J^k &= \frac{\hbar}{2mi}\left[\Psi^*(\partial_k - \frac{ie}{\hbar c}A^k)\Psi - \Psi(\partial_k + \frac{ie}{\hbar c}A^k)\Psi^*\right] \\ &= -\frac{e\rho}{mc}\left(-\frac{\kappa}{|\kappa|}\frac{\hbar c}{2e}\epsilon^{kj}\partial j\ln\rho\right) \\ &= -\left(-\frac{\kappa}{|\kappa|}\frac{\hbar}{2m}\right)\epsilon^{kj}\partial_j\rho. \end{aligned} \tag{60}$$



(a) Radial plot



(b) Vector field plot

Figure 3: Plots show the electric field of the JP solutions

This non-vanishing $\vec{J}$ is what makes this solution interesting. By inserting $\vec{J}$ into the second gauge field equations (6b), we obtain a non-vanishing electric field

$$E^i = -\left(-\frac{\kappa}{|\kappa|}\frac{e\hbar}{2mc\kappa}\right)\epsilon^{ij}\epsilon^{jk}\partial_k\rho = -\frac{e\hbar}{2mc|\kappa|}\partial_i\rho, \tag{61}$$

The radial and vector field plot on the plane of this electric field are shown in Figure 3 in which we set all constants to unity and the parameters $r_0 = 3, n = 2$.

Now we will show that this self-dual ansatz do make the equations consistent. First, we consider the Hamiltonian of the system

we rewrite the Hamiltonian (50) using the identity

$$|\vec{D}\Psi|^2 = \left|\left(D_1 - \frac{\kappa}{|\kappa|}iD_2\right)\Psi\right|^2 - \frac{\kappa}{|\kappa|}\frac{m}{\hbar}(\nabla\times\vec{J}) - \frac{\kappa}{|\kappa|}\frac{e}{\hbar c}B(\Psi^*\Psi). \tag{62}$$

The Hamiltonian then becomes

$$\mathcal{H} = \frac{\hbar^2}{2m}\left|\left(D_1 - \frac{\kappa}{|\kappa|}iD_2\right)\Psi\right|^2 - \frac{\kappa}{|\kappa|}\frac{\hbar}{2}(\nabla\times\vec{J}) - \left(\frac{g}{2} - \frac{\kappa}{|\kappa|}\frac{e^2\hbar}{2mc\kappa}\right)(\Psi^*\Psi)^2. \tag{63}$$

For a well-behaved matter field throughout the plane, $\nabla\times\vec{J}$ vanishes after integrating over space. Due to the arbitrariness of g, we can also set $g = e^2\hbar/(mc|\kappa|)$ (shortly, we will see that this is in fact the natural choice for g) so that the only first term survives.

$$\mathcal{H} = \frac{\hbar^2}{2m}\left|\left(D_1 - \frac{\kappa}{|\kappa|}iD_2\right)\Psi\right|^2.$$

Thus, The minimum energy can be obtained if it satisfies the ansatz (54). We see that they are truly the zero mode as we first claim from the ansatz.

Next, this self-dual ansatz enable us to rewrite $\vec{D}^2$ as

$$\vec{D}^2 = D_1^2 + D_2^2 = \frac{\kappa}{|\kappa|}\frac{e}{\hbar c}B, \tag{64}$$

so the static equation (52) can become

$$0 = \left[-\frac{\kappa}{|\kappa|}\frac{\hbar e}{2mc}B + eA_0 - g(\Psi^*\Psi)\right]\Psi.$$

Substituting g and $\Psi^*\Psi$ in terms of B to get

$$\begin{aligned} 0 &= \left[-\frac{\kappa}{|\kappa|}\frac{\hbar e}{2mc}B + eA_0 - \left(\frac{\kappa}{|\kappa|}\frac{\hbar e^2}{mc\kappa}\right)\left(-\frac{\kappa}{e}\right)B\right]\Psi \\ A_0 &= -\frac{\kappa}{|\kappa|}\frac{\hbar}{2mc}B. \end{aligned} \tag{65}$$

We can now directly calculate for $\vec{E}$

$$\begin{aligned} E^i &= -\partial_i A_0 \\ &= -\left[-\frac{\kappa}{|\kappa|}\frac{\hbar}{2mc}\partial_i\epsilon^{jk}\partial_j\left(-\frac{\kappa}{|\kappa|}\frac{\hbar c}{2e}\epsilon^{kl}\partial_l \ln\rho\right)\right] \\ &= \frac{\hbar^2}{4me}\partial_i\nabla^2\ln\rho \\ &= -\frac{\hbar e}{2mc|\kappa|}\partial_i\rho, \end{aligned} \tag{66}$$

which agrees with the electric field we previously found from the gauge equation. Therefore, the Schrödinger equation holds true.

Before we end this section, let us show the consequence of these solutions to the Schrödinger equation. Using the results we found, (52) can be rewritten as

$$\begin{aligned} i\hbar\,\partial_t\Psi &= \left[-\frac{\hbar^2}{2m}\vec{D}^2 + e\left(-\frac{\kappa}{|\kappa|}\frac{e\hbar}{2mc}\right)B + \left(\frac{\kappa}{|\kappa|}\frac{e\hbar}{mc}\right)B\right]\Psi \\ &= \left[-\frac{\hbar^2}{2m}\vec{D}^2 + e\left(\frac{\kappa}{|\kappa|}\frac{e\hbar}{2mc}\right)B\right]\Psi. \end{aligned} \tag{67}$$

The last term has manifested its nature. With the right choice of κ, it will represent the Zeeman term of the spin-1/2 particle with the right electron-spin g-factor. However, because the JP theory starts from a scalar field and this last term is actually the contribution from two different physics, one is the electric potential and one is the self-interaction from the nonlinear term, thus, we may say that these contributions create a pseudo effect which attach spin magnetic moment to the scalar particle.

We have shown how to obtain the electric field step-by-step from the JP theory, in the next section, we will repeat the same procedure with our Chern-Simons Electrodynamics in 2+1 dimensions and see where the result differs.

4.2 Vortex Solution by Duval, Horváthy and Palla

To compare with Jackiw and Pi case, we consider our system in the non-relativistic limit. This has already been studied by Duval, Horváthy and Palla in the subject of non-relativistic spinor fields in 2+1 dimensions. In this system, they choose the convention $\eta^{\mu\nu} = (-,+,+)$ and charge $-e$. They first consider the non-relativistic limit of the decoupled Dirac equation (12b), (12c)

$$(ic\gamma^{\alpha}_{\pm} D_{\alpha} - m)\psi_{\pm} = 0, \tag{68}$$

where we set the 3-dimensional mass m to zero and redefine $p_+ = p_- = m$, $\psi_{\pm}$ represent the right-handed and left-handed spinor respectively with $\gamma^{\alpha}_{\pm} = (\pm\sigma^3, i\sigma^1, i\sigma^2)$. These gamma matrices are slightly different from CKP case because they rotate the coordinates by 90 degrees so that $x \to -y, y \to x$.

By setting

$$\psi_+ = e^{-imc^2t}\begin{pmatrix}\Psi_+\\ \tilde{\chi}_+\end{pmatrix} \quad , \quad \psi_- = e^{-imc^2t}\begin{pmatrix}\tilde{\chi}_-\\ \Psi_-\end{pmatrix}, \tag{69}$$

(68) becomes

$$iD_t\Phi - c\vec{\sigma}\cdot\vec{D}\tilde{\chi} = 0,$$
$$iD_t\chi - c\vec{\sigma}\cdot\vec{D}\Phi - 2mc^2\tilde{\chi} = 0,$$

where $\Phi = \begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}$ and $\tilde{\chi} = \begin{pmatrix}\tilde{\chi}_-\\ \tilde{\chi}_+\end{pmatrix}$. In this non-relativistic limit, the last term of the second equation dominates the time derivative term so we can drop it out. The final equations is the Lévy-Leblond equations which describe Dirac particles in the non-relativistic limit.

$$\begin{aligned} D_t\Phi + i(\vec{\sigma}\cdot\vec{D})\chi &= 0,\\ (\vec{\sigma}\cdot\vec{D})\Phi + 2m\chi &= 0, \end{aligned} \tag{70}$$

where we redefine $\tilde{\chi}$ as $\chi = \tilde{\chi}/c$.

The equations tell us that Φ, χ represent two-component spinors that interact with the Chern-Simons gauge field through the current J^μ which is defined as

$$J^\mu = (\rho, \vec{J}) = (|\Phi|^2, i(\Phi^\dagger \vec{\sigma} \chi - \chi^\dagger \vec{\sigma} \Phi)). \tag{71}$$

By re-arranging the form of the Lévy-Leblond equations with the use of the identity

$$(\vec{D} \cdot \vec{\sigma})^2 = \vec{D}^2 + eB\sigma_3, \tag{72}$$

we can write one of the equation to solely depend on one spinor. The first equation then becomes

$$iD_t\Phi = -\frac{1}{2m}(\vec{D}^2 + eB\sigma_3)\Phi. \tag{73}$$

Because χ is related to Φ by $\chi = -(1/2m)(\vec{\sigma} \cdot \vec{D})\Phi$, thus, this one equation is enough to solve for a solution.

