ตัวแบบความเสี่ยงแบบเวลาไม่ต่อเนื่องบนฐานของอนุกรมเวลาปัวซงที่มีศูนย์เฟ้อ

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DISCRETE TIME RISK MODEL BASED ON ZERO INFLATED POISSON TIME SERIES

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การพัฒนาเครื่องมือและแบบจำลองเป็นสิ่งสำคัญของงานทางคณิตศาสตร์ประกันภัย สำหรับผลงานของบริษัทประกันภัยและในการพัฒนาผลิตภัณฑ์ทางประกันภัย ความเสี่ยงเป็นเครื่องมือหนึ่งที่สำคัญในการที่จะบอกนักคณิตศาสตร์ประกันภัยหรือผู้จัดการความเสี่ยงเกี่ยวกับระดับความเสี่ยงได้ ในการที่จะวัดความเสี่ยงให้มีความถูกต้องและแม่นยำ เราจำเป็นที่จะต้องมีแบบจำลองที่เหมาะสมในการนับจำนวนครั้งในการเรียกร้องค่าสินไหม โดยปกติแบบจำลองส่วนใหญ่ที่ใช้ในการนับจำนวนครั้งในการเรียกร้องค่าสินไหมจะถูกสร้างมาจากแบบจำลองปัวซง แต่เนื่องจากข้อมูลทางประกันภัยส่วนใหญ่มีข้อมูลที่เป็นศูนย์อยู่จำนวนมากซึ่งทำให้ข้อมูลมีการกระจายมากเกินไปหรือเรียกว่าโอเวอร์ดิซเพอชั่น ซึ่งข้อมูลแบบนี้จะขัดแย้งสมมติฐานของแบบจำลองปัวซง ดังนั้นเราจึงศึกษาแบบจำลองอื่น ๆ ที่สามารถนับจำนวนศูนย์เข้าไปในแบบจำลองได้ จากวารสารทางวิชาการ แบบจำลองปัวซงที่มีศูนย์เฟ้อเป็นที่ใช้กันอย่างแพร่หลายสำหรับการนับจำนวนศูนย์ในข้อมูล ดังนั้นในการศึกษานี้เราจะประยุกต์ใช้แบบจำลองปัวซงกรณีที่มีศูนย์เฟ้อกับอนุกรมเวลาเพื่อที่จะใช้ในการนับจำนวนครั้งในการเรียกร้องค่าสินไหม และนำมาประยุกต์ใช้กับตัวแบบความเสี่ยงเพื่อที่จะสร้างตัวแบบความเสี่ยงของเวลาไม่ต่อเนื่องบนฐานของอนุกรมเวลาปัวซงที่มีศูนย์เฟ้อ โดยในงานนี้ผู้ศึกษายังได้ศึกษาคุณสมบัติเชิงความน่าจะเป็นและการประมาณของความน่าจะเป็นที่จะล้มละลาย และนำเสนอการวิเคราะห์เชิงตัวเลขของการคำนวณค่าประมาณของความน่าจะเป็นที่จะล้มละลายและมูลค่าความเสี่ยงและมูลค่าความเสี่ยงส่วนเกิน

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An important goal of actuary is to develop models for company portfolio and insurance products. Risk measurement is one of the essential measures that inform actuaries and risk managers about the degree to which the risk bearing entity. To have precise risk measure, we require an appropriate claim count process. The common claim count processes are usually constructed from the Poisson distribution. However, insurance data have generally excess zeros which causes the overdispersion. This violates the assumption of the Poisson distribution. Therefore, alternative distributions accommodating zero count are explored in literature. The zero inflated Poisson distribution is one of the distributions widely used for zero count data. In this study, we apply the zero inflated Poisson distribution to construct an integer valued time series for claim counts. The model is then applied to construct risk models based on the zero inflated Poisson time series. We derive some properties and the approximation of the value of the ruin probability of the constructed models. In addition, we also perform some calculations of the value of the ruin probability, the value at risk, and the tail value at risk.

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CHAPTER I

INTRODUCTION

Risk measurement is one of the essential measures that can inform actuaries and risk managers about the degree to which the risk bearing entity. The insurance's portfolio can be called as the amount of surplus in the classical risk model. The amount of surplus process can be expressed by taking account of the inflow of premiums and the outflow of claim payments and starting with initial reserve. Therefore, we can measure risk of insurance company through many risk measures such as the ruin probability, the value at risk and the tail value at risk. In recent years, a majority of researches in actuarial science focuses on the development of risk models for different underlying distributions of the arrival of claims, claim sizes and particularly for the claim counts. Several distributions of claim counts have been explored such as Poisson distribution and Negative Binomial distribution. Besides the classical distributions, integer valued time series for claim counts are also introduced into the risk models.

Time series is a sequence of data points measured over time. Two common structures of time series models are the autoregressive (AR) and the moving average (MA) structures. The autoregressive structure assumes that the current value of the series can be explained as a linear regression of past values. The moving average structure assumes that the current value can be explained as a regression of past values of stochastic terms called white noises. The original autoregressive and moving average time series models are mostly studied under the normality assumption and applied to continuous variables of interest such as stock price markets. Later, the concepts of autoregressive moving average models were generalized to accommodate time series of counts. For example, McKenzie (1985) introduced the first autoregressive process integer valued AR(1) model as a counting process. The properties of the integer valued autoregressive (INAR) and integer valued moving average (INMA) are studied in Al-Osh and Alzaid (1987) and Alzaid and Al-Osh (1988). These models can be used as claim count models based on binomial thining operator proposed in Steutel and van Harn (1979).

Cossette and Marceau (2000) introduced the concept of time series of counts to the context of insurance risk models. In their study, they introduced the discrete time risk model with correlated classes of business and studied the impact on the finite time ruin probability and on the adjustment coefficient. Since then, several studies of the risk models based on integer valued time series have been intensively studied in literature. For example, Cossette et al. (2011) studied the classical risk models based on time series process based on the Poisson distribution. The Poisson distribution is one of extensive used distributions for count data. The one important characteristic of the Poisson distribution is that its expectation and variance are the same. This property may not be applicable for the data that exhibit overdispersion or underdispersion such as the insurance claim counts. Therefore, the alternative distributions have been proposed, for instance, Laphudomsakda and Suntornchost (2018) introduced the discrete risk model based on negative binomial moving average (NBMA) model.

However, none of these distributions is suitable for the data with excess of zeros. For instance, the insurance claim counts having small claims with deductibles and no claim discounts of automobile portfolio with excess zeros in data. Therefore, alternative distributions to accommodate zero counts have been proposed through the concept of zero inflated first introduced by Lambert (1992). The definition of zero inflated Poisson distribution is stated as follows

$$P(X = k) = \begin{cases} p + (1-p)e^{-\lambda}, & \text{if } k = 0, \\ (1-p)\dfrac{e^{-\lambda}\lambda^k}{k!}, & \text{if } k = 1, 2, \ldots \end{cases}$$

where $p \in (0,1)$ and $\lambda > 0$. In addition, parameter p represents for the proportion of zero and if $p = 0$, we obtain the Poisson distribution. Later on, the zero inflated Poisson has been applied to many applications such as a manual handling injury prevention strategy trialled (Yau and Lee, 2001), the credibility premiums (Boucher and Denuit, 2008) and the number of accidents (Boucher et al., 2009). Among these applications, one well known application of the zero inflated Poisson is to model claim counts. For example, Yip and Yau (2005) proposed proposed the zero inflated Poisson distribution for the excess zeros in insurance claim count data. Zhu (2012) proposed zero inflated Poisson time series of counts (ZIP-INGARCH). Aghababaei Jazi et al. (2012) introduced the first order integer valued AR process with zero inflated Poisson distribution (ZIP-INAR) to model the count of events in consecutive points of time. Sarul and Sahin (2015) proposed the zero inflated Poisson distribution as a claim count model to take account excess zeros in data.

In this study, we apply the zero inflated Poisson to construct new discrete time risk models based on the zero inflated Poisson moving average and the zero inflated Poisson autoregressive models. Moreover, we derive probabilistic properties of the new constructed risk models, the upper bound of the ruin probability and the risk measures.

The organization of this thesis is as follows. In Chapter 2, we introduce the background knowledge used throughout this thesis. In Chapter 3, we introduce risk models based on the first order zero inflated Poisson moving average and q^{th} order zero inflated Poisson moving average models, and related quantities such as the adjustment coefficient function, approximations to the value at risk and tail value at risk. Numerical results studying the trend of the ruin probability and the risk measures are also presented in Chapter 3. In Chapter 4, we introduce risk models based on the first order zero inflated Poisson autoregressive model, and related quantities and numerical results are presented. In Chapter 5, we give discussions and conclusion of this thesis.

CHAPTER II

BACKGROUND KNOWLEDGE

In this chapter, we provide some definitions and properties that will be used throughout this thesis.

2.1 Random Variables and Probabilistic Properties

In this section, we give useful theorems and definitions and some handful techniques to obtain the probabilistic properties for random variables.

Definition 2.1. Consider a random experiment whose sample space is S. A random variable X is a function from the sample sapce S into the set or real number $\mathbb{R}$ such that for each interval I in $\mathbb{R}$, the set $\{s \in S \,|\, X(s) \in I\}$ is an event in S.

Definition 2.2. The set $\{x \in \mathbb{R} \,|\, x = X(s),\, s \in S\}$ is called the space of random variable X.

Definition 2.3. If the space of random variable X is countable,then X is called a discrete random variable.

Definition 2.4. Let R_X be the space of discrete random variable X. The function $f : R_X \to \mathbb{R}$ defined by

$$f(x) \quad = \quad P(X = x)$$

is called the probability mass function (pmf) of X.

Theorem 2.5. If X is a discrete random variable with space R_X and the probability mass function $f(\cdot)$, then

(a) $f(x) \geq 0$ for all x in R_X, and

(b) $\displaystyle\sum_{x \in R_X} f(x) = 1.$

Definition 2.6. The cumulative distribution function $F(\cdot)$ of a random variable X is defined as

$$F(x) \quad = \quad P(X \leq x),$$

for all real number x.

Theorem 2.7. If X is a random variable with the space R_X and $f(\cdot)$ is the probability mass function of X , then the cumulative distribution $F(\cdot)$ can be defined as

$$F(x) \quad = \quad \sum_{t \leq x} f(t),$$

for $x \in R_X$.

Theorem 2.8. The cumulative distribution function $F(\cdot)$ of a random variable X has the following properties.

(a) $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$,

(b) $F(x)$ is a non decreasing function, that is if $x < y$,then $F(x) \leq F(y)$ for all real numbers x, y,

(c) $F(x)$ is right continuous for all $x_0 \in \mathbb{R}$ and $\lim_{x\to x_0^+} F(x) = F(x_0)$.

Definition 2.9. The n^{th} moment about the origin of a discrete random variable X, as denoted by $\mathrm{E}(X^n)$, is defined to be

$$\mathrm{E}(X^n) \quad = \quad \sum_{x \in R_X} x^n f(x), \tag{2.1}$$

for $n = 0, 1, 2, \ldots$, provided the right side converges absolutely and $f(\cdot)$ is the probability mass function of X.

Furthermore, If $n = 1$, then $\mathrm{E}(X)$ is called the first moment about the origin, or the expectation. If $n = 2$, then $\mathrm{E}(X^2)$ is called the second moment of random variable X about the origin.

Definition 2.10. Let X be a discrete random variable with space R_X and probability density mass function $f(\cdot)$. The expectation or the expected value of the random variable X is defined as

$$\mathrm{E}(X) \quad = \quad \sum_{x \in R_X} x f(x),$$

The expectation is also called mean of the random variable X, denoted by μ_X.

Theorem 2.11. If a and b are any two real numbers,then

$$\mathrm{E}(aX + b) \quad = \quad a\mathrm{E}(X) + b.$$

Definition 2.12. Let X be a random variable with mean μ_X. The variance of X, denoted by $\mathrm{Var}(X)$, is defined as

$$\mathrm{Var}(X) \quad = \quad \mathrm{E}\left(X - \mu_X\right)^2.$$

Theorem 2.13. If X is a random variable with mean μ_X, then

$$\mathrm{Var}(X) \quad = \quad \mathrm{E}(X^2) - (\mu_X)^2.$$

Theorem 2.14. If X is a random variable with variance $\mathrm{Var}(X)$ then

$$\mathrm{Var}(aX + b) \quad = \quad a^2\mathrm{Var}(X),$$

where a and b are arbitrary real constants.

Definition 2.15. Let X and Y be random variables with means μ_X and μ_Y, respectively. The covariance function between X and Y, denoted by $\mathrm{Cov}(X,Y)$, is defined as

$$\mathrm{Cov}(X,Y) \;=\; \mathrm{E}\left((X-\mu_X)(Y-\mu_Y)\right) \;=\; \mathrm{E}(XY)-\mu_X\mu_Y.$$

The correlation function between X and Y, denoted by $\mathrm{Corr}(X,Y)$, is defined as

$$\mathrm{Corr}(X,Y) \;=\; \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}.$$

Definition 2.16. Let X be a discrete random variable whose probability mass function $f(\cdot)$ with space R_X. The function $m_X : R_X \to \mathbb{R}$ defined by

$$m_X(t) \;=\; \mathrm{E}\left(e^{tX}\right) \;=\; \sum_{x\,\in\,R_X} e^{tx}f(x),$$

for $t \in \mathbb{R}$ and $m_X(\cdot)$ is called the moment generating function of X.

Definition 2.17. Let X be a discrete random variable whose probability mass function is $f(\cdot)$ with space R_X. The function $G_X : R_X \to \mathbb{R}$ defined by

$$G_X(t) \;=\; \mathrm{E}(t^X) \;=\; \sum_{x\,\in\,R_X} t^x f(x),$$

for $t \in \mathbb{R}$ and $G_X(\cdot)$ is called the probability generating function (p.g.f.) of X.

Definition 2.18. Let X and Y be discrete random variables defined on the same sample space. The function $F_{X,Y} : \mathbb{R}^2 \to [0,1]$ defined by

$$F_{X,Y}(x,y) \;=\; P\left(X \le x, Y \le y\right),$$

for all real numbers x and y and $F_{X,Y}(\cdot,\cdot)$ is called the joint cumulative distribution function of X and Y. The function $f_{X,Y} : \mathbb{R}^2 \to [0,\infty)$ defined by

$$f_{X,Y}(x,y) \;=\; P(X = x, Y = y),$$

for all real numbers x and y and $f_{X,Y}(\cdot,\cdot)$ is called the joint probability mass function of X and Y.

Definition 2.19. Let X and Y be discrete random variables with the joint probability mass function $f_{X,Y}(\cdot,\cdot)$. The marginal probability mass of Y, $f_Y : \mathbb{R} \to [0,\infty)$ defined by

$$f_Y(y) \;=\; \sum_{x\,\in\,\mathrm{Im}X} f_{X,Y}(x,y),$$

for all real number y.

Definition 2.20. Let X and Y be discrete random variables with the joint probability mass function $f_{X,Y}(\cdot,\cdot)$ and $f_Y(\cdot)$ is the marginal probability mass function of the random variable Y. The conditional probability mass function of X, given $Y = y$ for all values y such that $f_Y(\cdot) > 0$, defined by

$$f_{X|Y}(x|y) \;=\; \frac{f_{X,Y}(x,y)}{f_Y(y)},$$

for all $x \in R_x$.

Definition 2.21. Let X be a discrete random variable and $f_{X|Y}(x|y)$ be the condition probability mass function of X, given $Y = y$. The conditional expectation of X, given $Y = y$ defined by

$$\mathrm{E}(X|Y = y) \;=\; \sum_{x\,\in\,R_x} x f_{X|Y}(x|y).$$

Definition 2.22. Let $X_1, X_2, \ldots, X_n$ be discrete random variables with the probability mass functions $f_{X_1}(\cdot), f_{X_2}(\cdot), \ldots, f_{X_n}(\cdot)$. They are said to be identically distributed random variables if and only if

$$f_{X_1}(x) \;=\; f_{X_2}(x) \;=\; \ldots \;=\; f_{X_n}(x),$$

for $x \in \mathbb{R}$.

Definition 2.23. The discrete random variables $X_1, X_2, \ldots, X_n$, are said to be independent random variables if and only if the joint probability mass function $f_{X_1,X_2,\ldots,X_n}(\cdot,\cdot,\ldots,\cdot)$ can be written as

$$f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n),$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$, where $f_{X_i}(\cdot)$ is the probability mass of X_i $(i = 1, 2, \ldots, n)$.

The random variables $X_1, X_2, \ldots, X_n$ are said to be independent and identically distributed (i.i.d.) if random variables $X_1, X_2, \ldots, X_n$ have the same probability mass function and are mutually independent.

Lemma 2.24. Let X be a discrete random variable with probability generating function $G_X(\cdot)$, the probabilistic properties of X are listed as follows

(a) $\mathrm{E}(X) = \frac{d}{dt} G_X(t)\Big|_{t=1}$,

(b) $\mathrm{Var}(X) = \frac{d^2}{dt^2} G_X(t)\Big|_{t=1} + \mathrm{E}(X) - (\mathrm{E}(X))^2$,

(c) $\mathrm{Var}(X) = \mathrm{E}(\mathrm{Var}(X|U)) + \mathrm{Var}(\mathrm{E}(X|U))$, where U is any random variable,

(d) The skewness $Sk = \dfrac{\mathrm{E}(X^3) - 3\mathrm{E}(X)\mathrm{Var}(X) - \mathrm{E}^3(X)}{\mathrm{Var}(X)^{3/2}}$.

Proof. (a) Note that

$$\begin{aligned} \frac{d}{dt} G_X(t) &= \frac{d}{dt} \mathrm{E}(t^X) \\ &= \frac{d}{dt} \sum_{x \in R_X} t^x f(x) \\ &= \sum_{x \in R_X} x t^{x-1} f(x). \end{aligned}$$

Taking $t = 1$, so we can obtain E(X).

(b) Consider

$$\begin{aligned}\frac{d^2}{dt^2}G_X(t) &= \frac{d^2}{dt^2}\mathrm{E}(t^X)\\ &= \frac{d^2}{dt^2}\sum_{x\in R_X} t^x f(x)\\ &= \sum_{x\in R_X} x(x-1)t^{x-2}f(x)\\ &= \sum_{x\in R_X} x^2t^{x-2}f(x) - \sum_{x\in R_X} xt^{x-2}f(x).\end{aligned}$$

Taking $t = 1$,then we obtain $\mathrm{E}(X^2) - \mathrm{E}(X)$. Then, we add $\mathrm{E}(X) - (\mathrm{E}(X))^2$ into $\mathrm{E}(X^2) - \mathrm{E}(X)$, then we obtain

$$\mathrm{E}(X^2) - \mathrm{E}(X) + \mathrm{E}(X) - (\mathrm{E}(X))^2 = \mathrm{Var}(X).$$

(c) Note that,

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2\\ &= \mathrm{E}(\mathrm{E}(X^2|U)) - \mathrm{E}^2(\mathrm{E}(X|U))\\ &= \mathrm{E}(\mathrm{E}(X^2|U)) - \mathrm{E}^2(\mathrm{E}(X|U)) - \mathrm{E}(\mathrm{E}^2(X|U)) + \mathrm{E}(\mathrm{E}^2(X|U))\\ &= \mathrm{E}(\mathrm{E}(X^2|U) - \mathrm{E}^2(X|U)) + \mathrm{E}(\mathrm{E}^2(X|U)) - \mathrm{E}^2(\mathrm{E}(X|U))\\ &= \mathrm{E}(\mathrm{Var}(X|U)) + \mathrm{Var}(\mathrm{E}(X|U)).\end{aligned}$$

(d) Note that the formula of skewness is defined as

$$Sk = \frac{\mathrm{E}\left(X - \mathrm{E}(X)\right)^3}{\mathrm{Var}(X)^{3/2}}.$$

Then, we expand the numerator to derive another version that can be calculated more easily as follows

$$\begin{aligned}Sk &= \frac{\mathrm{E}\left(X - \mathrm{E}(X)\right)^3}{\mathrm{Var}(X)^{3/2}}\\ &= \frac{\mathrm{E}(X^3) - 3\mathrm{E}(X^2)\mathrm{E}(X) + 3\mathrm{E}(X)\mathrm{E}^2(X) - \mathrm{E}^3(X)}{\mathrm{Var}(X)^{3/2}}\\ &= \frac{\mathrm{E}(X^3) - 3\mathrm{E}(X)\left(\mathrm{E}(X^2) - \mathrm{E}^2(X)\right) - \mathrm{E}^3(X)}{\mathrm{Var}(X)^{3/2}}\\ &= \frac{\mathrm{E}(X^3) - \mathrm{E}(X)\mathrm{Var}(X) - \mathrm{E}^3(X)}{\mathrm{Var}(X)^{3/2}}.\end{aligned}$$

□

Definition 2.25. Let $\{\delta_j,\ j = 1, 2, \ldots\}$ be a sequence of independent and identically distributed random variables, X be an non-negative integer valued random variable which is independent of $\{\delta_j,\ j = 1, 2, \ldots\}$. Then the random variable

$$N = \sum_{j=1}^{X} \delta_j$$

is called a compound random variable.

Lemma 2.26. Let $N_i = \sum_{j=1}^{X_i} \delta_{i,j}$ for $i = 1, 2$ are compound random variables defined in Definition 2.25 where $\{\delta_{1,j}\, j = 1, 2, \ldots\}$ and $\{\delta_{2,j}\, j = 1, 2, \ldots\}$ are two mutually independent sequences of random variables and are independent of X_1 and X_2, respectively. The probabilistic properties of $N_i\ (i = 1, 2)$ are provided as follows.

(a) $\mathrm{E}(N_i) = \mathrm{E}(X_i)\mathrm{E}(\delta_i)$,

(b) $\mathrm{Var}(N_i) = \mathrm{E}(X_i)\mathrm{Var}(X_i) + \mathrm{Var}(X_i)(\mathrm{E}(\delta_i))^2$,

(c) $\mathrm{Cov}(N_i, X_i) = \mathrm{E}(\delta_i)\mathrm{Var}(X_i)$,

(d) $\mathrm{Cov}(N_1, N_2) = \mathrm{E}(\delta_1)\mathrm{E}(\delta_2)\mathrm{Cov}(X_1, X_2)$,

where $\mathrm{E}(\delta_i)$ is the mean of $\{\delta_{i,j}\, j = 1, 2, \ldots\}$ and $i = 1, 2$.

Proof. (a) For $i = 1, 2$, we know that $\{\delta_{i,j}\, j = 1, 2, \ldots\}$ are identically distributed, then $\mathrm{E}(\delta_{i,1}) = \mathrm{E}(\delta_{i,2}) = \ldots = \mathrm{E}(\delta_i)$. Thus, consider

$$\begin{aligned}
\mathrm{E}(N_i) &= \mathrm{E}\left(\sum_{j=1}^{X_i} (\delta_{i,j})\right) \\
&= \mathrm{E}\left(\mathrm{E}(\delta_{i,1} + \delta_{i,2} + \cdots + \delta_{i,X_i} \,|\, X_i)\right) \\
&= \mathrm{E}\left(\sum_{j=1}^{X_i} \mathrm{E}(\delta_{i,j})\right) \\
&= \mathrm{E}(X_i \mathrm{E}(\delta_i)) \\
&= \mathrm{E}(X_i)\mathrm{E}(\delta_i).
\end{aligned}$$

(b) Using Lemma 2.24 (c),we obtain

$$\begin{aligned}
\mathrm{Var}(N_i) &= \mathrm{E}\left(\mathrm{Var}\left(\sum_{j=1}^{X_i}\delta_{i,j}\middle|X_i\right)\right) + \mathrm{Var}\left(\mathrm{E}\left(\sum_{j=1}^{X_i}\delta_{i,j}\middle|X_i\right)\right)\\
&= \mathrm{E}(X_i\mathrm{Var}(\delta_i)) + \mathrm{Var}(X_i\mathrm{E}(\delta_i))\\
&= \mathrm{E}(X_i)\mathrm{Var}(\delta_i) + \mathrm{Var}(X_i)(\mathrm{E}(\delta_i))^2.
\end{aligned}$$

(c) Note that,

$$\begin{aligned}
\mathrm{Cov}(N_i, X_i) &= \mathrm{E}(N_iX_i) - \mathrm{E}(N_i)\mathrm{E}(X_i)\\
&= \mathrm{E}\left(X_i\sum_{j=1}^{X_i}\delta_{i,j}\right) - \mathrm{E}(X_i)\mathrm{E}(X_i)\mathrm{E}(\delta_i)\\
&= \mathrm{E}\left(X_i\mathrm{E}\left(\sum_{j=1}^{X_i}\delta_{i,j}\middle|X_i\right)\right) - (\mathrm{E}(X_i))^2\mathrm{E}(\delta_i)\\
&= \mathrm{E}(X_i^2\mathrm{E}(\delta_i)) - (\mathrm{E}(X_i))^2\mathrm{E}(\delta_i)\\
&= \mathrm{E}(X^2)\mathrm{E}(\delta_i) - (\mathrm{E}(X_i))^2\mathrm{E}(\delta_i)\\
&= \mathrm{E}(\delta_i)\left(\mathrm{E}(X_i^2) - (\mathrm{E}(X_i))^2\right)\\
&= \mathrm{E}(\delta_i)\mathrm{Var}(X_i).
\end{aligned}$$

(d) Note that,

$$\begin{aligned}
\mathrm{Cov}(N_1, N_2) &= \mathrm{Cov}\left(\sum_{j=1}^{X_1}\delta_{1,j}, \sum_{j=1}^{X_2}\delta_{2,j}\right)\\
&= \mathrm{E}\left(\mathrm{E}\left(\sum_{j=1}^{X_1}\delta_{1,j}, \sum_{j=1}^{X_2}\delta_{2,j}\middle|X_1, X_2\right)\right) - \mathrm{E}\left(\sum_{j=1}^{X_1}\delta_{1,j}\right)\mathrm{E}\left(\sum_{j=1}^{X_2}\delta_{2,j}\right)\\
&= \mathrm{E}(X_1\mathrm{E}(\delta_1)X_2\mathrm{E}(\delta_2)) - \mathrm{E}(X_1)\mathrm{E}(\delta_1)\mathrm{E}(X_2)\mathrm{E}(\delta_2) \qquad (2.2)\\
&= \mathrm{E}(\delta_1)\mathrm{E}(\delta_2)(\mathrm{E}(X_1X_2) - \mathrm{E}(X_1)\mathrm{E}(X_2))\\
&= \mathrm{E}(\delta_1)\mathrm{E}(\delta_2)\mathrm{Cov}(X_1, X_2),
\end{aligned}$$

where we use the fact that $\{\delta_{1,j}\, j = 1, 2, \ldots\}$ and $\{\delta_{2,j}\, j = 1, 2, \ldots\}$ are mutually independent to obtain (2.2). □

Theorem 2.27. The moment generating function of the compound random variable $S = X_1 + \cdots + X_N$ is

$$m_S(r) = G_N(m_X(r)), \tag{2.3}$$

where $G_N(\cdot)$ is the probability generating function of N.

Proof. Note that

$$\begin{aligned} \mathrm{E}(e^{rS}|N = n) &= \mathrm{E}\left(e^{r(X_1+\cdots+X_n)}\right) \\ &= (m_X(r))^n, \end{aligned}$$

so that $\mathrm{E}(e^{rS} \mid N) = (m_X(r))^N$, using the conditional expectation, then we obtain

$$\begin{aligned} m_S(r) &= \mathrm{E}(\mathrm{E}\left(e^{rS} \mid N\right)) \\ &= \mathrm{E}\left((m_X(r))^N\right) \\ &= G_N(m_X(r)). \end{aligned}$$

□

2.2 Zero Inflated Poisson Distribution

In this section, we first introduce the concept of zero inflated distribution and the properties of zero inflated Poisson distribution used in this study. We follow the probability mass function proposed by Lambert (1992).

The concept of zero inflated model is to allow more flexibility in modeling the distribution to accommodate zero counts into model. Zero inflated model, added the probability of being zero and can be formulated with the number of distributions. Zero inflated model can be expressed as the following

$$P(X = k) = pI_{(k)} + (1-p)f(k), \; k = 0, 1, 2, \ldots,$$

where

$$I_{(w)} = \begin{cases} 1, & w = 0, \\ 0, & w \neq 0, \end{cases}$$

and p is the proportion of zero, $f(\cdot)$ is probability mass function of Y where Y is a random variable, taking value $0, 1, 2, \ldots$, then X is called zero inflated version of random variable Y.

Definition 2.28. Let X be a zero inflated Poisson random variable with parameters p and λ, denoted by $X \sim ZIP(p, \lambda)$. The probability mass function of X defined as

$$P(X = k) = \begin{cases} p + (1-p)e^{-\lambda}, & \text{if } k = 0, \\ (1-p)\dfrac{e^{-\lambda}\lambda^k}{k!}, & \text{if } k = 1, 2, \ldots, \end{cases}$$

where $p \in (0, 1)$ and $\lambda > 0$.

Lemma 2.29. The zero inflated Poisson random variable $X \sim ZIP(p, \lambda)$, defined as Definition 2.28 has the following properties.

(a) The probability generating function: $G_X(t) = p + (1-p)e^{-\lambda(1-t)}$, for $t \in \mathbb{R}$,

(b) The expectation: $\mathrm{E}(X) = \lambda(1-p)$,

(c) The variance: $\mathrm{Var}(X) = \lambda(1-p)(1+\lambda p)$,

(d) The skewness: $Sk = \dfrac{1 + 3\lambda p + 2\lambda^2 p^2 - \lambda^2 p}{\sqrt{(1-p)\lambda(1+\lambda p)^3}}$.

Proof. (a) Using Definition 2.17 and the probability mass function as in Definition 2.28, we can obtain

$$\begin{aligned} G_X(t) &= \mathrm{E}(t^X) \\ &= \sum_{k\geq 0} t^k P(X=k) \\ &= P(X=0) + \sum_{k\geq 1} t^k P(X=k) \\ &= p + (1-p)e^{-\lambda}t^0 + \sum_{k\geq 1} t^k (1-p)\frac{e^{-\lambda}\lambda^k}{k!} \\ &= p + \sum_{k\geq 0} t^k (1-p)\frac{e^{-\lambda}\lambda^k}{k!} \\ &= p + (1-p)e^{-\lambda}\sum_{k\geq 0}\frac{(t\lambda)^k}{k!} \\ &= p + (1-p)e^{-\lambda}e^{\lambda t} \\ &= p + (1-p)e^{-\lambda(1-t)}, \end{aligned}$$

for $t \in \mathbb{R}$.

(b) Using Lemma 2.24 (a), we obtain $\mathrm{E}(X)$ as follows

$$\begin{aligned} \mathrm{E}(X) &= \frac{d}{dt}G_X(t)\Big|_{t=1} \\ &= \frac{d}{dt}(p + (1-p)e^{-\lambda(1-t)})\big|_{t=1} \\ &= \Big((1-p)e^{-\lambda(1-t)}\lambda\Big)\Big|_{t=1} \\ &= \lambda(1-p). \end{aligned}$$

(c) Using Lemma 2.24 (a), we first find $\frac{d^2}{dt^2}G_X(t)\Big|_{t=1}$ as follows

$$\begin{aligned} \frac{d^2}{dt^2}G_X(t)\Big|_{t=1} &= \frac{d}{dt}\Big(p + (1-p)e^{-\lambda(1-t)}\lambda\Big)\Big|_{t=1} \\ &= \Big(\lambda(1-p)e^{-\lambda(1-t)}\lambda\Big)\big|_{t=1} \\ &= \lambda^2(1-p). \end{aligned}$$

Then, we obtain $\mathrm{Var}(X)$ as follows

$$\begin{aligned}\mathrm{Var}(X) &= \frac{d^2}{dt^2}G_X(t)\Big|_{t=1} + \mathrm{E}(X) - (\mathrm{E}(X))^2 \\ &= \lambda^2(1-p) + \lambda(1-p) - (\lambda(1-p))^2 \\ &= \lambda(1-p)(\lambda + 1 - \lambda(1-p)) \\ &= \lambda(1-p)(1+\lambda p).\end{aligned}$$

(d) Using Lemma 2.24 (d), we first consider $\mathrm{E}(X^3)$ by applying probability generating function as follows

$$\frac{d^3}{dt^3}G_X(t) = \mathrm{E}\left(X(X-1)(X-2)t^{X-3}\right).$$

Let $t = 1$, then

$$G_X^{'''}(1) = \mathrm{E}\left(X(X-1)(X-2)\right).$$

Then,we have

$$\mathrm{E}(X^3) = G_X^{'''}(1) + 3\mathrm{E}(X^2) - 2\mathrm{E}(X),$$

and from Lemma 2.24, we have that $G_X^{''}(1) = \mathrm{E}(X^2) - \mathrm{E}(X)$.
Finally, we can have

$$\mathrm{E}(X^3) = G_X^{'''}(1) + 3G_X^{''}(1) + \mathrm{E}(X).$$

Moreover, we know $G_X^{''}(1) = \lambda^2(1-p)$ and $G_X^{'''}(1) = \lambda^3(1-p)$.
Hence,

$$\mathrm{E}(X^3) = \lambda^3(1-p) + 3\lambda^2(1-p) + \lambda(1-p).$$

Therefore, the skewness is

$$
\begin{aligned}
Sk &= \frac{\mathrm{E}(X^3) - \mathrm{E}(X)\mathrm{Var}(X) - \mathrm{E}^3(X)}{\mathrm{Var}(X)^{3/2}} \\
&= \frac{\lambda^3(1-p) + 3\lambda^2(1-p) + \lambda(1-p) - 3\lambda^2(1-p)^2(1+\lambda p) - (\lambda(1-p))^3}{(\lambda(1-p)(1+\lambda p))^{3/2}} \\
&= \frac{\lambda^2 + 3\lambda + 1 - 3\lambda(1-p)(1+\lambda p) - (\lambda(1-p))^2}{\sqrt{\lambda(1-p)(1+\lambda p)^3}} \\
&= \frac{\lambda^2 + 3\lambda + 1 - 3\lambda(1+\lambda p - p - \lambda p^2) - \lambda^2(1-2p+p^2)}{\sqrt{\lambda(1-p)(1+\lambda p)^3}} \\
&= \frac{1 + 3\lambda p + 2\lambda^2 p^2 - \lambda^2 p}{\sqrt{\lambda(1-p)(1+\lambda p)^3}}.
\end{aligned}
$$

□

2.3 Binomial Thining Operator

McKenzie (1985) and Alzaid and Al-Osh (1988) have proposed the model known as integer valued autoregressive (INAR) and moving average (INMA) processes. The models are constructed by using binomial thining operator. In this section, we first introduce the definition of binomial thining operator proposed by Steutel and van Harn (1979).