We see from the equation that Φ satisfies the Pauli equation. This starts to look similar to the Schrödinger equation (52). The similarity can be achieved more by considering the static system. Together with the gauge field equations, the governing equations are

$$0 = -\frac{1}{2m}(\vec{D}^2 + eB\sigma_3)\Phi + eA_t\Phi, \tag{74a}$$

$$\vec{J} = -\frac{\kappa}{e}\vec{\nabla} \times A_t, \tag{74b}$$

$$\kappa B = -e\rho. \tag{74c}$$

To tackle the equations, first, we assume the self-dual ansatz

$$\left(D_1 - \frac{\kappa}{|\kappa|}iD_2\right)\Phi = 0. \tag{75}$$

This self-dual equation helps us rewrite $\vec{D}^2$ as $(\kappa/|\kappa|)eB$. The Pauli equation (73) can then be written as

$$\begin{gathered}\left[-\frac{1}{2m}eB\left(\frac{\kappa}{|\kappa|} + \sigma_3\right) + eA_t\right]\Phi = 0 \\ \left[-\frac{1}{2m}eB\begin{pmatrix}\kappa/|\kappa| + 1 & 0 \\ 0 & \kappa/|\kappa| - 1\end{pmatrix} + eA_t\right]\begin{pmatrix}\Psi_+ \\ \Psi_-\end{pmatrix} = 0.\end{gathered} \tag{76}$$

This equation suggests us that it can be solved with vanishing A_t and Φ with only one component. This means

$$\Phi_{-\kappa} = \begin{pmatrix}\Psi_{-\kappa} \\ 0\end{pmatrix}, \qquad \Phi_{+\kappa} = \begin{pmatrix}0 \\ \Psi_{+\kappa}\end{pmatrix}, \tag{77}$$

will solve the equations where $\Phi_{-\kappa}, \Phi_{+\kappa}$ correspond to negative κ and positive κ respectively. Thus, $\Phi_{-\kappa}, \Phi_{+\kappa}$ will each solve the self-dual equation (75)

$$(D_1 + iD_2)\Phi_{-\kappa} = 0, \tag{78a}$$
$$(D_1 - iD_2)\Phi_{+\kappa} = 0. \tag{78b}$$

Moreover, both make χ from the second relation of (70) vanishes. Therefore, it leads to

$$\begin{aligned}
\psi_+ &= e^{-imc^2t}\begin{pmatrix}\Psi_+ \\ \chi_+/c\end{pmatrix} &&= e^{-imc^2t}\begin{pmatrix}\Psi_{-\kappa} \\ 0\end{pmatrix}, \\
\psi_- &= e^{-imc^2t}\begin{pmatrix}\chi_-/c \\ \Psi_-\end{pmatrix} &&= 0,
\end{aligned} \tag{79}$$

for negative κ case, and

$$\begin{aligned}
\psi_+ &= e^{-imc^2t}\begin{pmatrix}\Psi_+ \\ \chi_+/c\end{pmatrix} &&= 0, \\
\psi_- &= e^{-imc^2t}\begin{pmatrix}\chi_-/c \\ \Psi_-\end{pmatrix} &&= e^{-imc^2t}\begin{pmatrix}0 \\ \Psi_{+\kappa}\end{pmatrix},
\end{aligned} \tag{80}$$

for positive κ case.

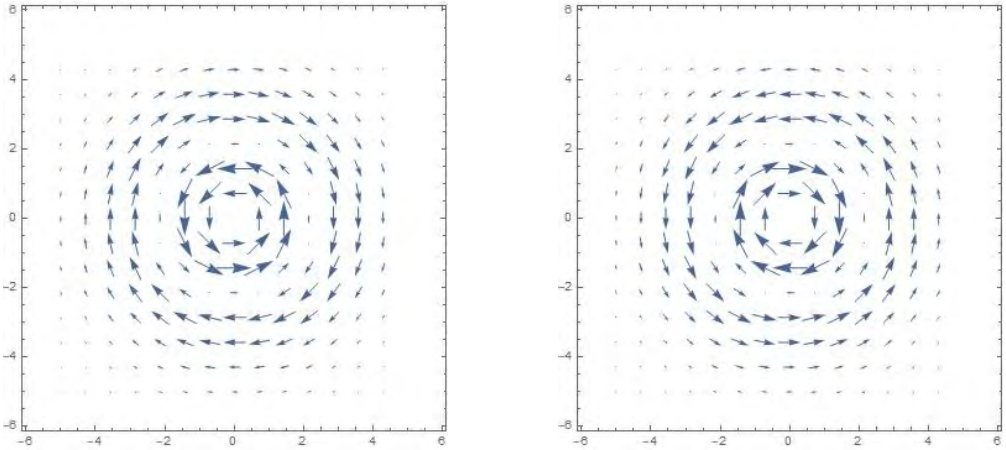
This means negative κ case will lead to the right-handed spinor with spin up solution and positive κ case will lead to the left-handed spinor with spin down solution.

By substituting $\Phi_{\pm\kappa}$ and B in terms of the mass density $\rho = |\Phi|^2$ into the Pauli equation (73), the final form becomes

$$iD_t\Psi_{\pm\kappa} = \left[-\frac{D^2}{2m} - \frac{\kappa}{|\kappa|}\lambda(\Psi^\dagger_{\pm\kappa}\Psi_{\pm\kappa})\right]\Psi_{\pm\kappa}, \tag{81}$$

where $\lambda = e^2/(2m\kappa)$.

This is almost identical to the gauged, nonlinear Schrodinger equation (52) except for the value of the non-linear coupling λ. It is exactly half of the value g used by Jackiw and Pi which becomes the reason why the spinor solution is purely magnetic. Remind that in Jackiw and Pi case, the electric field is needed such that A_t would cancel out half of the non-linear term. In Duval, Horváthy and Palla case, The nonlinear coupling is already half compared to the first theory so there is no need for A_t in the first place.



(a) 1st current term (b) 2nd current term

Figure 4: Plots show each current term of the DHP solutions

We can also dig deeper on this electric property by considering $\vec{J}$ using the definition (71) given by the Lévy-Leblond equation itself. By writing it in terms of Φ alone. It becomes

$$\vec{J} = \frac{1}{2im}[\Phi^\dagger \vec{D}\Phi - (\vec{D}\Phi)^\dagger\Phi] + \vec{\nabla} \times \left(\frac{1}{2m}\Phi^\dagger\sigma_3\Phi\right). \tag{82}$$

There is an additional divergenceless term compared to the current in the Jackiw and Pi construction. This term represents current from spin of the spin-1/2 particle. And by substituting the vortex solutions into the current $\vec{J}$, we see that

$$\begin{aligned}\vec{J} =& \frac{\kappa}{2m|\kappa|}\nabla \times (\Psi^*_{\pm\kappa}\Psi_{\pm\kappa}) - \frac{\kappa}{2m|\kappa|}\nabla \times (\Psi^*_{\pm\kappa}\Psi_{\pm\kappa}) \\ =& 0,\end{aligned} \tag{83}$$

the latter term cancel out the former term completely. Therefore, we could say that for these vortex solutions, the spin current is responsible for the vanishing total current overall, the probability fluid stop flowing, no current observed throughout the plane and thus no electric field could be produced. The vector field plots of the current from each term in (83) are shown in Figure 4 where we set all constants to unity and the parameters $r_0 = 3, n = 2$.

But what if we want to force our solutions to produce an electric field in the spinor case, is it still possible? The answer is shown in the next section.

4.3 Electromagnetic Vortex Solutions from the DHP Theory

Looking back at (76) and we see that we can induce an electric field by swapping the component of the spinor Φ with respect to the sign of κ. There will be two possible solutions that still satisfy the self-dual equations, is static, and do not require vanishing A_t. They are

$$\Phi_{-\kappa} = \begin{pmatrix} 0 \\ \Psi_{-\kappa} \end{pmatrix} \quad , \quad \Phi_{+\kappa} = \begin{pmatrix} \Psi_{+\kappa} \\ 0 \end{pmatrix}, \tag{84}$$

will solve the equations where $\Phi_{-\kappa}, \Phi_{+\kappa}$ correspond to negative κ and positive κ respectively. And as the same as DHP case, $\Phi_{-\kappa}, \Phi_{+\kappa}$ will each solve the self-dual equation (75)

$$(D_1 + iD_2)\Phi_{-\kappa} = 0, \tag{85a}$$
$$(D_1 - iD_2)\Phi_{+\kappa} = 0. \tag{85b}$$

However, unlike DHP case, χ from the second relation of (70) will not vanish. They will have the value

$$\begin{aligned} \chi_{-\kappa} =& -\frac{1}{2m}(\sigma \cdot \vec{D})\Phi_{-\kappa} \\ =& -\frac{1}{2m}\begin{pmatrix} 0 & D_1 - iD_2 \\ D_1 + iD_2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \Psi_{-\kappa} \end{pmatrix} \\ =& -\frac{1}{2m}\begin{pmatrix} (D_1 - iD_2)\Psi_{-\kappa} \\ 0, \end{pmatrix} \end{aligned} \tag{86}$$

for negative κ case, and

$$\begin{aligned} \chi_{+\kappa} =& -\frac{1}{2m}(\sigma \cdot \vec{D})\Phi_{-\kappa} \\ =& -\frac{1}{2m}\begin{pmatrix} 0 & D_1 - iD_2 \\ D_1 + iD_2 & 0 \end{pmatrix}\begin{pmatrix} \Psi_{+\kappa} \\ 0 \end{pmatrix} \\ =& -\frac{1}{2m}\begin{pmatrix} 0 \\ (D_1 + iD_2)\Psi_{+\kappa} \end{pmatrix}, \end{aligned} \tag{87}$$

for positive κ case.

Therefore, it leads to

$$\begin{aligned} \psi_+ = e^{-imc^2t}\begin{pmatrix} \Psi_+ \\ \chi_+/c \end{pmatrix} &= e^{-imc^2t}\begin{pmatrix} \Psi_{-\kappa} \\ 0 \end{pmatrix}, \\ \psi_- = e^{-imc^2t}\begin{pmatrix} \chi_-/c \\ \Psi_- \end{pmatrix} &= e^{-imc^2t}\begin{pmatrix} (D_1 - iD_2)\Psi_{-\kappa} \\ 0 \end{pmatrix}, \end{aligned} \tag{88}$$

for negative κ case, and

$$\begin{aligned}
\psi_+ &= e^{-imc^2t}\begin{pmatrix}\Psi_+\\ \chi_+/c\end{pmatrix} &&= e^{-imc^2t}\begin{pmatrix}0\\ (D_1+iD_2)\Psi_{+\kappa}\end{pmatrix},\\
\psi_- &= e^{-imc^2t}\begin{pmatrix}\chi_-/c\\ \Psi_-\end{pmatrix} &&= e^{-imc^2t}\begin{pmatrix}0\\ \Psi_{+\kappa}\end{pmatrix},
\end{aligned} \tag{89}$$

for positive κ case.

Now our solution from both cases give us both positive and negative chiral spinors. Thus, they are different from the solutions solved by DHP.

To solve for the solutions, first, we substitute the ansatz $\Psi_{\pm\kappa} = \rho^{1/2}$ into the self-dual equation. For simplicity, we will only consider for negative κ case. The second component of the self dual equation (85a) then becomes

$$\begin{aligned}
\left[(\partial_1 - ieA_1) + i\,(\partial_2 - ieA_2)\right]\rho^{1/2} &=0\\
\left(\frac{1}{2}\rho^{-1/2}\partial_1\rho + eA_2\rho^{1/2}\right) + i\left(\frac{1}{2}\rho^{-1/2}\partial_2\rho - eA_1\rho^{1/2}\right) &=0\\
\left(\frac{1}{2e}\rho^{-1}\partial_1\rho + A_2\right) + i\left(\frac{1}{2e}\rho^{-1}\partial_2\rho - eA_1\right) &=0\\
\rightarrow\quad A_i &=\frac{1}{2e}\epsilon^{ij}\partial_j\ln\rho.
\end{aligned} \tag{90}$$

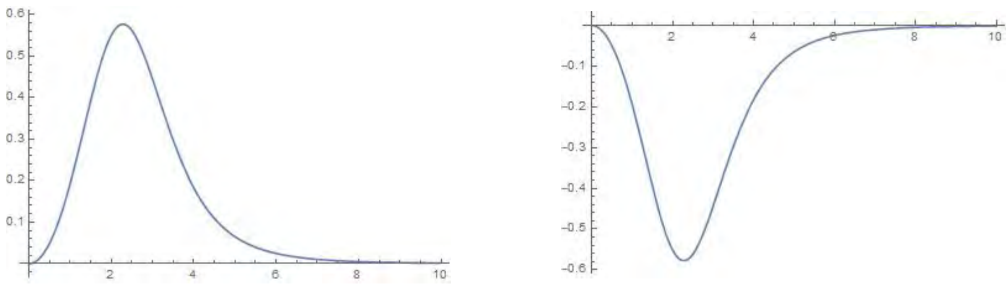
Substituting A_i into the gauge field equation (74c) will give us

$$\begin{aligned}
B &= -\frac{e}{\kappa}\rho\\
\partial_1A_2 - \partial_2A_1 &= -\frac{e}{\kappa}\rho\\
\frac{1}{2e}(\epsilon^{21}\partial_1^2\, ln\rho - \epsilon^{12}\partial_2^2\, ln\rho) &= -\frac{e}{(-|\kappa|)}\rho\\
\nabla^2\,\ln\rho &= -\frac{2e^2}{|\kappa|}\rho.
\end{aligned} \tag{91}$$

This is just the same Liouville equation (57) as in JP case. Thus, we immediately obtain the axially symmetric solution ρ in the form

$$\rho = \frac{4n^2|\kappa|}{e^2r^2}\left[\left(\frac{r_0}{r}\right)^n + \left(\frac{r}{r_0}\right)^n\right]^{-2}. \tag{92}$$

For the magnetic field B, It is linearly related to ρ (74c). Thus, It is not different from JP case and CKP case. The radial plots of ρ and B are shown in



(a) ρ radial plot (b) B radial plot

Figure 5: Plots show the radial plots of matter field ρ and magnetic field B of the electromagnetic vortex solutions from the DHP Theory

Figure 5 where we set all the constants to unity and the parameters $r_0 = 3, n = 2$.

Next we compute the electric field. By substituting $\Phi_{-\kappa}$ and $\Phi_{+\kappa}$ into (76). $\Phi_{-\kappa}$ will give us

$$\begin{aligned} \left[-\frac{1}{2m}eB\begin{pmatrix} -1+1 & 0 \\ 0 & -1-1 \end{pmatrix} + eA_t\right]\begin{pmatrix} 0 \\ \Psi_{-\kappa} \end{pmatrix} &= 0 \\ -\frac{eB}{2m}\begin{pmatrix} 0 \\ (-2)\Psi_{-\kappa} \end{pmatrix} + eA_t\begin{pmatrix} 0 \\ \Psi_{-\kappa} \end{pmatrix} &= 0 \\ \rightarrow \quad A_t &= -\frac{B}{m}, \end{aligned} \tag{93}$$

with the corresponding electric field

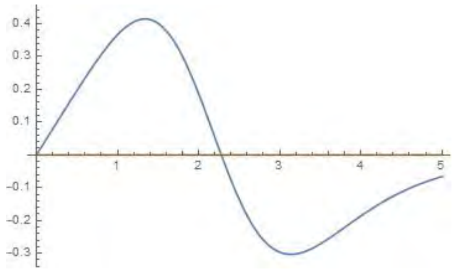
$$E^i = -\partial_i A_t = -\left[-\frac{1}{m}\partial_i\left(-\frac{e\rho}{\kappa}\right)\right] = \frac{e}{m|\kappa|}\partial_i\rho. \tag{94}$$

And $\Phi_{+\kappa}$ will give us

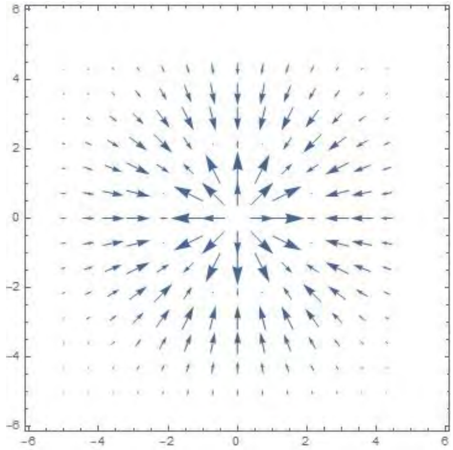
$$\begin{aligned} \left[-\frac{1}{2m}eB\begin{pmatrix} -1+1 & 0 \\ 0 & -1-1 \end{pmatrix} + eA_t\right]\begin{pmatrix} \Psi_{+\kappa} \\ 0 \end{pmatrix} &= 0 \\ -\frac{eB}{2m}\begin{pmatrix} 0 \\ (-2)\Psi_{-\kappa} \end{pmatrix} + eA_t\begin{pmatrix} \Psi_{+\kappa} \\ 0 \end{pmatrix} &= 0 \\ \rightarrow \quad A_t &= \frac{B}{m}, \end{aligned} \tag{95}$$

with the corresponding electric field

$$E^i = -\partial_i A_t = -\left[\frac{1}{m}\partial_i\left(-\frac{e\rho}{\kappa}\right)\right] = \frac{e}{m|\kappa|}\partial_i\rho. \tag{96}$$



(a) Radial plot



(b) Vector field plot

Figure 6: Plots show the electric field of the electromagnetic vortex solutions from the DHP Theory

Notice that this electric field does not depend on which sign of κ we choose for the system just like (66) in JP case. However, it has double value compared to JP case. The radial plot and vector plot of this electric field are shown in Figure 6 where we set all the constants to unity and the parameter $r_0 = 3, n = 2$.

The interesting feature is the attractive closed loop at a finite distance for both solutions in our case. if there is a test charge in this vicinity, it would fall into this closed loop. Even though we still not know the mechanism behind it, we suspect that both spinors play a crucial role in this phenomenon. Also, we note again that this system is in the long-distance limit. Thus, this closed loop might change its radius or even disappear once we consider the additional effect from the Maxwell term.

Another thing that is worth mentioning is their energy. But to calculate it, we must state the Hamiltonian of the system first. It is

$$H = \int \left\{ \frac{1}{2m} |\vec{D}\Phi|^2 + \lambda |\Phi|^2 \Phi^\dagger \sigma_3 \Phi \right\} d^2x. \tag{97}$$

We can repeat the same procedure as in JP case by using the identity (62). For simplicity, we will only compute for negative κ case. The solution by DHP then gives the Hamiltonian

$$
\begin{aligned}
H_{DHP} =& \int \left\{ \frac{1}{2m} [\, |(D_1 + iD_2)\Psi_{-\kappa}|^2 + eB|\Psi_{-\kappa}|^2 \,] \right. \\
&\left. + \frac{e^2}{2m\kappa} |\Psi_{-\kappa}|^2 \left[\begin{pmatrix} \Psi^*_{-\kappa} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \Psi_{-\kappa} \\ 0 \end{pmatrix} \right] \right\} d^2x \\
=& \int \left\{ \frac{e}{2m} B|\Psi_{-\kappa}|^2 + \frac{e^2}{2m\kappa} \rho(|\Psi_{-\kappa}|^2) \right\} d^2x \\
=& \int \left\{ \frac{e}{2m} \left(-\frac{e\rho}{\kappa} \right) \rho + \frac{e^2}{2m\kappa} \rho^2 \right\} d^2x \\
=& 0.
\end{aligned} \tag{98}
$$

where in the first line, the first term vanishes due to the self-dual property and the second term vanishes on the boundary. The last two terms cancel each other in the third line and we get the zero energy as the result. Thus, the DHP solutions are the zero modes of the system.