Definition 2.30. Let X be a non-negative integer valued random variable. For $\alpha \in (0,1)$, the '$\alpha\circ$' thining operator is defined as

$$
\alpha \circ X = \sum_{i=1}^{X} \delta_i,
$$

where $\{\delta_i,\ i = 1, 2, \ldots\}$ is a sequence of i.i.d. Bernoulli random variables with mean α and is independent from X.

Lemma 2.31 (Properties of binomial thining operator)**.** Let X_i , $i = 1, 2, \ldots$ be a non negative integer valued random variables, $\{\delta_{i,j}\, i, j = 1, 2, \ldots\}$ be a sequence of i.i.d. Bernoulli random variables with mean α_i and is independent from X_i. Then for $i, j = 1, 2, \ldots$, has the following properties.

(a) $\mathrm{E}(\alpha_i \circ X_i) = \alpha_i \mathrm{E}(X_i)$,

(b) $\mathrm{E}((\alpha_i \circ X_i)X_k) = \alpha_i \mathrm{E}(X_i X_k)$ for $i \neq k$,

(c) $\mathrm{Var}(\alpha_i \circ X_i) = \alpha_i(1-\alpha_i)\mathrm{E}(X_i) + \alpha_i^2 \mathrm{Var}(X_i)$,

(d) $\mathrm{Cov}(\alpha_i \circ X_i, X_k) = \alpha_i \mathrm{Cov}(X_i, X_k)$,

(e) $\mathrm{Cov}(\alpha_i \circ X_i, \alpha_k \circ X_k,) = \alpha_i \alpha_k \mathrm{Cov}(X_i, X_k)$.

Proof. (a) Since $\{\delta_{i,j}\, i, j = 1, 2, \ldots\}$ is a sequence of i.i.d. Bernoulli random variables with mean α_i, then $\mathrm{E}(\delta_{1,j}) = \mathrm{E}(\delta_{2,j}) = \mathrm{E}(\delta_{i,j}) = \alpha_i$. From Lemma 2.26 (a), we obtain

$$\begin{aligned} \mathrm{E}(\alpha_i \circ X_i) &= \mathrm{E}\left(\sum_{j=1}^{X_i} \delta_{i,j}\right) \\ &= \mathrm{E}(\delta_i)\mathrm{E}(X_i) \\ &= \alpha_i \mathrm{E}(X_i). \end{aligned}$$

(b) For $i \neq k$,

$$\begin{aligned} \mathrm{E}((\alpha_i \circ X_i)X_k) &= \mathrm{E}\left(X_k \sum_{j=1}^{X_i} \delta_{i,j}\right) \\ &= \mathrm{E}\left(\mathrm{E}\left(X_k \sum_{j=1}^{X_i} \delta_{i,j} \middle| X_i\right)\right) \\ &= \mathrm{E}\left(X_k \mathrm{E}\left(\sum_{j=1}^{X_i} \delta_{i,j} \middle| X_i\right)\right) \\ &= \mathrm{E}\left(X_i X_k \mathrm{E}(\delta_i)\right) \\ &= \mathrm{E}(\delta_i)\mathrm{E}(X_i X_k) \\ &= \alpha_i \mathrm{E}(X_i X_k). \end{aligned}$$

(c) Using Lemma 2.26 (b), then

$$\begin{aligned}\mathrm{Var}(\alpha_i \circ X_i) &= \mathrm{Var}\left(\sum_{j=1}^{X_i} \delta_{i,j}\right)\\ &= \mathrm{E}(X_i)\mathrm{Var}(\delta_i) + \mathrm{Var}(X_i)(\mathrm{E}(\delta_i))^2\\ &= \alpha_i(1-\alpha_i)\mathrm{E}(X_i) + \alpha_i^2\mathrm{Var}(X_i).\end{aligned}$$

(d) Using Lemma 2.26 (a) and (b), we obtain

$$\begin{aligned}\mathrm{Cov}(\alpha_i \circ X_i, X_k) &= \mathrm{E}((\alpha_i \circ X_i)X_k) - \mathrm{E}(\alpha_i \circ X_i)\mathrm{E}(X_k)\\ &= \alpha_i\mathrm{E}(X_iX_k) - \alpha_i\mathrm{E}(X_i)\mathrm{E}(X_k)\\ &= \alpha_i(\mathrm{E}(X_iX_k) - \mathrm{E}(X_i)\mathrm{E}(X_k))\\ &= \alpha_i\mathrm{Cov}(X_i, X_k).\end{aligned}$$

(e) Since $\{\delta_{i,j}\, i,j = 1,2,\ldots\}$ and $\{\delta_{k,l}\, k,l = 1,2,\ldots\}$ are two mutually independent sequences of i.i.d. Bernoulli random variables with means α_i and α_k, respectively,

$$\begin{aligned}\mathrm{Cov}(\alpha_i \circ X_i, \alpha_k \circ X_k) &= \mathrm{Cov}\left(\sum_{j=1}^{X_i}\delta_{i,j}, \sum_{l=1}^{X_k}\delta_{k,l}\right)\\ &= \mathrm{E}\left(\mathrm{E}\left(\sum_{j=1}^{X_i}\delta_{i,j}, \sum_{l=1}^{X_k}\delta_{k,l}\middle| X_i, X_k\right)\right) - \mathrm{E}\left(\sum_{j=1}^{X_i}\delta_{i,j}\right)\mathrm{E}\left(\sum_{l=1}^{X_k}\delta_{k,l}\right)\\ &= \mathrm{E}\left(X_i\mathrm{E}(\delta_i X_k \mathrm{E}(\delta_k))\right) - \mathrm{E}(X_i)\mathrm{E}(\delta_i)\mathrm{E}(X_k)\mathrm{E}(\delta_k)\\ &= \mathrm{E}(\delta_i)\mathrm{E}(\delta_k)\left(\mathrm{E}(X_iX_k) - \mathrm{E}(X_i)\mathrm{E}(X_k)\right)\\ &= \alpha_i\alpha_k\mathrm{Cov}(X_i, X_k).\end{aligned}$$

□

Definition 2.32. Let $\{a_n\; n = 0,1,2,\ldots\}$ be a sequence of real numbers. The function $G : R_x \to \mathbb{R}$ defined by

$$G(t) = \sum_{n=0}^{\infty} a_n t^n,$$

for $t \in \mathbb{R}$ and $G(\cdot)$ is called the generating function of a sequence $\{a_n\; n = 0,1,2,\ldots\}$.

Moreover , we will derive the joint probability generating function (joint p.g.f.) of the variables with thining operator which will be used in this study.

Lemma 2.33. Let $N_1, N_2, \ldots, N_n$ be compound random variables defined as Definition 2.25 ,then

$$N_i \;=\; \alpha_i \circ X_i \;=\; \sum_{j=1}^{X_i} \delta_{i,j},$$

for $i = 1, 2, \ldots, n$ and $\{\delta_{i,j},\; i, j = 1, 2, \ldots\}$ is a sequence of i.i.d. random variables with the p.g.f. $G_{\delta_i}(\cdot)$, and independent from X_i. Thus, the joint p.g.f. is given as follows

$$\begin{aligned}
G_{N_1,N_2,\ldots,N_n}(z_1, z_2, \ldots, z_n) &= \mathrm{E}(z_1^{\alpha_1 \circ X_1} z_2^{\alpha_2 \circ X_2} \cdots z_n^{\alpha_n \circ X_n}) \\
&= \mathrm{E}\left(z_1^{\sum_{j=1}^{X_1} \delta_{1,j}} z_2^{\sum_{j=1}^{X_2} \delta_{2,j}} \cdots z_n^{\sum_{j=1}^{X_n} \delta_{n,j}} \right) \\
&= \mathrm{E}\left(\mathrm{E}\left(z_1^{\sum_{j=1}^{X_1} \delta_{1,j}} z_2^{\sum_{j=1}^{X_2} \delta_{2,j}} \cdots z_n^{\sum_{j=1}^{X_n} \delta_{n,j}} \big| X_1, X_2, \ldots, X_n \right)\right) \\
&= \mathrm{E}\left(\prod_{j=1}^{X_1} \mathrm{E}(z_1^{\delta_{1,j}}) \prod_{j=1}^{X_2} \mathrm{E}(z_2^{\delta_{2,j}}) \cdots \prod_{j=1}^{X_n} \mathrm{E}(z_n^{\delta_{n,j}}) \right) \\
&= \mathrm{E}\left(G_{(\delta_1)}^{X_1}(z_1) G_{(\delta_2)}^{X_2}(z_2) \cdots G_{(\delta_n)}^{X_n}(z_n) \right) \\
&= G_{X_1,X_2,\ldots,X_n}(G_{\delta_1}(z_1) G_{\delta_2}(z_2) \cdots G_{\delta_n}(z_n)).
\end{aligned}$$

CHAPTER III

DISCRETE TIME RISK MODELS BASED ON THE ZERO INFLATED POISSON MOVING AVERAGE

In this chapter, we first introduce the definition of the discrete time surplus process. In Section 3.1, we introduce to the ruin probability which provides a definition of the time of ruin and a method of how to obtain the approximation of the ruin probability.

In Section 3.2, we discuss the discrete time risk models based on the first order zero inflated Poisson moving average (ZIPMA(1)) model and derive its properties. The definition of the first order zero inflated Poisson moving average model is given in Definition 3.3, the model properties are defined in Lemma 3.4. The derivation of the adjustment coefficient function of ZIPMA(1) is presented in Theorem 3.6 to obtain the Lundberg adjustment coefficient to approximate the ruin probability. The proof of the unique positive solution of zero root of the adjustment coefficient is presented in Lemma 3.7. Afterward, we obtain the estimated ruin probability. Moreover, we introduce risk measurements, such as the value at risk and the tail value at risk for a better decision when conjoins with the ruin probability. Section 3.2.3 shows the numerical experiments of the ruin probability and the risk measurements.

Moreover, we extend the model of claim counts which is the first order zero inflated moving average model to reach the q^{th} order zero inflated moving average (ZIPMA(q)) model in Section 3.3. In this section, we give detail of the derivation and proof to obtain the properties, the adjustment coefficient and the unique positive solution for ZIPMA(q). In Section 3.3.3, we show the numerical experiments of the ruin probability and the risk measurements in the cases of ZIPMA(2) and ZIPMA(3) risk models.

Definition 3.1. Let R_n be the discrete time surplus process defined as

$$R_n \quad = \quad u + n\pi - \sum_{i=1}^{n}\sum_{j=1}^{N_i} C_{i,j}, \tag{3.1}$$

where

- u is the positive initial reserve of the business;
- π is the premium rate per period;
- the sequence $C_{i,j}$ is the sequence of claim sizes in period i and individuals j and the sequence is independent and identically distributed distribution with moment generating function, $m_C(\cdot)$;
- N_i is the claim number in period i.

We also denote that

- $N_{(n)} = \sum_{i=1}^{n} N_i$ is the aggregate claim number for n periods;
- $W_i = \sum_{j=1}^{N_i} C_{i,j}$ is the aggregate claim size for period i;
- $S_n = \sum_{i=1}^{n} W_i$ is the net loss process.

3.1 Approximation to the Ruin Probability of Discrete Time Risk Model

In this section, we first give the definition of the first time of ruin and the definition of ruin probability and the methods that are applied to approximation to the ruin probability.

Definition 3.2. Let T be the time of ruin, the first time that the surplus becomes negative. Then T is defined as follows.

$$T \quad = \quad \inf\{n \in \mathbb{N}^+ \,|\, R_n \leq 0\}. \tag{3.2}$$

The ruin probability as a function of the initial capital u is defined as

$$\Psi(u) = P\{T < \infty \,|\, R_0 = u\}. \tag{3.3}$$

The ruin probability is generally difficult to calculate, we then approximate to the ruin probability which is normally applied in many researches. For example, Gray and Pitts (2012) proposed the approximation to ruin probability by using asymptotic Lundberg-type result

$$\lim_{u\to\infty} -\frac{\ln(\Psi(u))}{u} = R,$$

where R is the Lundberg adjustment coefficient. Thus, we determine R by using function called the adjustment coefficient function. Following Nyrhinen (1998) and Müller and Pflug (2001), let the adjustment coefficient function $c(\cdot)$ is defined as

$$c(z) = \lim_{n\to\infty} \frac{1}{n} c_n(z),$$

where $c_n(\cdot)$ is the logarithm function of the cumulative generating function of the aggregate net loss profit process defined by

$$c_n(z) = \ln E(e^{z(S_n - n\pi)}).$$

They claimed that if we can find the unique $R > 0$ such that $c(R) = 0$, then the positive zero root, R, is the Lunberg adjustment coefficient. Then the ruin probability $\Psi(u)$ is approximated by

$$\Psi(u) \simeq e^{-Ru}. \tag{3.4}$$

Hence, the main work of this study is to find the adjustment coefficient function, $c(\cdot)$, and the positve zero root, R, from the surplus process.

3.2 Discrete Time Risk Model based on the First Order Zero Inflated Poisson Moving Average (1) process

In this section, we provide the definition of the first order zero inflated Poisson moving average (ZIPMA(1)) model and derive its probabilistic properties. We firstly consider the discrete time surplus defined in Definition 3.1,

$$R_n = u + n\pi - \sum_{i=1}^{n}\sum_{j=1}^{N_i} C_{i,j},$$

when the claim counts, $\{N_i,\ i \in \mathbb{N}\}$, are modelled by the first order zero inflated Poisson moving average model. The definition of ZIPMA(1) and its probabilistic properties are provided in Definitions 3.3 and Lemma 3.4, respectively. In Section 3.2.1, we derive the adjustment coefficient function and the approximation to the ruin probability of the ZIPMA(1) risk model. We also provide the special case of the adjustment coefficient function when the claim sizes are exponentially distributed. In Section 3.2.2, we propose the approximation to the value at risk (VaR) of the ZIPMA(1) net loss process.
Next, we will use the zero inflated Poisson random variable with the binomial thinning operator to define the ZIPMA(1) model.

Definition 3.3. Let $\{N_i,\ i \in \mathbb{N}\}$ be the ZIPMA(1) model defined as

$$N_i = \alpha \circ \epsilon_{i-1} + \epsilon_i, \quad \text{for } i = 1, 2, \ldots, \tag{3.5}$$

where $\{\epsilon_t\ ,\ k = 0, 1, \ldots\}$ is a sequence of i.i.d. zero inflated Poisson random variables with parameters p and λ. The $\alpha\circ$ thining operator is defined in Definition 3.3 as

$$\alpha \circ \epsilon_{i-1} = \sum_{j=1}^{\epsilon_{i-1}} \delta_{(i-1),j},$$

where $\{\delta_{(i-1),j}\ ,\ i, j = 1, 2, \ldots\}$ is a sequence of i.i.d. Bernoulli random variables with mean α.

The concept of the first order zero inflated Poisson moving average model is that we apply the moving average model to consider the number of insured where ϵ_i is represented as the number of new insured in period i and α is the probability of that the insured will reclaim. Thus, $\alpha \circ \epsilon_{i-1}$ represents that the number of claims in period i from the new claims in period $i-1$, where the probability of reclaim is α. Hence, the interpretation of N_i is that the number of insured in period i based on the summation of the number of new insured in period i and the number of reclaims from new insured in period $i-1$, where the new claims follow the zero inflated Poisson distribution.

Lemma 3.4. Let $\{N_i\,,\, i \in \mathbb{N}\}$ be a ZIPMA(1) model defined in Definition 3.3, then $\{N_i\,,\, i \in \mathbb{N}\}$ has the following properties.

(a) The sequence $\{N_i\,,\, i \in \mathbb{N}\}$ is a stationary process with the probability generating function of N_i, $G_{N_i}(z) = \left(p + (1-p)e^{-\lambda(1-z)}\right)\left(p + (1-p)e^{-\lambda\alpha(1-z)}\right)$ for $i \in \mathbb{N}$ and $z \in \mathbb{R}$.

(b) The expectation of N_i is $\mathrm{E}(N_i) = \lambda(1-p)(1+\alpha)$ for $i \in \mathbb{N}$.

(c) The variance of N_i is $\mathrm{Var}(N_i) = \lambda(1-p)\left((1+p\lambda) + \alpha(1+p\lambda\alpha)\right)$ for $i \in \mathbb{N}$.

(d) The covariance function between N_i and N_{i-m},

$$\mathrm{Cov}(N_i, N_{i-m}) = \begin{cases} \lambda\alpha(1-p)(1+p\lambda)\ , & \text{for } m = 1, \\ 0 & , \text{for } m > 1. \end{cases}$$

(e) The correlation function between N_i and N_{i-m},

$$\mathrm{Corr}(N_i, N_{i-m}) = \begin{cases} \dfrac{\alpha(1+p\lambda)}{(1+p\lambda) + \alpha(1+p\lambda\alpha)}\ , & \text{for } m = 1, \\ 0 & , \text{for } m > 1. \end{cases}$$

Proof. To prove (a),we consider the of $\{N_i\,,\, i \in \mathbb{N}\}$. Since $\{\epsilon_t,\, t = 1, 2, \ldots\}$ is a sequence of i.i.d. zero inflated Poisson random variables with parameters p and λ, by Lemma 2.29, the probability generating function of N_i is

$$\begin{aligned} G_{N_i}(z) &= \mathrm{E}(z^{N_i}) \\ &= \mathrm{E}(z^{\alpha\circ\epsilon_{i-1}+\epsilon_i}) \\ &= \mathrm{E}(z^{\epsilon_i})\mathrm{E}(z^{\alpha\circ\epsilon_{i-1}}) \qquad (3.6) \\ &= G_{\epsilon_i}(z)G_{\epsilon_{i-1}}(1-\alpha+\alpha z) \\ &= \left(p+(1-p)e^{-\lambda(1-z)}\right)\left(p+(1-p)e^{-\lambda\alpha(1-z)}\right), \qquad (3.7) \end{aligned}$$

for $z \in \mathbb{R}$, where we apply the independence between $\alpha \circ \epsilon_{i-1}$ and ϵ_i to obtain (3.6) and apply Lemma 2.29 to obtain (3.7), respectively. Since the generating function $G_{N_i}(\cdot)$ does not depend on i then $G_{N_1}(\cdot) = G_{N_2}(\cdot) = \ldots = G_{N_i}(\cdot)$. Therefore, $\{N_i\,,\, i \in \mathbb{N}\}$ is a stationary process. In addition, the probability generating function of $\{N_i\,,\, i \in \mathbb{N}\}$ is given by

$$G_{N_i}(z) = \left(p+(1-p)e^{-\lambda(1-z)}\right)\left(p+(1-p)e^{-\lambda\alpha(1-z)}\right),$$

for all $i \in \mathbb{N}$.

(b) The expectation of N_i can be obtained by evaluating the derivative of $G_{N_i}(z)$ at $z = 1$ as follows.

$$\begin{aligned} \mathrm{E}(N_i) &= \frac{d}{dz}G_{N_i}(z)\Big|_{z=1} \\ &= \frac{d}{dz}\left(p+(1-p)e^{-\lambda(1-z)}\right)\left(p+(1-p)e^{-\lambda\alpha(1-z)}\right)\Big|_{z=1} \\ &= \left((p+(1-p)e^{-\lambda(1-z)})((1-p)e^{-\lambda\alpha(1-z)}\lambda\alpha)\right)\Big|_{z=1} \\ &\quad + \left((p+(1-p)e^{-\lambda\alpha(1-z)})((1-p)e^{-\lambda(1-z)}\lambda)\right)\Big|_{z=1} \\ &= ((p+1-p)(1-p)\lambda\alpha) + ((p+1-p)(1-p)\lambda) \\ &= \lambda(1-p)(1+\alpha). \end{aligned}$$

(c) Note that, $\mathrm{Var}(N_i) = \mathrm{E}(N_i^2) - \mathrm{E}^2(N_i)$. Therefore by applying the properties of the probability generating function in Lemma 2.24 as

$$\mathrm{E}(N_i^2) = \frac{d^2}{dz^2}G_{N_i}(z)\Big|_{z=1} + \frac{d}{dz}G_{N_i}(z)\Big|_{z=1}.$$

Note that,

$$
\begin{aligned}
\left.\frac{d^2}{dz^2}G_{N_i}(z)\right|_{z=1} &= \frac{d^2}{dz^2}\left(p+(1-p)e^{-\lambda(1-z)}\right)\left(p+(1-p)e^{-\lambda\alpha(1-z)}\right) \\
&= \frac{d}{dz}\Big((p+(1-p)e^{-\lambda(1-z)})((1-p)e^{-\lambda\alpha(1-z)}\lambda\alpha) \\
&\quad +(p+(1-p)e^{-\lambda\alpha(1-z)})((1-p)e^{-\lambda(1-z)}\lambda)\Big) \\
&= \left.\left(p+(1-p)e^{-\lambda(1-z)}\right)\left(\lambda\alpha(1-p)e^{-\lambda\alpha(1-z)}\lambda\alpha\right)\right|_{z=1} \\
&\quad + \left.\left(\lambda\alpha(1-p)e^{-\lambda\alpha(1-z)}\right)\left((1-p)e^{-\lambda(1-z)}\lambda\right)\right|_{z=1} \\
&\quad + \left.\left(p+(1-p)e^{-\lambda\alpha(1-z)}\right)\left(\lambda(1-p)e^{-\lambda(1-z)}\lambda\right)\right|_{z=1} \\
&\quad + \left.\left(\lambda(1-p)e^{-\lambda(1-z)}\right)\left((1-p)e^{-\lambda\alpha(1-z)}\lambda\alpha\right)\right|_{z=1} \\
&= (\lambda\alpha)^2(1-p)+\lambda^2\alpha(1-p)^2+\lambda^2(1-p)+\lambda^2\alpha(1-p)^2 \\
&= \lambda^2(1-p)\left(\alpha^2+2\alpha(1-p)+1\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathrm{E}(N_i^2) &= \left.\frac{d^2}{dz^2}G_{N_i}(z)\right|_{z=1} + \left.\frac{d}{dz}G_{N_i}(z)\right|_{z=1} \\
&= \lambda^2(1-p)\left(\alpha^2+2\alpha(1-p)+1\right)+\lambda(1-p)(1+\alpha).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\mathrm{Var}(N_i) &= \lambda^2(1-p)\left(\alpha^2+2\alpha(1-p)+1\right)+\lambda(1-p)(1+\alpha) \\
&\quad -(\lambda(1-p)(1+\alpha))^2 \\
&= \lambda(1-p)\left(\lambda(1+\alpha)^2-2p\lambda\alpha+(1+\alpha)-\lambda(1-p)(1+\alpha)^2\right) \\
&= \lambda(1-p)\left(\lambda(1+\alpha)^2-2p\lambda\alpha+(1+\alpha)-\lambda(1+\alpha)^2+p\lambda(1+\alpha)^2\right) \\
&= \lambda(1-p)\left(-2p\lambda\alpha+(1+\alpha)+p\lambda(1+\alpha)^2\right) \\
&= \lambda(1-p)\left(p\lambda((1+\alpha)^2-2\alpha)+(1+\alpha)\right) \\
&= \lambda(1-p)\left(p\lambda(1+\alpha^2)+(1+\alpha)\right) \\
&= \lambda(1-p)\left((1+p\lambda)+\alpha(1+p\lambda\alpha)\right).
\end{aligned}
$$

(d) To obtain the covariance function between N_i and N_{i-m}, $\text{Cov}(N_i, N_{i-m})$, we consider into two cases: $m = 1$ and $m > 1$ as follows.

For $m = 1$,

$$\begin{aligned}
\text{Cov}(N_i, N_{i-1}) &= \text{Cov}(\alpha \circ \epsilon_{i-1} + \epsilon_i, \alpha \circ \epsilon_{i-2} + \epsilon_{i-1}) \\
&= \text{Cov}(\alpha \circ \epsilon_{i-1}, \alpha \circ \epsilon_{i-2}) + \text{Cov}(\alpha \circ \epsilon_{i-1}, \epsilon_{i-1}) \\
&\quad + \text{Cov}(\epsilon_i, \alpha \circ \epsilon_{i-2}) + \text{Cov}(\epsilon_i, \epsilon_{i-1}) \\
&= \text{Cov}(\alpha \circ \epsilon_{i-1}, \epsilon_{i-1}) \\
&= \alpha \text{Var}(\epsilon_{i-1}) \\
&= \alpha\lambda(1-p)(1+p\lambda),
\end{aligned}$$

where we use the fact that ϵ_{i-1} is the zero inflated Poisson random variable and Lemma 2.29 to obtain the last equation.

For $m > 1$, by using the property that $\{\epsilon_i, i = 1, 2, \ldots\}$ is a sequence of independent random variables,

$$\begin{aligned}
\text{Cov}(N_i, N_{i-m}) &= \text{Cov}(\alpha \circ \epsilon_{i-1} + \epsilon_i, \alpha \circ \epsilon_{i-m-1} + \epsilon_{i-m}) \\
&= 0.
\end{aligned}$$

(e) From Lemma 2.29 and (d), then

$$\begin{aligned}
\text{Corr}(N_i, N_{i-m}) &= \frac{\text{Cov}(N_i, N_{i-m})}{\sqrt{\text{Var}(N_i)\text{Var}(N_{i-m})}} \\
&= \frac{\text{Cov}(N_i, N_{i-m})}{\text{Var}(N_i)} \\
&= \begin{cases} \dfrac{\alpha(1+p\lambda)}{(1+p\lambda)+\alpha(1+p\lambda\alpha)} & , \text{ for } m = 1, \\ 0 & , \text{ for } m > 1. \end{cases}
\end{aligned}$$

□

3.2.1 Adjustment coefficient function of ZIPMA(1)

In the previous section, we have provided the definition of the discrete time surplus process based on ZIPMA(1) model. In this section, we derive the adjustment coefficient function of ZIPMA(1) by applying the method from Section 3.1 to obtain the Lundberg adjustment coefficient. Afterward, we provide a proof of the unique positive solution of zero root of the adjustment coefficient. The risk model based on ZIPMA(1) is described below

Definition 3.5. The risk model based on ZIPMA(1) can be expressed as

$$R_n = u + n\pi - \sum_{i=1}^{n}\sum_{j=1}^{N_i} C_{i,j},$$

where u is the positive initial reserve, π is the premium rate per period, N_i is modelled by ZIPMA(1) defined in (3.5) and $\{C_{i,j}\}$ is the sequence of independent and identically distributed distribution.

Theorem 3.6. Let R_n be the discrete time surplus process defined in Definition 3.5. The adjustment coefficient function $c(z)$ of R_n is defined as

$$c(z) = \log\Big(p + (1-p)e^{-\lambda(1-m_C(z)(1-\alpha+\alpha m_C(z)))}\Big) - \pi z, \tag{3.8}$$

for $z \in \mathbb{R}^+$.

Proof. Let $z \in \mathbb{R}^+$. We denote that $\{C_{i,j},\, i,j = 1,2,\ldots\}$ is a sequence of i.i.d. random variables whose the moment generating function of $\{C_{i,j},\, i,j = 1,2,\ldots\}$ is defined as $m_C(\cdot)$ and the net loss process S_n whose the moment generating function of S_n is defined as $m_{S_n}(\cdot)$. We then simplify the form of the aggregate net loss profit process $c_n(\cdot)$ to obtain $c(\cdot)$ as

$$
\begin{aligned}
c_n(z) &= \log \mathrm{E}(e^{z(Sn-n\pi)}) \\
&= \log \mathrm{E}\left(\frac{e^{zS_n}}{e^{n\pi z}}\right) \\
&= \log\left(\frac{\mathrm{E}(e^{zS_n})}{e^{n\pi z}}\right) \\
&= \log m_{S_n}(z) - n\pi z, \qquad (3.9)
\end{aligned}
$$

then

$$
c(z) = \lim_{n\to\infty} \frac{1}{n}\log m_{S_n}(z) - \pi z.
$$

Next, we consider the moment generating function of S_n,

$$
\begin{aligned}
m_{S_n}(z) &= \mathrm{E}(e^{zS_n}) \\
&= \mathrm{E}\left(e^{z\sum_{i=1}^{n}\sum_{j=1}^{N_i} C_{i,j}}\right) \\
&= \mathrm{E}\left(e^{z\sum_{j=1}^{N_1} C_{1,j}+z\sum_{j=1}^{N_2} C_{2,j}+\cdots+z\sum_{j=1}^{N_n} C_{n,j}}\right) \\
&= \mathrm{E}\left(\mathrm{E}\left(e^{z\sum_{j=1}^{N_1} C_{1,j}+z\sum_{j=1}^{N_2} C_{2,j}+\cdots+z\sum_{j=1}^{N_n} C_{n,j}}\middle| N_1, N_2, \ldots, N_n\right)\right) \\
&= \mathrm{E}\left(\prod_{j=1}^{N_1}\mathrm{E}(e^{zC_{1,j}})\prod_{j=1}^{N_2}\mathrm{E}(e^{zC_{2,j}})\cdots\prod_{j=1}^{N_n}\mathrm{E}(e^{zC_{n,j}})\right) \\
&= \mathrm{E}\left(m_C^{N_1}(z)m_C^{N_2}(z)\cdots m_C^{N_n}(z)\right) \\
&= \mathrm{E}\left(m_C^{N(n)}(z)\right) \\
&= G_{N(n)}\left(m_C(z)\right). \qquad (3.10)
\end{aligned}
$$

Consequently,

$$
m_{S_n}(z) = G_{N(n)}\left(m_C(z)\right), \qquad (3.11)
$$

where $G_{N(n)}(\cdot)$ is the probability generating function of $N_{(n)}$.
To obtain (3.11), we first derive the probability generating function $G_{N(n)}(\cdot)$. Since $\{\epsilon_t,\, t = 0, 1, \ldots\}$ is a sequence of i.i.d. zero inflated Poisson random variables with parameters p and λ,

$$
\begin{aligned}
G_{N(n)}(z) &= \mathrm{E}(z^{N_1+N_2+\cdots+N_n}) \\
&= \mathrm{E}\left(z^{(\alpha\circ\epsilon_0+\epsilon_1)+(\alpha\circ\epsilon_1+\epsilon_2)+\cdots+(\alpha\circ\epsilon_{n-1}+\epsilon_n)}\right) \\
&= \mathrm{E}(z^{\epsilon_n})\mathrm{E}(z^{\alpha\circ\epsilon_0})\prod_{i=1}^{n-1}\mathrm{E}(z^{\alpha\circ\epsilon_i+\epsilon_i}) \\
&= \mathrm{E}\left(z^{\epsilon_n}\right)\mathrm{E}\left(z^{\sum_{j=1}^{\epsilon_0}\delta_{0,j}}\right)\prod_{i=1}^{n-1}\mathrm{E}\left(z^{\sum_{j=1}^{\epsilon_i}\delta_{i,j}+\epsilon_i}\right). \qquad (3.12)
\end{aligned}
$$

Using Lemma 2.29, we obtain the first term of (3.12) as

$$
\mathrm{E}(z^{\epsilon_n}) \;=\; p+(1-p)e^{-\lambda(1-z)} \quad \text{for } z \in \mathbb{R}^+. \qquad (3.13)
$$

Since $\{\delta_{i,j},\, i,j = 1,2,\ldots\}$ is a sequence of i.i.d. Bernoulli random variables with mean α and Lemma 2.29, the probability generating function $\mathrm{E}(z^{\delta_{i,1}}) = \mathrm{E}(z^{\delta_{i,2}}) = \ldots = \mathrm{E}(z^{\delta_{i,j}}) = 1-\alpha+\alpha z$, the second term of (3.12) is derived as follows.

$$
\begin{aligned}
\mathrm{E}\left(z^{\sum_{j=1}^{\epsilon_0}\delta_{0,j}}\right) &= \mathrm{E}\left(\mathrm{E}\left(z^{\sum_{j=1}^{\epsilon_0}\delta_{0,j}}\middle|\epsilon_0\right)\right) \\
&= \mathrm{E}\left(\prod_{j=1}^{\epsilon_0}\mathrm{E}(z^{\delta_{0,j}})\right) \\
&= \mathrm{E}\left((1-\alpha+\alpha z)^{\epsilon_0}\right) \\
&= G_{\epsilon_0}(1-\alpha+\alpha z) \\
&= p+(1-p)e^{-\lambda\alpha(1-z)}, \qquad (3.14)
\end{aligned}
$$

for $z \in \mathbb{R}^+$.