Next, we consider our electromagnetic vortex (EMV) solutions, the Hamiltonian is

$$
\begin{aligned}
H_{EMV} =& \int \left\{ \frac{1}{2m} [\, |(D_1 + iD_2)\Psi_{-\kappa}|^2 + eB|\Psi_{-\kappa}|^2 \,] \right. \\
&\left. + \frac{e^2}{2m\kappa} |\Psi_{-\kappa}|^2 \left[\begin{pmatrix} 0 & \Psi^*_{-\kappa} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ \Psi_{-\kappa} \end{pmatrix} \right] \right\} d^2x \\
=& \int \left\{ \frac{e}{2m} B|\Psi_{-\kappa}|^2 + \frac{e^2}{2m\kappa} \rho(-|\Psi_{-\kappa}|^2) \right\} d^2x \\
=& \int \left\{ \frac{e}{2m} \left(-\frac{e\rho}{\kappa} \right) \rho - \frac{e^2}{2m\kappa} \rho^2 \right\} d^2x \\
=& \frac{e}{m|\kappa|} \int \rho^2 \, d^2x.
\end{aligned} \tag{99}
$$

Numerical values we obtain after all the parameters are determined tell us that the Hamiltonian depends on both parameters r_0 and n and they are always greater than zero. Therefore, we can conclude that these solutions are different kind of mode from the zero modes in the original paper in which the energy depend on the size and number of solitons in the vortex.

Now, we do have possible non-vanishing electric field solutions for spinor in non-relativistic limit. Can we extend it to the relativistic case where dynamics of both two spinors come to play? Two things we can say are that the form of $\vec{J}$ and (70) will surely be different so even if we consider the static system and the self-dual equation is satisfied, there is no guarantee that the equation could be solved exactly like in the non-relativistic limit. And if it could be solved numerically, the additional term that manifest only in the relativistic regime will not admit simple solutions like the one we get in the non-relativistic regime. We have tried some ansatz to the Dirac equations but so far, none has a simple form enough to be able to solve (even numerically) except the one we have shown in section 3. However, the solution is not well-defined at a finite distance so the validity of the solution is still questionable.

5 Conclusions

All the solutions from fermionic system coupled to the gauge field via Chern-Simons interaction in relativistic (CKP), non-relativistic limit (DHP) and from the Chern-Simons gauged non-linear Schrödinger equation (JP) shares many properties together even though they come from different theories. The reason is because they are solved from the same self-dual ansatz, the same Liouville equation. Thus, the same solutions should be expected. However, the underlying structure from each theory still leads to different solutions directly or indirectly. The most interesting one is the presence of radial electric field in the JP theory. We know that its presence contributes to half of the Zeeman effect and give us the right factor for the spin-electron g-factor. It is thus interesting to find the non-vanishing electric field solutions in the fermionic system in the presence of the Chern-Simons coupling.

In the relativistic regime, we obtain one numerical solution from a specific setting, we believe that it is unphysical because the solution blows up at a finite distance, In the non-relativistic regime, we obtain two exact solutions by exchanging the component of the original spinors $\Phi_{\pm\kappa}$ with respect to the sign of Chern-Simons coupling κ. The electromagnetic vortices we found have the electric field of the same form as the JP solutions except that their amplitude are half of the value.

For possible physical applications, Chern-Simons vortices in the non-relativistic regime could be used to explain condensed matter phenomena such as fractional quantum Hall effect [5] or high temperature superconductivity [8]. For the vortices in the relativistic regime, the situation is ambiguous because there are not many relativistic phenomena that are confined in 2+1 dimensions. The condensed matter system is also intrinsically non-relativistic. Thus, an application from these relativistic vortices is still unknown.

6 Appendices

In this section, we present the calculation in details for the equations we used in this paper.

Appendix A: Calculation Details from Section 2

(2) → (3) ;

First, we rewrite (2) in terms of $\tilde{F}^{\mu} = \frac{1}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho}$

$$\begin{aligned}
-(\kappa e^2)\tilde{F}^{\nu} =& \partial_\mu F^{\mu\nu} \\
=& \frac{1}{2}\partial_\mu(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha)F^{\alpha\beta} \\
=& \frac{1}{2}\partial_\mu(\epsilon^{\mu\nu\rho}\epsilon_{\rho\alpha\beta})F^{\alpha\beta} \\
=& \frac{1}{2}\eta_{\rho\sigma}\epsilon^{\mu\nu\sigma}\partial_\mu(\epsilon^{\rho\alpha\beta}F_{\alpha\beta}) \\
=& \eta_{\rho\sigma}\epsilon^{\mu\nu\sigma}\partial_\mu\tilde{F}^{\rho}.
\end{aligned} \tag{100}$$

Multiplying above equation by κe^2 and apply (100) into the equation again to get

$$\begin{aligned}
-(\kappa e^2)^2\tilde{F}^{\nu} =& \eta_{\rho\sigma}\epsilon^{\mu\nu\sigma}\partial_\mu(\kappa e^2\tilde{F}^{\rho}) \\
=& -\eta_{\rho\sigma}\eta_{\lambda\beta}\epsilon^{\mu\nu\sigma}\epsilon^{\alpha\rho\beta}\partial_\mu\partial_\alpha\tilde{F}^{\lambda} \\
=& -\epsilon^{\mu\nu\sigma}\epsilon_{\lambda\alpha\sigma}\partial_\mu\partial^\alpha\tilde{F}^{\lambda} \\
=& -(\delta^\mu_\lambda\delta^\nu_\alpha - \delta^\mu_\alpha\delta^\nu_\lambda)\partial_\mu\partial^\alpha\tilde{F}^{\sigma} \\
=& -(\partial_\mu\partial_\nu\tilde{F}^{\mu} - \partial_\mu\partial^\mu\tilde{F}^{\nu}).
\end{aligned}$$

By choosing the gauge choice $\partial_\mu\tilde{F}^{\mu} = 0$, we finally get the equation of motion of the form

$$[\partial_\mu\partial^\mu + (\kappa e^2)^2]\tilde{F}^{\nu} = 0. \tag{101}$$

(8) → (11) ;

Substituting (9) into (8) to obtain the Lagrangian

$$
\begin{aligned}
\mathcal{L} =& -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \begin{pmatrix} e^{-ip_+z}\psi_+^\dagger & e^{-ip_-z}\psi_-^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&\times \left[i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D_0 + i\begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} D_i - m \right] \begin{pmatrix} e^{ip_+z}\psi_+ \\ e^{ip_-z}\psi_- \end{pmatrix} \\
=& -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \begin{pmatrix} e^{-ip_+z}\psi_+^\dagger & e^{-ip_-z}\psi_-^\dagger \end{pmatrix} \\
&\times \left[i\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D_0 + i\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} D_i - m\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \begin{pmatrix} e^{ip_+z}\psi_+ \\ e^{ip_-z}\psi_- \end{pmatrix} \\
=& -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \begin{pmatrix} e^{-ip_+z}\psi_+^\dagger & e^{-ip_-z}\psi_-^\dagger \end{pmatrix} \\
&\times \begin{pmatrix} ie^{ip_+z}D_0\psi_+ + i\sigma^i D_i e^{ip_+z}\psi_+ - me^{ip_-z}\psi_- \\ ie^{ip_-z}D_0\psi_- - i\sigma^i D_i e^{ip_-z}\psi_- - me^{ip_+z}\psi_+ \end{pmatrix} \\
=& -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\psi_+^\dagger D_0\psi_+ + i\psi_-^\dagger D_0\psi_- + \sum_{j=1}^{2}(i\psi_+^\dagger\sigma^j D_j\psi_+ - i\psi_-^\dagger\sigma^j D_j\psi_-) \\
&+ i\psi_+^\dagger\sigma^3 ip_+\psi_+ - i\psi_-^\dagger\sigma^3 ip_-\psi_- + (i\psi_+^\dagger\sigma^3 ieA_z\psi_+ - i\psi_-^\dagger\sigma^3 ieA_z\psi_-) \\
&= -m(e^{-i(p_+-p_-)z}\psi_+^\dagger\psi_- + e^{i(p_+-p_-)z}\psi_-^\dagger\psi_+).
\end{aligned}
$$

After performing the dimensional reduction by integrating out the z-dependence term and neglecting the 7th and 8th terms from $\mathcal{L}$ because they are irrelevant in 2+1 dimensions (A_z does not contribute to the electric field on the plane or the magnetic field perpendicular to the plane), the Lagrangian becomes the effective QED Lagrangian in 2+1 dimensions

$$
\begin{aligned}
\mathcal{L}_{2+1} =& -\frac{1}{4}F^{\rho\sigma}F_{\rho\sigma} + i\psi_+^\dagger(\sigma^3\sigma^3 D_0 + i\sigma^3\sigma^2 D_1 - i\sigma^3\sigma^1 D_2)\psi_+ \\
&- i\psi_-^\dagger(-\sigma^3\sigma^3 D_0 + i\sigma^3\sigma^2 D_1 - i\sigma^3\sigma^1 D_2)\psi_- - p_+\psi_+^\dagger\sigma^3\psi_+ + p_-\psi_-^\dagger\sigma^3\psi_- \\
&- m(\psi_+^\dagger\psi_- + \psi_-^\dagger\psi_+),
\end{aligned}
$$

where we also assume that p_+ and p_- are equal so that the mass terms survive after dimensional reduction.

To make it looks simpler we define $\bar{\psi}_\pm = \pm\psi^\dagger\sigma^3$ and $\gamma_\pm^\rho = (\pm\sigma^3, i\sigma^2, -i\sigma^1)$ so that

$$
\begin{aligned}
\mathcal{L}_{2+1} =& -\frac{1}{4}F^{\rho\sigma}F_{\rho\sigma} + \bar{\psi}_+(i\gamma_+^\rho D_\rho - p_+)\psi_+ + \bar{\psi}_-(i\gamma_-^\rho D_\rho - p_-)\psi_- \\
&- m(\psi_+^\dagger\psi_- + \psi_-^\dagger\psi_+)
\end{aligned}
$$

Lastly, we induce the Chern-Simons term from the first order loop correction. The final Lagrangian becomes

$$\begin{aligned}\mathcal{L}_{CSQED(2+1)} = & -\frac{1}{4}F^{\rho\sigma}F_{\rho\sigma} + \frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} + \bar{\psi}_+(i\gamma^\rho_+ D_\rho - p_+)\psi_+ \\ & + \bar{\psi}_-(i\gamma^\rho_- D_\rho - p_-)\psi_- - m(\psi^\dagger_+\psi_- + \psi^\dagger_-\psi_+)\end{aligned}$$

$\underline{(11) \to (12)}$;

The equations of motion can be extracted from the Lagrangian (11) by using the Euler-Lagrange equations, which are in the form

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}_+} - \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \bar{\psi}_+)}\right) = 0 \tag{102a}$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}_-} - \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \bar{\psi}_-)}\right) = 0 \tag{102b}$$

$$\frac{\partial \mathcal{L}}{\partial A_\beta} - \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)}\right) = 0 \tag{102c}$$

Substituting (11) into (102a),(102b) and (102c) to get

(102a) ; $\quad m\sigma^3\psi_- = (i\gamma^\mu_+ D_\mu - p_+)\psi_+$

(102b) ; $\quad -m\sigma^3\psi_+ = (i\gamma^\mu_- D_\mu - p_-)\psi_-$

(102c) ;

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial A_\beta} - \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)}\right) &= \frac{\kappa}{4}\epsilon^{\beta\nu\rho}F_{\nu\rho} - e(\bar{\psi}_+\gamma^\beta_+\psi_+ + \bar{\psi}_-\gamma^\beta_-\psi_-) - \frac{\kappa}{2}\epsilon^{\beta\mu\alpha}\partial_\alpha A_\mu \\ \frac{\kappa}{2}\epsilon^{\beta\nu\rho}F_{\nu\rho} &= e(\bar{\psi}_+\gamma^\beta_+\psi_+ + \bar{\psi}_-\gamma^\beta_-\psi_-)\end{aligned}$$

$\underline{(12) \to (14)}$;

Substituting the ansatz (13) into the equations of motion (12a) to obtain

$$
\begin{aligned}
m\sigma_3\psi_- =& i\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\partial_0 + i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}(\partial_x + ieA_x) - i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}(\partial_y + ieA_y)\right]\psi_+\\
&- p_+\psi_+ - m\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\psi_-\\
=&\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}(-iE_+)\psi_+\\
&+\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\left(\frac{\partial\rho}{\partial x}\partial_\rho + \frac{\partial\phi}{\partial x}\partial_\phi + ie\frac{\partial\rho}{\partial x}A_\rho + ie\frac{\partial\phi}{\partial x}A_\phi\right)\psi_+\\
&-\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\left(\frac{\partial\rho}{\partial y}\partial_\rho + \frac{\partial\phi}{\partial y}\partial_\phi + ie\frac{\partial\rho}{\partial y}A_\rho + ie\frac{\partial\phi}{\partial y}A_\phi\right)\psi_+ + ip_+\psi_+\\
&+ im\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\psi_-\\
=&-i\begin{pmatrix}(E_+ - p_+)f_+e^{ik_+\phi}\\ (E_+ + p_+)(-ig_+)e^{il_+\phi}\end{pmatrix}\\
&+\left(\cos\phi\partial\rho - \frac{\sin\phi}{\rho}\partial_\phi - ie\frac{\sin\phi}{\rho}A\right)\begin{pmatrix}ig_+e^{il_+\phi}\\ -f_+e^{ik_+\phi}\end{pmatrix}\\
&-\left(i\sin\phi\partial\rho + i\frac{\cos\phi}{\rho}\partial_\phi - e\frac{\cos\phi}{\rho}\right)\begin{pmatrix}ig_+e^{il_+\phi}\\ f_+e^{ik_+\phi}\end{pmatrix} + im\begin{pmatrix}f_-e^{ik_-\phi}\\ -ig_-e^{il_-\phi}\end{pmatrix}\\
=&-i\begin{pmatrix}(E_+ - p_+)f_+e^{ik_+\phi}\\ (E_+ + p_+)(-ig_+)e^{il_+\phi}\end{pmatrix}\\
&+\begin{pmatrix}(e^{-i\phi}\partial_\rho - i\frac{e^{-i\phi}}{\rho}\partial_\phi + \frac{e^{-i\phi}}{\rho}eA)ig_+e^{il_+\phi}\\ (-e^{i\phi}\partial_\rho - i\frac{e^{i\phi}}{\rho}\partial_\phi + \frac{e^{i\phi}}{\rho}eA)f_+e^{ik_+\phi}\end{pmatrix} + im\begin{pmatrix}f_-e^{ik_-\phi}\\ -ig_-e^{il_-\phi}\end{pmatrix}.
\end{aligned}
$$

We finally obtain two equations out of (12a)

$$
\begin{aligned}
0 &= -i(E_+ - p_+)f_+e^{ik_+\phi} + imf_-e^{ik_-\phi} + i\left(\partial_\rho g_+ + \frac{l_+}{\rho}g_+ + \frac{eA}{\rho}g_+\right)e^{i(l_+-1)\phi},\\
0 &= -(E_+ - p_+)g_+e^{il_+\phi} + mg_-e^{il_-\phi} - i\left(\partial_\rho f_+ - \frac{k_+}{\rho}f_+ - \frac{eA}{\rho}f_+\right)e^{i(k_+-1)\phi}.
\end{aligned}
$$

By setting $k_+ = k_- = l_+ - 1 = l_- - 1,\ E_+ = E_- = E,\ p_+ = p_- = p$ we can simplify the equations to the form

$$
\begin{aligned}
\partial_\rho g_+ &= -\left(\frac{k_+ + 1 + eA}{\rho}\right)g_+ + (E-p)f_+ - mf_-,\\
\partial_\rho f_+ &= \left(\frac{k_+ + eA}{\rho}\right)f_+ - (E+p)g_+ + mg_-.
\end{aligned}
$$

We can rewrite (12b) in the same manner as (12a), the equations will be similar except that all the subscript $+,-$ are exchanged and the sign of E and m are exchanged too so we get

$$\begin{aligned}\partial_\rho g_- &= -\left(\frac{k_- + 1 + eA}{\rho}\right) g_- - (E+p)f_- + mf_+,\\ \partial_\rho f_- &= \left(\frac{k_- + eA}{\rho}\right) f_- + (E-p)g_- - mg_+.\end{aligned}$$

Next, for the last equation (12c), we consider each component

$\beta = 0 \quad ;$

$$\begin{aligned}\frac{\kappa}{2e}(\epsilon^{012}F_{12} + \epsilon^{021}F_{21}) &= \begin{pmatrix} f_+^* e^{-ik_+\phi} & -ig_+^* e^{-il_+\phi}\end{pmatrix} \sigma^3\sigma^3 \begin{pmatrix} f_+ e^{ik_+\phi} \\ ig_+ e^{il_+\phi}\end{pmatrix}\\ &\quad + \begin{pmatrix} f_-^* e^{-ik_-\phi} & -ig_-^* e^{-il_-\phi}\end{pmatrix} (-\sigma^3)(-\sigma^3) \begin{pmatrix} f_- e^{ik_-\phi} \\ ig_- e^{il_-\phi}\end{pmatrix}\\ -\frac{\kappa}{e\rho}\partial_\rho A &= |f_+|^2 + |f_-|^2 + |g_+|^2 + |g_-|^2\end{aligned}$$

$\beta = 1 \quad ;$

$$\begin{aligned}\frac{\kappa}{2e}(\epsilon^{120}F_{20} + \epsilon^{102}F_{02}) &= \begin{pmatrix} f_+^* e^{-ik_+\phi} & -ig_+^* e^{-il_+\phi}\end{pmatrix} \sigma^3 i\sigma^2 \begin{pmatrix} f_+ e^{ik_+\phi} \\ ig_+ e^{il_+\phi}\end{pmatrix}\\ &\quad + \begin{pmatrix} f_-^* e^{-ik_-\phi} & -ig_-^* e^{-il_-\phi}\end{pmatrix} (-\sigma^3)(i\sigma^2) \begin{pmatrix} f_- e^{ik_-\phi} \\ ig_- e^{il_-\phi}\end{pmatrix}\\ 0 &= \begin{pmatrix} f_+^* e^{-ik_+\phi} & -ig_+^* e^{-il_+\phi}\end{pmatrix} \begin{pmatrix} ig_+ e^{il_+\phi} \\ f_+ e^{ik_+\phi}\end{pmatrix}\\ &\quad - \begin{pmatrix} f_-^* e^{-ik_-\phi} & -ig_-^* e^{-il_-\phi}\end{pmatrix} \begin{pmatrix} ig_- e^{il_-\phi} \\ f_- e^{ik_-\phi}\end{pmatrix}\\ 0 &= if_+^* g_+ e^{i\phi} - ig_+^* f_+ e^{-i\phi} - if_-^* g_- e^{i\phi} + ig_-^* f_- e^{-i\phi}\\ 0 &= (f_+^* g_+ - f_-^* g_-)e^{i\phi} + (g_-^* f_- - g_+^* f_+)e^{-i\phi}\end{aligned}$$