For the third term of (3.12), we have that $\{\delta_{i,j},\, i,j = 1,2,\ldots\}$ is a sequence of i.i.d. Bernoulli random variables with mean α and Lemma 2.29, then we obtain

$$\begin{aligned}
\mathrm{E}\left(z^{\sum_{j=1}^{\epsilon_i}\delta_{i,j}+\epsilon_i}\right) &= \mathrm{E}\left(\mathrm{E}\left(z^{\sum_{j=1}^{\epsilon_i}\delta_{i,j}+\epsilon_i}\middle|\epsilon_i\right)\right) \\
&= \mathrm{E}\left(z^{\epsilon_i}\prod_{j=1}^{\epsilon_i}\mathrm{E}(z^{\delta_{i,j}})\right) \\
&= \mathrm{E}\left(z^{\epsilon_i}(1-\alpha+\alpha z)^{\epsilon_i}\right) \\
&= \mathrm{E}\left((z(1-\alpha+\alpha z))^{\epsilon_i}\right) \\
&= G_{\epsilon_i}(z(1-\alpha+\alpha z)) \\
&= p+(1-p)e^{-\lambda(1-z(1-\alpha+\alpha z))}, \qquad (3.15)
\end{aligned}$$

for $z \in \mathbb{R}^+$. Substituting (3.13)-(3.15) into (3.12), we obtain

$$\begin{aligned}
G_{N(n)}(z) &= \left(p+(1-p)e^{-\lambda(1-z)}\right)\left(p+(1-p)e^{-\lambda\alpha(1-z)}\right) \\
&\quad \left(p+(1-p)e^{-\lambda(1-z(1-\alpha+\alpha z))}\right)^{n-1}, \qquad (3.16)
\end{aligned}$$

where $z \in \mathbb{R}^+$.

Therefore, we apply the result obtained in (3.16) into (3.11)

$$\begin{aligned}
m_{S_n}(z) &= \left(p+(1-p)e^{-\lambda(1-m_C(z))}\right)\left(p+(1-p)e^{-\lambda\alpha(1-m_C(z))}\right) \\
&\quad \left(p+(1-p)e^{-\lambda(1-m_C(z)(1-\alpha+\alpha m_C(z)))}\right)^{n-1}, \qquad (3.17)
\end{aligned}$$

for $z \in \mathbb{R}^+$. Consequently, we obtain $m_{S_n}(\cdot)$ from (3.17), then we put into (3.9) as the following.

$$\begin{aligned}
c_n(z) &= \log\left(p+(1-p)e^{-\lambda(1-m_C(z))}\right)+\log\left(p+(1-p)e^{-\lambda\alpha(1-m_C(z))}\right) \\
&\quad +(n-1)\log\left(p+(1-p)e^{-\lambda(1-m_C(z)(1-\alpha+\alpha m_C(z)))}\right)-n\pi z,
\end{aligned}$$

for $z \in \mathbb{R}^+$.

Hence, we obtain the adjustment coefficient function $c(\cdot)$ is given by

$$\begin{aligned} c(z) &= \lim_{n\to\infty} \frac{1}{n} c_n(z) - \pi z \\ &= \log\left(p + (1-p)e^{-\lambda(1-m_C(z)(1-\alpha+\alpha m_C(z)))}\right) - \pi z, \end{aligned}$$

for $z \in \mathbb{R}^+$. □

Since the premium per period, π, followed the net profit condition (NPC)(Thomas, 2009) condition and premium calculation followed the expectation value principle (EVP)(Gray and Pitts, 2012)

$$\begin{aligned} \pi &= \mathrm{E}(W)(1+\theta) \\ &= \mathrm{E}(N)\mathrm{E}(C)(1+\theta) \\ &= \lambda(1-p)(1+\alpha)\mathrm{E}(C)(1+\theta), \end{aligned} \tag{3.18}$$

for a security loading $\theta > 0$, $\mathrm{E}(W)$ is the expectation of the aggregate claim size, $\mathrm{E}(N)$ is the expectation of the claim number and $\mathrm{E}(C)$ is the expectation of claim size. Next, we will show that the adjustment coefficient has the unique positive zero root in D where $D = \{z \in \mathbb{R}^+\}$.

Lemma 3.7. The equation $c(z) = 0$ has the unique positive solution in D, where $c(z)$ is the adjustment coefficient function defined in Theorem 3.6.

Proof. To prove the Lemma,we will show that

(a) $c(0) = 0$,

(b) $\left.\dfrac{d}{dz}c(z)\right|_{z=0} < 0$,

(c) $\dfrac{d^2}{dz^2}c(z) > 0$ for $z \in D$,

(d) $\lim_{z\to+\infty} c(z) = +\infty$.

(a) Note that

$$c(z) \quad = \quad \log\left(p + (1-p)e^{-\lambda(1-m_C(z)(1-\alpha+\alpha m_C(z)))}\right) - \pi z.$$

We substitute $z = 0$ into $c(z)$ defined in Theorem 3.6, then we obtain

$$\begin{aligned} c(0) &= \log\left(p + (1-p)e^{-\lambda(1-m_C(0)(1-\alpha+\alpha m_C(0)))}\right) - \pi(0) \\ &= \log\left(p + (1-p)\right) \\ &= 0. \end{aligned}$$

(b) Consider

$$\frac{d}{dz}c(z) \quad = \quad \frac{(1-p)e^{-\lambda(1-m_C(z)(1-\alpha+\alpha m_C(z)))}(-\lambda m_C^{'}(z)(-1+\alpha-2\alpha m_C(z)))}{p + (1-p)e^{-\lambda(1-m_C(z)(1-\alpha+\alpha m_C(z)))}} - \pi.$$

Since we have $\pi = \lambda(1-p)\mathrm{E}(C)(1+\alpha)(1+\theta)$, then, for $\theta > 0$,

$$\begin{aligned} \left.\frac{d}{dz}c(z)\right|_{z=0} &= \frac{(1-p)e^{-\lambda(1-(1-\alpha+\alpha))}(-\lambda\mathrm{E}(C)(-1+\alpha-2\alpha))}{p + (1-p)e^{-\lambda(1-(1-\alpha+\alpha))}} - \pi \\ &= \lambda(1-p)\mathrm{E}(C)(1+\alpha) - (\lambda(1-p)\mathrm{E}(C)(1+\alpha)(1+\theta)) \\ &= \lambda(1-p)\mathrm{E}(C)(1+\alpha)(1-(1+\theta)) \\ &= -\theta\lambda(1-p)\mathrm{E}(C)(1+\alpha) \\ &< 0. \end{aligned}$$

Then, we obtain that $\left.\frac{d}{dz}c(z)\right|_{z=0} < 0.$

(c) Note that,

$$
\begin{aligned}
\frac{d^2}{dz^2}c(z) &= \frac{p(1-p)e^{-\lambda(1-m_C(z)(1-\alpha+\alpha m_C(z)))}\lambda(m_C^{''}(z)(1-\alpha))}{\left(p+(1-p)e^{-\lambda(1-m_C(z)(1-\alpha+\alpha m_C(z)))}\right)^2} \\
&+\frac{p(1-p)e^{-\lambda(1-m_C(z)(1-\alpha+\alpha m_C(z)))}2\lambda\alpha((m_C^{'}(z))^2+m_C(z)m_C^{''}(z)))}{\left(p+(1-p)e^{-\lambda(1-m_C(z)(1-\alpha+\alpha m_C(z)))}\right)^2} \\
&+\frac{p(1-p)e^{-\lambda(1-m_C(z)(1-\alpha+\alpha m_C(z)))}\left(\lambda m_C^{'}(z)(1-\alpha+\alpha m_C(z))\right)^2}{\left(p+(1-p)e^{-\lambda(1-m_C(z)(1-\alpha+\alpha m_C(z)))}\right)^2} \\
&+\frac{\left((1-p)e^{-\lambda(1-m_C(z)(1-\alpha+\alpha m_C(z)))}\right)^2\lambda(m_C^{''}(z)(1-\alpha))}{\left(p+(1-p)e^{-\lambda(1-m_C(z)(1-\alpha+\alpha m_C(z)))}\right)^2} \\
&+\frac{\left((1-p)e^{-\lambda(1-m_C(z)(1-\alpha+\alpha m_C(z)))}\right)^2 2\lambda\alpha((m_C^{'}(z))^2+m_C(z)m_C^{''}(z)))}{\left(p+(1-p)e^{-\lambda(1-m_C(z)(1-\alpha+\alpha m_C(z)))}\right)^2}.
\end{aligned}
$$

By the properties that $m_C(z) > 0$, $m_C^{'}(z) > 0$, $m_C^{''}(z) > 0$ and $\alpha \in (0,1)$, then $\frac{d^2}{dz^2}c(z) > 0$.

(d) We can show that the limit of $c(z)$ reaches to $+\infty$ as z approaches $+\infty$. Let us first consider

$$
\begin{aligned}
f(z) &= \lambda(m_C(z)(1-\alpha+\alpha m_C(z))-1) \\
&\propto \lambda m_C(z)(1-\alpha+\alpha m_C(z)) \\
&\propto \lambda\alpha m_C^2(z),
\end{aligned}
$$

for $z \in D$. We know that $m_C(z)$ is the monotonically increasing function and continuous function in D, then $m_C^2(z)$ is growing up to $+\infty$ with the exponential rate, then we can conclude that $f(z)$ will grow to infinity with exponential rate which is faster than any linear trend. Hence, we obtain that

$$
\lim_{z\to+\infty}\left(\log\left(p+(1-p)e^{\lambda(m_C(z)-1)(1-\alpha+\alpha m_C(z))}\right)-\pi z\right) = +\infty.
$$

□

Example 3.1. In this part, we consider a special case when the claim amounts follow an exponential distribution. That is, $\{C_{i,j},\ i \in \mathbb{N},\ j = 1,2,\ldots\}$ is a sequence of i.i.d. exponentially distributed random variables with parameter $\beta > 0$. The moment generating function of $\{C_{i,j},\ i \in \mathbb{N},\ j = 1,2,\ldots\}$ is defined as $m_C(z) = \frac{1}{1-z/\beta}$ for $z < \beta$. Using Theorem (3.6), the adjustment coefficient function is defined as

$$c(z) = \log\left(p + (1-p)e^{-\lambda(1-\frac{1}{1-z/\beta}(1-\alpha+\alpha\frac{1}{1-z/\beta}))}\right) - \pi z, \tag{3.19}$$

where $\pi = \lambda(1-p)(1+\alpha)\mathrm{E}(C)(1+\theta),\ 0 < z < \beta$.

3.2.2 Approximation to the value at risk and tail value at risk of ZIPMA(1)

The value at risk at the confidence level γ, $\mathrm{VaR}_\gamma(S_n)$, for ZIPMA(1) process in the $(1-\gamma)$ quantile of S_n that refers to the amount of the net loss. So, the more value of $\mathrm{VaR}_\gamma(S_n)$ the higher risk of the surplus.

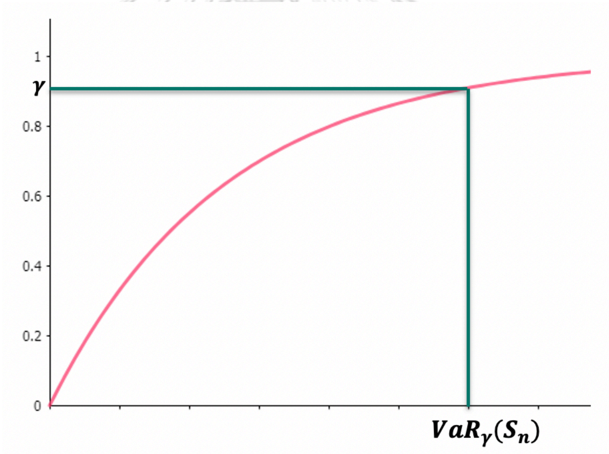

Figure 3.1: The graph of value at risk at confidence level γ.

As Figure 3.1, the red line represents the cumulative distribution of S_n, then we can see that at confidence level γ, we can obtain the value of the value at risk that can inform us about the estimated loss that the company may confront at confidence level γ or a $(1-\gamma)$ probability that the loss maybe greater than the approximated value. Note that $S_n = \sum_{i=1}^{n}\sum_{j=1}^{N_i} C_{i,j}$ be the net loss process and N_i be a ZIPMA(1) process. The $\mathrm{VaR}_\gamma(S_n)$ is defined as

$$\mathrm{VaR}_\gamma(S_n) = \inf\{k \in \mathbb{R} | F_{S_n}(k) > \gamma\}, \tag{3.20}$$

where $F_{S_n}(\cdot)$ is the cumulative distribution function of S_n. It is generally difficult to obtain the distribution of S_n from the moment generating function given in (3.17). Therefore, we apply the Fast Fourier Transform (FFT) algorithm (Gray and Pitts, 2012) to obtain an approximation of the density function of $F_{S_n}(\cdot)$ which can be described as the following. From (3.11), we know that

$$m_{S_n}(z) \;=\; G_{N(n)}\left(m_C(z)\right),$$

where $m_C(\cdot)$ is the moment generating function of $C_{i,j}$ for all $i, j = 1, 2, \ldots$
Since C is exponentially distributed with pamameter β, so we first discretise distribution of C, then for a given discretisation parameter h, we have

$$f_0 \;=\; \Pr(0 < C \leq h/2)) \;=\; 1 - e^{-2h\beta},$$

and for $k = 1, 2, \ldots,$

$$\begin{aligned} f_k &= \Pr\left((k-0.5)h < C \leq (k+0.5)h\right) \\ &= e^{-(k-0.5)h\beta}(1-e^{-h\beta}). \end{aligned}$$

Thus, check that $\sum_{k=0}^{\infty} f_k = 1$, then $(f_0, f_1, \ldots)$ is a discrete approximation to the distribution of X_1.

Let $\phi_C(\cdot)$ be the characteristic function of $C_{i,j}\,(i, j = 1, 2, \ldots)$. Therefore, we apply the FFT algorithm to approximate the characteristic function $\phi_C(\cdot)$ of $C_{i,j}$. We can calculate the characteristic function of S_n as follows.

$$\begin{aligned} \phi_{S_n}(x) &= G_{N(n)}(\phi_C(x)) \\ &= \left(p + (1-p)e^{-\lambda(1-\phi_C(x))}\right)\left(p + (1-p)e^{-\lambda\alpha(1-\phi_C(x))}\right) \\ &\quad \times \left(p + (1-p)e^{-\lambda(1-\phi_C(x)(1-\alpha+\alpha\phi_C(x)))}\right)^{n-1}, \end{aligned}$$

where $x \in \mathbb{R}^+$.
Applying the inverse FFT algorithm, we can approximate to the density of S_n, and the $F_{S_n}(\cdot)$. Finally, we can calculate the value of $\mathrm{VaR}_\gamma(S_n)$.

The value at risk is usually applied by banks or a company that want to measure risk over a short time, then the tail value at risk is risk measure that is in many ways superior than the value at risk. The tail value at risk is basically a standard risk measurement which is applied in insurance companies as effective over a year or more.

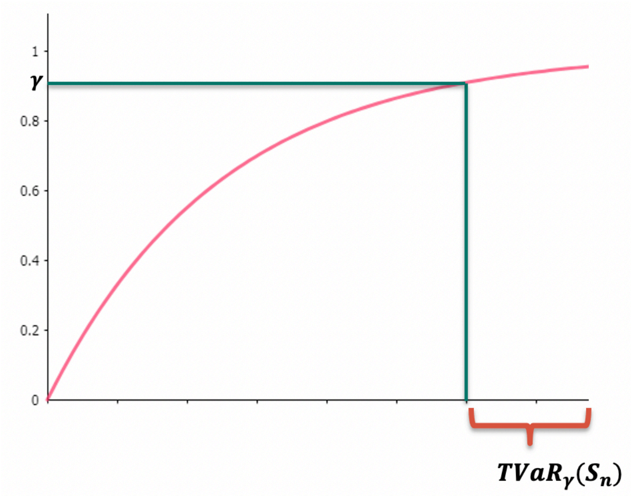

Figure 3.2: The graph of tail value at risk at confidence level γ.

As Figure 3.2, it can be seen that the tail value at risk can inform about the behavior of loss or the average of loss beyond the value at risk. According to the value of the tail value at risk, insurance companies can apply these values to be one of many decisive options about the strategies and financial planning. The risk measure VaR_γ is a merely cutoff point and does not describe the tail behavior beyond the VaR_γ threshold. The tail value at risk at the confidence level γ, $\mathrm{TVaR}_\gamma(S_n)$ is defined as follows.

$$\mathrm{TVaR}_\gamma = \frac{1}{1-\gamma}\int_\gamma^1 VaR_w(S_n)dw \tag{3.21}$$

where VaR_w is the value at risk at confidence level w. It is difficult to directly calculate the integral form, we then apply Riemann sum to approximate the value of the tail value at risk.

3.2.3 Numerical experiments of risk model based on ZIPMA(1)

In this section, we present examples to calculate the adjustment coefficient and approximation to the ruin probability of risk model based on ZIPMA(1) claim count process. We also provide the calculation of the value at risk and tail value at risk of the 12^{th} periods of time at the confidence levels 0.9 and 0.95.

3.2.4 Calculation of the adjustment coefficient of risk model based on ZIPMA(1)

Let R_n be the discrete time surplus process defined in (3.1), and $\{N_i,\ i = 1, 2, \dots n\}$ is ZIPMA(1) model as claim counts process as defined in Definition 3.3. Let $\{C_{i,j},\ i, j = 1, 2, \dots\}$ is a sequence of i.i.d. exponentially distributed random variables with parameter β and we obtain $c(z)$ as in Example 3.1. The parameters setting are $u = 2$, $(\lambda, p) = (1.5, 0.2)$, $\beta \in \{0.5, 1, 2, 4, 32\}$ and $\theta = 0.3$. Figure 3.3 shows the graph of the unique positive zero root or the adjustment coefficient. Table 3.1, Figure 3.4 and Figure 3.5 show the adjustment coefficient z_0 and the approximation of the ruin probability of R_n, $\Psi_{R_n}(u) = \exp(-z_0 u)$ in parentheses, for different values of $\alpha \in \{0, 0.25, 0.5, 0.75, 1\}$.

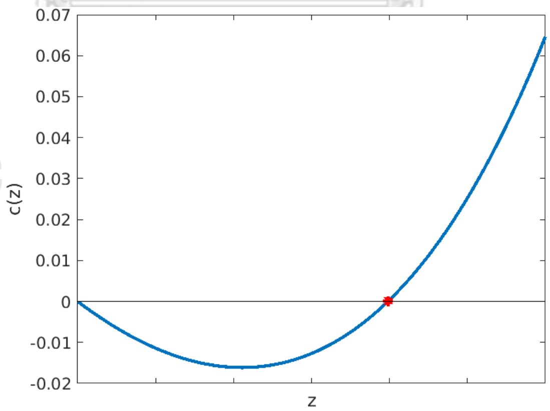

Figure 3.3: The unique positive zero root of the adjustment coefficient for ZIPMA(1).

Table 3.1: The adjustment coefficient z_0 and the approximation of $\Psi_{R_n}(u)$.

β \ α	0	0.25	0.5	0.75	1
0.5	0.1017	0.0849	0.0762	0.0708	0.0669
	(0.8159)	(0.8438)	(0.8586)	(0.8679)	(0.8747)
1	0.2035	0.1698	0.1525	0.1416	0.1339
	(0.6656)	(0.7120)	(0.7371)	(0.7533)	(0.7650)
2	0.4070	0.3396	0.3051	0.2832	0.2678
	(0.4431)	(0.5070)	(0.5432)	(0.5675)	(0.5853)
4	0.8140	0.6793	0.6102	0.5665	0.5357
	(0.1963)	(0.2570)	(0.2951)	(0.3220)	(0.3425)
32	6.5125	5.4349	4.8819	4.5327	4.2858
	(0.000002)	(0.000019)	(0.000057)	(0.000116)	(0.000189)

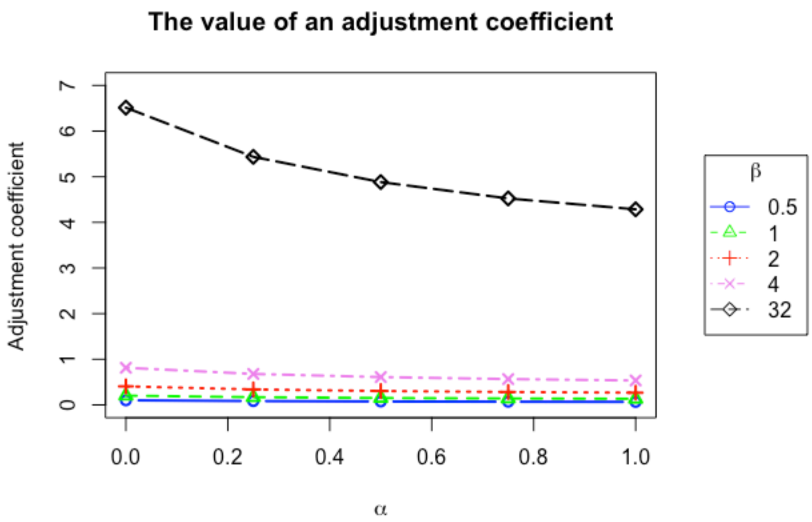

Figure 3.4: The trend of the adjustment coefficient when α increases and the claim size decreases of ZIPMA(1).

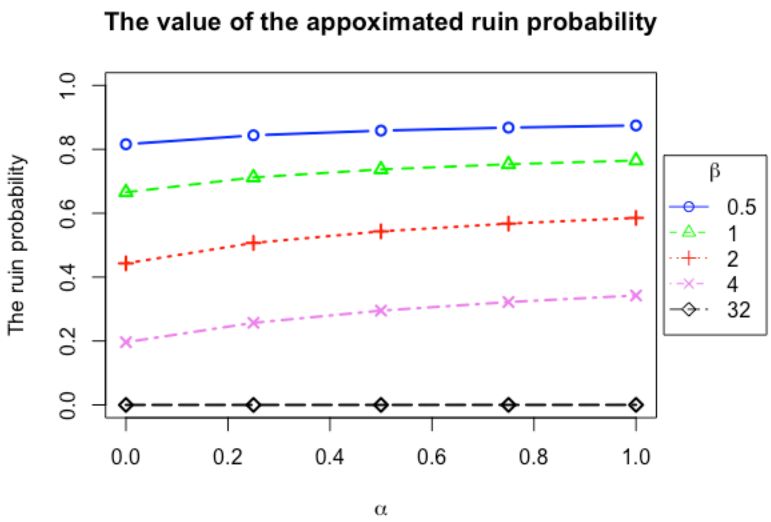

Figure 3.5: The trend of the ruin probability when α increases and the claim size decreases of ZIPMA(1).

Table 3.1 shows that the value of the ruin probability increases along with the increase of the values of α, but the adjustment coefficient decreases when the value of α grows up. In addition, the value of the ruin probability decreases and the adjustment coefficient increases with the increase of the values of β. This result is satisfied because the greater value of α which is regarding to the increasing of the number of claims and the greater value of β which is regarding to the decreasing of claim sizes. Figure 3.3 shows the unique positive zero root of $c(z)$ in case of $\beta = 4$ and $\alpha = 0.25$, which is the red point on the blue line and it satisfies 4 statements in Lemma 3.7 that is the trend of $c(z)$ surge to positive infinity. Figures 3.4 - 3.5 show the trend of the value of the adjustment coefficient and the ruin probability along with the increase of the values of α and β.

3.2.5 Calculation of the value at risk and the tail value at risk for risk model based on ZIPMA(1)

Let the time period n be 12 and divide the domain of $\{C_{i,j},\ i, j = 1, 2, \ldots\}$ which $\beta = 0.5$ to be 5×10^5 parts with the length of steps are 0.0005 for the FFT distribution approximation. For the Riemann sum approximation of tail value at risk, we divide the length of steps of value at risk as 5×10^{-6}. Table 3.2, Figure 3.6 show $\mathrm{VaR}_\gamma(S_{12})$ and $\mathrm{TVaR}_\gamma(S_{12})$ for the confidence levels $\gamma = 0.90$ and 0.95, respectively.

Table 3.2: The value of the value at risk and the tail value at risk of ZIPMA(1).

α	0	0.25	0.5	0.75	1
$\mathrm{VaR}_{0.90}(S_{12})$	44.1375	54.6690	64.7550	74.5600	84.1695
$\mathrm{VaR}_{0.95}(S_{12})$	49.4405	61.0660	72.0510	82.647	92.9765
$\mathrm{TVaR}_{0.90}(S_{12})$	51.2812	63.2719	74.5464	85.3923	95.9468
$\mathrm{TVaR}_{0.95}(S_{12})$	56.0299	68.9843	81.0389	92.5661	103.7379

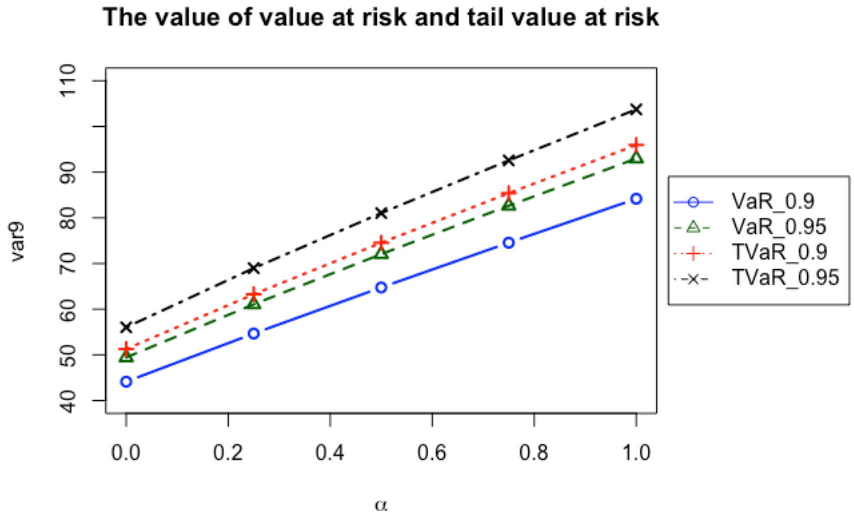

Figure 3.6: The trend of the value at risk and tail value at risk when α increases at the confidence level 0.90 and 0.95 of ZIPMA(1).

From Table 3.2 and Figure 3.6, we can see that the $\mathrm{VaR}_\gamma(S_n)$ increases as α increases. Similarly, $\mathrm{VaR}_\gamma(S_n)$ increases as γ increases. The great value of α represents that there is more probability that the new customers from the previous year will reclaim this year, it means that either company will gain more profits or face the huge loss occurred by insured. The value at risk can inform the estimated loss at confidence level γ and the meaning of γ is that a $(1-\gamma)$ probability that the loss will fall in value by greater than the estimated loss.

3.3 Discrete Time Risk Model based on q^{th} Order Zero Inflated Poisson Moving Average (ZIPMA(q))

In this section, we extend the ZIPMA(1) risk model to the ZIPMA(q) risk model where the discrete time surplus process is in the same form as Definition (3.1)

$$R_n = u + n\pi - \sum_{i=1}^{n}\sum_{j=1}^{N_n} C_{i,j}.$$

However, the claim counts, $\{N_n,\, n \in \mathbb{N}\}$, are modelled by the q^{th} order zero inflated Poisson moving average model denoted by ZIPMA(q). The definition of ZIPMA(q) and probabilistic properties are provided in Definition 3.8 and Lemma 3.9, respectively. In Section 3.3.1, we derive the adjustment coefficient function and the approximation to the ruin probability of the ZIPMA(q) risk model. We also provide the special case of the adjustment coefficient function when the claim sizes are exponentially distributed. Next, we will use the zero inflated Poisson random variable with the binomial thinning operator to get the ZIPMA(q) model.

Definition 3.8. Let $\{N_n,\, n \in \mathbb{N}\}$ be the ZIPMA(q) model defined as follows.

$$N_n = \epsilon_n + \alpha_1 \circ \epsilon_{n-1} + \alpha_2 \circ \epsilon_{n-2} + \cdots + \alpha_q \circ \epsilon_{n-q},$$

where $\{\epsilon_t,\, t = 1, 2, \ldots\}$ is a sequence of i.i.d. zero inflated Poisson random variables with parameters p and λ. The $\alpha\circ$ thining operator is defined in Definition 3.8 as

$$\alpha_i \circ \epsilon_{n-1} = \sum_{j=1}^{\epsilon_{n-1}} \delta_{i,j}^{(n-1)}$$

where for any $n \in \mathbb{N}$, $\{\delta_{i,j}^{(n-1)}\ i = 1, 2, \ldots, q,\ n \in \mathbb{N},\ j = 1, 2, \ldots\}$ is a sequence of i.i.d. Bernoulli random variables with mean α_i.

In ZIPMA(1) model, we consider only the number of claims in period i as a consequence of new claims in period i and $i-1$. However, in real situation, the number of new claims in period i could depend on new claims from other previous periods. Therefore, we extend the first order zero inflated Poisson moving average model to a more general model, the zero inflated Poisson q^{th} order moving average model ZIPMA(q) where $q \in \mathbb{N}$. The terms $\alpha_i \circ \epsilon_{n-i}$ represents that the number of claims from the number of claims in period $n-i$, where the probability of reclaim is α_i. Hence, N_n is the number of insured in period n based on the summation of the number of reclaims from period $n-1, n-2, \ldots, n-q$ and the number of new claims in period i.

Lemma 3.9. Let $\{N_n,\, n \in \mathbb{N}\}$ be the ZIPMA(q) process defined in Definition 3.8, then $\{N_n,\, n \in \mathbb{N}\}$ has the following properties.

(a) The sequence $\{N_n,\, n \in \mathbb{N}\}$ is a stationary process with the probability generating function of N_n, $G_{N_n}(z) = \prod_{i=0}^{q}\left(p + (1-p)e^{-\lambda\alpha_i(1-z)}\right)$ where $\alpha_0 = 1$ and for $n \in \mathbb{N}$.

(b) The expectation of N_n is $\mathrm{E}(N_n) = \lambda(1-p)\left(\sum_{i=0}^{q}\alpha_i\right)$ where $\alpha_0 = 1$.

(c) The variance of N_n, $\mathrm{Var}(N_n) = \lambda(1-p)(1+\lambda p)\left(\sum_{i=0}^{q}\alpha_i^2\right) + \lambda(1-p)\sum_{i=0}^{q}\alpha_i(1-\alpha_i)$ where $\alpha_0 = 1$.

(d) The covariance function between N_n and N_{n-m},

$$\mathrm{Cov}(N_n, N_{n-m}) = \begin{cases} \lambda(1-p)(1+\lambda p)\left(\alpha_m + \sum_{i=1}^{q-m}\alpha_i\alpha_{i+m}\right) & , \text{ for } 1 \le m \le q, \\ 0 & , \text{ for } m > q. \end{cases}$$

(e) The correlation function between N_n and N_{n-m} where $m < n$,

$$\mathrm{Corr}(N_n, N_{n-m}) = \begin{cases} \frac{\lambda(1-p)(1+\lambda p)\left(\alpha_m + \sum_{i=1}^{q-m}\alpha_i\alpha_{i+m}\right)}{\lambda(1-p)(1+\lambda p)(\sum_{i=0}^{q}\alpha_i^2) + \lambda(1-p)\sum_{i=0}^{q}(\alpha_i(1-\alpha_i))} & , 1 \le m \le q, \\ 0 & , m > q, \end{cases}$$

where $\alpha_0 = 1$.