$\beta = 2 \quad ;$

$$\frac{\kappa}{2e}(\epsilon^{201}F_{01} + \epsilon^{210}F_{10}) = \begin{pmatrix} f_+^* e^{-ik_+\phi} & -ig_+^* e^{-il_+\phi} \end{pmatrix} \sigma^3(-i\sigma^1) \begin{pmatrix} f_+ e^{ik_+\phi} \\ ig_+ e^{il_+\phi} \end{pmatrix}$$
$$+ \begin{pmatrix} f_-^* e^{-ik_-\phi} & -ig_-^* e^{-il_-\phi} \end{pmatrix} (-\sigma^3)(-i\sigma^1) \begin{pmatrix} f_- e^{ik_-\phi} \\ ig_- e^{il_-\phi} \end{pmatrix}$$
$$0 = \begin{pmatrix} f_+^* e^{-ik_+\phi} & -ig_+^* e^{-il_+\phi} \end{pmatrix} \begin{pmatrix} g_+ e^{il_+\phi} \\ if_+ e^{ik_+\phi} \end{pmatrix}$$
$$- \begin{pmatrix} f_-^* e^{-ik_-\phi} & -ig_-^* e^{-il_-\phi} \end{pmatrix} \begin{pmatrix} g_- e^{il_-\phi} \\ if_- e^{ik_-\phi} \end{pmatrix}$$
$$0 = f_+^* g_+ e^{i\phi} + g_+^* f_+ e^{-i\phi} - f_-^* g_- e^{i\phi} - g_-^* f_- e^{-i\phi}$$
$$0 = (f_+^* g_+ - f_-^* g_-)e^{i\phi} - (g_-^* f_- - g_+^* f_+)e^{-i\phi}$$

The last two components give us two constraints of the system. But notice that if only $g_+^* f_+ - g_-^* f_- = 0$, Both constraints will be satisfied immediately. Thus, we now obtain all the equations of (14).

$\underline{(16) \rightarrow (17)} \quad ;$

Starting with (16)

$$\frac{1}{\rho}\frac{d^2A}{d\rho} - (n + eA + \frac{1}{2})\frac{2}{\rho^2}\frac{dA}{d\rho} = 0.$$

Without a knowledge to solve for the exact solution of this non-linear differential equation, we instead using the DSolve operation in Mathematica to solve this equation, The solution is

$$A(\rho) = \frac{1}{e}[-1 - n + \sqrt{-1 - 2n - n^2 - 2e^2C_1}$$
$$\times \tan[\sqrt{-1 - 2n - n^2 - 2e^2C_1}(C_2 + \ln\rho)]].$$

Redefine

$$-iC_3 \equiv \sqrt{-1 - 2n - n^2 - 2e^2C_1}, \quad C_4 \equiv C_3C_2.$$
$$; \;\tan(-iC_4 - iC_{3ln\rho}) = -i\left(\frac{e^{C_4}e^{C_3ln\rho} - e^{-C_4}e^{-C_3ln\rho}}{e^{C_4}e^{C_3ln\rho} + e^{-C_4}e^{-C_3ln\rho}}\right)$$
$$= -i(\frac{\rho^{C_3} - e^{-2C_4}\rho^{-C_3}}{\rho^{C_3} + e^{-2C_4}\rho^{-C_3}})$$
$$= -i(\frac{\rho^{2C_3} - \lambda^2}{\rho^{2C_3} + \lambda^2}) \qquad ; \quad \lambda \equiv e^{-C_4}$$
$$\rightarrow \; A(\rho) = \frac{1}{e}\left[-1 - n - C_3\left(\frac{\rho^{2C_3} - \lambda^2}{\rho^{2C_3} + \lambda^2}\right)\right].$$

Next, we apply the boundary condition $A(0) = A_0$

$$\begin{aligned}
A(0) &= \frac{1}{e}(-1-n-C_3), \\
C_3 &= eA_0+n+1 \quad \rightarrow \quad eA_0+n+1>0 \\
\rightarrow \quad A(\rho) &= \frac{1}{e}\left[-1-n\left(\frac{\rho^{2C_3}+\lambda^2}{\rho^{2C_3}+\lambda^2}\right)-(eA_0+n+1)\left(\frac{\rho^{2C_3}-\lambda^2}{\rho^{2C_3}+\lambda^2}\right)\right] \\
&= -\frac{2(n+1)}{e}\left(\frac{\rho^{2(eA_0+n+1)}}{\rho^{2(eA_0+n+1)}+\lambda^2}\right)-A_0\left(\frac{\rho^{2(eA_0+n+1)}-\lambda^2}{\rho^{2(eA_0+n+1)}+\lambda^2}\right).
\end{aligned}$$

Knowing $A(\rho)$, we can now find f_+ by substituting A into (15b) to get

$$f_+ = \frac{(eA_0+n+1)\lambda}{e}\left[2\kappa\left(\frac{E+p}{E}\right)\right]^{1/2}\left(\frac{\rho^{eA_0+n}}{\rho^{2(eA_0+n+1)}+\lambda^2}\right).$$

$\underline{(25) \rightarrow (26)}$;

In this calculation, we will show the energy and angular momentum for case $\kappa > 0$. The total energy is defined as the integration of the component 00 of the energy-momentum tensor over the plane

$$\begin{aligned}
\mathcal{E} &= \int d^2x T^{00} \\
&= \int d^2x i[\bar{\psi}_+(\sigma^3 D^0)\psi_+ - \bar{\psi}_-(\sigma^3 D^0)\psi_-] \\
&= \int d^2x i\left[\begin{pmatrix} f_+^* e^{-in\phi} & -ig_+^* e^{-i(n+1)\phi}\end{pmatrix}\begin{pmatrix}(-iE)f_+e^{in\phi} \\ (-iE)ig_+e^{i(n+1)\phi}\end{pmatrix}\right. \\
&\quad \left. + \begin{pmatrix} f_-^* e^{-in\phi} & -ig_-^* e^{-i(n+1)\phi}\end{pmatrix}\begin{pmatrix}(-iE)f_-e^{in\phi} \\ (-iE)ig_-e^{i(n+1)\phi}\end{pmatrix}\right] \\
&= \int d^2x(|f_+|^2+|f_-|^2+|g_+|^2+|g_-|^2)E \\
&= -\frac{\kappa}{e}E(\int d^2xB) \\
&= \frac{q}{e}E.
\end{aligned}$$

Next, the total angular momentum is defined by

$$\mathcal{J} = \int d^2x \epsilon_{ij}x^iT^{0j}.$$

But before we directly calculate $\mathcal{J}$, we will separate $T^{\mu\rho}$ into two parts. Separately computing and combining it together later will make the calculation looks easier than calculating it all at once.

$$\begin{aligned}
T^{\mu\rho} =& \frac{i}{2}[\bar{\psi}_+(\gamma_+^\mu\partial^\rho + \gamma_+^\rho\partial^\mu)\psi_+ + \bar{\psi}_-(\gamma_-^\mu\partial^\rho + \gamma_-^\rho\partial^\mu)\psi_-] \\
=& -\frac{e}{2}[\bar{\psi}_+(\gamma_+^\mu A^\rho + \gamma_+^\rho A^\mu)\psi_+ + \bar{\psi}_-(\gamma_-^\mu A^\rho + \gamma_-^\rho A^\mu)\psi_-] \\
=& \frac{i}{2}T_1^{\mu\rho} + \frac{\kappa}{4}(\epsilon^{\mu\beta\gamma}F_{\beta\gamma}A^\rho + \epsilon^{\rho\beta\gamma}F_{\beta\gamma}A^\mu) \\
=& \frac{i}{2}T_1^{\mu\rho} + \frac{\kappa}{4}T_2^{\mu\rho},
\end{aligned}$$

which means

$$\begin{aligned}
\mathcal{J} =& \int d^2x\epsilon_{ij}x^i\frac{i}{2}T_1^{0\rho} + \int d^2x\epsilon_{ij}x^i\frac{\kappa}{4}T_2^{0\rho} \\
=& \mathcal{J}_1 + \mathcal{J}_2.
\end{aligned}$$

Next we consider each component from each part of $T^{\mu\rho}$ that contribute to the angular momentum

$$\begin{aligned}
T_1^{0y} =& e^{iEt}\begin{pmatrix} f_+^* e^{-in\phi} & 0 \end{pmatrix}\sigma^3[\sigma^3\partial^y + (-i\sigma^1)\partial^0]e^{-iEt}\begin{pmatrix} f_+ e^{in\phi} \\ 0 \end{pmatrix} \\
&+ e^{iEt}\begin{pmatrix} f_-^* e^{-in\phi} & 0 \end{pmatrix}\sigma^3[\sigma^3\partial^y + (-i\sigma^1)\partial^0]e^{-iEt}\begin{pmatrix} f_- e^{in\phi} \\ 0 \end{pmatrix} \\
=& \sum_{j=\pm} f_j^*\left[-\frac{\partial\rho}{\partial y}\partial_\rho f_j - \frac{\partial\phi}{\partial y}f_j(in)\right] \\
=& \sum_{j=\pm} -f_j^*\left(\sin\phi\partial\rho f_j + \frac{\cos\phi}{\rho}inf_j\right), \\
T_1^{0x} =& e^{iEt}\begin{pmatrix} f_+^* e^{-in\phi} & 0 \end{pmatrix}\sigma^3[\sigma^3\partial^x + i\sigma^2\partial^0]e^{-iEt}\begin{pmatrix} f_+ e^{in\phi} \\ 0 \end{pmatrix} \\
&+ e^{iEt}\begin{pmatrix} f_-^* e^{-in\phi} & 0 \end{pmatrix}\sigma^3[\sigma^3\partial^x + i\sigma^2\partial^0]e^{-iEt}\begin{pmatrix} f_- e^{in\phi} \\ 0 \end{pmatrix} \\
=& \sum_{j=\pm} f_j^*\left[-\frac{\partial\rho}{\partial x}\partial_\rho f_j - \frac{\partial\phi}{\partial x}f_j(in)\right] \\
=& \sum_{j=\pm} -f_j^*\left(\sin\phi\partial\rho f_j + \frac{\cos\phi}{\rho}inf_j\right), \\
xT_1^{0y} - yT_1^{0x} =& \sum_{j=\pm} -f_j^*(\rho\cos\phi\sin\phi\partial_\rho f_j + \cos^2\phi ikf_j - \rho\cos\phi\sin\phi\partial_\rho f_j + \sin^2\phi ikf_j) \\
=& -ik(|f_+|^2 + |f_-|^2).
\end{aligned}$$