Proof. To prove (a) we consider the probability generating function of $\{N_n,\, n \in \mathbb{N}\}$. Since $\{\epsilon_t,\, t = 1, 2, \ldots\}$ is a sequence of i.i.d. zero inflated Poisson random variables with parameters p and λ, the probability generating function of N_n can be completed as

$$
\begin{aligned}
G_{N_n}(z) &= \mathrm{E}\left(z^{\epsilon_n+\sum_{j=1}^{\epsilon_{n-1}}\delta_{1,j}^{(n-1)}+\sum_{j=1}^{\epsilon_{n-2}}\delta_{2,j}^{(n-2)}+\cdots+\sum_{j=1}^{\epsilon_{n-q}}\delta_{q,j}^{(n-q)}}\right)\\
&= \mathrm{E}\left(z^{\epsilon_n} z^{\sum_{j=1}^{\epsilon_{n-1}}\delta_{1,j}^{(n-1)}} z^{\sum_{j=1}^{\epsilon_{n-2}}\delta_{2,j}^{(n-2)}}\ldots z^{\sum_{j=1}^{\epsilon_{n-q}}\delta_{q,j}^{(n-q)}}\right)\\
&= \mathrm{E}\left(z^{\epsilon_n}\right)\mathrm{E}\left(z^{\sum_{j=1}^{\epsilon_{n-1}}\delta_{1,j}^{(n-1)}}\right)\cdots\left(z^{\sum_{j=1}^{\epsilon_{n-q}}\delta_{q,j}^{(n-q)}}\right)\\
&= \mathrm{E}(z^{\epsilon_n})\mathrm{E}\left(\mathrm{E}\left(z^{\sum_{j=1}^{\epsilon_{n-1}}\delta_{1,j}^{(n-1)}}\middle|\epsilon_{n-1}\right)\right)\cdots\mathrm{E}\left(\mathrm{E}\left(z^{\sum_{j=1}^{\epsilon_{n-q}}\delta_{q,j}^{(n-q)}}\middle|\epsilon_{n-q}\right)\right)\\
&= \mathrm{E}(z^{\epsilon_n})\prod_{i=1}^{q}\mathrm{E}\left(\mathrm{E}\left(z^{\sum_{j=1}^{\epsilon_{n-i}}\delta_{i,j}^{(n-i)}}\middle|\epsilon_{n-i}\right)\right)\\
&= \mathrm{E}(z^{\epsilon_n})\prod_{i=1}^{q}\mathrm{E}\left(\prod_{j=1}^{\epsilon_{n-i}}\mathrm{E}(z^{\delta_{i,j}^{(n-i)}}\right)\\
&= \mathrm{E}(z^{\epsilon_n})\prod_{i=1}^{q}G_{\epsilon_{n-i}}(G_{\delta_{i,1}}(z))\\
&= \left(p+(1-p)e^{-\lambda(1-z)}\right)\prod_{i=1}^{q}\left(p+(1-p)e^{-\lambda\alpha_i(1-z)}\right),
\end{aligned}
$$

for $z \in \mathbb{R}$. Since $G_{N_n}(\cdot)$ does not depend on n then $G_{N_1}(\cdot) = G_{N_2}(\cdot) = \ldots = G_{N_n}(\cdot)$. Therefore, $\{N_n\ n \in \mathbb{N}\}$ is a stationary process. Furthermore, the probability generating function of $\{N_n\ n \in \mathbb{N}\}$ is given by

$$
G_{N_n}(z) = \left(p+(1-p)e^{-\lambda(1-z)}\right)\prod_{i=1}^{q}\left(p+(1-p)e^{-\lambda\alpha_i(1-z)}\right),
$$

for all $n \in \mathbb{N}$.

(b) From Lemma 2.31 and $\{\epsilon_t,\ t = 1, 2, \ldots\}$ is a sequence of i.i.d. zero inflated Poisson random variables, then we obtain

$$
\begin{aligned}
\mathrm{E}(N_n) &= \mathrm{E}\left(\epsilon_n + \alpha_1\circ\epsilon_{n-1} + \alpha_2\circ\epsilon_{n-2} + \cdots + \alpha_q\circ\epsilon_{n-q}\right)\\
&= \mathrm{E}\left(\epsilon_n\right) + \sum_{i=1}^{q}\mathrm{E}(\alpha_i\circ\epsilon_{n-i})\\
&= \lambda(1-p) + \sum_{i=1}^{q}\alpha_i\mathrm{E}(\epsilon_{n-i})\\
&= \lambda(1-p)\left(1+\sum_{i=1}^{q}\alpha_i\right).
\end{aligned}
$$

(c) Using Lemma 2.31 and $\{\epsilon_t,\, t = 1, 2, \ldots\}$ is a sequence of i.i.d. zero inflated Poisson random variables, then we have

$$
\begin{aligned}
\mathrm{Var}(N_n) &= \mathrm{Var}\left(\epsilon_n + \alpha_1 \circ \epsilon_{n-1} + \alpha_2 \circ \epsilon_{n-2} + \cdots + \alpha_q \circ \epsilon_{n-q}\right) \\
&= \mathrm{Var}(\epsilon_n) + \sum_{i=1}^{q} \mathrm{Var}(\alpha_i \circ \epsilon_{n-i}) \\
&= \mathrm{Var}(\epsilon_n) + \sum_{i=1}^{q} \big(\alpha_i(1-\alpha_i)\mathrm{E}(\epsilon_{n-i}) + \alpha_i^2 \mathrm{Var}(\epsilon_{n-i})\big) \\
&= \mathrm{Var}(\epsilon_n) + \sum_{i=1}^{q} \alpha_i^2 \mathrm{Var}(\epsilon_{n-i}) + \sum_{i=1}^{q} \alpha_i(1-\alpha_i)\mathrm{E}(\epsilon_{n-i}) \\
&= \mathrm{Var}(\epsilon_n)\left(1 + \sum_{i=1}^{q} \alpha_i^2\right) + \mathrm{E}(\epsilon_n) \sum_{i=1}^{q} \alpha_i(1-\alpha_i) \qquad (3.22) \\
&= \lambda(1-p)(1+\lambda p)(\sum_{i=0}^{q} \alpha_i) + \lambda p \sum_{i=0}^{q} \alpha_i(1-\alpha_i),
\end{aligned}
$$

where we apply the fact that $\{\epsilon_t\,,\, t = 1, 2, \ldots\}$ is independent and identically distributed random variables to obtain (3.22).

(d) Note that $\{\epsilon_t,\, t = 1, 2, \ldots\}$ is a sequence of i.i.d. zero inflated Poisson random variables with parameters p and λ.

For $m = 1$, using Lemma 2.31, then

$$
\begin{aligned}
\mathrm{Cov}(N_n, N_{n-1}) &= \mathrm{Cov}\big(\epsilon_n + \alpha_1 \circ \epsilon_{n-1} + \alpha_2 \circ \epsilon_{n-2} + \cdots + \alpha_q \circ \epsilon_{n-q}, \\
&\qquad \epsilon_{n-1} + \alpha_1 \circ \epsilon_{n-2} + \alpha_2 \circ \epsilon_{n-3} + \cdots + \alpha_q \circ \epsilon_{n-(q+1)}\big) \\
&= \mathrm{Cov}(\alpha_1 \circ \epsilon_{n-1}, \epsilon_{n-1}) + \mathrm{Cov}(\alpha_2 \circ \epsilon_{n-2}, \alpha_1 \circ \epsilon_{n-2}) \\
&\qquad + \cdots + \mathrm{Cov}(\alpha_q \circ \epsilon_{n-q}, \alpha_{q-1} \circ \epsilon_{n-q}) \\
&= \alpha_1 \mathrm{Cov}(\epsilon_{n-1}, \epsilon_{n-1}) + \alpha_1\alpha_2 \mathrm{Cov}(\epsilon_{n-2}, \epsilon_{n-2}) \\
&\qquad + \cdots + \alpha_q\alpha_{q-1} \mathrm{Cov}(\epsilon_{n-q}, \epsilon_{n-q}) \\
&= \mathrm{Var}(\epsilon_{n-1})(\alpha_1 + \sum_{i=1}^{q-1} \alpha_i\alpha_{i+1}) \\
&= \lambda(1-p)(1+\lambda p)(\alpha_1 + \sum_{i=1}^{q-1} \alpha_i\alpha_{i+1}), \qquad (3.23)
\end{aligned}
$$

where we use Lemma 2.29 (c) to obtain the last equation.

For $m \leq q$, we obtain

$$\begin{aligned}
\mathrm{Cov}(N_n, N_{n-m}) &= \mathrm{Cov}\big(\epsilon_n + \alpha_1 \circ \epsilon_{n-1} + \alpha_2 \circ \epsilon_{n-2} + \cdots + \alpha_q \circ \epsilon_{n-q}, \\
&\quad \epsilon_{n-m} + \alpha_1 \circ \epsilon_{n-(m+1)} + \alpha_2 \circ \epsilon_{n-(m+2)} + \cdots + \alpha_q \circ \epsilon_{n-(q+m)}\big) \\
&= \mathrm{Cov}(\alpha_m \circ \epsilon_{n-m}, \epsilon_{n-m}) + \mathrm{Cov}(\alpha_{m+1} \circ \epsilon_{n-(m+1)}, \alpha_1 \circ \epsilon_{n-(m+1)}) \\
&\quad + \cdots + \mathrm{Cov}(\alpha_{q-m} \circ \epsilon_{n-(q+m)}, \alpha_q \circ \epsilon_{n-(q+m)}) \\
&= \alpha_m \mathrm{Cov}(\epsilon_{n-m}, \epsilon_{n-m}) + \alpha_1\alpha_{m+1}\mathrm{Cov}(\epsilon_{n-(m+1)}, \epsilon_{n-(m+1)}) \\
&\quad + \cdots + \alpha_q\alpha_{q-m}\mathrm{Cov}(\epsilon_{n-(q+m)}, \epsilon_{n-(q+m)}) \\
&= \mathrm{Var}(\epsilon_{n-2})(\alpha_2 + \sum_{i=1}^{q-2} \alpha_i\alpha_{i+2}) \\
&= \lambda(1-p)(1+\lambda p)(\alpha_m + \sum_{i=1}^{q-m} \alpha_i\alpha_{i+m}),
\end{aligned}$$

where we use Lemma 2.29 (c) to obtain the last equation.

For $m > q$, we obtain

$$\begin{aligned}
\mathrm{Cov}(N_n, N_{n-m}) &= \mathrm{Cov}\big(\epsilon_n + \alpha_1 \circ \epsilon_{n-1} + \alpha_2 \circ \epsilon_{n-2} + \cdots + \alpha_q \circ \epsilon_{n-q}, \\
&\quad \epsilon_{n-m} + \alpha_1 \circ \epsilon_{n-(m+1)} + \alpha_2 \circ \epsilon_{n-(m+2)} + \cdots + \alpha_q \circ \epsilon_{n-(q-m))}\big) \\
&= 0.
\end{aligned}$$

(e) From Lemma 2.29 and (d) we know that $\mathrm{Var}(N_n)$ does not depend on n. Then,

$$\begin{aligned}
\mathrm{Corr}(N_n, N_{n-m}) &= \frac{\mathrm{Cov}(N_n, N_{n-m})}{\sqrt{\mathrm{Var}(N_n)\mathrm{Var}(N_{n-m})}} \\
&= \frac{\mathrm{Cov}(N_n, N_{n-m})}{\mathrm{Var}(N_n)}.
\end{aligned}$$

Then, we get

$$\mathrm{Corr}(N_n, N_{n-m}) = \frac{\lambda(1-p)(1+\lambda p)(\alpha_m + \sum_{i=1}^{q-m} \alpha_i\alpha_{i+m})}{\lambda(1-p)(1+\lambda p)(\sum_{i=0}^{q} \alpha_i^2) + \lambda(1-p)\sum_{i=0}^{q}(\alpha_i(1-\alpha_i))},$$

for $m \leq q$,

$$\mathrm{Corr}(N_n, N_{n-m}) = 0,$$

for $m > q$. □

3.3.1 Adjustment coefficient function of ZIPMA(q)

In the previous section, we have provided the definition of the discrete time surplus process based on ZIPMA(q) model. In this section, we derive the adjustment coefficient function $c(\cdot)$, of ZIPMA(q) surplus process using the method from Section 3.1 to obtain the Lundberg adjustment coefficient. Afterward, we provide a proof of the unique positive solution of zero root of the adjustment coefficient. The risk model based on ZIPMA(q) is described as the following.

Definition 3.10. The risk model based on ZIPMA(q) can be expressed as

$$R_n = u + n\pi - \sum_{m=1}^{n}\sum_{j=1}^{N_m} C_{m,j},$$

where u is the positive initial reserve, π is the premium rate per period, N_m is modelled by zero inflated Poisson q^{th} order moving average (ZIPMA(q)) defined in Definition 3.8 and $\{C_{m,j}\}$ is the sequence of independent and identically distributed random variables.

Lemma 3.11. Let N_i, $i \in \mathbb{N}$ be the ZIPMA(q) defined in Definition 3.8, then the joint probability generating function of $(N_1, N_2, \ldots, N_n)$ can be expressed as

$$\begin{aligned} G_{N_1,N_2,\ldots,N_n}(z_1, z_2, \ldots, z_n) &= \Big(p + (1-p)e^{-\lambda\alpha_q(1-z_1)}\Big) \times \cdots \\ &\times \Big(p + (1-p)e^{-\lambda(1-(1-\alpha_1+\alpha_1 z_1)\cdots(1-\alpha_q+\alpha_q z_q))}\Big) \\ &\times \prod_{i=1}^{n-q}\Big(p + (1-p)e^{-\lambda(1-z_i(1-\alpha_1+\alpha_1 z_{i+1})\cdots(1-\alpha_q+\alpha_q z_{i+q}))}\Big) \\ &\times \Big(p + (1-p)e^{-\lambda(1-z_{n-1}(1-\alpha_1+\alpha_1 z_n))}\Big) \times \cdots \\ &\times \Big(p + (1-p)e^{-\lambda(1-z_{n-(q-1)}(1-\alpha_1+\alpha_1 z_{n+1-(q-1)})\cdots(1-\alpha_{q-1}+\alpha_{q-1} z_n))}\Big) \\ &\times \Big(p + (1-p)e^{-\lambda(1-z_n)}\Big), \end{aligned}$$

for $z_1, z_2, \ldots, z_n \in \mathbb{R}^+$.

Proof. The moment generating function of S_n, $m_{S_n}(\cdot)$, from (3.10) defined as

$$\begin{aligned} m_{S_n}(z) &= \mathrm{E}\left(e^{zS_n}\right) \\ &= \mathrm{E}\left(e^{z(W_1+W_2+\cdots+W_3)}\right) \\ &= m_{W_1,W_2,\ldots,W_n}(z,z,\ldots,z), \end{aligned}$$

for $z \in \mathbb{R}^+$ and $W_i = \sum_{j=1}^{N_i} C_{i,j}$ defined in Definition 3.1. Then, the joint probability generating function of $(N_1, N_2, \ldots, N_n)$ is given by

$$G_{N_1,N_2,\ldots,N_n}(z_1, z_2, \ldots, z_n) = \mathrm{E}\left(z_1^{N_1} z_2^{N_2} \cdots z_n^{N_n}\right),$$

for $z_1, z_2, \ldots, z_n \in \mathbb{R}^+$. The multivariate of the moment generating function, $m_{S_n}(z_1, z_2, \ldots, z_n)$ of $(W_1, W_2, \ldots, W_n)$ can be expressed as the joint probability generating function of $(N_1, N_2, \ldots, N_n)$ and the moment generating function of $\{C_{i,j}\}$ denoted by $m_C(\cdot)$, then we obtain

$$\begin{aligned} m_{S_n}(z_1, z_2, \ldots, z_n) &= m_{W_1,W_2,\ldots,W_n}(z_1, z_2, \ldots, z_n) \\ &= G_{N_1,N_2,\ldots,N_n}(m_C(z_1), m_C(z_2), \ldots, m_C(z_n)). \end{aligned} \quad (3.24)$$

Then, to obtain (3.24), we find the expression for the probability generating function, $G_{N_1,N_2,\ldots,N_n}(z_1, z_2, \ldots, z_n)$. Since $\{\epsilon_t,\ t = 1, 2, \ldots\}$ is a sequence of i.i.d. random variables, we firstly consider the the joint probability generating function of $(N_1, N_2, \ldots, N_n)$ as follows.

$$
\begin{aligned}
G_{N_1,N_2,\ldots,N_n}(z_1,z_2,\ldots,z_n) &= \mathrm{E}\left(z_1^{N_1} z_2^{N_2}\cdots z_n^{N_n}\right)\\
&= \mathrm{E}\Big(z_1^{\epsilon_1+\alpha_1\circ\epsilon_{1-1}+\alpha_2\circ\epsilon_{1-2}+\cdots+\alpha_q\circ\epsilon_{1-q}}\\
&\quad\times\ z_2^{\epsilon_2+\alpha_1\circ\epsilon_{2-1}+\alpha_2\circ\epsilon_{2-2}+\cdots+\alpha_q\circ\epsilon_{2-q}}\\
&\qquad\qquad\vdots\\
&\quad\times\ z_n^{\epsilon_n+\alpha_1\circ\epsilon_{n-1}+\alpha_2\circ\epsilon_{n-2}+\cdots+\alpha_q\circ\epsilon_{n-q}}\Big)\\
&= \mathrm{E}(z_1^{\alpha_q\circ\epsilon_{1-q}})\mathrm{E}(z_1^{\alpha_{q-1}\circ\epsilon_{2-q}} z_2^{\alpha_q\circ\epsilon_{2-q}})\times\cdots\\
&\quad\times\mathrm{E}(z_1^{\alpha_1\circ\epsilon_{1-1}} z_2^{\alpha_2\circ\epsilon_{2-2}}\cdots z_q^{\alpha_q\circ\epsilon_{n-n}})\\
&\quad\times\prod_{i=1}^{n-q}\mathrm{E}\left(z_i^{\epsilon_i} z_{i+1}^{\alpha_1\circ\epsilon_i}\cdots z_{i+q}^{\alpha_q\circ\epsilon_i}\right)\\
&\quad\times\prod_{i=1}^{q-1}\mathrm{E}(z_{n-i}^{\epsilon_{n-i}} z_{n+1-i}^{\alpha_1\circ\epsilon_{n-i}}\cdots z_n^{\alpha_{q-i}\circ\epsilon_{n-i}})\\
&\quad\times\mathrm{E}(z_n^{\epsilon_n}). \qquad (3.25)
\end{aligned}
$$

For the first q terms of (3.25), we apply Lemma 2.29 and the fact that $\{\epsilon_t,\ t = 1,2,\ldots\}$ is a sequence of i.i.d. zero inflated Poisson random variables with parameters p and λ to consider the first q terms, we start with the first term as the following.

$$
\begin{aligned}
\mathrm{E}(z_1^{\alpha_q\circ\epsilon_{1-q}}) &= \mathrm{E}\left(\mathrm{E}\left(z^{\alpha_q\circ\epsilon_{1-q}}|\epsilon_{1-q}\right)\right)\\
&= \mathrm{E}\left(\prod_{j=1}^{\epsilon_{1-q}}\mathrm{E}\left(z_1^{\delta_{q,j}^{(1-q)}}\right)\right)\\
&= \mathrm{E}\left((1-\alpha_q+\alpha_q z_1)^{\epsilon_{1-q}}\right)\\
&= G_{\epsilon_{1-q}}(1-\alpha_q+\alpha_q z_1)\\
&= p+(1-p)e^{-\lambda\alpha_q(1-z_1)}. \qquad (3.26)
\end{aligned}
$$

For the second term, note that

$$\begin{aligned}
\mathrm{E}(z_1^{\alpha_{q-1}\circ\epsilon_{2-q}} z_2^{\alpha_q\circ\epsilon_{2-q}}) &= \mathrm{E}\left(\mathrm{E}\left(z_1^{\alpha_{q-1}\circ\epsilon_{2-q}} z_2^{\alpha_q\circ\epsilon_{2-q}}|\epsilon_{2-q}\right)\right) \\
&= \mathrm{E}\left(\prod_{j=1}^{\epsilon_{2-q}} \mathrm{E}\left(z_1^{\delta_{q-1,j}^{(2-q)}}\right) \prod_{j=1}^{\epsilon_{2-q}} \mathrm{E}\left(z_2^{\delta_{q,j}^{(2-q)}}\right)\right) \\
&= \mathrm{E}\left(((1-\alpha_{q-1}+\alpha_{q-1}z_1)(1-\alpha_q+\alpha_q z_2))^{\epsilon_{2-q}}\right) \\
&= G_{\epsilon_{2-q}}\left((1-\alpha_{q-1}+\alpha_{q-1}z_1)(1-\alpha_q+\alpha_q z_2)\right) \\
&= p+(1-p)e^{-\lambda(1-(1-\alpha_{q-1}+\alpha_{q-1}z_1)(1-\alpha_q+\alpha_q z_2))}. \qquad (3.27)
\end{aligned}$$

Finally, we can apply the same technique as in (3.26) and (3.27) to formulate the q term as follows.

$$\begin{aligned}
\mathrm{E}(z_1^{\alpha_1\circ\epsilon_0} z_2^{\alpha_2\circ\epsilon_0} \cdots z_q^{\alpha_q\circ\epsilon_0}) &= G_{\epsilon_0}\left((1-\alpha_1+\alpha_1 z_1)\cdots(1-\alpha_q+\alpha_q z_q)\right) \\
&= p+(1-p)e^{-\lambda(1-(1-\alpha_1+\alpha_1 z_1)\cdots(1-\alpha_q+\alpha_q z_q))}. \qquad (3.28)
\end{aligned}$$

For $\prod_{i=1}^{n-q} \mathrm{E}\left(z_i^{\epsilon_i} z_{i+1}^{\alpha_1\circ\epsilon_i} \cdots z_{i+q}^{\alpha_q\circ\epsilon_i}\right)$, we know that $\{\epsilon_t, t = 1, 2, \ldots\}$ is a sequence of i.i.d. zero inflated Poisson random variables with parameters p and λ. First, consider the case $i = 1$, we obtain

$$\begin{aligned}
\mathrm{E}\left(z_1^{\epsilon_1} z_2^{\alpha_1\circ\epsilon_1} \cdots z_{1+q}^{\alpha_q\circ\epsilon_1}\right) &= \mathrm{E}\left(\mathrm{E}\left(z_1^{\epsilon_1} z_2^{\alpha_1\circ\epsilon_1} \cdots z_{1+q}^{\alpha_q\circ\epsilon_1}|\epsilon_1\right)\right) \\
&= \mathrm{E}\left(z_1^{\epsilon_1} \prod_{j=1}^{\epsilon_1} \mathrm{E}\left(z_2^{\delta_{1,j}^{(1)}}\right) \cdots \prod_{j=1}^{\epsilon_1} \mathrm{E}\left(z_{1+q}^{\delta_{q,j}^{(1)}}\right)\right) \\
&= \mathrm{E}\left(z_1(1-\alpha_1+\alpha_1 z_2)\cdots(1-\alpha_q+\alpha_q z_{1+q})\right)^{\epsilon_1} \\
&= G_{\epsilon_1}(z_1(1-\alpha_1+\alpha_1 z_2)\cdots(1-\alpha_q+\alpha_q z_{1+q})) \\
&= p+(1-p)e^{-\lambda(1-z_1(1-\alpha_1+\alpha_1 z_2)\cdots(1-\alpha_q+\alpha_q z_{1+q}))}. \qquad (3.29)
\end{aligned}$$

As a consequence, we apply the same technique as in (3.29) for $i = 2, 3, \ldots, n-q$, then we obtain

$$\prod_{i=1}^{n-q} \mathrm{E}\left(z_i^{\epsilon_i} z_{i+1}^{\alpha_1 \circ \epsilon_i} \cdots z_{i+q}^{\alpha_q \circ \epsilon_i}\right) = \prod_{i=1}^{n-q} \left(p + (1-p)e^{-\lambda(1 - z_i(1-\alpha_1+\alpha_1 z_{i+1})\cdots(1-\alpha_q+\alpha_q z_{i+q}))}\right). \tag{3.30}$$

For $\prod_{i=1}^{q-1} \mathrm{E}(z_{n-i}^{\epsilon_{n-i}} z_{n+1-i}^{\alpha_1 \circ \epsilon_{n-i}} \cdots z_n^{\alpha_{q-i} \circ \epsilon_{n-i}})$, we apply the similar technique as in (3.28). First, we start with $i = 1$,

$$\begin{aligned} \mathrm{E}(z_{n-1}^{\epsilon_{n-1}} z_n^{\alpha_1 \circ \epsilon_{n-1}}) &= G_{\epsilon_{n-1}}(z_{n-1}(1-\alpha_1+\alpha_1 z_n)) \\ &= p + (1-p)e^{-\lambda(1 - z_{n-1}(1-\alpha_1+\alpha_1 z_n))}. \end{aligned} \tag{3.31}$$

Consequently, we can apply to obtain the general form for $i = 2, 3, \ldots, q-1$ as follows.

$$\begin{aligned} \prod_{i=1}^{q-1} \mathrm{E}(z_{n-i}^{\epsilon_{n-i}} z_{n+1-i}^{\alpha_1 \circ \epsilon_{n-i}} \cdots z_n^{\alpha_{q-i} \circ \epsilon_{n-i}}) &= \prod_{i=1}^{q-1} \left(G_{\epsilon_{n-i}}(z_{n-i}(1-\alpha_1+\alpha_1 z_{n+1-i})\cdots(1-\alpha_i+\alpha_i z_n))\right) \\ &= \prod_{i=1}^{q-1} \left(p + (1-p)e^{-\lambda(1 - z_{n-i}(1-\alpha_1+\alpha_1 z_{n+1-i})\cdots(1-\alpha_i+\alpha_i z_n))}\right). \end{aligned} \tag{3.32}$$

Finally, the last term of (3.25), we have that $\{\epsilon_t, t = 1, 2, \ldots\}$ is a sequence of i.i.d. zero inflated Poisson random variables with parameters p and λ, then we obtain

$$\mathrm{E}(z_n^{\epsilon_n}) = p + (1-p)e^{-\lambda(1-z_n)}. \tag{3.33}$$

Substituting (3.26) - (3.33) into (3.25),

$$\begin{aligned}
\mathrm{E}\left(z_1^{N_1} z_2^{N_2} \cdots z_n^{N_n}\right) &= \left(p + (1-p)e^{-\lambda\alpha_q(1-z_1)}\right) \times \cdots \\
&\times \left(p + (1-p)e^{-\lambda(1-(1-\alpha_1+\alpha_1 z_1)\cdots(1-\alpha_q+\alpha_q z_q))}\right) \\
&\times \prod_{i=1}^{n-q} \left(p + (1-p)e^{-\lambda(1-z_i(1-\alpha_1+\alpha_1 z_{i+1})\cdots(1-\alpha_q+\alpha_q z_{i+q}))}\right) \\
&\times \left(p + (1-p)e^{-\lambda(1-z_{n-1}(1-\alpha_1+\alpha_1 z_n))}\right) \times \cdots \\
&\times \left(p + (1-p)e^{-\lambda(1-z_{n-(q-1)}(1-\alpha_1+\alpha_1 z_{n+1-(q-1)})\cdots(1-\alpha_{q-1}+\alpha_{q-1} z_n))}\right) \\
&\times \left(p + (1-p)e^{-\lambda(1-z_n)}\right). \qquad (3.34)
\end{aligned}$$

□

Theorem 3.12. Let R_n be the discrete time surplus process defined in Definition 3.10. The adjustment coefficient function $c(\cdot)$ of R_n is defined as

$$c(z) = \log(p + (1-p)e^{-\lambda(1-m_C(z)(1-\alpha_1+\alpha_1 m_C(z))\cdots(1-\alpha_q+\alpha_q m_C(z)))}) - \pi z, \quad (3.35)$$

for $z \in \mathbb{R}^+$ and $\alpha_0 = 1$.

Proof. We denote that $\{C_{i,j},\, i,j = 1,2,\ldots\}$ is a sequence of i.i.d. random variables whose the moment of generating function, $m_C(\cdot)$.

Note that,

$$c(z) = \lim_{n\to\infty} \frac{1}{n} \log m_{S_n}(z) - \pi z.$$

From Lemma 3.11, we obtain

$$\begin{aligned}
\mathrm{E}\left(z^{N_1}z^{N_2}\cdots z^{N_n}\right) &= \left(p+(1-p)e^{-\lambda\alpha_q(1-z)}\right)\times\cdots \\
&\times\left(p+(1-p)e^{-\lambda(1-(1-\alpha_1+\alpha_1 z)\cdots(1-\alpha_q+\alpha_q z))}\right) \\
&\times\left(p+(1-p)e^{-\lambda(1-z(1-\alpha_1+\alpha_1 z)\cdots(1-\alpha_q+\alpha_q z))}\right)^{n-q} \\
&\times\left(p+(1-p)e^{-\lambda(1-z(1-\alpha_1+\alpha_1 z))}\right)\times\cdots \\
&\times\left(p+(1-p)e^{-\lambda(1-z(1-\alpha_1+\alpha_1 z)\cdots(1-\alpha_{q-1}+\alpha_{q-1} z))}\right) \\
&\times\left(p+(1-p)e^{-\lambda(1-z)}\right). \qquad (3.36)
\end{aligned}$$

Hence, from (3.24), we can obtain the moment generating function of S_n, by replacing z by $m_C(z)$ in (3.36) as

$$\begin{aligned}
m_{S_n}(z) &= \left(p+(1-p)e^{-\lambda\alpha_q(1-m_C(z))}\right)\times\cdots \\
&\times\left(p+(1-p)e^{-\lambda(1-(1-\alpha_1+\alpha_1 m_C(z))\cdots(1-\alpha_q+\alpha_q m_C(z)))}\right) \\
&\times\left(p+(1-p)e^{-\lambda(1-m_C(z)(1-\alpha_1+\alpha_1 m_C(z))\cdots(1-\alpha_q+\alpha_q m_C(z)))}\right)^{n-q} \\
&\times\left(p+(1-p)e^{-\lambda(1-m_C(z)(1-\alpha_1+\alpha_1 m_C(z)))}\right)\times\cdots \\
&\times\left(p+(1-p)e^{-\lambda(1-m_C(z)(1-\alpha_1+\alpha_1 m_C(z))\cdots(1-\alpha_{q-1}+\alpha_{q-1} m_C(z)))}\right) \\
&\times\left(p+(1-p)e^{-\lambda(1-m_C(z))}\right). \qquad (3.37)
\end{aligned}$$

Consequently, we obtain $m_{S_n}(\cdot)$ from (3.37), then we put into the adjustment coefficient function as follows.

$$\begin{aligned}
c(z) &= \lim_{z\to+\infty}\frac{1}{n}\log m_{S_n}(z)-\pi z \\
&= \log\left(p+(1-p)e^{-\lambda(1-m_C(z)(1-\alpha_1+\alpha_1 m_C(z))\cdots(1-\alpha_q+\alpha_q m_C(z)))}\right)-\pi z.
\end{aligned}$$

□

The premium per period, π, follows the explanation in (3.18). Let $D=\{z\in\mathbb{R}^+\}$. We will show that the adjustment coefficient has the unique positive zero root in D for $q\geq 1$.

Lemma 3.13. Let $q \geq 1$, the adjustment coefficient function of ZIPMA(q) has the unique positive solution of the equation $c(z) = 0$ in D.

Proof. To simplify the notation,

$$A_i(z) \;:=\; 1 - \alpha_i + \alpha_i m_C(z).$$

Then, we obtain

$$\begin{aligned} A_i(0) &= 1, \\ A_i^{'}(z) &= \alpha_i m_C^{'}(z), \\ A_i^{''}(z) &= \alpha_i m_C^{''}(z), \end{aligned}$$

and

$$A_i^{'}(0) = \alpha_i \mathrm{E}(C),$$

where $\alpha_0 = 1$.

We can simplify the adjustment coefficient function defined in Theorem 3.12 as

$$\begin{aligned} c(z) &= \log\left(p + (1-p)e^{-\lambda(1-m_C(z)(1-\alpha_1+\alpha_1 m_C(z))\cdots(1-\alpha_q+\alpha_q m_C(z)))}\right) - \pi z \\ &= \log\left(p + (1-p)e^{-\lambda(1-A_0(z)A_1(z)\cdots A_q(z))}\right) - \pi z \\ &= \log\left(p + (1-p)e^{-\lambda(1-\prod_{i=0}^{q} A_i(z))}\right) - \pi z. \end{aligned} \tag{3.38}$$

Similar to Lemma 3.7 to prove the Lemma, then we will show that

(a) $c(0) = 0$,

(b) $\left.\dfrac{d}{dz}c(z)\right|_{z=0} < 0$,

(c) $\dfrac{d^2}{dz^2}c(z) > 0$ for $z \in D$,

(d) $\lim\limits_{z \to +\infty} c(z) = +\infty$.

(a) Note that

$$c(z) \;=\; \log\left(p+(1-p)e^{-\lambda(1-\prod_{i=0}^{q}A_i(z))}\right)-\pi z.$$

We substitute $z = 0$ into $c(z)$, then we obtain

$$\begin{aligned} c(0) &= \log\left(p+(1-p)e^{-\lambda(1-\prod_{i=1}^{q}A_i(0))}\right)-\pi(0) \\ &= \log(p+(1-p)) \\ &= 0. \end{aligned}$$

(b) Note that

$$\frac{d}{dz}c(z) \;=\; \frac{(1-p)e^{-\lambda(1-\prod_{i=0}^{q}A_i(z))}\lambda\left(\sum_{s=0}^{q}\prod_{i=0\,,\,i\neq s}^{q}A'_s(z)A_i(z)\right)}{p+(1-p)e^{-\lambda(1-\prod_{i=0}^{q}A_i(z))}}-\pi.$$

Since we have $\pi = \lambda(1-p)\mathrm{E}(C)(\sum_{i=0}^{q}\alpha_i)(1+\theta)$, then for $\theta > 0$,

$$\begin{aligned} \left.\frac{d}{dz}c(z)\right|_{z=0} &= \frac{(1-p)e^{-\lambda(1-\prod_{i=0}^{q}A_i(0))}\lambda\left(\sum_{s=0}^{q}\prod_{i=0\,,\,i\neq s}^{q}A'_s(0)A_i(0)\right)}{p+(1-p)e^{-\lambda(1-\prod_{i=0}^{q}A_i(0))}}-\pi \\ &= \frac{(1-p)e^{-\lambda(1-1)}\lambda(\sum_{s=0}^{q}\alpha_s\mathrm{E}(C))}{p+(1-p)e^{\lambda(1-1)}}-\pi \\ &= \lambda(1-p)\mathrm{E}(C)\sum_{s=0}^{q}\alpha_s-\lambda(1-p)\mathrm{E}(C)(1+\theta)\sum_{s=0}^{q}\alpha_s \\ &= \lambda(1-p)\mathrm{E}(C)\left(\sum_{s=0}^{q}\alpha_s-\sum_{s=0}^{q}\alpha_s-\theta\sum_{s=0}^{q}\alpha_s\right) \\ &= -\lambda(1-p)\mathrm{E}(C)\left(\theta\sum_{s=0}^{q}\alpha_s\right) \\ &< 0. \end{aligned}$$

Then, we obtain that $\left.\frac{d}{dz}c(z)\right|_{z=0} < 0.$

(c) Since $z \in D$, $A_i(z) > 0$, $A_i^{'}(z) > 0$ and $A_i^{''}(z) > 0$, then we obtain

$$
\begin{aligned}
\frac{d^2}{dz^2}c(z) \;=\; & \frac{p(1-p)e^{-\lambda(1-\prod_{i=0}^{q}A_i(z))}\left(2\lambda\sum_{x=0}^{q}\sum_{y=x+1}^{q}\prod_{i=0,i\neq x,y}^{q}A_x^{'}(z)A_y^{'}(z)A_i(z)\right)}{\left(p+(1-p)e^{-\lambda(1-\prod_{i=0}^{q}A_i(z))}\right)^2} \\
& + \frac{p(1-p)e^{-\lambda(1-\prod_{i=0}^{q}A_i(z))}\left(\lambda\sum_{s=0}^{q}\prod_{i=0,i\neq s}^{q}A_s^{''}(z)A_i\right)}{\left(p+(1-p)e^{-\lambda(1-\prod_{i=0}^{q}A_i(z))}\right)^2} \\
& + \frac{p(1-p)e^{-\lambda(1-\prod_{i=0}^{q}A_i(z))}\left(\lambda\sum_{s=0}^{q}\prod_{i=0,i\neq s}^{q}A_s^{'}(z)A_i\right)^2}{\left(p+(1-p)e^{-\lambda(1-\prod_{i=0}^{q}A_i(z))}\right)^2} \\
& + \frac{\left(1-p)e^{-\lambda(1-\prod_{i=0}^{q}A_i(z))}\right)^2\left(2\lambda\sum_{x=0}^{q}\sum_{y=x+1}^{q}\prod_{i=0,i\neq x,y}^{q}A_x^{'}(z)A_y^{'}(z)A_i(z)\right)}{\left(p+(1-p)e^{-\lambda(1-\prod_{i=0}^{q}A_i(z))}\right)^2} \\
& + \frac{\left(1-p)e^{-\lambda(1-\prod_{i=0}^{q}A_i(z))}\right)^2\left(\lambda\sum_{s=0}^{q}\prod_{i=0,i\neq s}^{q}A_s^{''}(z)A_i\right)}{\left(p+(1-p)e^{-\lambda(1-\prod_{i=0}^{q}A_i(z))}\right)^2}.
\end{aligned}
$$

Thus, we can conclude that $\frac{d^2}{dz^2}c(z) > 0$.

(d) We can show that the limit of $c(z)$ reaches to $+\infty$ as z approaches $+\infty$. Let us first consider

$$
\begin{aligned}
f(z) &= \lambda(\prod_{i=0}^{q}A_i(z)-1), \\
&\propto \lambda\prod_{i=0}^{q}A_i(z) \\
&\propto \lambda m_C^{q+1}(z)\prod_{i=0}^{q}\alpha_i,
\end{aligned}
$$

for $z \in D$. We know that $m_C(z)$ is the monotonically increasing function and continuous function in D, then $m_C^{q+1}(z)$ is growing up to $+\infty$ with the exponential rate, then we can conclude that $f(z)$ will grow with exponential rate which is faster than any linear trend. Hence, we can make the conclusion as

$$
\lim_{z\to+\infty}\left(\log\left(p+(1-p)e^{-\lambda(1-\prod_{i=0}^{q}A_i(z))}\right)-\pi z\right) \;=\; +\infty.
$$

□

Example 3.2. In this part, we consider a special case when the claim amounts follow an exponential distribution. That is $\{C_{i,j},\ i \in \mathbb{N},\ j = 1, 2, \ldots\}$ is a sequence of i.i.d. exponentially distributed random variables with parameter $\beta > 0$. The moment generating function of $\{C_{i,j},\ i \in \mathbb{N},\ j = 1, 2, \ldots\}$ is defined as $m_C(z) = \frac{1}{1-z/\beta}$ for $z < \beta$. Using Theorem 3.6, the adjustment coefficient function is defined as

$$c(z) \quad = \quad \log\left(p + (1-p)e^{-\lambda(1-\prod_{i=0}^{q} A_i(z))}\right) - \pi z, \tag{3.39}$$

where $A_i(z) = 1 - \alpha_i + \frac{\alpha_i}{1-z/\beta}$ and $\pi = \lambda(1-p)(\sum_{i=0}^{q} \alpha_i)\mathrm{E}(C)(1+\theta),\ \ 0 < z < \beta.$

3.3.2 Approximate to the value at risk and tail value at risk of ZIPMA(q)

The value at risk at the confidence level γ, $\mathrm{VaR}_\gamma(S_n)$ and the tail value at risk at the confidence level γ, $\mathrm{TVaR}_\gamma(S_n)$ for ZIPMA(q) process can be approximated by the similar technique as in ZIPMA(1). Therefore, we consider the characteristic function of S_n as follows.

$$\begin{aligned}
\phi_{S_n}(x) &= G_{N(n)}(\phi_C(x)) \\
&= \left(p + (1-p)e^{-\lambda\alpha_q(1-\phi_C(x))}\right) \times \cdots \\
&\quad \times \left(p + (1-p)e^{-\lambda(1-(1-\alpha_1+\alpha_1\phi_C(x))\cdots(1-\alpha_q+\alpha_q\phi_C(x)))}\right) \\
&\quad \times \left(p + (1-p)e^{-\lambda(1-\phi_C(x)(1-\alpha_1+\alpha_1\phi_C(x))\cdots(1-\alpha_q+\alpha_q\phi_C(x)))}\right)^{n-q} \\
&\quad \times \left(p + (1-p)e^{-\lambda(1-\phi_C(x)(1-\alpha_1+\alpha_1\phi_C(x)))}\right) \times \cdots \\
&\quad \times \left(p + (1-p)e^{-\lambda(1-\phi_C(x)(1-\alpha_1+\alpha_1\phi_C(x))\cdots(1-\alpha_{q-1}+\alpha_{q-1}\phi_C(x)))}\right) \\
&\quad \times \left(p + (1-p)e^{-\lambda(1-\phi_C(x))}\right),
\end{aligned}$$

where $x \in \mathbb{R}^+$.

3.3.3 Numerical experiments of risk model based on ZIPMA(q)

In this section, we show examples to calculate the adjustment coefficient and approximation to the ruin probability of risk model based on ZIPMA(q) claim count process where we consider a special case when $q = 2$ and $q = 3$. That is the ZIPMA(2) and ZIPMA(3), respectively. In addition, the two risk measurements of 12^{th} period of time at the confidence levels 0.9 and 0.95 are also provided.

3.3.4 Calculation of the adjustment coefficient of risk model based on ZIPMA(2)

Let R_n be the discrete time surplus process defined in (3.1), and $\{N_i,\ i = 1, 2, \ldots n\}$ is a sequence of ZIPMA(2) claฟim count process defined in Definition 3.8. Let $D = \{z \in \mathbb{R}^+\}$ and $z < \beta$, and $\{C_{i,j},\ i, j = 1, 2, \ldots\}$ is a sequence of i.i.d. random variables with the exponential distribution with parameter β and we obtain $c(z)$ as in Example 3.2. The parameters setting are $u = 2$, $(\lambda, p) = (1.5, 0.2)$, $\beta = 4$ and $\theta = 0.3$. Table 3.3 shows the adjustment coefficient z_0 for different values of $\alpha_1, \alpha_2 \in \{0, 0.25, 0.50, 0.75, 1\}$ and the value of upper bound of the ruin probability of R_n, $\Psi_{R_n}(u) = \exp(-z_0 u)$ in parentheses. Figure 3.8 - 3.9 show the trend of the adjustment coefficient and the value of upper bound of the ruin probability, respectively.

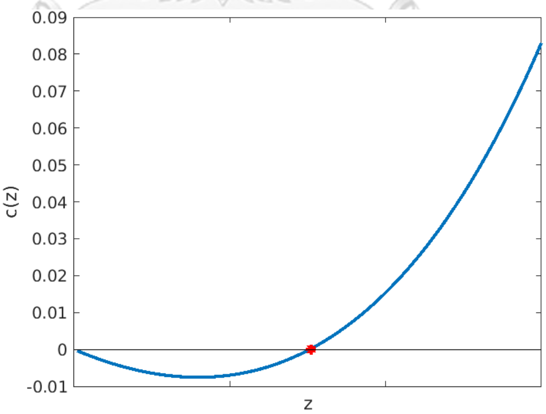

Figure 3.7: The unique positive zero root of the adjustment coefficient for ZIPMA(2).

Table 3.3: The adjustment coefficient z_0 and the approximation of $\Psi_{R_n}(u)$ of ZIPMA(2).

α_1 \ α_2	0	0.25	0.5	0.75	1
0	0.8140 (0.1963)	0.6793 (0.2570)	0.6102 (0.2951)	0.5665 (0.3220)	0.5357 (0.3425)
0.25	0.6793 (0.2570)	0.5927 (0.3056)	0.5418 (0.3383)	0.5074 (0.3624)	0.4821 (0.3812)
0.5	0.6102 (0.2951)	0.5418 (0.3383)	0.4988 (0.3687)	0.4687 (0.3916)	0.4460 (0.4098)
0.75	0.5665 (0.3220)	0.5074 (0.3624)	0.4687 (0.3916)	0.4408 (0.4141)	0.4196 (0.4320)
1	0.5357 (0.3425)	0.4821 (0.3813)	0.4460 (0.4098)	0.4196 (0.4320)	0.3992 (0.4500)

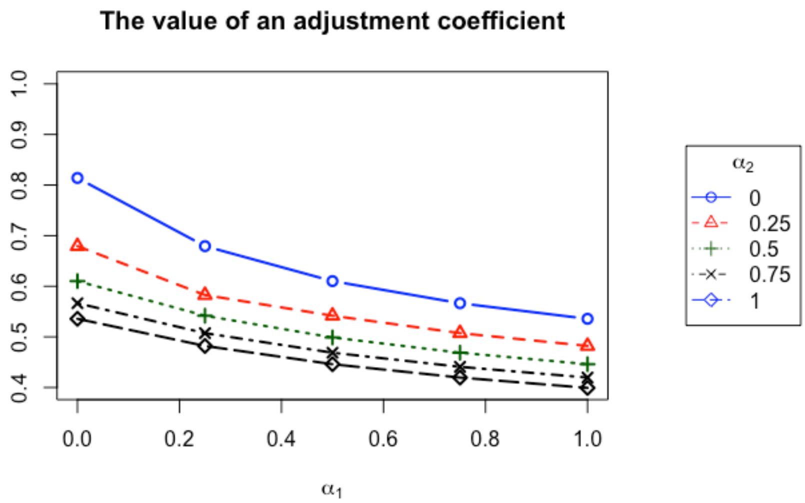

Figure 3.8: The trend of the adjustment coefficient according to the changes of α_1 and α_2 of ZIPMA(2).

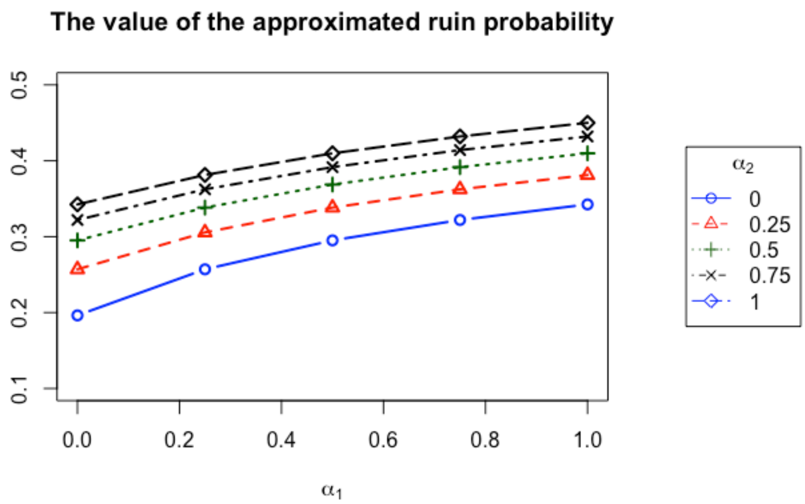

Figure 3.9: The trend of the ruin probability according to the changes of α_1 and α_2 of ZIPMA(2).

Figure 3.7 shows the unique positive zero root of $c(z)$ in the case of $\beta = 4$, $\alpha_1 = 0.25$ $\alpha_2 = 0$, which is the red point on the blue line and it satisfies 4 statements in Lemma 3.13 that is the trend of $c(z)$ surge to positive infinity. Table 3.3 and Figures 3.8-3.9 show that the value of ruin probability increases while the adjustment coefficient decreases. Besides, the ruin probability dependently grows as a function of the level α_i , $i = 1, 2$. Therefore, the ZIPMA(2) risk model with two periods of claim count seems to have a high value of the ruin probability than the ruin probability from ZIPMA(1) risk model.

3.3.5 Calculation of the value at risk and the tail value at risk for risk model based on ZIPMA(2)

In this part, we show calculations of the value at risk and tail value at risk of a risk model based on ZIPMA(q) when $q = 2$. Let the time period n be 12 and divide the domain of $\{C_{i,j},\ i, j = 1, 2, \ldots\}$ which $\beta = 4$ to be 5×10^5 parts with the length of steps are 0.0005 for the FFT distribution approximation. For the Riemann sum approximation of tail value at risk, we divided the length of steps of value at risk as 5×10^{-6}. Tables 3.4 - 3.5 show $\mathrm{VaR}_\gamma(S_{12})$ for the different values of $\alpha_1, \alpha_2 \in \{0, 0.25, 0.5, 0.75, 1\}$ and the value of $\mathrm{TVaR}_\gamma(S_{12})$ (in parentheses) for the confidence levels $\gamma = 0.90$ and 0.95, respectively.

Table 3.4: The value of the value at risk and tail value at risk at confidence level 0.90 of ZIPMA(2).

$\alpha_1 \backslash \alpha_2$	0	0.25	0.5	0.75	1
0	5.5200	6.8200	8.0600	9.2800	10.4600
	(6.40828)	(7.8785)	(9.26798)	(10.6081)	(11.9146)
0.25	6.8400	8.1200	9.3600	10.5600	11.7600
	(7.90664)	(9.34731)	(10.726)	(12.0633)	(13.3708)
0.5	8.1000	9.3600	10.6000	11.8200	13.0200
	(9.31545)	(10.7463)	(12.1235)	(13.4635)	(14.7757)
0.75	9.3200	10.5800	11.8200	13.0400	14.2400
	(10.6707)	(12.0995)	(13.4796)	(14.825)	(16.1438)
1	10.5200	11.8000	13.0400	14.2600	15.4600
	(11.9897)	(13.4204)	(14.8055)	(16.1574)	(17.4835)

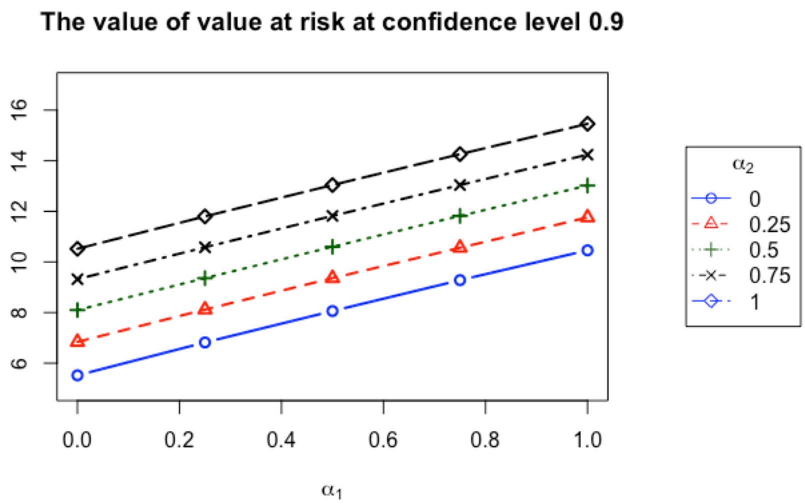

Figure 3.10: The trend of the value at risk according to the changes of α_1 and α_2 at the confidence level 0.90 of ZIPMA(2).

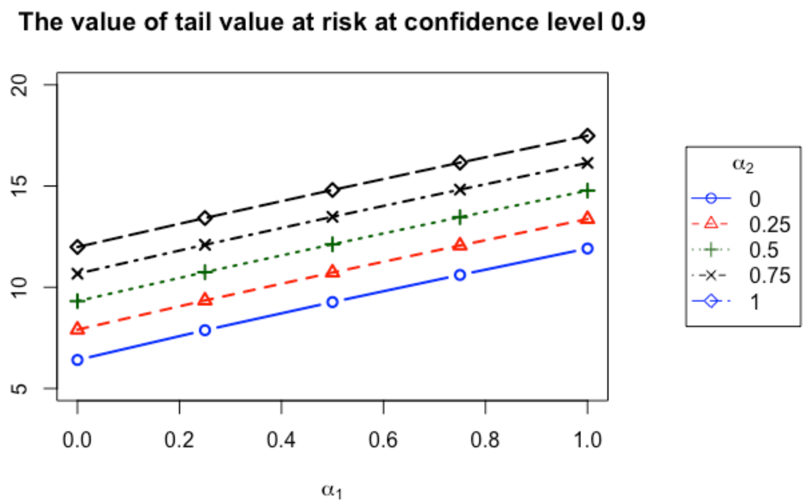

Figure 3.11: The trend of the tail value at risk according to the changes of α_1 and α_2 at the confidence level 0.90 of ZIPMA(2).

Table 3.5: The value of the value at risk and tail value at risk at confidence level 0.95 of ZIPMA(2).

α_1 \ α_2	0	0.25	0.5	0.75	1
0	6.1800	7.6000	8.9600	10.2800	11.5600
	(7.00104)	(8.58557)	(10.0686)	(11.4909)	(12.8722)
0.25	7.6400	9.0400	10.3800	11.6800	12.9600
	(8.6197)	(10.1672)	(11.6373)	(13.0564)	(14.439)
0.5	9.0000	10.4000	11.7400	13.0400	14.3400
	(10.1259)	(11.6616)	(13.1303)	(14.553)	(15.9414)
0.75	10.3200	11.7200	13.0600	14.3800	15.6600
	(11.5663)	(13.0996)	(14.572)	(16.0014)	(17.398)
1	11.6200	13.0200	14.3600	15.6800	16.9800
	(12.9624)	(14.4982)	(15.9769)	(17.4142)	(18.8199)

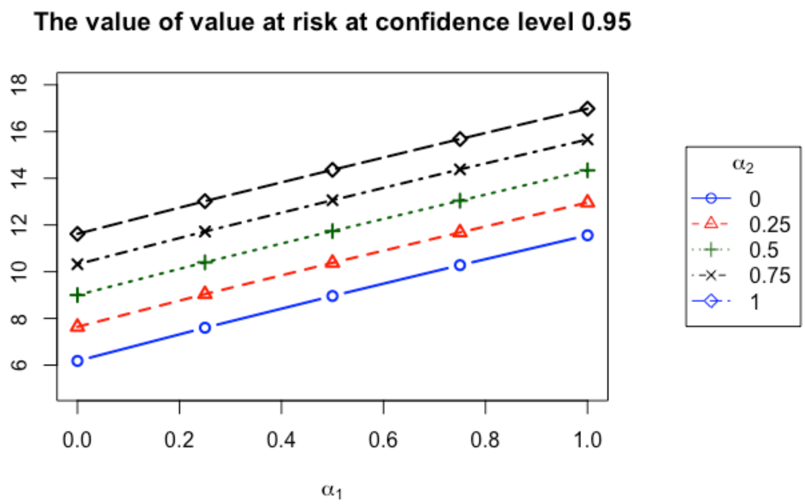

Figure 3.12: The trend of the value at risk according to the changes of α_1 and α_2 at the confidence level 0.95 of ZIPMA(2).

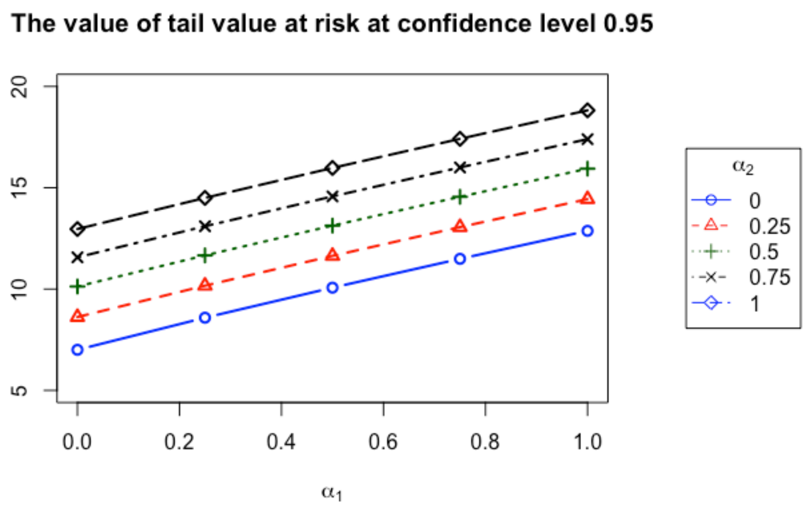

Figure 3.13: The trend of the tail value at risk according to the changes of α_1 and α_2 at the confidence level 0.95 of ZIPMA(2).

Tables 3.4 - 3.5 and Figures 3.10 - 3.13 show that the value of VaR_γ and TVaR_γ are increasing together with the increase of the values of α_1, α_2 and confidence level γ. In the other words, the increasing of α_1 and α_2 which means that there are more the number of new claims will continuously claim in the current year. Consequently, the company will receive either high earned premiums or massive claims. The confidence level γ can inform us about the probability that the loss will undergo over the estimated loss with a probability $(1-\gamma)$.

3.3.6 Calculation of the adjustment coefficient of risk model based on ZIPMA(3)

Let R_n be the discrete time surplus process defined in (3.1), and $\{N_i,\ i = 1, 2, \ldots n\}$ be a sequence of ZIPMA(2) claim count process defined in Definition 3.8. Let $D = \{z \in \mathbb{R}^+\}$ and $z < \beta$, and $\{C_{i,j},\ i, j = 1, 2, \ldots\}$ is a sequence of i.i.d. random variables with the exponential distribution with parameter β and we obtain $c(z)$ as in Example 3.2. The parameters setting are $u = 2$, $(\lambda, p) = (1.5, 0.2)$, $\beta = 4$ and $\theta = 0.3$. Figures 3.15 - 3.19 show the trend of the adjustment coefficient z_0 for the different values of $\alpha_1, \alpha_2, \alpha_3 \in \{0, 0.25, 0.50, 0.75, 1\}$ and the value of upper bound of the ruin probability of R_n, $\Psi_{R_n}(u) = \exp(-z_0 u)$. Table 3.6 shows the value of the adjustment coefficient z_0 and the value of upper bound of the ruin probability in parentheses.

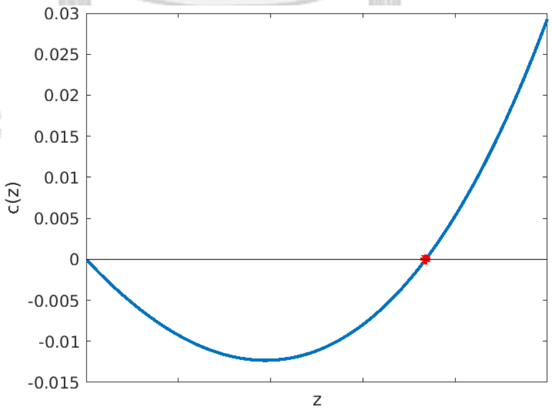

Figure 3.14: The unique positive zero root of the adjustment coefficient for ZIPMA(3).

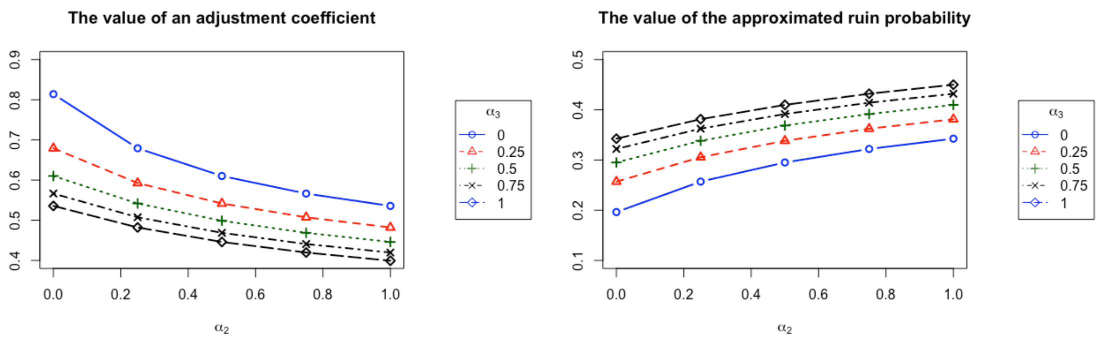

Figure 3.15: The trend of the adjustment coefficient and the approximated ruin probability when fixed $\alpha_1 = 0$ and either α_2 or α_3 increases of ZIPMA(3).

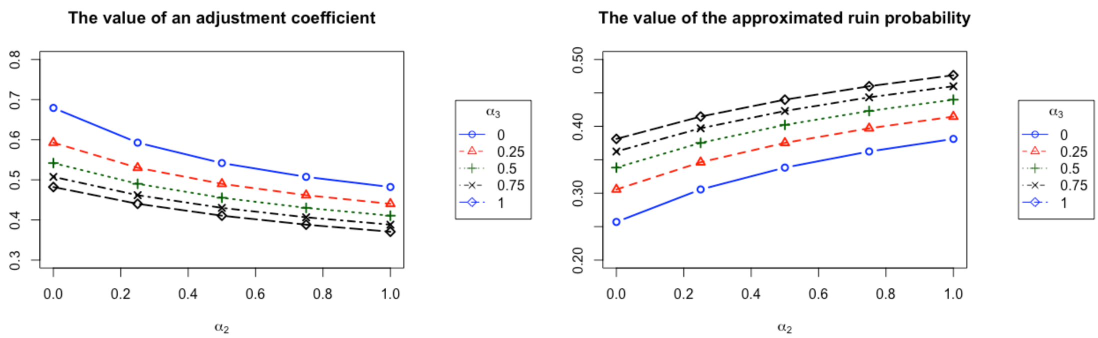

Figure 3.16: The trend of the adjustment coefficient and the approximated ruin probability when fixed $\alpha_1 = 0.25$ and either α_2 or α_3 increases of ZIPMA(3).

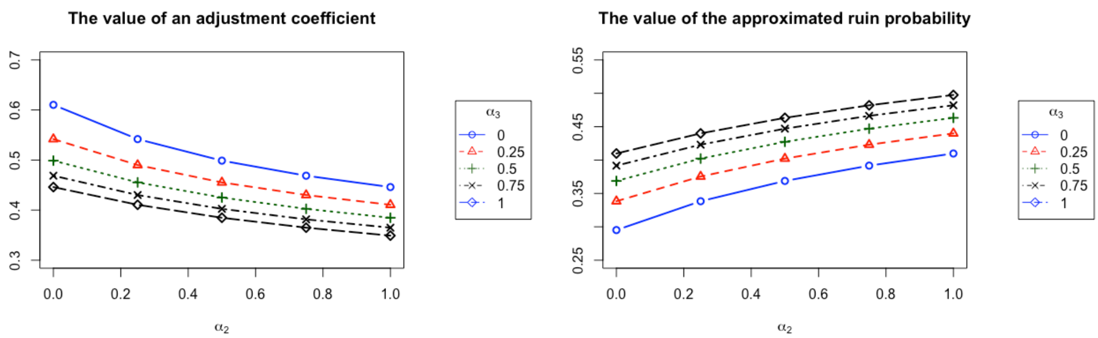

Figure 3.17: The trend of the adjustment coefficient and the approximated ruin probability when fixed $\alpha_1 = 0.5$ and either α_2 or α_3 increases of ZIPMA(3).

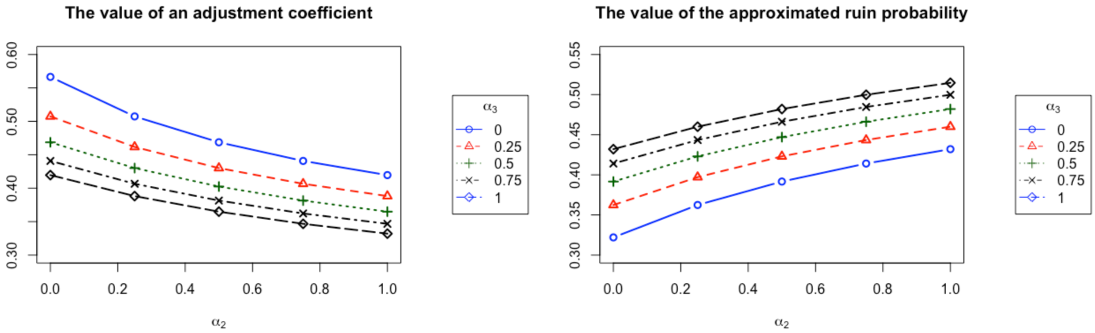

Figure 3.18: The trend of the adjustment coefficient and the approximated ruin probability when fixed $\alpha_1 = 0.75$ and either α_2 or α_3 increases of ZIPMA(3).

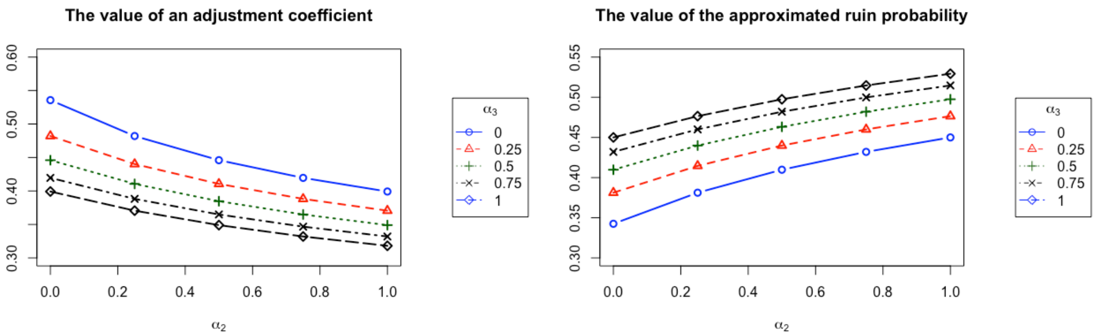

Figure 3.19: The trend of the adjustment coefficient and the approximated ruin probability when fixed $\alpha_1 = 1$ and either α_2 or α_3 increases of ZIPMA(3).