$$
\begin{aligned}
T_2^{0y} =& \epsilon^{0xy} F_{xy} A^y + \epsilon^{0yx} F_{yx} A^y + \epsilon^{y\beta\gamma} F_{\beta\gamma} A^0 \\
=& 2\frac{\partial_\rho A}{\rho}(-1)\left(\frac{\cos\phi}{\rho} A\right), \\
T_2^{0x} =& 2\frac{\partial_\rho A}{\rho}(-1)\left(\frac{-\sin\phi}{\rho} A\right), \\
xT_2^{0y} - yT_2^{0x} =& -2\frac{\partial_\rho A}{\rho}(\cos^2\phi + \sin^2\phi) A \\
=& -2(\partial_\rho A) A.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathcal{J}_1 =& \frac{i}{2}\int d^2x (xT_1^{0y} - yT_1^{0x}) \\
=& \frac{n}{2}\int d^2x (|f_+|^2 + |f_-|^2) \\
\mathcal{J}_1 =& \frac{n}{2}\left(-\frac{\kappa\Phi}{e}\right), \\
\mathcal{J}_2 =& \frac{\kappa}{4}\int d^2x (xT_2^{0y} - yT_2^{0x}) \\
=& -\frac{\kappa}{4}(2)(2\pi)\int A\partial_\rho A d\rho \\
=& -\frac{\kappa}{4}(2\pi) A_\infty^2 \\
=& -\frac{\kappa}{4}\Phi\left[-\frac{2}{e}(n+1)\right] \\
=& \left(\frac{\kappa\Phi}{e}\right)\left(\frac{n+1}{2}\right), \\
\mathcal{J} =& \mathcal{J}_1 + \mathcal{J}_2 \\
=& \frac{n}{2}\left(-\frac{\kappa\Phi}{e}\right) + \left(\frac{\kappa\Phi}{e}\right)\left(\frac{n+1}{2}\right) \\
=& \frac{\kappa\Phi}{2e}.
\end{aligned}
$$

Appendix B: Calculation Details from Section 4.1

<u>Verifying that (47) satisfies (46)</u> ;

$$
\begin{aligned}
\nabla_r \times \vec{A}(t,r) =& \frac{1}{\kappa}\int d^2r'[\nabla_r \times \vec{G}(r-r')]\rho(t,r') \\
=& -\frac{1}{\kappa}\int d^2r'\delta^2(r-r')\rho(t,r') \\
B =& -\frac{\rho(t,r)}{\kappa}.
\end{aligned}
$$

$$
\begin{aligned}
\nabla_r A_0(t,r) =& \frac{1}{\kappa}\int d^2r'\nabla_r[\vec{G}(r-r')\cdot\vec{j}(t,r')] \\
=& \frac{1}{\kappa}\int d^2r'[(\vec{j}(t,r')\cdot\nabla_r)\vec{G}(r-r') + \vec{j}(t,r')\times(\nabla_r\times\vec{G}(r-r')), \\
\partial_t\vec{A}(t,r) =& \frac{1}{\kappa}\int d^2r'\vec{G}(r-r')(-\nabla_{r'}\cdot\vec{j}(t,r')) \\
=& \frac{1}{\kappa}\int d^2r'(\vec{j}(t,r')\cdot\nabla_{r'})\vec{G}(r-r'), \\
E^i =& -\partial_t\vec{A} - \partial_i A_0 \\
=& \frac{1}{\kappa}\epsilon^{ij}\int d^2r'\frac{1}{\kappa}\epsilon^{ij}\int d^2r' j_j(t,r')\delta^2(r-r').
\end{aligned}
$$

<u>(45) → (50)</u> ;

The Hamiltonian and the Lagrangian are related by the Legendre transformation

$$
H_{JP} = \int d^2r \left\{ \dot{\Psi}\frac{\partial\mathcal{L}}{\partial\dot{\Psi}} + \dot{\Psi}^*\frac{\partial\mathcal{L}}{\partial\dot{\Psi}^*} + \dot{A}^i\frac{\partial\mathcal{L}}{\partial\dot{A}^i} \right\} - L_{JP}. \tag{103}
$$

To find the form of H_{JP}, notice that by considering only the gauge terms, we can rewritten them as

$$
\begin{aligned}
\dot{A}^i\frac{\partial\mathcal{L}}{\partial\dot{A}^i} - \frac{\kappa}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} =& \frac{\kappa}{2}(A_2\dot{A}_1 - A_a\dot{A}_2) - \frac{\kappa}{2}[\epsilon^{012}A_0(-\partial_1 A_2 + \partial_2 A_1) \\
&+ \epsilon^{120}A_1(\partial_2 A_0 + \dot{A}_2) + \epsilon^{201}A_2(-\dot{A}_1 - \partial_1 A_0)] \\
=& -\frac{\kappa}{2}[A_0(-\partial_1 A_2 + \partial_2 A_1) + A_0(\partial_2 A_1 - \partial_1 A_2] \\
=& \kappa A_0 B,
\end{aligned}
$$

up to integration over space.

Substituting it into H_{JP} to get

$$
\begin{aligned}
H_{JP} &= \int d^2r \left\{ i\hbar\Psi^*\dot{\Psi} + \kappa A_0 B - i\hbar\Psi^*\dot{\Psi} + eA_0|\Psi|^2 + \frac{\hbar^2}{2m}|\vec{D}\Psi|^2 - \frac{g}{2}(\Psi^*\Psi)^2 \right\} \\
&= \int d^2r \left\{ \frac{\hbar^2}{2m}|\vec{D}\Psi|^2 - \frac{g}{2}(\Psi^*\Psi)^2 \right\}.
\end{aligned}
$$

<u>(51) $\rightarrow$ (52) using (50)</u> ;

First, we decompose $|\vec{D}\Psi|^2$ into each term (also set natural constants to unity)

$$
\begin{aligned}
|\vec{D}\Psi|^2 &= \sum_{i=1}^{2}(D_i\Psi)(D_i\Psi)^* \\
&= \sum_{i=1}^{2}[(\partial_i - ieA_i)\Psi][(\partial_i + ieA_i)\Psi^*] \\
&= \sum_{i=1}^{2}[\partial_i\Psi\partial_i\Psi^* + ieA_i(\Psi^*\partial_i\Psi - \Psi\partial_i\Psi^*) + e^2A_i^2\Psi^*\Psi] \\
&= \sum_{i=1}^{2}[\partial_i(\Psi^*\partial_i\Psi) - \Psi^*\partial_i^2\Psi + ieA_i(\Psi^*\partial_i\Psi - \Psi\partial_i\Psi^*) + e^2A_i^2\Psi^*\Psi].
\end{aligned}
$$

For simplicity, we will consider the functional derivative respect to $\Psi^*(r')$ on each term separately.

3rd term ;

$$
\begin{aligned}
&\frac{\delta}{\delta\Psi^*(r')}\int d^2r A_i(r)\Psi^*(r)\partial_i\Psi(r) \\
&= \int d^2r'' \frac{\delta}{\delta A_i(r'')}\left[\int d^2r A_i(r)\Psi^*(r)\partial_i\Psi(r)\right]\frac{\delta A_i(r'')[\Psi^*]}{\delta\Psi^*(r')} + A_i(r')\partial_i\Psi(r') \\
&= \int d^2r''\Psi^*(r'')\partial_i\Psi(r'')\frac{e}{\kappa}G_i(r''-r')\Psi(r') + A_i(r')\partial_i\Psi(r') \\
&= -\Psi(r')\frac{e}{\kappa}\int d^2r'' G_i(r'-r'')\Psi^*(r'')\partial_i\Psi(r'') + A_i(r')\partial_i\Psi(r')
\end{aligned}
$$

4th term ;

$$\frac{\delta}{\delta\Psi^*(r')}\int d^2r A_i(r)\Psi(r)\partial_i\Psi^*(r)$$
$$= \frac{\delta}{\delta\Psi^*(r')}\int d^2r[-\partial_i A_i(r)\Psi(r)\Psi^*(r) - A_i(r)\partial_i\Psi(r)\Psi^*(r) + \partial_i(A_i(r)\Psi(r)\Psi^*(r))]$$
$$= -\partial_i A_i(r')\Psi(r') - A_i(r')\partial_i\Psi(r')$$
$$- \int d^2r''\frac{\delta}{\delta A_i(r'')}\left[\int d^2r A_i(r)\Psi^*(r)\partial_i\Psi(r)\right]\frac{\delta A_i(r'')[\Psi^*]}{\delta\Psi^*(r')}$$
$$= -\partial_i A_i(r')\Psi(r') - A_i(r')\partial_i\Psi(r') + \Psi(r')\frac{e}{\kappa}\int d^2r'' G_i(r'-r'')\Psi^*(r'')\partial_i\Psi(r'')$$

5th term ;

$$\frac{\delta}{\delta\Psi^*(r')}\int d^2r A_i^2(r)\Psi^*(r)\Psi(r)$$
$$= \int d^2r''\frac{\delta}{\delta A_i(r'')}\left[\int d^2r A_i^2(r)\Psi^*(r)\Psi(r)\right]\frac{\delta A_i(r'')[\Psi^*]}{\delta\Psi(r')} + A_i^2(r')\Psi(r')$$
$$= -2\int d^2r'' A_i(r'')\Psi^*(r'')\Psi(r'')\frac{e}{\kappa}G_i(r'-r'')\Psi(r') + A_i^2(r')\Psi(r')$$

Now, we substitute all terms into $|\vec{D}\Psi(r)|^2$ to obtain

$$\frac{\delta}{\delta\Psi^*(r')}\frac{1}{2m}\int d^2r|\vec{D}\Psi(r)|^2$$
$$=\frac{1}{2m}\sum_{i=1}^{2}\{-\partial_i^2\Psi(r') + ieA_i(r')\partial_i\Psi(r') + ie\partial_i A_i(r')\Psi(r') + ieA_i(r')\partial_i\Psi(r')$$
$$+e^2A_i^2(r')\Psi(r') - ie\Psi(r')\left[\frac{1}{\kappa}\int d^2r'' G_i(r'-r'')\left[\Psi^*(r'')\partial_i\Psi(r'') + \partial_i\Psi(r'')\Psi^*(r'')\right.\right.$$
$$\left.\left.-2ieA_i(r'')\Psi^*(r'')\Psi(r'')\right]\right]\}$$
$$= -\frac{1}{2m}\vec{D}^2\Psi(r')$$
$$+\sum_{i=1}^{2}e\left(\frac{1}{2mi}\right)\Psi(r')\left[\frac{1}{\kappa}\int d^2r'' G_i(r'-r'')[\Psi^*(r'')D_i\Psi(r'') - \Psi(r'')(D_i\Psi(r''))^*]\right]$$
$$= -\frac{1}{2m}\vec{D}^2\Psi(r') + eA_0\Psi(r').$$

Substitute into the right hand side of (51) to obtain the final equation

$$i\partial_t\Psi = -\frac{1}{2m}\vec{D}^2\Psi + eA_0\Psi - g(\Psi^*\Psi)\Psi.$$

Proof of the identity (62)

To show explicitly, we will decompose each term and then compare between the left hand side (LHS) and the right hand side(RHS) of the equation.

1st term LHS ;

$$
\begin{aligned}
\sum_{i=1}^{2}|D_i\Psi|^2 =& \sum_{i=1}^{2}(\partial_i\Psi - ieA_i\Psi)(\partial_i\Psi^* + ieA_i\Psi^*) \\
=& \sum_{i=1}^{2}\partial_i\Psi\partial_i\Psi^* + ieA_i(\Psi^*\partial_i\Psi - \Psi\partial_i\Psi^*) + e^2A_i^2|\Psi|^2
\end{aligned}
$$

1st term RHS ;

$$
\begin{aligned}
|(D_1+iD_2)\Psi|^2 =& |(\partial_1 - ieA_1 + i\partial_2 + eA_2)\Psi|^2 \\
=& [(\partial_1 + eA_2 + i(+\partial_2 - eA_1))\Psi][(\partial_1 + eA_2 - i(+\partial_2 - eA_1))\Psi^*] \\
=& \big[|(\partial_1 + eA_2)\Psi|^2 + |(\partial_2 - eA_1)\Psi|^2 + i[(\partial_2 - eA_1)\Psi][(\partial_1 + eA_2)\Psi^*] \\
& -i[(\partial_1 + eA_2)\Psi][(\partial_2 - eA_1)\Psi^*]\big] \\
=& |D_1\Psi|^2 + |D_2\Psi|^2 + eA_2(\Psi\partial_1\Psi^* + \Psi^*\partial_1\Psi) - eA_1(\Psi\partial_2\Psi^* + \Psi^*\partial_2\Psi) \\
& + i\partial_2\Psi\partial_1\Psi^* + i\partial_1\Psi\partial_2\Psi^*
\end{aligned}
$$

2nd term RHS ;

$$
\begin{aligned}
\partial_1 J_2 =& \frac{1}{2i}[\partial_1\Psi^*(\partial_2 - ieA_2)\Psi - \partial_1\Psi(\partial_2 + ieA_2)\Psi^* + \Psi^*(\partial_2 - ieA_2)\partial_1\Psi \\
& - \Psi(\partial_2 + ieA_2)\partial_1\Psi^* - ie(2\partial_1 A_2|\Psi|^2)]
\end{aligned}
$$

$$
\begin{aligned}
\partial_1 J_2 - \partial_2 J_1 =& \frac{1}{2mi}[2(\partial_1\Psi^*\partial_2\Psi - \partial_1\Psi\partial_2\Psi^*) - 2ieA_2(\Psi\partial_1\Psi^* + \Psi^*\partial_1\Psi) \\
& + (\Psi^*\partial_2\partial_1\Psi - \Psi\partial_2\partial_1\Psi^*) - (\Psi^*\partial_1\partial_2\Psi - \Psi\partial_1\partial_2\Psi^*) \\
& + 2ieA_1(\Psi\partial_2\Psi^* + \Psi^*\partial_2\Psi) - ie2B|\Psi|^2] \\
m(\partial_1 J_2 - \partial_2 J_1) =& -i(\partial_1\Psi^*\partial_2\Psi - \partial_1\Psi\partial_2\Psi^*) - eA_2(\Psi\partial_1\Psi^* + \Psi^*\partial_1\Psi) + \\
& eA_1(\Psi\partial_2\Psi^* + \Psi^*\partial_1\Psi) - eB|\Psi|^2
\end{aligned}
$$

Gathering all terms from the right hand side together to obtain

$$
|(D_1+iD_2)\Psi|^2 + m(\partial_1 J_2 - \partial_2 J_1) + eB|\Psi|^2 = |\vec{D}\Psi|^2.
$$

Appendix C: Calculation Details from Section 4.2

(68) $\rightarrow$ (70) ;

For the right handed spinor equation (ψ_+)

$$\begin{aligned}0 =& (ic\gamma_+^\alpha D_\alpha - m)\psi_+ \\ =& (i\sigma^3 D_0 - \sigma^1 D_1 - \sigma^2 D_2 - m)\psi_+ \\ =& e^{-imc^2 t}\left[i\begin{pmatrix} D_t\Psi_+ \\ -D_t\tilde{\chi}_+ \end{pmatrix} + m\begin{pmatrix} \Psi_+ \\ -\tilde{\chi}_+ \end{pmatrix} - c\vec{\sigma}\cdot\vec{D}\begin{pmatrix} \Psi_+ \\ \tilde{\chi}_+ \end{pmatrix} - m\begin{pmatrix} \Psi_+ \\ \tilde{\chi}_+ \end{pmatrix}\right] \\ 0 =& i\begin{pmatrix} D_t\Psi_+ \\ -D_t\tilde{\chi}_+ \end{pmatrix} - c\vec{\sigma}\cdot\vec{D}\begin{pmatrix} \Psi_+ \\ \tilde{\chi}_+ \end{pmatrix} - 2m\begin{pmatrix} 0 \\ \tilde{\chi}_+ \end{pmatrix}.\end{aligned} \tag{104}$$

For the left handed spinor equation (ψ_-)

$$\begin{aligned}0 =& -i\begin{pmatrix} D_t\tilde{\chi}_- \\ -D_t\Psi_- \end{pmatrix} - m\begin{pmatrix} \tilde{\chi}_- \\ -\Psi_- \end{pmatrix} - c\vec{\sigma}\cdot\vec{D}\begin{pmatrix} \tilde{\chi}_- \\ \Psi_- \end{pmatrix} - m\begin{pmatrix} \tilde{\chi}_- \\ \Psi_- \end{pmatrix} \\ 0 =& -i\begin{pmatrix} D_t\tilde{\chi}_- \\ -D_t\Psi_- \end{pmatrix} - c\vec{\sigma}\cdot\vec{D}\begin{pmatrix} \tilde{\chi}_- \\ \Psi_- \end{pmatrix} - 2m\begin{pmatrix} \tilde{\chi}_- \\ 0 \end{pmatrix}.\end{aligned} \tag{105}$$

By swapping the lower component between two equations, (104) becomes

$$\begin{aligned}iD_t\begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} - c\vec{\sigma}\cdot\vec{D}\begin{pmatrix} \tilde{\chi}_- \\ \tilde{\chi}_+ \end{pmatrix} =& 0 \\ iD_t\Phi - c\vec{\sigma}\cdot\vec{D}\tilde{\chi} =& 0.\end{aligned}$$

and (105) becomes

$$\begin{aligned}-iD_t\begin{pmatrix} \tilde{\chi}_- \\ \tilde{\chi}_+ \end{pmatrix} + c\vec{\sigma}\cdot\vec{D}\begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} + 2m\begin{pmatrix} \tilde{\chi}_- \\ \tilde{\chi}_+ \end{pmatrix} =& 0 \\ -iD_t\tilde{\chi} + c\vec{\sigma}\cdot\vec{D}\Phi + 2m\tilde{\chi} =& 0.\end{aligned}$$

By approximating in the non-relativistic limit, we can drop the time derivative term out of the second equation so the equation becomes

$$c\vec{\sigma}\cdot\vec{D}\Phi + 2m\tilde{\chi} = 0.$$

Proof of (72)

$$\begin{aligned}(\vec{\sigma}\cdot\vec{D})(\vec{\sigma}\cdot\vec{D}) =& \sigma_i(\partial_i - ieA_i)\sigma_j(\partial_j - ieA_j) \\ =& \vec{D}^2 + i\epsilon_{ijk}\sigma_k\partial_i\partial_j - ie(A_i\partial_j + \partial_i[A_j) - e^2 A_i A_j \\ =& \vec{D}^2 + e\sigma_k\epsilon_{ijk}A_i\partial_j + \partial_i A_j + A_j\partial_i \\ =& \vec{D}^2 + e\sigma_3 B\end{aligned}$$

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