Table 3.6: The adjustment coefficient z_0 and the approximation of $\Psi_{R_n}(u)$ of ZIPMA(3)

	α_2 \ α_3	0	0.25	0.5	0.75	1
$\alpha_1 = 0$	0	0.8140	0.6793	0.6102	0.5665	0.5357
		(0.1963)	(0.2570)	(0.2951)	(0.3220)	(0.3425)
	0.25	0.6793	0.5927	0.5418	0.5074	0.4821
		(0.2570)	(0.3056)	(0.3383)	(0.3624)	(0.3812)
	0.5	0.6102	0.5418	0.4988	0.4687	0.4460
		(0.2951)	(0.3383)	(0.3687)	(0.3916)	(0.4098)
	0.75	0.5665	0.5074	0.4687	0.4408	0.4196

Continued

Table 3.6: (continued) The adjustment coefficient z_0 and the approximation of $\Psi_{R_n}(u)$ of ZIPMA(3)

	α_2 \ α_3	0	0.25	0.5	0.75	1
		(0.3220)	(0.3624)	(0.3916)	(0.4141)	(0.4320)
	1	0.5357	0.4821	0.4460	0.4196	0.3992
		(0.3425)	(0.3813)	(0.4098)	(0.4320)	(0.4500)
$\alpha_1 = 0.25$	0	0.6793	0.5927	0.5418	0.5074	0.4821
		(0.2570)	(0.3056)	(0.3383)	(0.3624)	(0.3812)
	0.25	0.5927	0.5301	0.4900	0.4617	0.4402
		(0.3056)	(0.3463)	(0.3753)	(0.3971)	(0.4146)
	0.5	0.5418	0.4900	0.4553	0.4301	0.4106
		(0.3383)	(0.3753)	(0.4022)	(0.4230)	(0.4399)
	0.75	0.5074	0.4617	0.4301	0.4066	0.3882
		(0.3624)	(0.3971)	(0.4230)	(0.4434)	(0.4600)
	1	0.4821	0.4402	0.4106	0.3882	0.3706
		(0.3812)	(0.4146)	(0.4399)	(0.4600)	(0.4765)
$\alpha_1 = 0.5$	0	0.6102	0.5418	0.4988	0.4687	0.4460
		(0.2951)	(0.3383)	(0.3687)	(0.3916)	(0.4098)
	0.25	0.5418	0.4900	0.4553	0.4301	0.4106
		(0.3383)	(0.3753)	(0.4022)	(0.4230)	(0.4399)
	0.5	0.4988	0.4553	0.4251	0.4025	0.3847
		(0.3687)	(0.4022)	(0.4273)	(0.4470)	(0.4632)
	0.75	0.4687	0.4301	0.4025	0.3815	0.3649
		(0.3916)	(0.4230)	(0.4470)	(0.4662)	(0.4820)
	1	0.4460	0.4106	0.3847	0.3649	0.3490
		(0.4098)	(0.4399)	(0.4632)	(0.4820)	(0.4975)
$\alpha_1 = 0.75$	0	0.5665	0.5074	0.4687	0.4408	0.4196

Continued

Table 3.6: (continued) The adjustment coefficient z_0 and the approximation of $\Psi_{R_n}(u)$ of ZIPMA(3)

	α_2 \ α_3	0	0.25	0.5	0.75	1
		(0.3220)	(0.3624)	(0.3916)	(0.4141)	(0.4320)
	0.25	0.5074	0.4617	0.4301	0.4066	0.3882
		(0.3624)	(0.3971)	(0.4230)	(0.4434)	(0.4600)
	0.5	0.4687	0.4301	0.4025	0.3815	0.3649
		(0.3916)	(0.4230)	(0.4470)	(0.4662)	(0.4820)
	0.75	0.4408	0.4066	0.3815	0.3622	0.3467
		(0.4141)	(0.4434)	(0.4662)	(0.4846)	(0.4998)
	1	0.4196	0.3882	0.3649	0.3467	0.3320
		(0.4320)	(0.4600)	(0.4820)	(0.4998)	(0.5147)
$\alpha_1 = 1$	0	0.5357	0.4821	0.4460	0.4196	0.3992
		(0.3425)	(0.3812)	(0.4098)	(0.4320)	(0.4500)
	0.25	0.4821	0.4402	0.4106	0.3882	0.3706
		(0.3812)	(0.4146)	(0.4399)	(0.4600)	(0.4765)
	0.5	0.4460	0.4106	0.3847	0.3649	0.3490
		(0.4098)	(0.4399)	(0.4632)	(0.4820)	(0.4975)
	0.75	0.4196	0.3882	0.3649	0.3467	(0.3320)
		(0.4320)	(0.4600)	(0.4820)	(0.4998)	(0.5147)
	1	0.3992	0.3706	0.3490	0.3320	0.3181
		(0.4500)	(0.4765)	(0.4975)	(0.5147)	(0.5293)

Figure 3.14 shows the unique positive zero root of $c(z)$ in case $\beta = 4$, $\alpha_1 = 0.5$ and $\alpha_2, \alpha_3 = 0$, which is the red point on the blue line and it satisfies 4 statements in Lemma 3.13 that is the trend of $c(z)$ surge to positive infinity. Figures 3.15 - 3.19 shows the similar trend to Figures 3.8-3.9 that the ruin prob-

ability is increasing while the adjustment coefficient is decreasing along with the increasing of level α_i.

3.3.7 Calculation of the value at risk and the tail value at risk for risk model based on ZIPMA(3)

In this part, we show a calculation of the value at risk and tail value at risk of a risk model based on ZIPMA(q) when $q = 3$. Let the time period n be 12 and divide the domain of $\{C_{i,j},\, i,j = 1,2,\ldots\}$ which $\beta = 4$ to be 5×10^5 parts with the length of steps are 0.0005 for the FFT distribution approximation. For the Riemann sum approximation of tail value at risk, we divide the length of steps of value at risk as 5×10^{-6}. Figures 3.20 - 3.29 show the trend of $\mathrm{VaR}_\gamma(S_{12})$ and $\mathrm{TVaR}_\gamma(S_{12})$ for the different values of $\alpha_1, \alpha_2, \alpha_3 \in \{0, 0.25, 0.5, 0.75, 1\}$ at the confidence levels $\gamma = 0.90$ and 0.95, respectively. Table 3.7 - 3.8 show the the value of $\mathrm{VaR}_\gamma(S_{12})$ and $\mathrm{TVaR}_\gamma(S_{12})$ in parentheses at the confidence levels $\gamma = 0.90$ and 0.95, respectively.

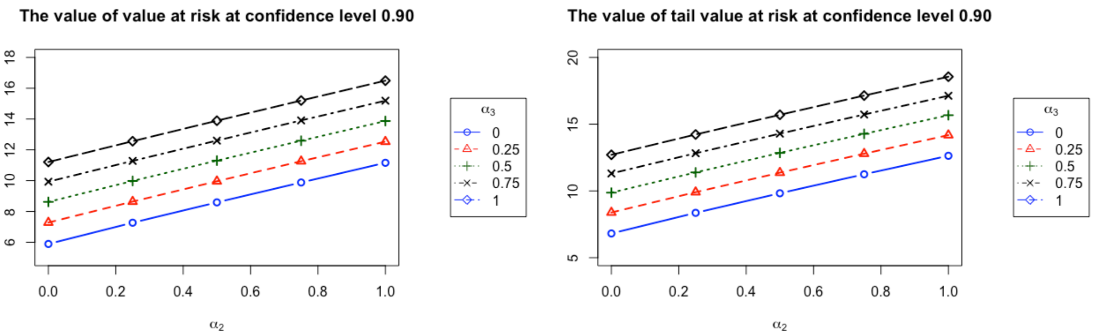

Figure 3.20: The trend of the value at risk and the tail value at risk when fixed $\alpha_1 = 0$ and either α_2 or α_3 increases at confidence level 0.90 of ZIPMA(3).

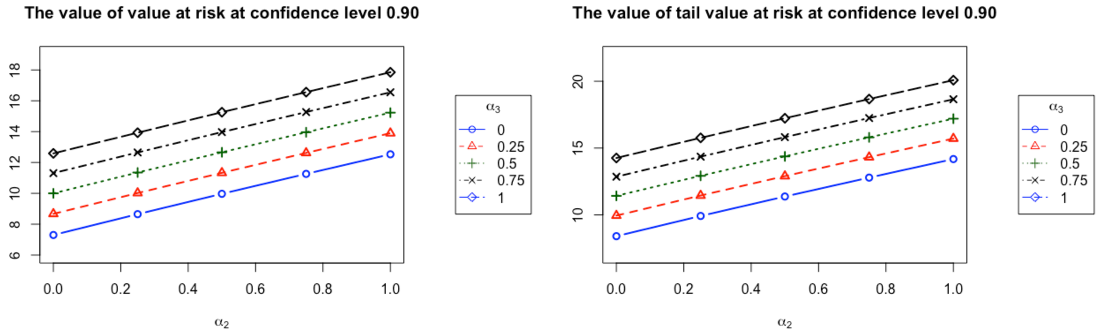

Figure 3.21: The trend of the value at risk and the tail value at risk when fixed $\alpha_1 = 0.25$ and either α_2 or α_3 increases at confidence level 0.90 of ZIPMA(3).

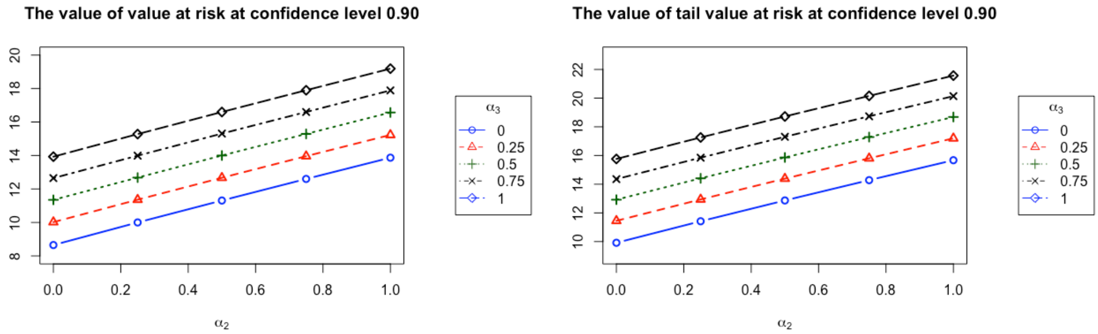

Figure 3.22: The trend of the value at risk and the tail value at risk when fixed $\alpha_1 = 0.50$ and either α_2 or α_3 increases at confidence level 0.90 of ZIPMA(3).

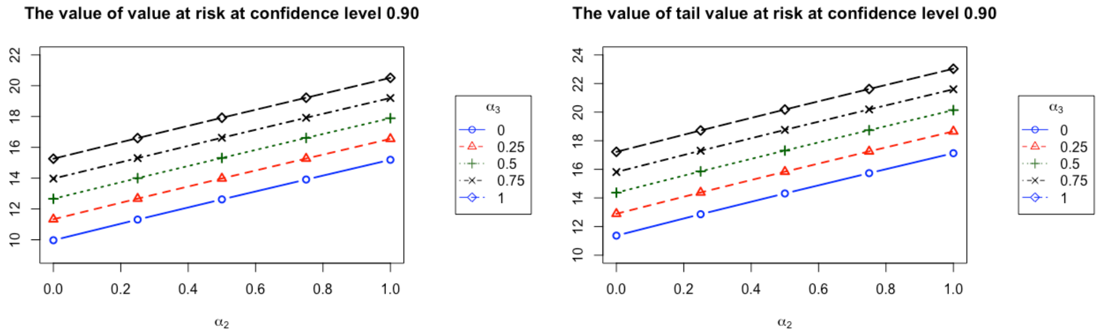

Figure 3.23: The trend of the value at risk and the tail value at risk when fixed $\alpha_1 = 0.75$ and either α_2 or α_3 increases at confidence level 0.90 of ZIPMA(3).

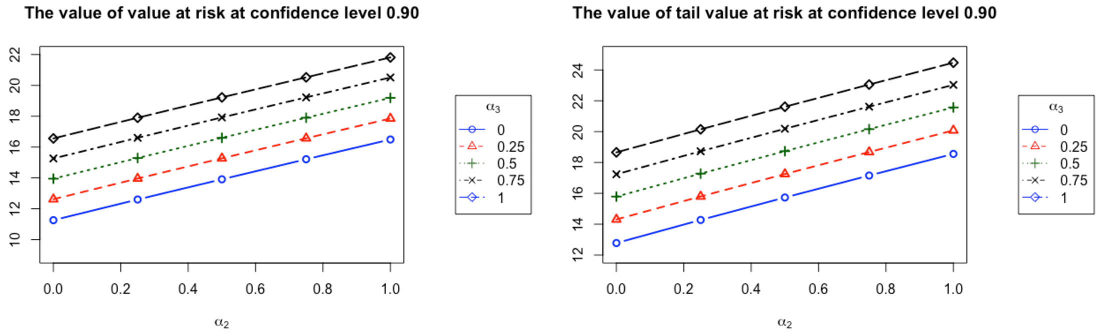

Figure 3.24: The trend of the value at risk and the tail value at risk when fixed $\alpha_1 = 1$ and either α_2 or α_3 increases at confidence level 0.90 of ZIPMA(3).

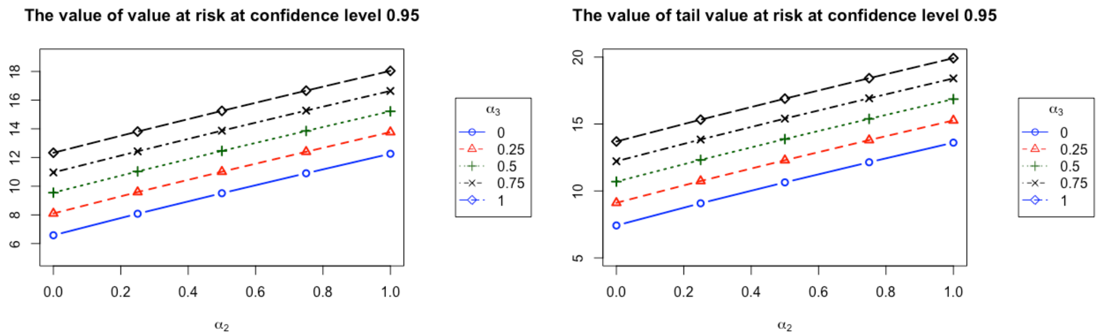

Figure 3.25: The trend of the value at risk and the tail value at risk when fixed $\alpha_1 = 0$ and either α_2 or α_3 increases at confidence level 0.95 of ZIPMA(3).

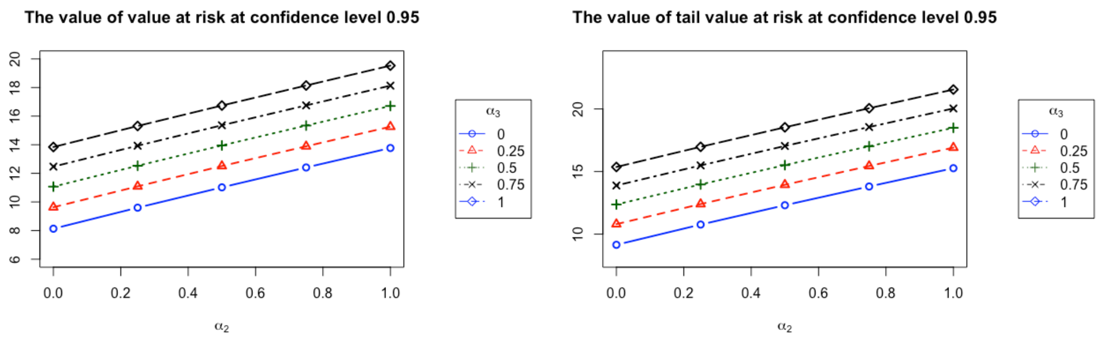

Figure 3.26: The trend of the value at risk and the tail value at risk when fixed $\alpha_1 = 0.25$ and either α_2 or α_3 increases at confidence level 0.95 of ZIPMA(3).

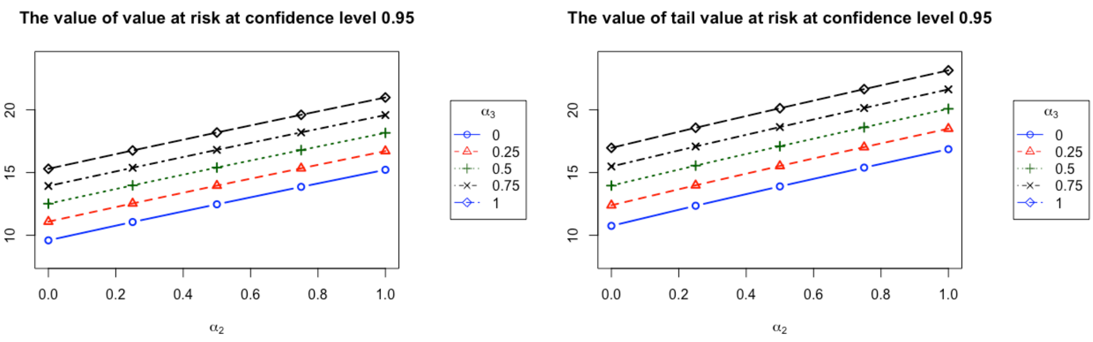

Figure 3.27: The trend of the value at risk and the tail value at risk when fixed $\alpha_1 = 0.50$ and either α_2 or α_3 increases at confidence level 0.95 of ZIPMA(3).

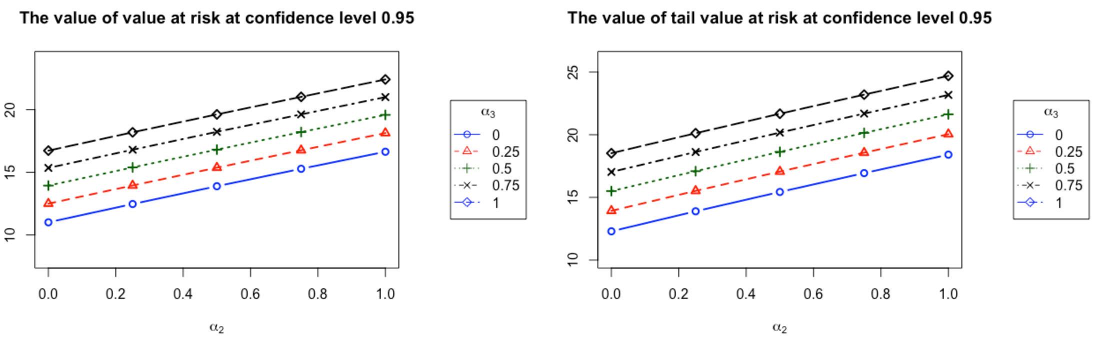

Figure 3.28: The trend of the value at risk and the tail value at risk when fixed $\alpha_1 = 0.75$ and either α_2 or α_3 increases at confidence level 0.95 of ZIPMA(3).

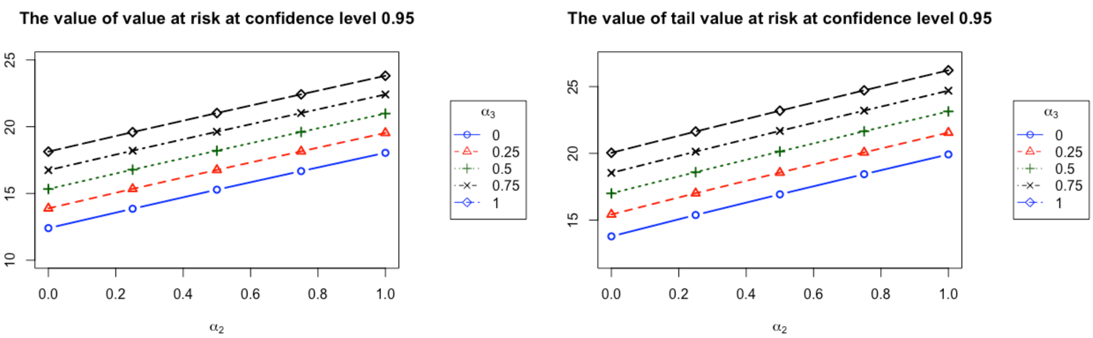

Figure 3.29: The trend of the value at risk and the tail value at risk when fixed $\alpha_1 = 1$ and either α_2 or α_3 increases at confidence level 0.95 of ZIPMA(3).

Table 3.7: The value of value at risk and tail value at risk at confidence level 0.90 of ZIPMA(3).

	α_2 \ α_3	0	0.25	0.5	0.75	1
$\alpha_1 = 0$	0	5.8900	7.2700	8.5900	9.8800	11.1600
		(6.81278)	(8.35703)	(9.82485)	(11.2452)	(12.6327)
	0.25	7.2800	8.6400	9.9600	11.2600	12.5300
		(8.38418)	(9.90613)	(11.3671)	(12.7875)	(14.1783)
	0.5	8.6200	9.9800	11.3000	12.5900	13.8700
		(9.87097)	(11.3868)	(12.8493)	(14.2745)	(15.672)
	0.75	9.9300	11.2800	12.6000	13.9100	15.1900
		(11.3061)	(12.8228)	(14.2902)	(15.7225)	(17.1278)
	1	11.2100	12.5600	13.8900	15.2000	16.4900
		(12.7057)	(14.2265)	(15.7009)	(17.1412)	(18.5552)
$\alpha_1 = 0.25$	0	7.3000	8.6500	9.9700	11.2600	12.5300
		(8.41112)	(9.92307)	(11.377)	(12.7919)	(14.1783)
	0.25	8.6700	10.0200	11.3300	12.6200	13.9000
		(9.95163)	(11.4522)	(12.9037)	(14.3204)	(15.711)
	0.5	10.0000	11.3500	12.6600	13.9600	15.2300
		(11.427)	(12.9256)	(14.3798)	(15.8017)	(17.1985)
	0.75	11.3100	12.6500	13.9700	15.2700	16.5500
		(12.859)	(14.36)	(15.8195)	(17.2481)	(18.6525)
	1	12.5900	13.9300	15.2600	16.5600	17.8500
		(14.2598)	(15.7656)	(17.2318)	(18.6679)	(20.0802)
$\alpha_1 = 0.5$	0	8.6500	10.0000	11.3100	12.6000	13.8700
		(9.91636)	(11.4162)	(12.8667)	(14.2823)	(15.672)
	0.25	10.0200	11.3600	12.6700	13.9600	15.2300
		(11.4463)	(12.9383)	(14.3875)	(15.8052)	(17.1985)

Continued

Table 3.7: (continued) The value of value at risk and tail value at risk at confidence level 0.90 of ZIPMA(3).

	α_2 \ α_3	0	0.25	0.5	0.75	1
	0.5	11.3500	12.6800	14.0000	15.2900	16.5700
		(12.9201)	(14.4112)	(15.8631)	(17.2854)	(18.6846)
	0.75	12.6500	13.9900	15.3100	16.6000	17.8900
		(14.3549)	(15.8485)	(17.3052)	(18.7336)	(20.1395)
	1	13.9300	15.2800	16.6000	17.9000	19.1900
		(15.7608)	(17.2587)	(18.7214)	(20.1566)	(21.5697)
$\alpha_1 = 0.75$	0	9.9700	11.3100	12.6200	13.9100	15.1900
		(11.3659)	(12.862)	(14.3138)	(15.7333)	(17.1278)
	0.25	11.3300	12.6700	13.9800	15.2700	16.5500
		(12.8936)	(14.3832)	(15.8336)	(17.2546)	(18.6525)
	0.5	12.6600	14.0000	15.3100	16.6100	17.8900
		(14.3704)	(15.8591)	(17.3116)	(18.7365)	(20.1395)
	0.75	13.9700	15.3000	16.6200	17.9200	19.2000
		(15.8107)	(17.3014)	(18.7581)	(20.1882)	(21.5969)
	1	15.2600	16.6000	17.9200	19.2200	20.5100
		(17.2233)	(18.7177)	(20.1796)	(21.6157)	(23.0309)
$\alpha_1 = 1$	0	11.2600	12.6000	13.9100	15.2100	16.4900
		(12.7775)	(14.2742)	(15.7297)	(17.1544)	(18.5552)
	0.25	12.6200	13.9600	15.2700	16.5700	17.8500
		(14.3072)	(15.7975)	(17.2512)	(18.6768)	(20.0802)
	0.5	13.9500	15.2900	16.6000	17.9000	19.1900
		(15.7892)	(17.2781)	(18.7334)	(20.1622)	(21.5697)
	0.75	15.2600	16.6000	17.9200	19.2200	20.5100
		(17.2363)	(18.7267)	(20.1852)	(21.6183)	(23.0309)

Continued

Table 3.7: (continued) The value of value at risk and tail value at risk at confidence level 0.90 of ZIPMA(3).

	α_2 \ α_3	0	0.25	0.5	0.75	1
	1	16.5600	17.9000	19.2200	20.5200	21.8100
		(18.6565)	(20.1498)	(21.6126)	(23.0511)	(24.4692)

Table 3.8: The value of value at risk and tail value at risk at confidence level 0.95 of ZIPMA(3).

	α_2 \ α_3	0	0.25	0.5	0.75	1
$\alpha_1 = 0$	0	6.5800	8.0800	9.5100	10.9000	12.2600
		(7.42384)	(9.08114)	(10.6428)	(12.1461)	(13.6092)
	0.25	8.1000	9.5900	11.0100	12.4000	13.7700
		(9.11413)	(10.7437)	(12.2971)	(13.8006)	(15.2679)
	0.5	9.5500	11.0300	12.4600	13.8500	15.2200
		(10.6986)	(12.3208)	(13.8763)	(15.3857)	(16.8615)
	0.75	10.9600	12.4300	13.8700	15.2700	16.6400
		(12.2194)	(13.8428)	(15.4045)	(16.9228)	(18.408)
	1	12.3300	13.8100	15.2500	16.6600	18.0400
		(13.6969)	(15.3255)	(16.8958)	(18.4239)	(19.9196)
$\alpha_1 = 0.25$	0	8.1300	9.6000	11.0200	12.4100	13.7700
		(9.14689)	(10.7641)	(12.309)	(13.8059)	(15.2679)
	0.25	9.6300	11.0900	12.5100	13.8900	15.2600
		(10.7987)	(12.4019)	(13.9439)	(15.4431)	(16.9103)
	0.5	11.0700	12.5300	13.9500	15.3400	16.7100
		(12.3693)	(13.9702)	(15.5158)	(17.0214)	(18.4962)
	0.75	12.4700	13.9300	15.3600	16.7500	18.1300
		(13.8866)	(15.4905)	(17.0426)	(18.5565)	(20.0407)

Continued

Table 3.8: (continued) The value of value at risk and tail value at risk at confidence level 0.95 of ZIPMA(3).

	α_2 \ α_3	0	0.25	0.5	0.75	1
	1	13.8400	15.3100	16.7400	18.1400	19.5300
		(15.3658)	(16.9755)	(18.536)	(20.059)	(21.5527)
α_1 = 0.5	0	9.6000	11.0600	12.4700	13.8600	15.2200
		(10.7533)	(12.356)	(13.897)	(15.3952)	(16.8615)
	0.25	11.0900	12.5400	13.9600	15.3400	16.7100
		(12.3924)	(13.9854)	(15.525)	(17.0255)	(18.4962)
	0.5	12.5200	13.9800	15.4000	16.7900	18.1600
		(13.9615)	(15.5533)	(17.0962)	(18.6025)	(20.0804)
	0.75	13.9300	15.3800	16.8100	18.2000	19.5800
		(15.4824)	(17.0773)	(18.6262)	(20.14)	(21.626)
	1	15.3000	16.7600	18.1900	19.6000	20.9800
		(16.9679)	(18.5681)	(20.1244)	(21.6464)	(23.1412)
α_1 = 0.75	0	11.0100	12.4700	13.8900	15.2800	16.6400
		(12.2913)	(13.8898)	(15.4327)	(16.9356)	(18.408)
	0.25	12.5000	13.9500	15.3700	16.7600	18.1300
		(13.9279)	(15.5181)	(17.0594)	(18.5642)	(20.0407)
	0.5	13.9400	15.3900	16.8100	18.2100	19.5800
		(15.5008)	(17.0899)	(18.6337)	(20.1433)	(21.626)
	0.75	15.3500	16.8000	18.2300	19.6200	21.0000
		(17.0286)	(18.6202)	(20.1691)	(21.6852)	(23.1748)
	1	16.7300	18.1900	19.6200	21.0200	22.4100
		(18.5224)	(20.1185)	(21.6738)	(23.1971)	(24.6946)
α_1 = 1	0	12.4000	13.8500	15.2800	16.6700	18.0400
		(13.7831)	(15.3824)	(16.93)	(18.4396)	(19.9196)

Continued

Table 3.8: (continued) The value of value at risk and tail value at risk at confidence level 0.95 of ZIPMA(3).

α_2 \ α_3	0	0.25	0.5	0.75	1
0.25	13.8800	15.3400	16.7600	18.1500	19.5300
	(15.4223)	(17.0135)	(18.5589)	(20.0695)	(21.5527)
0.5	15.3300	16.7800	18.2000	19.6000	20.9800
	(17.0016)	(18.5912)	(20.1386)	(21.6529)	(23.1412)
0.75	16.7400	18.2000	19.6200	21.0200	22.4100
	(18.5378)	(20.129)	(21.6804)	(23.2002)	(24.6946)
1	18.1300	19.5900	21.0200	22.4200	23.8100
	(20.0411)	(21.6358)	(23.1924)	(24.7187)	(26.2200)

Figures 3.20 - 3.29 and Table 3.7 - 3.8 show that the value of VaR_γ and TVaR_γ are increasing together with the increase of the values of $\alpha_1, \alpha_2, \alpha_3$ and confidence levels γ.

CHAPTER IV

DISCRETE TIME RISK MODEL BASED ON THE ZERO INFLATED POISSON AUTOREGRESSIVE

In this chapter, we give the definition of the discrete time surplus process as in Definition 4.1. In Section 4.1, we apply the another prospective model of time series, which is the autoregressive model. In this section, we provide details of the definition of the zero inflated Poisson autoregressive model in Definition 4.2, its probabilistic properties in Lemma 4.3, the adjustment coefficient in Theorem 4.5 and the proof of the unique positive solution in Lemma 4.6. Finally, the numerical experiments of the ruin probability and risk measurements are shown in Section 4.1.5.

Definition 4.1. Let R_n be the discrete time surplus process defined as

$$R_n = u + n\pi - \sum_{i=1}^{n} \sum_{j=1}^{N_i} C_{i,j}, \tag{4.1}$$

where

- u is the positive initial reserve of the business;
- π is the premium rate per period;
- the sequence $C_{i,j}$ is the sequence of claim sizes in period i and individuals j and the sequence is independent and identically distributed distribution with moment generating function, $m_C(\cdot)$;
- N_i is the claim number in period i.

We also denote that

- $N_{(n)} = \sum_{i=1}^{n} N_i$ is the aggregate claim number for n periods;
- $W_i = \sum_{j=1}^{N_i} C_{i,j}$ is the aggregate claim size for period i;
- $S_n = \sum_{i=1}^{n} W_i$ is the net loss process.

4.1 Discrete time risk model based on first order zero inflated Poisson autoregressive

. In this section, we provide the definition of zero inflated Poisson autoregressive (ZIPAR) model and derive its probabilistic properties. We consider the discrete time surplus defined in Definition 4.1, when the claim counts, $\{N_i,\ i \in \mathbb{N}\}$, are modelled by the zero inflated Poisson first order autoregressive denoted by ZIPAR(1). The definition of ZIPAR(1) and its probabilistic properties are provided in Definitions 4.2 and Lemma 4.3, respectively. In Section 4.1.1, we derive the adjustment coefficient function and the approximation to the ruin probability of the ZIPAR(1) risk model. We consider a special case of the adjustment coefficient function when the claim sizes are exponentially distributed. In Section 4.1.2, we derive an approximation to the value at risk (VaR) of the ZIPAR(1) net loss process.

The concepts of zero inflated Poisson first order autoregressive model is quite different from zero inflated Poisson moving average model. In the model of zero inflated Poisson autoregressive, we consider the number of claim where N_{i-1} is the number of claim in period $i-1$ and α is the reclaim probability. Thus, $\alpha \circ N_{i-1}$ is the number of insured in period $i-1$ will reclaim in period i with a probability α and ϵ_i is the number of new insured in period i. Hence, the number of insured in period i, N_i, is based on the summation of the number of new claims in period i and the number of reclaims from period $i-1$ when the reclaim probability is α.

The definition of the zero inflated Poisson first order autoregressive (ZIPAR(1)) is presented as follows.

Definition 4.2. The zero inflated Poisson first order autoregressive, $N = \{N_i, i \in \mathbb{N}\}$ is defined as

$$N_i \;=\; \alpha \circ N_{i-1} + \epsilon_i, \quad \text{for} \quad i \;=\; 1, 2, \ldots, \tag{4.2}$$

where N_1 follows the zero inflated Poisson with parameters p and λ, $\alpha\circ$ is the thining operator and $\{\epsilon_i, \in \mathbb{N}\}$ is a sequence of i.i.d. random variables.
We assume the probability generating function of $\{\epsilon_i\,,\, i \in \mathbb{N}\}$ is defined as

$$G_{\epsilon_i}(z) \;=\; \frac{p + (1-p)e^{-\lambda(1-z)}}{p + (1-p)e^{-\lambda\alpha(1-z)}},$$

where $p, \lambda > 0$ and $\alpha \in (0, 1)$ and $\{\epsilon_i,\, i \in \mathbb{N}\}$ is independent of N_i for every i.
The $\alpha\circ$ thining operator is defined as follows.

$$\alpha \circ N_{i-1} \;=\; \sum_{j=1}^{N_{i-1}} \delta_{(i-1)1j}.$$

Following Joe (1997), the dependence structure of the ZIPAR(1) model can be expressed as follows. First, note that for Z_i and Y_i follow the Bernoulli with parameters α_1 and α_2, respectively. Then,

$$Y_i Z_i \;=\; \begin{cases} 1 \;\text{ if } Y_i \;=\; 1, Z_i \;=\; 1 \;\text{ with probability } \alpha_1\alpha_2 \\ 0 \;\text{ otherwise, with probability } 1 - \alpha_1\alpha_2. \end{cases}$$

Hence $Y_i Z_i$ can be considered as a Bernoulli $(\alpha_1\alpha_2)$ random variable. Then,

$$(\alpha_1\alpha_2) \circ N \;\stackrel{d}{=}\; \sum_{i=1}^{N} X_i,$$

where N is the count random variable and X_i is the Bernoulli random variable with parameter $\alpha_1\alpha_2$. Furthermore,

$$\begin{aligned}
\alpha_1 \circ (\alpha_2 \circ N) &\stackrel{d}{=} \alpha_1 \circ \sum_{i=1}^{N} Y_i \\
&= \sum_{i=1}^{N} \alpha_1 \circ Y_i \\
&= \sum_{i=1}^{N} \sum_{j=1}^{Y_i} Z_j \\
&\stackrel{d}{=} \sum_{i=1}^{N} Y_i Z_j. \qquad (4.3)
\end{aligned}$$

where the last equation is obtained from

$$\begin{aligned}
\sum_{i=1}^{Y_i} Z_i &= \begin{cases} 1 & \text{if } Y_i = 1, Z_i = 1 \text{ with probability } \alpha_1\alpha_2 \\ 0 & \text{otherwise, with probability } 1 - \alpha_1\alpha_2 \end{cases} \\
&= Z_i Y_i.
\end{aligned}$$

Therefore, we can conclude that

$$(\alpha_1\alpha_2) \circ N \stackrel{d}{=} \alpha_1 \circ (\alpha_2 \circ N).$$

Consequently, the expressions of $N_2, N_3, \ldots$ are defined as

$$\begin{aligned}
N_2 &= \sum_{i=1}^{N_1} \delta_{21i} + \epsilon_2, \\
N_3 &= \sum_{i=1}^{N_1} \delta_{21i}\delta_{31i} + \sum_{i=1}^{\epsilon_2} \delta_{32i} + \epsilon_3, \\
&\vdots \\
N_n &= \sum_{i=1}^{N_1} \delta_{21i}\delta_{31i}\cdots\delta_{ni} + \sum_{j=2}^{n-1} \sum_{i=1}^{\epsilon_j} \prod_{k=j+1}^{n} \delta_{kji} + \epsilon_n.
\end{aligned}$$

The random variables $\{\delta_{21j}, \delta_{31j}, \delta_{32j}, \ldots, \delta_{n1j}, \delta_{n2j}, \ldots, \delta_{n(n-1)j}\ ,\ j = 1, 2, \ldots\}$ are i.i.d. Bernoulli random variables with mean α. Furthermore, we give details about the construction in the dependence structure by understanding the multiplication of the $\alpha\circ$ thining operator. Having defined the ZIPAR(1) process, its probabilistic properties can be obtained as in Lemma 4.3 below

Lemma 4.3. Let $\{N_i, i \in \mathbb{N}\}$ be the ZIPAR(1) model defined in Definition 4.2, then $\{N_i, i \in \mathbb{N}\}$ has the following. properties.

(a) The sequence $\{N_i\,,\ i\ \in\ \mathbb{N}\}$ is a stationary process with the probability generating function of N_i, $G_{N_i}(z) = p + (1-p)e^{-\lambda(1-z)}$ for $i \in \mathbb{N}$.

(b) The expectation of N_i is $\mathrm{E}(N_i) = \lambda(1-p)$ for $i \in \mathbb{N}$.

(c) The variance of N_i is $\mathrm{Var}(N_i) = \lambda(1-p)(1+\lambda p)$ for $i \in \mathbb{N}$.

(d) The covariance function between N_i and N_{i-m},

$$\mathrm{Cov}(N_i, N_{i-m}) = \alpha^m \lambda(1-p)(1+\lambda p),$$

for $m \in \mathbb{N}$.

(e) The correlation function between N_i and N_{i-m},

$$\mathrm{Corr}(N_i, N_{i-m}) = \alpha^m,$$

for $m \in \mathbb{N}$.

Proof. To prove (a), we consider the probability generating function of $\{N_i\,,\ i \in \mathbb{N}\}$, let $\{N_i\,,\ i\ \in\ \mathbb{N}\}$ and $\{\epsilon_i,\ i\ \in\ \mathbb{N}\}$ be the processes defined in Definition 4.2 and use the fact that N_i and ϵ_i are independent, then we obtain

$$\begin{aligned}
G_{N_i}(z) &= \mathrm{E}(z^{N_i}) \\
&= \mathrm{E}(z^{\alpha\circ N_{i-1}+\epsilon_i}) \\
&= \mathrm{E}(z^{\epsilon_i})\mathrm{E}(z^{\alpha\circ N_{i-1}}) \\
&= G_{\epsilon_i}(z)G_{N_{i-1}}(1-\alpha+\alpha z) \\
&= \left(\frac{p+(1-p)e^{-\lambda(1-z)}}{p+(1-p)e^{-\lambda\alpha(1-z)}}\right)\left(p+(1-p)e^{-\lambda(1-(1-\alpha+\alpha z))}\right) \\
&= p+(1-p)e^{-\lambda(1-z)},
\end{aligned}$$

for $z \in \mathbb{R}$. Since $G_{N_i}(\cdot)$ does not depend on i then $G_{N_1}(\cdot) = G_{N_2}(\cdot) = \ldots = G_{N_i}(\cdot)$. Therefore, $\{N_i\,,\, i \in \mathbb{N}\}$ is a stationary process. In addition, the probability generating function of $\{N_i\,,\, i \in \mathbb{N}\}$ is given by

$$G_{N_i}(z) = \left(p+(1-p)e^{-\lambda(1-z)}\right),$$

for all $i \in \mathbb{N}$.

(b) Since $G_{N_i}(z) = \mathrm{E}(z^{N_i})$ for all $i \in \mathbb{N}$, we can use the p.g.f. $G_{N_i}(z)$ obtained in (a) and the properties of the probability generating function to find $\mathrm{E}(N_i)$ as follows.

$$\begin{aligned}
\mathrm{E}(N_i) &= \frac{d}{dz}G_{N_i}(z)\bigg|_{z=1} \\
&= \left((1-p)e^{-\lambda(1-z)}\lambda\right)\bigg|_{z=1} \\
&= \lambda(1-p).
\end{aligned}$$

(c) To obtain the variance of N_i, we first compute the second moment $\mathrm{E}(N_i^2)$ by applying the properties of the probability generating function as the following.

$$\mathrm{E}(N_i^2) = \frac{d^2}{dz^2}G_{N_i}(z)\bigg|_{z=1} + \frac{d}{dz}G_{N_i}(z)\bigg|_{z=1}.$$

Note that,

$$\frac{d^2}{dz^2}G_{N_i}(z)\bigg|_{z=1} = \left((1-p)e^{-\lambda(1-z)}\lambda\right)(\lambda)\bigg|_{z=1} = (1-p)\lambda^2.$$

Thus,

$$\begin{aligned}\mathrm{E}(N_i^2) &= \frac{d^2}{dz^2}G_{N_i}(z)\bigg|_{z=1} + \frac{d}{dz}G_{N_i}(z)\bigg|_{z=1}\\ &= (1-p)\lambda^2 + (1-p)\lambda.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathrm{Var}(N_i) &= \mathrm{E}(N_i^2) - \mathrm{E}^2(N_i)\\ &= (1-p)\lambda^2 + (1-p)\lambda - ((1-p)\lambda)^2\\ &= (1-p)\lambda(\lambda + 1 - (1-p)\lambda)\\ &= \lambda(1-p)(1+\lambda p).\end{aligned}$$

(d) To obtain the formula for the covariance function by applying the independence of ϵ_i and N_i and use Lemma 2.29, for $i = 1, 2, \ldots$.

For $m = 1$, using Lemma 2.31

$$\begin{aligned}\mathrm{Cov}(N_i, N_{i-1}) &= \mathrm{Cov}(\alpha \circ N_{i-1} + \epsilon_i, N_{i-1})\\ &= \mathrm{Cov}(\alpha \circ N_{i-1}, N_{i-1}) + \mathrm{Cov}(\epsilon_i, N_{i-1}) \quad (4.4)\\ &= \mathrm{Cov}(\alpha \circ N_{i-1}, N_{i-1})\\ &= \alpha \mathrm{Var}(N_{i-1})\\ &= \alpha\lambda(1-p)(1+\lambda p), \quad (4.5)\end{aligned}$$

where we use the property that ϵ_i and N_{i-1} are independent to obtain (4.4), and use (c) to obtain (4.5).

For $m > 1$, by using the independence between ϵ_i and N_{i-m} for all i and $m > 0$, we obtain

$$\begin{aligned}
\text{Cov}(N_i, N_{i-m}) &= \text{Cov}(\alpha \circ N_{i-1} + \epsilon_i, N_{i-m}) \\
&= \text{Cov}(\alpha \circ N_{i-1}, N_{i-m}) \\
&= \text{Cov}(\alpha^2 \circ N_{i-2} + \alpha \circ \epsilon_{i-1}, N_{i-m}) \\
&= \text{Cov}(\alpha^2 \circ N_{i-2}, N_{i-m}) \\
&\vdots \\
&= \text{Cov}(\alpha^m \circ N_{i-m}, N_{i-m}) \\
&= \alpha^m \text{Var}(N_{i-m}) \\
&= \alpha^m \lambda(1-p)(1+\lambda p), \qquad (4.6)
\end{aligned}$$

where we apply (c) to obtain (4.6).

(e) From (c), we know that $\text{Var}(N_i)$ does not depend on i and the result from (d), then for $m \in \mathbb{N}$,

$$\begin{aligned}
\text{Corr}(N_i, N_{i-m}) &= \frac{\text{Cov}(N_i, N_{i-m})}{\sqrt{\text{Var}(N_i)\text{Var}(N_{i-m})}} \\
&= \frac{\text{Cov}(N_i, N_{i-m})}{\text{Var}(N_i)} \\
&= \frac{\alpha^m \text{Var}(N_{i-m})}{\text{Var}(N_i)} \\
&= \alpha^m.
\end{aligned}$$

□

4.1.1 Adjustment coefficient function of ZIPAR(1)

In this section, we derive the adjustment coefficient function of the zero inflated Poisson AR(1) by applying the method from Section 3.1 to obtain the Lundberg adjustment coefficient. Then, we provide a proof of the unique positive solution of zero root of the adjustment coefficient. The risk model based on ZIPAR(1) can be expressed as follows.

Definition 4.4. The risk model based on ZIPAR(1) can be expressed as

$$R_n = u + n\pi - \sum_{i=1}^{n}\sum_{j=1}^{N_i} C_{i,j},$$

where u is the positive initial reserve, π is the premium rate per period, N_i is modelled by zero inflated Poisson first order autoregressive (ZIPAR(1)) and $\{C_{i,j}\}$ is the sequence of independent and identically distributed random variables representing claim sizes in period i and individuals j.

Theorem 4.5. Let R_n be the discrete time surplus process defined in Definition 4.4. Under the condition that $\alpha m_C(z) < 1$, the adjustment coefficient function $c(\cdot)$ is defined as

$$c(z) = \log\left(\frac{p + (1-p)e^{-\lambda\left(1-\frac{\bar{\alpha} m_C(z)}{1-\alpha m_C(z)}\right)}}{p + (1-p)e^{-\lambda\alpha\left(1-\frac{\bar{\alpha} m_C(z)}{1-\alpha m_C(z)}\right)}}\right) - \pi z, \tag{4.7}$$

for $z \in \mathbb{R}^+$ and $\bar{\alpha} = 1 - \alpha$.

Proof. From (3.10), we have that

$$\begin{aligned} c(z) &= \lim_{n\to\infty} \frac{1}{n} \log m_{S_n}(z) - \pi z \\ &= \lim_{n\to\infty} \frac{1}{n} \log\left(G_{N(n)}(m_C(z))\right) - \pi z. \end{aligned}$$

Then to obtain the adjustment coefficient function, we will first obtain $G_{N(n)}(m_C(z))$ as the following. Since $\{\epsilon_i\,,\, i \in \mathbb{N}\}$ is independent and identically distributed and independent of N_1, we obtain

$$\begin{aligned} G_{N(n)}(z) &= \mathrm{E}(z^{N_1+N_2+\cdots+N_n}) \\ &= \mathrm{E}\left(z^{N_1+\alpha\circ N_1+\epsilon_2+\alpha\circ N_2+\epsilon_3+\cdots+\alpha\circ N_{n-1}+\epsilon_n}\right) \\ &= \mathrm{E}\left(z^{N_1+\alpha\circ N_1+\epsilon_2+\alpha^2\circ N_1+\alpha\circ\epsilon_2+\epsilon_3+\cdots+\alpha^{n-1}\circ N_1+\cdots+\alpha\circ\epsilon_{n-1}+\epsilon_n}\right) \\ &= \mathrm{E}\left(z^{N_1+\alpha\circ N_1+\cdots+\alpha^{n-1}\circ N_1}\right)\times \mathrm{E}\left(z^{\epsilon_2+\alpha\circ\epsilon_2+\cdots+\alpha^{n-2}\circ\epsilon_2}\right) \\ &\quad \times\cdots\times \mathrm{E}\left(z^{\epsilon_{n-1}+\alpha\circ\epsilon_{n-1}}\right)\times \mathrm{E}\left(z^{\epsilon_n}\right). \end{aligned} \tag{4.8}$$

We obtain the last term of (4.8) as we apply the p.g.f. of $\{\epsilon_i\,,\, i \in \mathbb{N}\}$ from Definition 4.2 as follows.

$$\mathrm{E}(z^{\epsilon_n}) = \frac{p+(1-p)e^{-\lambda(1-z)}}{p+(1-p)e^{-\lambda\alpha(1-z)}}, \quad \text{for } z \in \mathbb{R}^+. \tag{4.9}$$

We need to find the expression of $\mathrm{E}\left(z^{\sum_{i=0}^{n-1}\alpha^i\circ N_1}\right)$, we then apply $\{\delta_{ijk}\}_{i,j,k=1,2,\ldots}$ in Definition 4.2 and the p.g.f. of $\{\delta_{ijk}\}$ which is the sequence of i.i.d. Bernoulli random variables, $G_{\delta_{ijk}}(z) = \mathrm{E}(z^{\delta_{ijk}}) = \bar{\alpha}+\alpha z$ and N_1 follows the zero inflated Poisson with parameters p and λ to provide the development from periods $n = 1, 2, 3, 4$.

For $n = 1$, we obtain

$$\begin{aligned} \mathrm{E}\left(z^{\sum_{i=0}^{0}\alpha^i\circ N_1}\right) &= \mathrm{E}\left(z^{N_1}\right) \\ &= p+(1-p)e^{-\lambda(1-z)}. \end{aligned}$$

For $n = 2$, we obtain

$$\begin{aligned}
\mathrm{E}\left(z^{\sum_{i=0}^{1}\alpha^i\circ N_1}\right) &= \mathrm{E}\left(z^{N_1}z^{\alpha\circ N_1}\right)\\
&= \mathrm{E}\left(z^{N_1}\mathrm{E}\left(z^{\sum_{i=1}^{N_1}\delta_{21i}}\middle|N_1\right)\right)\\
&= \mathrm{E}\left(z^{N_1}\prod_{i=1}^{N_1}\mathrm{E}\left(z^{\delta_{21i}}\right)\right)\\
&= \mathrm{E}\left(z^{N_1}(\bar{\alpha}+\alpha z)^{N_1}\right)\\
&= G_{N_1}(z(\bar{\alpha}+\alpha z))\\
&= p+(1-p)e^{-\lambda(1-z(\bar{\alpha}+\alpha z))}.
\end{aligned}$$

For $n = 3$, we have

$$\begin{aligned}
\mathrm{E}\left(z^{\sum_{i=0}^{2}\alpha^i\circ N_1}\right) &= \mathrm{E}\left(z^{N_1}z^{\alpha\circ N_1}z^{\alpha^2\circ N_1}\right)\\
&= \mathrm{E}\left(z^{N_1}\mathrm{E}\left(z^{\sum_{i=1}^{N_1}\delta_{21i}}\mathrm{E}\left(z^{\sum_{i=1}^{N_1}\delta_{21i}\delta_{31i}}\middle|N_1,\delta_{21i}\right)\middle|N_1\right)\right)\\
&= \mathrm{E}\left(z^{N_1}\mathrm{E}\left(\prod_{i=1}^{N_1}z^{\delta_{21i}}\mathrm{E}\left(\left(z^{\delta_{21i}}\right)^{\delta_{31i}}\right)\middle|N_1\right)\right)\\
&= \mathrm{E}\left(z^{N_1}\mathrm{E}\left(\prod_{i=1}^{N_1}z^{\delta_{21i}}\left(\bar{\alpha}+\alpha z^{\delta_{21i}}\right)\middle|N_1\right)\right)\\
&= \mathrm{E}\left(z^{N_1}\mathrm{E}\left(\prod_{i=1}^{N_1}(\bar{\alpha}z^{\delta_{21i}}+\alpha z^{2\delta_{21i}})\middle|N_1\right)\right)\\
&= \mathrm{E}\left(z^{N_1}\prod_{i=1}^{N_1}\left(\bar{\alpha}(\bar{\alpha}+\alpha z)+\alpha(\bar{\alpha}+\alpha z^2)\right)\right)\\
&= \mathrm{E}\left(z^{N_1}\left(\bar{\alpha}(\bar{\alpha}+\alpha z)+\alpha(\bar{\alpha}+\alpha z^2)\right)^{N_1}\right)\\
&= G_{N_1}\left(z\left(\bar{\alpha}(\bar{\alpha}+\alpha z)+\alpha(\bar{\alpha}+\alpha z^2)\right)\right)\\
&= G_{N_1}\left(\bar{\alpha}z+\alpha\bar{\alpha}z^2+\alpha^2z^3\right)\\
&= p+(1-p)e^{-\lambda(1-(\bar{\alpha}z+\alpha\bar{\alpha}z^2+\alpha^2z^3))}.
\end{aligned}$$

For $n = 4$, we have

$$
\begin{aligned}
\mathrm{E}\left(z^{\sum_{i=0}^{3}\alpha^i\circ N_1}\right) &= \mathrm{E}\left(z^{N_1}z^{\alpha\circ N_1}z^{\alpha^2\circ N_1}z^{\alpha^3\circ N_1}\right)\\
&= \mathrm{E}\Bigg(z^{N_1}\mathrm{E}\Bigg(z^{\sum_{i=1}^{N_1}\delta_{21i}}\mathrm{E}\Bigg(z^{\sum_{i=1}^{N_1}\delta_{21i}\delta_{31i}}\times\\
&\qquad \mathrm{E}\left(z^{\sum_{i=1}^{N_1}\delta_{21i}\delta_{31i}\delta_{41i}}\Big|N_1,\delta_{21i},\delta_{31i}\right)\Big|N_1,\delta_{21i}\Bigg)\Big|N_1\Bigg)\Bigg)\\
&= \mathrm{E}\left(z^{N_1}\mathrm{E}\left(z^{\sum_{i=1}^{N_1}\delta_{21i}}\mathrm{E}\left(z^{\sum_{i=1}^{N_1}\delta_{21i}\delta_{31i}}\prod_{i=1}^{N_1}(\bar{\alpha}+\alpha z^{\delta_{21i}\delta_{31i}})\Big|N_1,\delta_{21i}\right)\Big|N_1\right)\right)\\
&= \mathrm{E}\left(z^{N_1}\mathrm{E}\left(z^{\sum_{i=1}^{N_1}\delta_{21i}}\mathrm{E}\left(\prod_{i=1}^{N_1}(\bar{\alpha}z^{\delta_{21i}\delta_{31i}}+\alpha z^{2\delta_{21i}\delta_{31i}})\Big|N_1,\delta_{21i}\right)\Big|N_1\right)\right)\\
&= \mathrm{E}\left(z^{N_1}\mathrm{E}\left(z^{\sum_{i=1}^{N_1}\delta_{21i}}\prod_{i=1}^{N_1}\left(\bar{\alpha}(\bar{\alpha}+\alpha z^{\delta_{21i}})+\alpha(\bar{\alpha}+\alpha z^{2\delta_{21i}})\right)\Big|N_1\right)\right)\\
&= \mathrm{E}\left(z^{N_1}\mathrm{E}\left(\prod_{i=1}^{N_1}\left(\bar{\alpha}^2z^{\delta_{21i}}+\bar{\alpha}\alpha z^{2\delta_{21i}}+\bar{\alpha}\alpha z^{\delta_{21i}}+\alpha^2z^{3\delta_{21i}}\right)\Big|N_1\right)\right)\\
&= \mathrm{E}\left(z^{N_1}\mathrm{E}\left(\prod_{i=1}^{N_1}\left(\bar{\alpha}z^{\delta_{21i}}+\bar{\alpha}\alpha z^{2\delta_{21i}}+\alpha^2z^{3\delta_{21i}}\right)\Big|N_1\right)\right)\\
&= \mathrm{E}\left(z^{N_1}\prod_{i=1}^{N_1}\left(\bar{\alpha}(\bar{\alpha}+\alpha z)+\bar{\alpha}\alpha(\bar{\alpha}+\alpha z^2)+\alpha^2(\bar{\alpha}+\alpha z^3)\right)\right)\\
&= \mathrm{E}\left(\prod_{i=1}^{N_1}\left(\bar{\alpha}z+\bar{\alpha}\alpha z^2+\bar{\alpha}\alpha^2z^3+\alpha^3z^4\right)\right)\\
&= \mathrm{E}\left(\left(\bar{\alpha}z+\bar{\alpha}\alpha z^2+\bar{\alpha}\alpha^2z^3+\alpha^3z^4\right)^{N_1}\right)\\
&= G_{N_1}\left(\bar{\alpha}z+\bar{\alpha}\alpha z^2+\bar{\alpha}\alpha^2z^3+\alpha^3z^4\right)\\
&= p+(1-p)e^{-\lambda\left(1-\left(\bar{\alpha}z+\bar{\alpha}\alpha z^2+\bar{\alpha}\alpha^2z^3+\alpha^3z^4\right)\right)}.
\end{aligned}
$$

Consequently, we deduce the following general form for case n. Since $\{\delta_{ijk}\}_{i,j,k=1,2,...}$ in Definition 4.2 and the p.g.f. of $\{\delta_{ijk}\}$ which is the sequence of i.i.d. Bernoulli random variables and N_1 follows the zero inflated Poisson with parameters p and λ, then we have

$$\begin{aligned}
\mathrm{E}\left(z^{\sum_{i=0}^{n-1}\alpha^i\circ N_1}\right) &= \mathrm{E}\left(z^{N_1}z^{\alpha\circ N_1}z^{\alpha^2\circ N_1}z^{\alpha^3\circ N_1}\cdots z^{\alpha^{n-1}\circ N_1}\right)\\
&= \mathrm{E}\left(z^{N_1}z^{\sum_{i=1}^{N_1}\delta_{21i}}\cdots z^{\sum_{i=1}^{N_1}\delta_{21i}\cdots\delta_{n1i}}\right)\\
&= \mathrm{E}\Bigg(z^{N_1}\mathrm{E}\Bigg(z^{\sum_{i=1}^{N_1}\delta_{21i}}\cdots\mathrm{E}\Bigg(z^{\sum_{i=1}^{N_1}\delta_{21i}\cdots\delta_{(n-1)1i}}\\
&\quad\mathrm{E}\left(z^{\sum_{i=1}^{N_1}\delta_{21i}\cdots\delta_{(n)1i}}\middle|N_1,\delta_{21i},\ldots,\delta_{(n-1)1i}\right)\cdots\Big|N_1\Bigg)\Bigg)\\
&= \mathrm{E}\Bigg(z^{N_1}\mathrm{E}\Bigg(z^{\sum_{i=1}^{N_1}\delta_{21i}}\cdots\mathrm{E}\Bigg(z^{\sum_{i=1}^{N_1}\delta_{21i}\cdots\delta_{(n-1)1i}}\\
&\quad\prod_{i=1}^{N_1}\left(\bar{\alpha}+\alpha z^{\delta_{21i}\cdots\delta_{(n-1)1i}}\right)\Big|N_1,\delta_{21i},\ldots,\delta_{(n-2)1i}\Bigg)\cdots\Big|N_1\Bigg)\Bigg)\\
&\quad\vdots\\
&= \mathrm{E}\left(z^{N_1}\prod_{i=1}^{N_1}\left(\bar{\alpha}+\bar{\alpha}\alpha z+\bar{\alpha}\alpha^2z^2+\cdots+\alpha^{n-1}z^{n-1}\right)\right)\\
&= \mathrm{E}\left(\left(\bar{\alpha}z+\bar{\alpha}\alpha z^2+\bar{\alpha}\alpha^2z^3+\cdots+\alpha^{n-1}z^n\right)^{N_1}\right)\\
&= \mathrm{E}\left(\left(\bar{\alpha}\left(\sum_{i=0}^{n-2}\alpha^iz^{i+1}\right)+\alpha^{n-1}z^n\right)^{N_1}\right)\\
&= G_{N_1}\left(\left(\bar{\alpha}\left(\sum_{i=0}^{n-2}\alpha^iz^{i+1}\right)+\alpha^{n-1}z^n\right)\right)\\
&= p+(1-p)e^{-\lambda\left(1-\left(\bar{\alpha}\left(\sum_{i=0}^{n-2}\alpha^iz^{i+1}\right)+\alpha^{n-1}z^n\right)\right)}. \qquad (4.10)
\end{aligned}$$

Thus, we obtain the first term and the last term of (4.8), then we will find the rest by applying the development of expression (4.10). We use the fact that the p.g.f. of $\{\epsilon_i\}$ follows Definition 4.2 and $\{\delta_{ijk}\}$ is the sequence of i.i.d. Bernoulli random variables defined in Definition 4.2. Let first consider

$$
\begin{aligned}
\mathrm{E}\left(z^{\sum_{i=0}^{n-2}\alpha^i\circ\epsilon_2}\right) &= \mathrm{E}\left(z^{\epsilon_2+\alpha\circ\epsilon_2+\cdots+\alpha^{n-2}\circ\epsilon_2}\right) \\
&= \mathrm{E}\left(z^{\epsilon_2}z^{\alpha\circ\epsilon_2}\cdots z^{\alpha^{n-2}\circ\epsilon_2}\right) \\
&= \mathrm{E}\left(z^{\epsilon_2}z^{\sum_{i=1}^{\epsilon_2}\delta_{22i}}\cdots z^{\sum_{i=1}^{\epsilon_2}\delta_{22i}\delta_{32i}\cdots\delta_{(n-1)2i}}\right) \\
&= \mathrm{E}\Big(z^{\epsilon_2}\mathrm{E}\Big(z^{\sum_{i=1}^{\epsilon_2}\delta_{22i}}\cdots \mathrm{E}\Big(z^{\sum_{i=1}^{\epsilon_2}\delta_{22i}\delta_{32i}\cdots\delta_{(n-1)2i}} \\
&\qquad \mathrm{E}\left(z^{\sum_{i=1}^{\epsilon_2}\delta_{22i}\delta_{32i}\cdots\delta_{(n-1)2i}}\Big|\epsilon_2,\delta_{22i},\ldots,\delta_{(n-2)2i}\right)\cdots\Big|\epsilon_2\Big)\Big) \\
&\quad \vdots \\
&= G_{\epsilon_2}\left(\bar{\alpha}\left(\sum_{i=0}^{n-3}\alpha^i z^{i+1}\right)+\alpha^{n-2}z^{n-1}\right) \\
&= \frac{p+(1-p)e^{-\lambda\left(1-\left(\bar{\alpha}\left(\sum_{i=0}^{n-3}\alpha^i z^{i+1}\right)+\alpha^{n-2}z^{n-1}\right)\right)}}{p+(1-p)e^{-\lambda\alpha\left(1-\left(\bar{\alpha}\left(\sum_{i=0}^{n-3}\alpha^i z^{i+1}\right)+\alpha^{n-2}z^{n-1}\right)\right)}}. \qquad (4.11)
\end{aligned}
$$

As a consequence, we also obtain terms of $\epsilon_3,\epsilon_4,\ldots,\epsilon_{n-1}$ by applying the technique in (4.11), then we obtain the general form for each ϵ_j for $j = 2,\ldots,n-1$ in (4.8) as follows.

$$
\mathrm{E}\left(z^{\sum_{i=0}^{n-j}\alpha^i\circ\epsilon_j}\right) = \frac{p+(1-p)e^{-\lambda\left(1-\left(\bar{\alpha}\left(\sum_{i=0}^{n-(j+1)}\alpha^i z^{i+1}\right)+\alpha^{n-j}z^{n-(j-1)}\right)\right)}}{p+(1-p)e^{-\lambda\alpha\left(1-\left(\bar{\alpha}\left(\sum_{i=0}^{n-(j+1)}\alpha^i z^{i+1}\right)+\alpha^{n-j}z^{n-(j-1)}\right)\right)}}. \qquad (4.12)
$$

Substituting (4.9)-(4.11) and (4.12) for $j = 2,3,\ldots,n-1$ into (4.8), then we obtain

$$
\begin{aligned}
G_{N(n)}(z) &= \left(p+(1-p)e^{-\lambda\left(1-\left(\bar{\alpha}\left(\sum_{i=0}^{n-2}\alpha^i z^{i+1}\right)+\alpha^{n-1}z^n\right)\right)}\right) \\
&\quad \times\prod_{j=2}^{n-1}\left(\frac{p+(1-p)e^{-\lambda\left(1-\left(\bar{\alpha}\left(\sum_{i=0}^{n-(j+1)}\alpha^i z^{i+1}\right)+\alpha^{n-j}z^{n-(j-1)}\right)\right)}}{p+(1-p)e^{-\lambda\alpha\left(1-\left(\bar{\alpha}\left(\sum_{i=0}^{n-(j+1)}\alpha^i z^{i+1}\right)+\alpha^{n-j}z^{n-(j-1)}\right)\right)}}\right) \\
&\quad \times\left(\frac{p+(1-p)e^{-\lambda(1-z)}}{p+(1-p)e^{-\lambda\alpha(1-z)}}\right). \qquad (4.13)
\end{aligned}
$$

Therefore, the moment generating function of S_n is defined as (4.13) by replacing $z = m_C(z)$ as

$$\begin{aligned} m_{S_n}(z) &= \left(p+(1-p)e^{-\lambda\left(1-\left(\bar{\alpha}\left(\sum_{i=0}^{n-2}\alpha^i m_C(z)^{i+1}\right)+\alpha^{n-1}m_C(z)^n\right)\right)}\right) \\ &\times \prod_{j=2}^{n-1}\left(\frac{p+(1-p)e^{-\lambda\left(1-\left(\bar{\alpha}\left(\sum_{i=0}^{n-(j+1)}\alpha^i m_C(z)^{i+1}\right)+\alpha^{n-j}m_C(z)^{n-(j-1)}\right)\right)}}{p+(1-p)e^{-\lambda\alpha\left(1-\left(\bar{\alpha}\left(\sum_{i=0}^{n-(j+1)}\alpha^i m_C(z)^{i+1}\right)+\alpha^{n-j}m_C(z)^{n-(j-1)}\right)\right)}}\right) \\ &\times \left(\frac{p+(1-p)e^{-\lambda(1-m_C(z))}}{p+(1-p)e^{-\lambda\alpha(1-m_C(z))}}\right). \end{aligned} \tag{4.14}$$

From (4.14), we obtain the adjustment coefficient function $c_n(z)$ as follows.

$$\begin{aligned} c_n(z) &= \log\Bigg(\left(p+(1-p)e^{-\lambda\left(1-\left(\bar{\alpha}\left(\sum_{i=0}^{n-2}\alpha^i m_C(z)^{i+1}\right)+\alpha^{n-1}m_C(z)^n\right)\right)}\right) \\ &\times \prod_{j=2}^{n-1}\left(\frac{p+(1-p)e^{-\lambda\left(1-\left(\bar{\alpha}\left(\sum_{i=0}^{n-(j+1)}\alpha^i m_C(z)^{i+1}\right)+\alpha^{n-j}m_C(z)^{n-(j-1)}\right)\right)}}{p+(1-p)e^{-\lambda\alpha\left(1-\left(\bar{\alpha}\left(\sum_{i=0}^{n-(j+1)}\alpha^i m_C(z)^{i+1}\right)+\alpha^{n-j}m_C(z)^{n-(j-1)}\right)\right)}}\right) \\ &\times \left(\frac{p+(1-p)e^{-\lambda(1-m_C(z))}}{p+(1-p)e^{-\lambda\alpha(1-m_C(z))}}\right)\Bigg) - n\pi z. \end{aligned}$$

By the assumption that $\alpha m_C(z) < 1$, thus we have $\sum_{i=0}^{n}(\alpha m_C(z))^i$ is the geometric sequence. Then we can rearrange the equation as follows.

$$\begin{aligned} c_n(z) &= \log\left(p+(1-p)e^{-\lambda\left(1-\left(\bar{\alpha}m_C(z)\left(\frac{1-(\alpha m_C(z))^{n-2}}{1-\alpha m_C(z)}\right)+m_C(z)(\alpha m_C(z))^{n-1}\right)\right)}\right) \\ &+ \sum_{j=2}^{n-1}\log\left(\frac{p+(1-p)e^{-\lambda\left(1-\left(\bar{\alpha}m_C(z)\left(\frac{1-(\alpha m_C(z))^{n-(j+1)}}{1-\alpha m_C(z)}\right)+m_C(z)(\alpha m_C(z))^{n-j}\right)\right)}}{p+(1-p)e^{-\lambda\alpha\left(1-\left(\bar{\alpha}m_C(z)\left(\frac{1-(\alpha m_C(z))^{n-(j+1)}}{1-\alpha m_C(z)}\right)+m_C(z)(\alpha m_C(z))^{n-j}\right)\right)}}\right) \\ &+ \log\left(\frac{p+(1-p)e^{-\lambda(1-m_C(z))}}{p+(1-p)e^{-\lambda\alpha(1-m_C(z))}}\right) - n\pi z. \end{aligned}$$

Finally, we thus obtain the adjustment coefficient function is given by

$$\begin{aligned}
c(z) &= \lim_{n\to\infty} \frac{1}{n} c_n(z) - \pi z \\
&= \lim_{n\to\infty} \frac{1}{n} \left(\log \left(p + (1-p) e^{-\lambda \left(1 - \left(\bar{\alpha} m_C(z) \left(\frac{1-(\alpha m_C(z))^{n-2}}{1-\alpha m_C(z)} \right) + m_C(z)(\alpha m_C(z))^{n-1} \right)\right)} \right) \right) \\
&\quad + \lim_{n\to\infty} \frac{1}{n} \sum_{j=2}^{n-1} \log \left(\frac{p + (1-p) e^{-\lambda \left(1 - \left(\bar{\alpha} m_C(z) \left(\frac{1-(\alpha m_C(z))^{n-(j+1)}}{1-\alpha m_C(z)} \right) + m_C(z)(\alpha m_C(z))^{n-j} \right)\right)}}{p + (1-p) e^{-\lambda\alpha \left(1 - \left(\bar{\alpha} m_C(z) \left(\frac{1-(\alpha m_C(z))^{n-(j+1)}}{1-\alpha m_C(z)} \right) + m_C(z)(\alpha m_C(z))^{n-j} \right)\right)}} \right) \\
&\quad + \lim_{n\to\infty} \frac{1}{n} \log \left(\frac{p + (1-p) e^{-\lambda(1-m_C(z))}}{p + (1-p) e^{-\lambda\alpha(1-m_C(z))}} \right) - \pi z.
\end{aligned}$$

Since $\alpha m_C(z) < 1$, then the limit of $(\alpha m_C(z))^n$ as n approaches to infinity is a zero value, for the first and third terms of $c(\cdot)$, their limit approach to zero and for the second term, we then apply the Cesàro mean theorem (Peyerimhoff, 1969). Hence, we obtain

$$c(z) = \log \left(\frac{p + (1-p) e^{-\lambda\left(1 - \frac{\bar{\alpha} m_C(z)}{1-\alpha m_C(z)}\right)}}{p + (1-p) e^{-\lambda\alpha\left(1 - \frac{\bar{\alpha} m_C(z)}{1-\alpha m_C(z)}\right)}} \right) - \pi z.$$

□

The premium per period, π, follows the net profit condition (NPC) (Thomas, 2009) condition and premium calculation followed the expectation value principle (EVP) (Gray and Pitts, 2012) as follows.

$$\begin{aligned}
\pi &= \mathrm{E}(W)(1+\theta) \\
&= \mathrm{E}(N)\mathrm{E}(C)(1+\theta) \\
&= \lambda(1-p)\mathrm{E}(C)(1+\theta),
\end{aligned}$$

for a security loading $\theta > 0$, $\mathrm{E}(W)$ is the expectation of the aggregate claim size, $\mathrm{E}(N)$ is the expectation of the claim number and $\mathrm{E}(C)$ is the expectation of claim size. Next, we will show that the adjustment coefficient has the unique positive zero root in $\mathbb{R}^+$.

Lemma 4.6. From the expression for the adjustment coefficient function of the ZIPAR(1), the equation $c(z) = 0$ has the unique positive solution in $\mathbb{R}^+$.

Proof. Similar to Lemma 3.7 to prove the Lemma, then we will show that

(a) $c(0) = 0$,

(b) $\left.\frac{d}{dz}c(z)\right|_{z=0} < 0$,

(c) $\frac{d^2}{dz^2}c(z) > 0$, for $z \in \mathbb{R}^+$,

(d) There exists $z^* \in D$ such that $\lim_{z\to z^*} c(z) = +\infty$.

(a) Note that

$$c(z) = \log\left(\frac{p+(1-p)e^{-\lambda\left(1-\frac{\bar{\alpha}m_C(z)}{1-\alpha m_C(z)}\right)}}{p+(1-p)e^{-\lambda\alpha\left(1-\frac{\bar{\alpha}m_C(z)}{1-\alpha m_C(z)}\right)}}\right) - \pi z.$$

We substitute $z = 0$ into $c(z)$ defined in Theorem 4.5, then we obtain

$$\begin{aligned} c(0) &= \log\left(\frac{p+(1-p)e^{-\lambda\left(1-\frac{\bar{\alpha}m_C(0)}{1-\alpha m_C(0)}\right)}}{p+(1-p)e^{-\lambda\alpha\left(1-\frac{\bar{\alpha}m_C(0)}{1-\alpha m_C(0)}\right)}}\right) - \pi(0) \\ &= \log\left(\frac{p+(1-p)}{p+(1-p)}\right) \\ &= 0. \end{aligned}$$

Before giving the proof of the statements (b), (c) and (d), we define the notations that helps to simplify the notations as follows.

$$\begin{aligned} E(z) &= (1-p)e^{-\lambda\left(1-\frac{\bar{\alpha}m_C(z)}{1-\alpha m_C(z)}\right)}, \\ E_\alpha(z) &= (1-p)e^{-\lambda\alpha\left(1-\frac{\bar{\alpha}m_C(z)}{1-\alpha m_C(z)}\right)}, \\ E'(z) &= \frac{d}{dz}E(z) \\ &= E(z)T(z). \end{aligned}$$

Therefore,

$$\begin{aligned}
E_\alpha^{'}(z) &= \frac{d}{dz}E_\alpha(z)\\
&= \alpha E_\alpha(z)T(z),\\
E^{''}(z) &= \frac{d^2}{dz^2}E(z)\\
&= E(z)T^{'}(z) + E^{'}(z)T(z)\\
&= E(z)T^{'}(z) + E(z)T^2(z),\\
E_\alpha^{''}(z) &= \frac{d^2}{dz^2}E_\alpha(z)\\
&= \alpha E_\alpha(z)T^{'}(z) + \alpha E_\alpha^{'}(z)T(z)\\
&= \alpha E_\alpha(z)T^{'}(z) + \alpha^2 E_\alpha(z)T^2(z),
\end{aligned}$$

where

$$\begin{aligned}
T(z) &= \frac{\bar{\alpha}m_C^{'}(z)\lambda}{(1-\alpha m_C(z))^2},\\
T^{'}(z) &= \frac{(1-\alpha m_C(z))\lambda\bar{\alpha}m_C^{''}(z) + 2\lambda\alpha\bar{\alpha}(m_C^{'}(z))^2}{(1-\alpha m_C(z))^3}.
\end{aligned}$$

Moreover, we notice that $E(z) > 0$, $E_\alpha(z) > 0$, $E^{'}(z) > 0$, $E_\alpha^{'}(z) > 0$, $E^{''}(z) > 0$ and $E_\alpha^{''}(z) > 0$ with $T(z) > 0$ and $T^{'}(z) > 0$.

(b) Consider

$$\begin{aligned}
\frac{d}{dz}c(z) &= \frac{d}{dz}\log\left(\frac{p+E(z)}{p+E_\alpha(z)}\right) - \pi\\
&= \left(\frac{p+E_\alpha(z)}{p+E(z)}\right)\left(\frac{(p+E_\alpha(z))E^{'}(z) - (p+E(z))E_\alpha^{'}(z)}{(p+E_\alpha(z))^2}\right) - \pi.
\end{aligned}$$

Since we have $\pi = \lambda(1-p)\mathrm{E}(C)(1+\theta)$, then for $\theta > 0$,

$$\begin{aligned}
\frac{d}{dz}c(z)\Big|_{z=0} &= \left(\frac{p+E_\alpha(0)}{p+E(0)}\right)\left(\frac{(p+E_\alpha(0))E'(0)-(p+E(0))E'_\alpha(0)}{(p+E_\alpha(0))^2}\right)-\pi \\
&= \left(\frac{p+E_\alpha(0)}{p+E(0)}\right)\left(\frac{(p+E_\alpha(0))E(0)T(0)-(p+E(0))\alpha E(0)T(0)}{(p+E_\alpha(0))^2}\right)-\pi \\
&= (1-p)\frac{\lambda\bar{\alpha}m'_C(0)}{\bar{\alpha}^2}-(1-p)\frac{\lambda\alpha\bar{\alpha}m'_C(0)}{\bar{\alpha}^2}-(1+\theta)\lambda(1-p)E(C) \\
&= \lambda(1-p)\mathrm{E}(C)-\lambda(1-p)\mathrm{E}(C)(1+\theta) \\
&= -\lambda(1-p)\mathrm{E}(C)\theta \\
&< 0.
\end{aligned}$$

Then, we obtain that $\frac{d}{dz}c(z)\Big|_{z=0} < 0$.

(c) Consider

$$
\begin{aligned}
\frac{d^2}{dz^2}c(z) &= \frac{d}{dz}\left(\frac{d}{dz}c(z)\right) \\
&= \frac{d}{dz}\left(\left(\frac{(p+E_\alpha(z))E^{'}(z)-(p+E(z))E_\alpha^{'}(z)}{(p+E_\alpha(z))(p+E(z))}\right)-\pi\right) \\
&= \frac{(p+E_\alpha(z))(p+E(z))\left(pE^{''}(z)+E_\alpha(z)E^{''}(z)-pE_\alpha^{''}(z)-E(z)E_\alpha^{''}\right)}{((p+E_\alpha(z))(p+E(z)))^2} \\
&\quad -\frac{(pE^{'}(z)+E^{'}(z)E_\alpha(z)-pE_\alpha^{'}(z)-E(z)E_\alpha^{'}(z))\left(pE^{'}(z)+pE_\alpha^{'}(z)\right)}{((p+E_\alpha(z))(p+E(z)))^2} \\
&\quad -\frac{(pE^{'}(z)+E^{'}(z)E_\alpha(z)-pE_\alpha^{'}(z)-E(z)E_\alpha^{'}(z))\left(E(z)E_\alpha^{'}(z)+E^{'}(z)E_\alpha(z)\right)}{((p+E_\alpha(z))(p+E(z)))^2} \\
&= \frac{p^3(E^{''}(z)-E_\alpha^{''}(z))+p^2(E_\alpha(z)E^{''}(z)-E(z)E_\alpha^{''}(z)+(z))}{((p+E_\alpha(z))(p+E(z)))^2} \\
&\quad +\frac{p^3(E(z)E^{''}(z)-E(z)E_\alpha^{''}(z)+E_\alpha(z)E^{''}(z))+p^2(-E_\alpha E_\alpha^{''}(z)-E^{'2}(z)+E_\alpha^{'2})}{((p+E_\alpha(z))(p+E(z)))^2} \\
&\quad +\frac{p(E(z)E_\alpha(z)E^{''}(z)-E^2(z)E_\alpha^{''}(z)+E_\alpha^2(z)E^{''}(z))}{((p+E_\alpha(z))(p+E(z)))^2} \\
&\quad +\frac{p(-E(z)E_\alpha(z)E_\alpha^{''}(z)+E(z)E_\alpha(z)E^{''}(z)-E(z)E_\alpha(z)E_\alpha^{''}(z))}{((p+E_\alpha(z))(p+E(z)))^2} \\
&\quad +\frac{p(-E^{'2}(z)E_\alpha(z)-E^{'2}(z)E_\alpha(z))}{((p+E_\alpha(z))(p+E(z)))^2} \\
&\quad +\frac{p(E(z)E_\alpha^{'2}(z)+E(z)E_\alpha^{'2}(z))+(E(z)E_\alpha^2(z)E^{''}(z)-E^2(z)E_\alpha(z)E_\alpha^{''}(z))}{((p+E_\alpha(z))(p+E(z)))^2} \\
&\quad +\frac{(-E^{'2}(z)E_\alpha^2(z))+E^2(z)E_\alpha^{'2}(z))}{((p+E_\alpha(z))(p+E(z)))^2} \\
&= \frac{p^3(E^{''}(z)-E_\alpha^{''}(z))+p^2(2E(z)E_\alpha(z)(T^{'}(z)+T^2(z)-\alpha(T^{'}(z)+\alpha T^2(z))))}{((p+E_\alpha(z))(p+E(z)))^2} \\
&\quad +\frac{p(T^{'}(z)(1-\alpha)(E^2(z)E_\alpha(z)+E(z)E_\alpha^2(z))+E(z)E_\alpha^2(z)T^2(z)(1-\alpha^2))}{((p+E_\alpha(z))(p+E(z)))^2} \\
&\quad +\frac{E(z)E_\alpha(z)\left(E(z)E_\alpha(z)T^{'}(z)(1-\alpha)-p\alpha^2T^2(z)(E(z)-E_\alpha(z))\right)}{((p+E_\alpha(z))(p+E(z)))^2} \\
&\quad +\frac{p(E(z)E_\alpha(z)T^{'}(z)(E(z)-\alpha E_\alpha(z))}{((p+E_\alpha(z))(p+E(z)))^2}.
\end{aligned}
$$

Since the assumption $\alpha m_C(z) < 1$ and $T(z)$, $T^{'}(z)$, $E(z)$ and $E_\alpha(z)$ which are increasing functions and we know that $E(z)-E_\alpha(z) > 0$. Then for $\alpha \in (0,1)$, we know that $1-\alpha$ and $1-\alpha^2$ are greater than 0. For the third term, we notice that $E(z) - E_\alpha(z)$ is close to zero when α is growing up and on top of that it is weighted by α^2 and p, then the third term is positive. Hence, we can conclude that $\frac{d^2}{dz^2}c(z) > 0$.

(d) We want to show that the limit of $c(z)$ reaches to $+\infty$ as z approaches to some $z^* \in \mathbb{R}^+$. Let us first consider

$$f(z) \;=\; \lambda\left(\frac{\bar{\alpha}m_C(z)}{1-\alpha m_C(z)} - 1\right) \qquad \text{for } z \in \mathbb{R}^+.$$

Next, we will show that $f(z)$ is the nonnegative function and the increasing function by considering as follows.

$$\frac{\bar{\alpha}m_C(z)}{1-\alpha m_C(z)} - 1 \;=\; \frac{m_C(z)-1}{1-\alpha m_C(z)}.$$

We then follow the assumption that $\alpha m_C(z) < 1$ and also hold $1 - \alpha m_C(z) > 0$, then we obtain $f(z)$ for $z \in \mathbb{R}^+$ as the nonnegative function. Since $m_C(z)$ is increasing function in $\mathbb{R}^+$ and $0 < m_C(z) < \frac{1}{\alpha}$. Thus, there exists $z^* \in D$ such that

$$\lim_{z \to z^*} m_C(z) \;=\; \frac{1}{\alpha}.$$

Then, we obtain that $1 - \alpha m_C(z)$ is decreasing and continuous function. We also obtain

$$\lim_{z \to z^*} 1 - \alpha m_C(z) \;=\; 0,$$

and $1 - \alpha m_C(z) \geq 0$ for all $0 \leq z \leq z^*$. Therefore,

$$\lim_{z \to z^*} f(z) \;=\; \infty.$$

Consequently,

$$\lim_{z \to z^*} \frac{p + (1-p)e^{f(z)}}{p + (1-p)e^{\alpha f(z)}} \;=\; \lim_{z \to z^*} e^{(1-\alpha)f(z)} \;=\; \infty,$$

then, we obtain

$$\lim_{z\to z^*}\log\left(\frac{p+(1-p)e^{f(z)}}{p+(1-p)e^{\alpha f(z)}}\right) = \infty.$$

Hence, we can conclude that

$$\lim_{z\to z^*}\log\left(\frac{p+(1-p)e^{f(z)}}{p+(1-p)e^{\alpha f(z)}}\right) - \pi z = \infty.$$

□

Example 4.1. We let the claim amounts follow the exponential distribution. That is $\{C_{i,j}, i, j \in \mathbb{N}\}$ is a sequence of i.i.d. exponentially distributed with parameter $\beta > 0$. The moment generating function of $\{C_{i,j}, i, j \in \mathbb{N}\}$ is denoted as $m_C(z) = \frac{1}{1-z/\beta}$ for $z < \beta$. By Theorem (4.5), the adjustment coefficient function is provided as follows.

$$c(z) = \log\left(\frac{p+(1-p)e^{-\lambda\left(1-\frac{\bar{\alpha}(1-z/\beta)}{1-\alpha(1-z/\beta)}\right)}}{p+(1-p)e^{-\lambda\alpha\left(1-\frac{\bar{\alpha}(1-z/\beta)}{1-\alpha(1-z/\beta)}\right)}}\right) - (1-p)\frac{\lambda}{\beta}(1+\theta)z. \quad (4.15)$$

4.1.2 Approximation to the value at risk and the tail value at risk of ZIPAR(1)

In this section, we conduct the approximation to the value at risk and the tail value at risk at confidence level γ for ZIPAR(1) process by the similar techniques as in ZIPMA(1). Therefore, we consider the characteristic function of S_n as follows.

$$\begin{aligned}
\phi_{S_n}(x) &= G_{N(n)}(\phi_C(x)) \\
&= \left(p+(1-p)e^{-\lambda\left(1-\left(\bar{\alpha}\left(\sum_{i=0}^{n-2}\alpha^i\phi_C(x)^{i+1}\right)+\alpha^{n-1}\phi_C(x)^n\right)\right)}\right) \\
&\quad \times \prod_{j=2}^{n-1}\left(\frac{p+(1-p)e^{-\lambda\left(1-\left(\bar{\alpha}\left(\sum_{i=0}^{n-(j+1)}\alpha^i\phi_C(x)^{i+1}\right)+\alpha^{n-j}\phi_C(x)^{n-(j-1)}\right)\right)}}{p+(1-p)e^{-\lambda\alpha\left(1-\left(\bar{\alpha}\left(\sum_{i=0}^{n-(j+1)}\alpha^i\phi_C(x)^{i+1}\right)+\alpha^{n-j}\phi_C(x)^{n-(j-1)}\right)\right)}}\right) \\
&\quad \times \left(\frac{p+(1-p)e^{-\lambda(1-\phi_C(x))}}{p+(1-p)e^{-\lambda\alpha(1-\phi_C(x))}}\right),
\end{aligned}$$

where $x \in \mathbb{R}^+$.

4.1.3 Numerical experiments of the risk model based on ZIPAR(1)

In this section, we show some examples of numerical calculations of the adjustment coefficient and approximation to the ruin probability of a risk model based on the ZIPAR(1) claim count process. In addition, the two risk measurements of 12^{th} period of time at the confidence levels 0.9 and 0.95 are also provided.

4.1.4 Calculation of the adjustment coefficient of the risk model based on ZIPAR(1)

We are setting the components of the risk model as follows; $\{N_i, i \in \mathbb{N}\}$ is the ZIPAR(1) model, $\{C_{i,j}, i, j \in \mathbb{N}\}$ is a sequence of i.i.d. exponentially distributed with parameter β and we obtain $c(z)$ as in Example 4.1. The parameters setting are $u = 2$, $(\lambda, p) = (1.5, 0.2)$ and the security loading $\theta = 0.3$. Table 4.1, Figures 4.1 - 4.2 show the adjustment coefficient z_0 and the approximation of of the ruin probability as $\Psi(u) = \exp(-z_0 u)$ in parentheses, for different values of $\alpha \in \{0, 0.25, 0.5, 0.75, 0.995\}$.

Table 4.1: The adjustment coefficient z_0 and the approximation of $\Psi_{R_n}(u)$ of ZIAR(1).

β \ α	0	0.25	0.5	0.75	0.995
0.5	0.1016	0.7401	0.0479	0.0235	0.0007
	(0.8160)	(0.8623)	(0.9085)	(0.9540)	(0.9986)
1	0.2032	0.1494	0.0957	0.0469	0.0013
	(0.6660)	(0.7415)	(0.8256)	(0.9103)	(0.9973)
2	0.4063	0.2989	0.1914	0.0938	0.0025
	(0.4436)	(0.5500)	(0.6818)	(0.8288)	(0.9948)
4	0.8125	0.5977	0.3828	0.1875	0.0050
	(0.1960)	(0.3025)	(0.4649)	(0.6871)	(0.9899)
32	6.5000	4.7813	3.0625	1.5001	0.0401
	(0.000002)	(0.00007)	(0.0022)	(0.0498)	(0.9229)

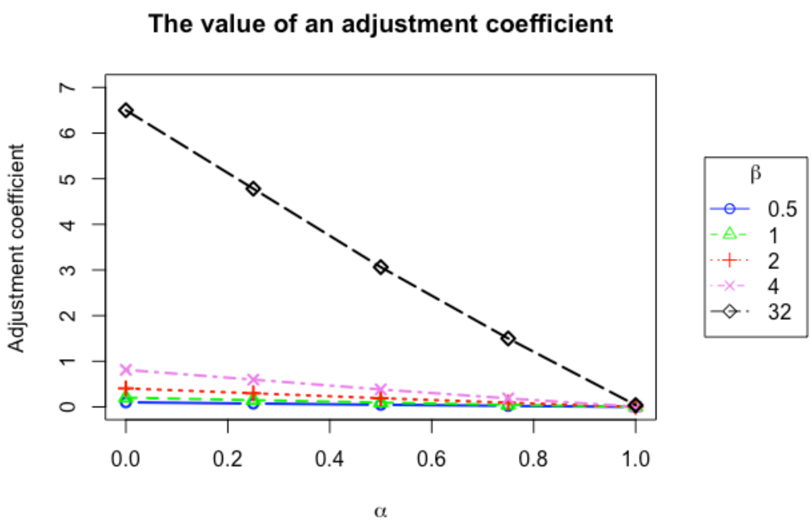

Figure 4.1: The trend of the adjustment coefficient when α increases and the claim size decreases of ZIPAR(1).

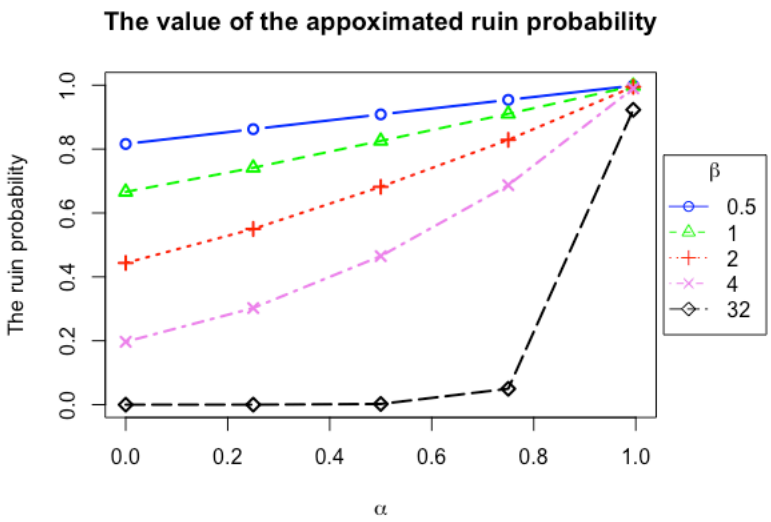

Figure 4.2: The trend of the ruin probability according to the changes of α_1 and α_2 of ZIPAR(1).

The results are as we would expect that the estimated ruin probability increases with the dependence parameter α is growing up. In other words, α is represented as the probability of the former portfolio will reclaim again in next year. The more value of α, the more impact on the current portfolio. Moreover, we are given the situations that claim sizes become smaller, then the approximate to the ruin probability decreases.

4.1.5 Calculation of the value at risk and the tail value at risk for the risk models based on ZIPAR(1)

We conduct the numerical calculation for the two risk measurements that are the value at risk and the tail value at risk. The setting parameters are the same as in section 4.1.4 with selecting $\beta = 4$. Table 4.2 and Figure 4.3 show $\mathrm{VaR}_\gamma(S_{12})$ and $\mathrm{TVaR}_\gamma(S_{12})$ for the confidence levels $\gamma = 0.90$ and 0.95, respectively.

Table 4.2: The value of the value at risk and the tail value at risk of ZIPAR(1).

α	0	0.25	0.5	0.75	1
$\mathrm{VaR}_{0.90}(S_{12})$	5.5200	5.8200	6.3000	7.1600	9.0600
$\mathrm{VaR}_{0.95}(S_{12})$	6.1800	6.6200	7.2800	8.5200	11.0600
$\mathrm{TVaR}_{0.90}(S_{12})$	6.4082	6.8863	7.6366	8.9965	11.7707
$\mathrm{TVaR}_{0.95}(S_{12})$	7.0010	7.5938	8.5287	10.2214	13.5804

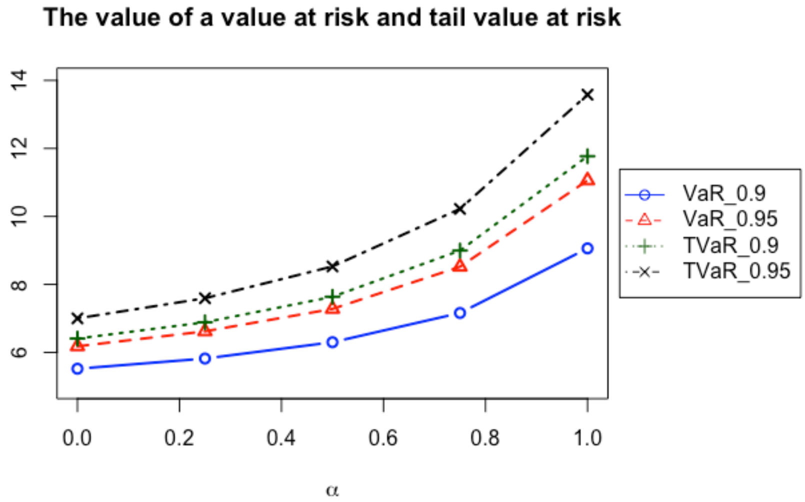

Figure 4.3: The trend of the value at risk and the tail value at risk according to the changes of α_1 and α_2 of ZIPAR(1).

From Table 4.2, we can see that the $\mathrm{VaR}_\gamma(S_n)$ increases as α increases. Similarly, $\mathrm{VaR}_\gamma(S_n)$ increases as γ increases. The interpretation of the increasing of value α and γ are likewise in ZIPMA(1) and ZIPMA(q).

CHAPTER V

CONCLUSIONS AND DISCUSSIONS

5.1 Conclusions

This research aims to construct the classical risk model based on zero inflated Poisson time series as a claim count process. According to the behavior of customers with deductible amount in contracts tend to not state the claims that less than or equal to deductible amount in order to get discount in premiums in the next year. Consequently, it generated more zero claims in data than expected. By analysing the insurance data issues in an excess zeros that caused overdispersion in the data, this thesis shows how to tackle this problems. To overcome this issues, we proposed the zero inflated Poisson time series such as the first order zero inflated Poisson moving average ZIPMA(1), the q^{th} order zero inflated Poisson moving average ZIPMA(q) and the first order zero inflated Poisson autoregressive ZIPAR(1) as claim counts model in the classical risk models and generally extended ZIPMA(1) to be more practical model as ZIPMA(q). We found that these new risk models are appropriate for the overdispersion data. Regarding to the variances that are greater than the expectations. We also provided the derivation of the adjustment coefficient functions of ZIPMA(1), ZIPMA(q) and ZIPAR(1) risk models and prove the existence of their unique positive solutions. We present a method for calculating the value of the ruin probability, the value at risk and the tail value at risk. Finally, we compare the result from ZIPMA(1), ZIPMA(2), ZIPMA(3) and ZIPAR(1). The value of $\alpha_{MA} = \alpha_{AR} = \{0, 0.25, 0.5, 0.75, 0.995\}$ and we set up the value of α_1,α_2 and α_3 from ZIPMA(2) and ZIPMA(3) are that $\alpha_1 = \alpha_2 = \alpha_3 = \{0, 0.25, 0.5, 0.75, 0.995\}$ in order to compare with ZIPMA(1) and ZIPAR(1).

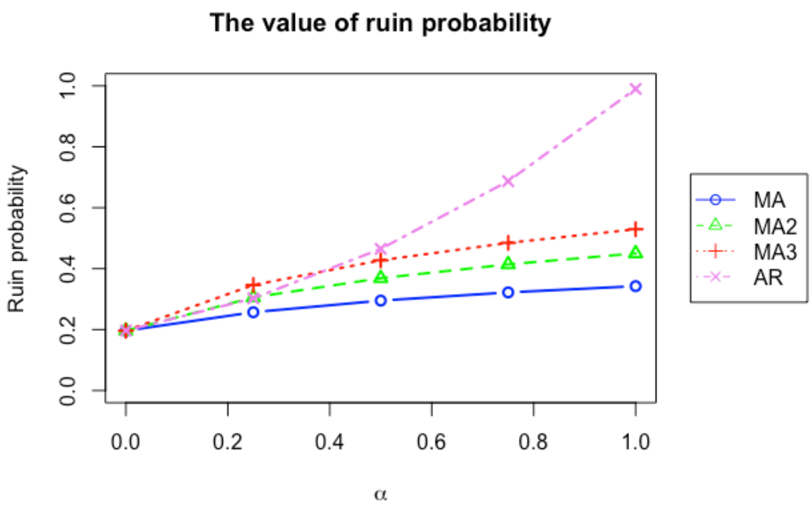

Figure 5.1: The ruin probability from ZIPMA versus ZIPAR

Figure 5.1 shows that the value of ruin probability from ZIPMA is growing up with the higher order. The higher order of ZIPMA means that we have the number of new claims from more previous periods and if we have the number of new claims from every previous periods in insurance data, then the whole data is applied, then it will result that in a higher order of ZIPMA, the value of ruin probability will approach to ZIPAR(1).

5.2 Future Work

Further research is needed to determine the risk sharing between 2 companies or more than 3 companies. Regarding to the real world, most of insurance business is basically doing activity such risk diversification as reinsurance. Then, if we can find the ruin probability between 2 companies or more than 3 companies such as the insurance company and reinsurance company, then it can be one the options to make a decision for financial planning or business strategies. Thus, one direction of future study is to consider multivariate zero inflated Poisson time series or another model to solve the data issues.

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APPENDIX

BIOGRAPHY

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